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LABELLING CAYLEY GRAPHS ON ABELIAN GROUPS*

SANMING ZHOU†

In memory of Xin-Bang Yan who turned on my interest in mathematics

Abstract. For given integers $j \geq k \geq 1$, an $L(j,k)$-labelling of a graph Γ is an assignment of labels—nonnegative integers—to the vertices of Γ such that adjacent vertices receive labels that differ by at least j, and vertices distance two apart receive labels that differ by at least k. The span of such a labelling is the difference between the largest and the smallest labels used, and the minimum span over all $L(j,k)$-labellings of Γ is denoted by $\lambda_{j,k}(\Gamma)$. The minimum number of labels needed in an $L(j,k)$-labelling of Γ is independent of j and k, and is denoted by $\mu(\Gamma)$. In this paper we introduce a general approach to $L(j,k)$-labelling Cayley graphs Γ over Abelian groups and deriving upper bounds for $\lambda_{j,k}(\Gamma)$ and $\mu(\Gamma)$. Using this approach we obtain upper bounds on $\lambda_{j,k}(\Gamma)$ and $\mu(\Gamma)$ for graphs Γ admitting a vertex-transitive Abelian group of automorphisms. Hypercubes Q_d are examples of such graphs, and as consequences we obtain upper bounds for $\lambda_{j,k}(Q_d)$ and $\mu(Q_d)$. We also obtain the exact values of $\lambda_{j,k}(\Gamma)$ $(2k \geq j \geq k)$ and $\mu(\Gamma)$ for some Hamming graphs Γ. The result shows that, under certain arithmetic conditions, these two invariants rely only on k and the orders of the two largest complete graph factors of the Hamming graph.

Key words. channel assignment, $L(j,k)$-labelling, $\lambda_{j,k}$-number, λ-number, radio chromatic number, Cayley graph, hypercube, Hamming graph

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1. Introduction.

In the *channel assignment problem* [13] one wishes to assign channels to the transmitters in a radio communication system such that interference is avoided as much as possible. For this purpose various constraints have been proposed [13, 22] to put on the channel separations between pairs of transmitters within certain distance. It is suggested [11] that “close” transmitters be assigned channels at least k apart, and “very close” transmitters be assigned channels at least j apart, where j and k are given integers with $j \geq k \geq 1$. Since bandwidth is a limited resource, a major concern is to minimize the span of channels used. Taking channels as nonnegative integers, this problem can be formulated as a labelling problem [7, 11] for the corresponding interference graph. More explicitly, for a graph $\Gamma = (V(\Gamma), E(\Gamma))$ with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, a mapping f from $V(\Gamma)$ to $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ is called [7, 11] an $L(j,k)$-*labelling* of Γ if, for any $u, v \in V(\Gamma)$,

$$d_\Gamma(u,v) = 1 \Rightarrow |f(u) - f(v)| \geq j$$

and

$$d_\Gamma(u,v) = 2 \Rightarrow |f(u) - f(v)| \geq k,$$

where $d_\Gamma(u,v)$ is the distance in Γ between u and v. The integer $f(u)$ is called the *label* of u under f, and $\mathrm{sp}(\Gamma; f) = \max_{u \in V(\Gamma)} f(u) - \min_{v \in V(\Gamma)} f(v)$ the *span* of

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http://www.siam.org/journals/sidma/19-4/40445.html
†Department of Mathematics and Statistics, The University of Melbourne, Parkville, VIC 3010, Australia (smzhou@ms.unimelb.edu.au).

f. Without loss of generality *we will always assume that the smallest label used by an* $L(j,k)$*-labelling is* 0. With this convention the span of f is equal to the largest label used by f, that is, $\mathrm{sp}(\Gamma; f) = \max_{u \in V(\Gamma)} f(u)$. The $\lambda_{j,k}$*-number* of Γ, denoted by $\lambda_{j,k}(\Gamma)$, is defined [7, 11] to be the minimum span of all $L(j,k)$-labellings of Γ. Usually, $\lambda_{2,1}(\Gamma)$ is called [11] the λ*-number* of Γ and is denoted by $\lambda(\Gamma)$.

A relevant invariant is the minimum number $\mu_{j,k}(\Gamma)$ of labels needed in an $L(j,k)$-labelling of Γ. This invariant is actually independent of choice of j and k [26], that is, for any $j \geq k \geq 1$,

$$\mu_{j,k}(\Gamma) = \mu_{1,1}(\Gamma). \tag{1}$$

In fact, since $j \geq k \geq 1$, any $L(j,k)$-labelling of Γ is an $L(1,1)$-labelling of Γ, and hence $\mu_{1,1}(\Gamma) \leq \mu_{j,k}(\Gamma)$. On the other hand, for any $L(1,1)$-labelling of Γ using $\mu_{1,1}(\Gamma)$ labels, by multiplying the label of each vertex by j we obtain an $L(j,k)$-labelling of Γ which uses $\mu_{1,1}(\Gamma)$ labels. Therefore, we have $\mu_{j,k}(\Gamma) \leq \mu_{1,1}(\Gamma)$ and (1) follows. In the following we will denote $\mu_{1,1}(\Gamma)$ by $\mu(\Gamma)$. Thus, in view of (1), $\mu_{j,k}(\Gamma)$ is equal to $\mu(\Gamma)$ for any $j \geq k \geq 1$. An $L(j,k)$-labelling of Γ is said to be *optimal for* $\lambda_{j,k}$ if its span is $\lambda_{j,k}(\Gamma)$, and *optimal for* μ if it uses $\mu(\Gamma)$ distinct labels. In particular, an $L(2,1)$-labelling of Γ is *optimal for* λ if its span is $\lambda(\Gamma)$. Note that an $L(j,k)$-labelling of Γ which is optimal for $\lambda_{j,k}$ is not necessarily optimal for μ and vice versa.

The $L(j,k)$-labelling problem, in particular in the $L(2,1)$ case, has been studied extensively in the past more than one decade; see [2, 3, 4, 6, 7, 8, 9, 10, 11, 19, 22, 24, 29, 30] for examples. The $L(2,1)$-labelling problem was proposed [11] initially by Roberts in a personal communication to Griggs. Interestingly, according to [15], essentially the same concept was also introduced by Harary in a private communication [14]. In fact, if we view labels as colors, then an $L(2,1)$-labelling is a *radio coloring* in the sense of [14, 15] and vice versa. In [14, 15], the minimum n for which there exists a radio coloring of Γ using colors from $\{1, 2, \ldots, n\}$ (not every color in $\{1, 2, \ldots, n\}$ needs to be used) is called the *radio coloring number* of Γ. Clearly, this number is exactly $\lambda(\Gamma) + 1$ for any graph Γ. In [14, 27] the minimum number of colors used in a radio coloring of Γ is called the *radio chromatic number* of Γ. From this definition it follows immediately that the radio chromatic number of Γ is exactly $\mu_{2,1}(\Gamma)$, and hence is equal to $\mu(\Gamma)$ by (1). Taking nonnegative integers as colors, a proper vertex coloring of the *square* Γ^2 of Γ is an $L(1,1)$-labelling of Γ and vice versa, where Γ^2 is defined to have vertex set $V(\Gamma)$ and edges joining distinct vertices of distance at most 2 in Γ. Thus, we have $\mu(\Gamma) = \chi(\Gamma^2)$, where χ is the chromatic number. Also, we notice that the invariant $\chi_{\bar{2}}(\Gamma)$ introduced in [28] is the same as $\mu(\Gamma)$.

In [11] Griggs and Yeh conjectured that $\lambda(\Gamma) \leq \Delta^2$ for any graph Γ with maximum degree $\Delta = \Delta(\Gamma) \geq 2$. In the same paper they proved that $\lambda(\Gamma) \leq \Delta^2 + 2\Delta$ for any graph Γ. This conjecture stimulated substantially the study of λ-number, and it was confirmed for quite a few classes of graphs, e.g., the class of graphs of diameter 2 considered in [11] and the class of chordal graphs [24]. For certain subclasses of chordal graphs the upper bound Δ^2 can be improved, as shown in [3]. For general graphs Γ, as far as we know, currently the best known bound is $\lambda(\Gamma) \leq \Delta^2 + \Delta - 1$ [20], which is an improvement of the bound $\lambda(\Gamma) \leq \Delta^2 + \Delta$ given in [3]. In the complexity aspect, Griggs and Yeh [11] proved that the $L(2,1)$-labelling problem is NP-complete for general graphs, and in contrast Chang and Kuo [3] gave a polynomial algorithm for trees.

The motivation of the present paper comes from the research [9, 29] on the λ-numbers of hypercubes and Hamming graphs. The *Cartesian product* $\Gamma_1 \Box \Gamma_2 \Box \cdots \Box \Gamma_d$

of $d \geq 2$ given graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_d$ is the graph with vertex set $V(\Gamma_1) \times V(\Gamma_2) \times \cdots \times V(\Gamma_d)$ such that $(\alpha_1, \alpha_2, \ldots, \alpha_d), (\beta_1, \beta_2, \ldots, \beta_d) \in V(\Gamma_1) \times V(\Gamma_2) \times \cdots \times V(\Gamma_d)$ are adjacent if and only if $\alpha_i \neq \beta_i$ for exactly one subscript i, and for such an i, α_i, β_i are adjacent in Γ_i. Let K_n denote the complete graph of order n. The Cartesian product

$$H_{n_1,n_2,\ldots,n_d} := K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_d}$$

of complete graphs is called a *Hamming graph*, where $n_i \geq 2$ for each $i = 1, 2, \ldots, d$. As a convention, when we write $H_{n_1,n_2,\ldots,n_d}$ we assume without loss of generality that

$$n_1 \geq n_2 \geq \cdots \geq n_d \geq 2.$$

In the case where $n_1 = n_2 = \cdots = n_d = n$, we use $H(d, n)$ in place of $H_{n_1,n_2,\ldots,n_d}$. Thus,

$$H(d, n) := K_n \Box K_n \Box \cdots \Box K_n \ \ (d \text{ factors}).$$

In particular,

$$Q_d := H(d, 2)$$

is called the *d-cube* (hypercube). By using coding theory, Whittlesey, Georges, and Mauro [29, Theorem 3.7] proved that, if $2^{n-1} \leq d \leq 2^n - q$ for some q between 1 and $n + 1$, then

$$\lambda(Q_d) \leq 2^n + 2^{n-q+1} - 2. \tag{2}$$

Recently, Georges, Mauro, and Stein [9] determined the λ-number of the Hamming graph $H(d, p^r)$ under the assumption that p is a prime, and either $d \leq p$ and $r \geq 2$, or $d < p$ and $r = 1$. They proved [9, Theorem 3.1] that, under these conditions,

$$\lambda(H(d, p^r)) = p^{2r} - 1. \tag{3}$$

In the same paper [9] they also determined the $\lambda_{j,k}$-number of H_{n_1,n_2}, and this work was extended to $H(3, n)$ in [8].

2. Main results. Stimulated by [9, 29], our initial attempt was to improve the bound (2) and determine the λ-number of general Hamming graphs $H_{n_1,n_2,\ldots,n_d}$. This led us to a general approach to $L(j, k)$-labelling Cayley graphs on Abelian groups, which can be used to produce upper bounds for the $\lambda_{j,k}$-number and the μ-number of such graphs. In this section we will outline this approach and present the main results of the paper; see Theorems 2.2, 2.5, and 2.9 and their corollaries below. We will leave a detailed discussion on the approach to section 3. The approach seems to be powerful enough to derive the exact value of, or good upper bounds for, the $\lambda_{j,k}$-number and the μ-number of some Cayley graphs. In this paper we will apply it to Hamming graphs and a family of graphs containing all hypercubes. As we will see, (2) and (3) are special cases of our much more general results for such graphs.

Let G be a group and X a subset of G. If $1 \notin X$ and $x \in X$ implies $x^{-1} \in X$, where 1 is the identity element of G, then we call X a *Cayley set* of G. For such an X, the *Cayley graph* of G with respect to X, denoted by $\Gamma(G, X)$, is the graph with vertices the elements of G in which $x, y \in G$ are adjacent if and only if $xy^{-1} \in X$. Since X is inverse-closed, $\Gamma(G, X)$ is well defined as an undirected simple graph. To exclude the less interesting case where $\Gamma(G, X) = K_{|G|}$ is a complete graph, *we will* *assume without mentioning explicitly that* $X \neq G - \{1\}$. As usual, for a normal subgroup H of G, we use G/H to denote the quotient group of G by H, and $|G:H|$ the order of G/H. For any subsets X, Y of G, denote $XY := \{xy : x \in X, y \in Y\}$ and set $X^2 := XX$. As usual we use $\langle X \rangle$ to denote the subgroup of G generated by X. Call X a *generating set* of G if $\langle X \rangle = G$.

The key concept for our approach is the following definition of avoidability. Note that, for any Cayley set X of a group G, we have $1 = xx^{-1} \in X^2$ by the assumption that X is closed under taking inverse.

DEFINITION 2.1. *Let G be a finite Abelian group and X a Cayley set of G. A subgroup H of G is said to* avoid X *if $H \cap X = \emptyset$ and $H \cap X^2 = \{1\}$.*

Regarding this concept a few observations will be given in Remark 3.1. The following theorem shows that, once a subgroup H avoiding X is known, we can obtain upper bounds for the $\lambda_{j,k}$-number and the μ-number of $\Gamma(G, X)$.

THEOREM 2.2. *Let $j \geq k \geq 1$ be integers. Let G be a finite Abelian group and X a Cayley set of G. Then, for any subgroup H of G which avoids X, we have*

$$\lambda_{j,k}(\Gamma(G,X)) \leq |G:H| \max\{k, \lceil j/2 \rceil\} + |G:\langle G - HX\rangle| \min\{j-k, \lfloor j/2 \rfloor\} - j \tag{4}$$

$$\mu(\Gamma(G,X)) \leq |G:H|. \tag{5}$$

A very important case occurs when $2k \geq j$. In this case we have $\max\{k, \lceil j/2 \rceil\} = k$, $\min\{j-k, \lfloor j/2 \rfloor\} = j-k$, and hence (4) becomes

$$\lambda_{j,k}(\Gamma(G,X)) \leq |G:H|k + |G:\langle G - HX\rangle|(j-k) - j. \tag{6}$$

In particular, for $L(2,1)$-labellings we have $2k = j = 2$ and hence Theorem 2.2 has the following consequence.

COROLLARY 2.3. *Let G be a finite Abelian group and X a Cayley set of G. Then, for any subgroup H of G which avoids X, we have*

$$\lambda(\Gamma(G,X)) \leq |G:H| + |G:\langle G - HX\rangle| - 2 \tag{7}$$

and

$$\mu(\Gamma(G,X)) \leq |G:H|.$$

An $L(j,k)$-labelling is called *no-hole* if the labels used by it consist of a set of consecutive integers. In the case where $G - HX$ is a generating set of G, we have $|G:\langle G - HX\rangle| = 1$, and Theorem 2.2 together with its proof implies the following result, which will be the main tool in our treatment of Hamming graphs.

COROLLARY 2.4. *Let $j \geq k \geq 1$ be integers. Let G be a finite Abelian group and X a Cayley set of G. Then, for any subgroup H of G which avoids X and is such that $G - HX$ generates G, we have*

$$\lambda_{j,k}(\Gamma(G,X)) \leq (|G:H| - 1)\max\{k, \lceil j/2 \rceil\}. \tag{8}$$

In particular,

$$\lambda(\Gamma(G,X)) \leq |G:H| - 1 \tag{9}$$

and $\Gamma(G,X)$ admits a no-hole $L(2,1)$-labelling using $|G:H|$ labels.

The class of Cayley graphs on Abelian groups is very large, and our results above apply to all such graphs universally. Because of this nature, it is unrealistic to expect that the bounds (4)–(9) are tight universally for all graphs in the class. However, as we will see later, for some Cayley graphs on Abelian groups they do produce sharp or near-sharp bounds for $\lambda_{j,k}$ and/or μ.

Let $\text{Aut}(\Gamma)$ denote the automorphism group of a graph Γ. A subgroup G of $\text{Aut}(\Gamma)$ is said to be *vertex-transitive* if, for any $\alpha, \beta \in V(\Gamma)$, there exists $g \in G$ such that g permutes α to β; such a group G is *regular* if there exists exactly one element g permuting α to β. The graph Γ is said to be *vertex-transitive* if $\text{Aut}(\Gamma)$ is vertex-transitive. Using our general approach we obtain the following theorem for the family of connected graphs with automorphism group containing a vertex-transitive Abelian subgroup. Hypercubes are examples of such graphs (see the discussion after Corollary 2.6); for existence and construction of other graphs in this family, the reader is referred to [16, 17, 18]. For any integer $d \geq 1$, denote

$$n := 1 + \lfloor \log_2 d \rfloor$$

and

$$t := \min\{2^n - d - 1, n\}.$$

Note that both n and t are functions of d. From the definition of n it follows that $2^{n-1} \leq d < 2^n$, that is, n is the smallest integer such that $d < 2^n$. This choice of n makes the following upper bounds (10)–(16) as small as possible.

THEOREM 2.5. *Let Γ be a connected graph whose automorphism group contains a vertex-transitive Abelian subgroup. Let d be the degree of vertices of Γ, and n, t be as above. Then, for any integers $j \geq k \geq 1$, we have*

$$\lambda_{j,k}(\Gamma) \leq 2^n \max\{k, \lceil j/2 \rceil\} + 2^{n-t} \min\{j-k, \lfloor j/2 \rfloor\} - j, \tag{10}$$

$$\mu(\Gamma) \leq 2^n. \tag{11}$$

As in (6), when $2k \geq j$, (10) becomes

$$\lambda_{j,k}(\Gamma) \leq 2^n k + 2^{n-t}(j-k) - j.$$

In particular, for $L(2,1)$-labellings, Theorem 2.5 implies the following corollary.

COROLLARY 2.6. *Let Γ and d be the same as in Theorem 2.5. Then*

$$\lambda(\Gamma) \leq 2^n + 2^{n-t} - 2 \tag{12}$$

and

$$\mu(\Gamma) \leq 2^n.$$

Note that Q_d is a Cayley graph on the elementary Abelian 2-group $\mathbb{Z}_2^d$ of order 2^d, namely $Q_d \cong \Gamma(\mathbb{Z}_2^d, X)$, where X is the set of elements of $\mathbb{Z}_2^d$ with exactly one nonzero coordinate. Thus, from [1, Lemma 16.3] it follows that Q_d admits $\mathbb{Z}_2^d$ as a vertex-transitive (regular, in fact) group of automorphisms. Since $\mathbb{Z}_2^d$ is Abelian, Theorem 2.5 and Corollary 2.6 imply the following two corollaries for Q_d.

COROLLARY 2.7. *Let d, j and k be integers with $d \geq 1$ and $j \geq k \geq 1$. Then*

$$\lambda_{j,k}(Q_d) \leq 2^n \max\{k, \lceil j/2 \rceil\} + 2^{n-t} \min\{j-k, \lfloor j/2 \rfloor\} - j \tag{13}$$

$$\mu(Q_d) \leq 2^n. \tag{14}$$

Moreover, the proof of Theorem 2.5 gives rise to a systematic way of generating $L(j,k)$-labellings of Q_d which use 2^n labels and have span the right-hand side of (13); see the last paragraph of section 4. Again, when $2k \geq j$, (13) becomes

$$\lambda_{j,k}(Q_d) \leq 2^n k + 2^{n-t}(j-k) - j.$$

In particular, for the λ-number of hypercubes, we have the following corollary.

COROLLARY 2.8. *For any integer $d \geq 1$, we have*

$$\lambda(Q_d) \leq 2^n + 2^{n-t} - 2 \quad ([29, \text{Theorem } 3.7]) \tag{15}$$

$$\mu(Q_d) \leq 2^n \quad ([28]). \tag{16}$$

The bounds (15) and (16) are equivalent to (2) and one of the main results of [28, line 12, pp. 185], respectively. To see this we distinguish the following two cases:

(i) $2^{n-1} \leq d \leq 2^n - n - 1$;

(ii) $2^n - n - 1 \leq d \leq 2^n - q$, for some q between 1 and n.

In case (i), $t = n$ and we may choose $q = n+1$ in (2); hence $t = q-1$ and (15) and (2) are identical. Also, in this case the upper bounds in (10) and (13) are $(2^n - 1)\max\{k, \lceil j/2 \rceil\}$, and that in (12) and (15) are $2^n - 1$. In case (ii), we have $q - 1 \leq 2^n - d - 1 \leq n$; hence $t = 2^n - d - 1$ and (15) and (2) are the same if we choose $q = 2^n - d$.

The bound (14) is tight when $d = 2^n - 1$. In fact, for any $d \geq 1$, since the d neighbors of the 0-labelled vertex of Q_d are distance two apart, they must be assigned distinct labels no less than j under any $L(j,k)$-labelling. Thus, $\mu(Q_d) \geq d+1$. In the case where $d = 2^n - 1$, we have $\mu(Q_d) \leq d+1$ by (14) and hence $\mu(Q_d) = d+1$, that is, (14) is sharp. Note that (15) implies $\lambda(Q_d) \leq 2d$, as noticed in [29, Theorem 3.8].

For Hamming graphs we obtain the following results by using Theorem 2.2.

THEOREM 2.9. *Let n_1, n_2, d be integers such that $n_1 > d \geq 2$, n_2 divides n_1, and each prime factor of n_1 is no less than d. Then, for any integers $j \geq k \geq 1$, and for any positive integers $n_3, \ldots, n_d$ which are less than or equal to n_2, we have*

$$\lambda_{j,k}(H_{n_1,n_2,\ldots,n_d}) \leq (n_1 n_2 - 1)\max\{k, \lceil j/2 \rceil\} \tag{17}$$

$$\mu(H_{n_1,n_2,\ldots,n_d}) = n_1 n_2 \tag{18}$$

and we can give explicitly an $L(j,k)$-labelling of $H_{n_1,n_2,\ldots,n_d}$ which has span $(n_1 n_2 - 1)\max\{k, \lceil j/2 \rceil\}$ and is optimal for μ. Furthermore, if in addition $2k \geq j$, then

$$\lambda_{j,k}(H_{n_1,n_2,\ldots,n_d}) = (n_1 n_2 - 1)k \tag{19}$$

and this $L(j,k)$-labelling is optimal for $\lambda_{j,k}$ and μ simultaneously.

Note that, in the case where $2k \geq j$, Theorem 2.9 gives the exact values of both $\lambda_{j,k}$ and μ for $H_{n_1,n_2,\ldots,n_d}$ above. It shows that the trivial lower bounds $\lambda_{j,k}(H_{n_1,n_2,\ldots,n_d}) \geq (n_1 n_2 - 1)k$ and $\mu(H_{n_1,n_2,\ldots,n_d}) \geq n_1 n_2$ (see Lemma 5.1) are both obtained. Another interesting feature is that both $\lambda_{j,k}$ and μ are irrelevant to j in this case: they rely on k, n_1, and n_2 only. In particular, for the $L(2,1)$ case we have the following corollary.

COROLLARY 2.10. *Let $n_1, n_2, n_3, \ldots, n_d$ and $d \geq 2$ be as in Theorem 2.9. Then*

$$\lambda(H_{n_1,n_2,\ldots,n_d}) = n_1 n_2 - 1 \tag{20}$$

$$\mu(H_{n_1,n_2,\ldots,n_d}) = n_1 n_2. \tag{21}$$

Moreover, we can give explicitly a no-hole $L(2,1)$-labelling of $H_{n_1,n_2,\ldots,n_d}$ which is optimal for λ and μ simultaneously.

For special Hamming graphs $H(d,n)$ (which is the graph K_n^d in [9]), Theorem 2.9 implies the following result.

COROLLARY 2.11. *Let $n = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$, where p_i is a prime and $r_i \geq 1$, for each $i = 1, 2, \ldots, t$. Let d be an integer such that $2 \leq d \leq p_i$ for each i and $\sum_{i=1}^{t}(p_i - d + r_i) \geq 2$. Then, for any integers $j \geq k \geq 1$, we have*

$$\lambda_{j,k}(H(d,n)) \leq (n^2 - 1)\max\{k, \lceil j/2 \rceil\}$$

and

$$\mu(H(d,n)) = n^2.$$

Moreover, if in addition $2k \geq j$, then

$$\lambda_{j,k}(H(d,n)) = (n^2 - 1)k. \tag{22}$$

The condition $\sum_{i=1}^{t}(p_i - d + r_i) \geq 2$ ensures that $n > d$, as required by Theorem 2.9. It is equivalent to either $t \geq 2$, or $t = 1$ and $p_1 - d + r_1 \geq 2$. In the latter case, $n = p^r$ is a prime power and (22) becomes $\lambda_{j,k}(H(d,p^r)) = (p^{2r} - 1)k$. For the $L(2,1)$ case, this gives $\lambda(H(d,p^r)) = p^{2r} - 1$, which is exactly (3). Also, we can get (3) from (20) directly. Thus, Corollaries 2.10–2.11 (and hence Theorem 2.9) generalize (3) to a wide extent.

Theorems 2.2, 2.5, and 2.9 will be proved in sections 3, 4, and 5, respectively. Remarks on the results above will be given in these sections as well. Concluding remarks and open questions arising from Theorem 2.9 will be offered in the last section.

3. Proof of Theorem 2.2. The terminology and notation for groups used in the paper are standard; see, for example, [25]. We will reserve the upper case English letters G, H for groups and the upper case Greek letters Γ, Σ for graphs. We will use certain lower case English letters such as g, h, u, v, w, x, y, z to denote elements of groups, but we reserve $d, i, j, k, \ell, m, n, r, s, t$ for integers. For two sets X and Y, $X - Y$ denotes the set $\{x \in X : x \notin Y\}$. For any graph Γ and a partition $\mathcal{P}$ of $V(\Gamma)$, the *quotient graph* $\Gamma_{\mathcal{P}}$ of Γ with respect to $\mathcal{P}$ is defined to have vertex set $\mathcal{P}$ in which two parts of $\mathcal{P}$ are adjacent if and only if there exists at least one edge of Γ joining a vertex in the first part to a vertex in the second part. In the case where each part of $\mathcal{P}$ is an independent set of Γ with ℓ vertices, for some integer $\ell \geq 1$, and the subgraph induced by two adjacent parts is a perfect matching of ℓ edges, the graph Γ is called an *ℓ-fold cover* of the quotient $\Gamma_{\mathcal{P}}$.

Let G be a finite group. For an element x of G, we will use $o(x)$ to denote the *order* of x in G, that is, the smallest positive integer n such that $x^n = 1$. The element x is called an *involution* if $o(x) = 2$. For a Cayley set X of G, from the definition of $\Gamma(G,X)$ it follows that $x, y \in G$ are connected by a path of $\Gamma(G,X)$ if and only if $xy^{-1} \in \langle X\rangle$; in particular $\Gamma(G,X)$ is a connected graph if and only if $\langle X\rangle = G$. Moreover, $\Gamma(G,X)$ is vertex-transitive and G is isomorphic to a regular subgroup of the automorphism group of $\Gamma(G,X)$ (see, e.g., [1, Theorem 16.4]). In particular, all vertices of $\Gamma(G,X)$ have the same degree, which is equal to $|X|$. For a normal subgroup H of G, the quotient group G/H gives rise to a natural partition of G with parts the cosets Hg of H in G. We will use the same notation G/H for this partition. Denote $X/H := \{Hx : x \in X\}$. Then

$$X/H = \{Hz \in G/H : Hz \cap X \neq \emptyset\}.$$

It should be noticed that X/H is not necessarily a subgroup of the quotient group G/H, and that $Hx \in X/H$ does not imply $x \in X$.

The idea behind our approach is rather natural: for a Cayley graph $\Gamma(G,X)$ on an Abelian group G, if we can find a subgroup H of G which "avoids" the Cayley set X, then we can label the elements in the same coset of H in G by the same label. In this way we get an $L(j,k)$-labelling of $\Gamma(G,X)$ and thus upper bounds for $\lambda_{j,k}(\Gamma(G,X))$ and $\mu(\Gamma(G,X))$. A very special case of this method for $L(2,1)$-labelling Hamming graphs $H(d,p^r)$ was used implicitly in the proof of [9, Theorem 3.1]. The approach proposed in the present paper is much more general and powerful. Before proceeding to the proof of Theorem 2.2, let us record the following observations about the concept of avoidability.

Remark 3.1. (a) The trivial subgroup $\{1\}$ avoids every Cayley set of G.

(b) The condition $H \cap X^2 = \{1\}$ implies that either $H \cap X = \emptyset$ or $H \cap X = \{x\}$ for an involution x of G. In fact, if $H \cap X \neq \emptyset$, then $xy = 1$ for any $x, y \in H \cap X$ (not necessarily distinct) since $xy \in H \cap X^2 = \{1\}$. That is, any two elements of $H \cap X$ are inverse of each other. From this it follows that $H \cap X = \{x\}$ for an involution x of G.

(c) Thus, if G contains no involutions, then H avoids X if and only if $H \cap X^2 = \{1\}$. This is the case in particular when, say, the order of G is odd.

To prove Theorem 2.2 we need some combinatorial properties of the Cayley graph $\Gamma(G,X)$ and its quotient graph $(\Gamma(G,X))_{G/H}$ with respect to the partition G/H, where $H \leq G$ avoids X. Define

$$\mathcal{G}_{H,X} := \{Hz \in G/H : Hz \cap X = \emptyset\}. \tag{23}$$

Since H avoids X, we have $H \cap X = \emptyset$, and hence $H \in \mathcal{G}_{H,X}$ and $H \subseteq G - HX$. (In fact, if $H \not\subseteq G - HX$, then $h_1 = h_2 x$ for some $h_1, h_2 \in H$, $x \in X$, and hence $h_2^{-1}h_1 = x \in H \cap X = \emptyset$, a contradiction.) Thus, $\mathcal{G}_{H,X} \neq \emptyset$ and $H \leq \langle G - HX\rangle$. Also, $Hz \in \mathcal{G}_{H,X}$ if and only if $x \notin Hz$ for all $x \in X$, which is true if and only if $Hz \neq Hx$ for all $x \in X$. Therefore, we have

$$\mathcal{G}_{H,X} = G/H - X/H = (G - HX)/H \tag{24}$$

and hence

$$\langle \mathcal{G}_{H,X}\rangle = \langle G - HX\rangle/H. \tag{25}$$

LEMMA 3.2. *Let G be a finite Abelian group and X a Cayley set of G. Let H be a subgroup of G which avoids X. Then the following* (a)–(d) *hold.*

(a) *The mapping ψ defined by $x \mapsto Hx$, for $x \in X$, is a bijection from X to X/H.*

(b) *Any two vertices in the same coset of* H *in* G *are at least distance three apart in* $\Gamma(G,X)$*; in particular each coset of* H *is an independent set of* $\Gamma(G,X)$.

(c) *Both* X/H *and* $\mathcal{G}_{H,X} - \{H\}$ *are Cayley sets of* G/H*; moreover, the corresponding Cayley graphs* $\Gamma(G/H, X/H)$, $\Gamma(G/H, \mathcal{G}_{H,X} - \{H\})$ *are complementary graphs with degrees* $|X|$, $|G : H| - |X| - 1$, *respectively.*

(d) $\Gamma(G/H, X/H) \cong (\Gamma(G,X))_{G/H}$, *and* $\Gamma(G,X)$ *is an* $|H|$*-fold cover of* $\Gamma(G/H, X/H)$.

Proof. (a) Clearly, ψ is surjective. If $Hx = Hy$ for distinct $x, y \in X$, then $1 \neq xy^{-1} \in H \cap X^2$, which contradicts the avoidability of H from X. Thus, ψ is also injective and hence is a bijection from X to X/H.

(b) For distinct $x, y \in G$ in the same coset of H, we have $xy^{-1} \in H - \{1\}$. Thus, since H avoids X, we have $xy^{-1} \notin X \cup X^2$. By the definition of $\Gamma(G,X)$, it is easy to see that the distance $d(x,y)$ in $\Gamma(G,X)$ between x and y is equal to the minimum number of elements of X whose product is xy^{-1}. Therefore, $xy^{-1} \notin X \cup X^2$ implies $d(x,y) \geq 3$, as required.

(c) Since X is a Cayley set of G, it is closed under taking inverse. This together with the fact that $(Hx)^{-1} = Hx^{-1}$ implies that X/H is closed under taking inverse as well. Also, since $H \cap X = \emptyset$, the identity H of G/H is not in X/H. Thus, X/H is a Cayley set of G/H. Since $\mathcal{G}_{H,X} - \{H\} = G/H - X/H - \{H\}$ by (24), this implies that $\mathcal{G}_{H,X} - \{H\}$ is a Cayley set of G/H as well. Note that X/H and $\mathcal{G}_{H,X} - \{H\}$ constitute a partition of G/H. Therefore, they give rise to complementary Cayley graphs of G/H. From (a) we have $|X/H| = |X|$, and hence $\Gamma(G/H, X/H)$ has degree $|X|$. Consequently, $\Gamma(G/H, \mathcal{G}_{H,X} - \{H\})$ has degree $|G : H| - |X| - 1$.

(d) We have $Hx, Hy \in G/H$ are adjacent in $\Gamma(G/H, X/H) \Leftrightarrow Hx(Hy)^{-1} \in X/H$ $\Leftrightarrow H(xy^{-1}) = Hz$ for some $z \in X \Leftrightarrow xy^{-1} = hz$ for some $z \in X$ and $h \in H \Leftrightarrow$ $x(hy)^{-1} = z$ for some $z \in X$ and $h \in H \Leftrightarrow x \in Hx$ and $hy \in Hy$ are adjacent in $\Gamma(G,X)$ for some $h \in H \Leftrightarrow gx \in Hx$ and $ghy \in Hy$ are adjacent in $\Gamma(G,X)$ for some $h \in H$ and any $g \in H \Leftrightarrow Hx, Hy$ are adjacent in the quotient graph $(\Gamma(G,X))_{G/H}$. (Here we used the assumption that G is Abelian.) Hence we have $\Gamma(G/H, X/H) \cong (\Gamma(G,X))_{G/H}$. Moreover, from the arguments above we see that, for adjacent cosets Hx and Hy, each element of Hx is adjacent to at least one element of Hy in $\Gamma(G,X)$. However, $\Gamma(G,X)$ and $\Gamma(G/H, X/H)$ have the same degree $|X|$. So the subgraph of $\Gamma(G,X)$ induced by $Hx \cup Hy$ is forced to be a perfect matching between Hx and Hy. Therefore, $\Gamma(G,X)$ is an $|H|$-fold cover of $\Gamma(G/H, X/H)$. □

In the case where in addition $\langle X \rangle = G$, one can check that $\Gamma(G/H, X/H)$ is the underlying undirected graph of the Schreier coset graph for (G, H, X), and in this case this Schreier coset graph has no loop or multiple arc. (For any group G with generating set X, and any subgraph H of G, the *Schreier coset graph* [12] for (G, H, X) is the directed graph with vertex set $G/H = \{Hz : z \in G\}$ and arcs (Hz, Hzx) for all Hz and $x \in X$, where loops and multiple arcs are allowed.)

A cycle (path, respectively) in a graph visiting all vertices is called a Hamiltonian cycle (Hamiltonian path, respectively). A graph is *Hamiltonian* if it contains a Hamiltonian cycle. The following result is well known; see, e.g., [21, Corollary 3.2].

LEMMA 3.3. *Every connected Cayley graph on a finite Abelian group of order at least three is Hamiltonian.*

An immediate consequence of this result is that every connected Cayley graph on any finite Abelian group contains a Hamiltonian path. This will be used in the following proof of Theorem 2.2.

Proof of Theorem 2.2. Let G be a finite Abelian group and X a Cayley set of

G. Let H be a subgroup of G which avoids X. For notational simplicity, we denote $\mathcal{G} = \langle \mathcal{G}_{H,X} \rangle$ and $\hat{x} = Hx$ for $x \in G$. Denote $r = |G : H|$ and $s = |G : \langle G - HX \rangle|$. Then $s = |(G/H) : \mathcal{G}| = r/|\mathcal{G}|$ by (25).

Let us first treat the degenerate case where $\mathcal{G}_{H,X} = \{H\}$. In this case we have $s = r$ and $X/H = G/H - \{H\}$, and hence $\Gamma(G/H, X/H)$ is a complete graph. Order linearly the cosets in G/H in an arbitrary way. Then assign label $(i-1)j$ to every element of the ith member of G/H, for $i = 1, 2, \ldots, r$. Using Lemma 3.2(b) and noting $j \geq k$, one can check that this labelling is an $L(j,k)$-labelling of $\Gamma(G,X)$. Clearly, it uses r labels and has span $(r-1)j$. Thus, we have $\lambda_{j,k}(\Gamma(G,X)) \leq (r-1)j$ and $\mu(\Gamma(G,X)) \leq r$. But, since $s = r$ and $\max\{k, \lceil j/2 \rceil\} + \min\{j-k, \lfloor j/2 \rfloor\} = j$, the right-hand side of (4) is exactly $(r-1)j$. Therefore, we have proved (4) and (5) in the case where $\mathcal{G}_{H,X} = \{H\}$.

In the following we deal with the general case where $\mathcal{G}_{H,X} - \{H\} \neq \emptyset$. Let

$$\mathcal{G}\hat{x}_1, \mathcal{G}\hat{x}_2, \ldots, \mathcal{G}\hat{x}_s$$

be representatives of distinct cosets of $\mathcal{G}$ in G/H, where we set $\mathcal{G}\hat{x}_1 = \mathcal{G}$. Then of course they consist of a partition of G/H. Recall from Lemma 3.2(c) that $\mathcal{G}_{H,X} - \{H\}$ is a Cayley set of G/H. By the definition of the Cayley graph $\Gamma(G/H, \mathcal{G}_{H,X} - \{H\})$, two cosets $\hat{x}, \hat{y}$ of H are connected by a path of $\Gamma(G/H, \mathcal{G}_{H,X} - \{H\})$ if and only if $\hat{x}(\hat{y})^{-1} = \widehat{xy^{-1}} \in \langle \mathcal{G}_{H,X} - \{H\} \rangle = \mathcal{G}$, which in turn is true if and only if $\hat{x}, \hat{y}$ are in the same coset $\mathcal{G}\hat{x}_i$ of $\mathcal{G}$ in G/H, for some i. Thus, for each $i = 1, 2, \ldots, s$, $\mathcal{G}\hat{x}_i$ induces a connected component of $\Gamma(G/H, \mathcal{G}_{H,X} - \{H\})$. In what follows we will denote this component by $\widehat{\Gamma}_i$. These components $\widehat{\Gamma}_i$, $i = 1, 2, \ldots, s$, are isomorphic to each other since as a Cayley graph $\Gamma(G/H, \mathcal{G}_{H,X} - \{H\})$ is vertex-transitive. Since $\mathcal{G}_{H,X} - \{H\}$ generates $\mathcal{G}$ and is a Cayley set of G/H (Lemma 3.2(c)), it is also a Cayley set of $\mathcal{G}$. Hence $\mathcal{G}_{H,X} - \{H\}$ gives rise to a connected Cayley graph $\Gamma(\mathcal{G}, \mathcal{G}_{H,X} - \{H\})$, which is exactly the connected component $\widehat{\Gamma}_1$ of $\Gamma(G/H, \mathcal{G}_{H,X} - \{H\})$ induced by $\mathcal{G}$. By Lemma 3.3, $\widehat{\Gamma}_1$ contains a Hamiltonian path, and hence so does each $\widehat{\Gamma}_i$ as $\widehat{\Gamma}_i \cong \widehat{\Gamma}_1$. Let

$$\hat{x}_{i,1}, \hat{x}_{i,2}, \ldots, \hat{x}_{i,t}$$

be a Hamiltonian path of $\widehat{\Gamma}_i$, where $t = |\mathcal{G}| = r/s$. Then any two consecutive members in this sequence are adjacent in $\Gamma(G/H, \mathcal{G}_{H,X} - \{H\})$, and hence are not adjacent in $\Gamma(G/H, X/H)$ by Lemma 3.2(c). Hence, for each $i = 1, 2, \ldots, s$, by Lemma 3.2(d) there is no edge of $\Gamma(G,X)$ joining any element of $\hat{x}_{i,\ell}$ and any element of $\hat{x}_{i,\ell+1}$, for $\ell = 1, 2, \ldots, t-1$. By Lemma 3.2(b) the elements of $\hat{x}_{i,\ell}$ are distance at least three apart in $\Gamma(G,X)$, for each i and $\ell = 1, 2, \ldots, t$.

Now we define f to be the labelling such that all the elements of $\hat{x}_{i,\ell}$ are labelled by

$$(i-1)\left((t-1)\max\{k, \lceil j/2 \rceil\} + j\right) + (\ell - 1)\max\{k, \lceil j/2 \rceil\}$$

for $i = 1, 2, \ldots, s$ and $\ell = 1, 2, \ldots, t$. Then, for any $\hat{x}_{i,\ell}$ and $\hat{x}_{i',\ell'}$ with $i \neq i'$, the labels of the elements of $\hat{x}_{i,\ell}$ and $\hat{x}_{i',\ell'}$ differ by at least j. For $\hat{x}_{i,\ell}$ and $\hat{x}_{i,\ell'}$ with the same first subscript, if an element of $\hat{x}_{i,\ell}$ is adjacent to an element of $\hat{x}_{i,\ell'}$ in $\Gamma(G,X)$, then $|\ell - \ell'| \geq 2$ by the discussion in the previous paragraph, and hence the labels of these two elements differ by at least $2\max\{k, \lceil j/2 \rceil\}$, which is obviously no less than j. Also, if an element of $\hat{x}_{i,\ell}$ is distance two apart from an element of $\hat{x}_{i,\ell'}$ in $\Gamma(G,X)$, then $\ell \neq \ell'$ by Lemma 3.2(b) and hence the labels of these two elements differ by at least $\max\{k, \lceil j/2 \rceil\}$, which is no less than k. Therefore, f is an $L(j,k)$-labelling of $\Gamma(G,X)$. Noting that $r = st$, this labelling uses r distinct labels and has span

$$\begin{aligned}\mathrm{sp}(\Gamma(G,X); f) &= (s-1)((t-1)\max\{k, \lceil j/2 \rceil\} + j) + (t-1)\max\{k, \lceil j/2 \rceil\} \\ &= r\max\{k, \lceil j/2 \rceil\} + s(j - \max\{k, \lceil j/2 \rceil\}) - j \\ &= r\max\{k, \lceil j/2 \rceil\} + s\min\{j-k, \lfloor j/2 \rfloor\} - j.\end{aligned}$$

Therefore, the upper bounds (4) and (5) follow and the proof is complete. ∎

Proof of Corollary 2.4. We use the notation in the proof of Theorem 2.2. Since H avoids X and $G - HX$ is a generating set of G, we have $\mathcal{G} = \langle \mathcal{G}_{H,X} \rangle = G/H$ by (25). Hence $s = 1$, $t = r = |G : H|$, and $\Gamma(G/H, \mathcal{G}_{H,X} - \{H\})$ is connected. Thus, by Lemma 3.3, $\Gamma(G/H, \mathcal{G}_{H,X} - \{H\})$ contains a Hamiltonian path $Hx_{1,1}, Hx_{1,2}, \dots, Hx_{1,r}$. From the proof of Theorem 2.2, the labelling f which assigns $(\ell - 1)\max\{k, \lceil j/2 \rceil\}$ to the elements of $Hx_{1,\ell}$ $(\ell = 1, 2, \dots, r)$ is an $L(j,k)$-labelling of $\Gamma(G,X)$. Since this labelling has span $(r-1)\max\{k, \lceil j/2 \rceil\}$, we obtain (8) immediately.

For the $L(2,1)$ case, we have $2k = j = 2$ and hence (9) follows from (8). Also, in this case the labelling f above uses labels $0, 1, 2, \dots, r-1$, and hence is a no-hole $L(2,1)$-labelling. This completes the proof. ∎

We conclude this section by giving the following remarks.

Remark 3.4. (a) The proof of Theorem 2.2 gives an explicit $L(j,k)$-labelling of $\Gamma(G,X)$ provided that a Hamiltonian cycle of $\Gamma(\mathcal{G}, \mathcal{G}_{H,X} - \{H\})$ is known, where $\mathcal{G} = \langle \mathcal{G}_{H,X} \rangle$ as above.

(b) A Cayley set X may be avoided by several subgroups H of G. To get a better upper bound for $\lambda_{j,k}(\Gamma(G,X))$, we will be interested in those H such that the right-hand side of (4) is as small as possible.

In the case where $G - HX$ is a generating set of G, we have by (8)

$$\lambda_{j,k}(\Gamma(G,X)) \le (|G : H| - 1)\max\{k, \lceil j/2 \rceil\} \le (|G| - 1)\max\{k, \lceil j/2 \rceil\}.$$

Note that the second equality occurs precisely when $H = \{1\}$. On the other hand, if $H = \{1\}$, then $G - HX$ is a generating set of $G \Leftrightarrow G - X$ is a generating set of $G \Leftrightarrow$ the complement graph of $\Gamma(G,X)$ is connected $\Leftrightarrow$ the complement graph of $\Gamma(G,X)$ has a Hamiltonian path $\Leftrightarrow$ the elements of G can be ordered as $x_1, x_2, \dots, x_{|G|}$ such that any two consecutive elements in this sequence are nonadjacent in $\Gamma(G,X)$. In this simplest case, (9) gives the bound $\lambda(\Gamma(G,X)) \le |G| - 1$, which is the same as the one obtained by using [10, Theorem 1.1(a)]. The reader can easily find examples which show that even in this somewhat "worst" case the bound $|G| - 1$ can be the actual value of the λ-number of $\Gamma(G,X)$.

(c) The bound (7) can be improved as

$$\lambda(\Gamma(G,X)) \le |G : \langle G - HX \rangle|(\lambda_0 + 2) - 2, \tag{26}$$

where $\lambda_0 = \lambda(\Gamma(\mathcal{G}, \mathcal{G} - \mathcal{G}_{H,X}))$. In fact, in the proof of Theorem 2.2 for the $L(2,1)$ case, we assigned t $(= |\mathcal{G}| = |\langle G - HX \rangle : H|)$ distinct labels to the vertices of $\widehat{\Gamma}_i$. But $\lambda_0 + 1$ labels will be enough, and so replacing t by $\lambda_0 + 1$ in the proof of Theorem 2.2 will give the proof of (26). Note that $\lambda_0 + 1 \le t$ since the complementary graph $\Gamma(\mathcal{G}, \mathcal{G}_{H,X} - \{H\})$ of $\Gamma(\mathcal{G}, \mathcal{G} - \mathcal{G}_{H,X})$ contains a Hamiltonian path. Hence (26) does imply (7), and it is better than (7) in the case where $\lambda_0 + 1$ is strictly less than

t. The inequality (26) establishes a connection between the λ-numbers of $\Gamma(G, X)$ and $\Gamma(\mathcal{G}, \mathcal{G} - \mathcal{G}_{H,X})$, the latter being an induced subgraph of the quotient graph $\Gamma(G/H, X/H)$ of $\Gamma(G, X)$.

(d) From (25) one can see that (4) and (7) can be rewritten as

$$\lambda_{j,k}(\Gamma(G, X)) \leq |G : H| \left(\max\{k, \lceil j/2 \rceil\} + \frac{\min\{j - k, \lfloor j/2 \rfloor\}}{|\langle \mathcal{G}_{H,X} \rangle|} \right) - j \tag{27}$$

$$\lambda(\Gamma(G, X)) \leq |G : H| \left(1 + \frac{1}{|\langle \mathcal{G}_{H,X} \rangle|} \right) - 2, \tag{28}$$

respectively. As in (6), if $2k \geq j$, then $\max\{k, \lceil j/2 \rceil\}$ in (27) can be replaced by k.

(e) From (24) it follows that $\mathcal{G}_{H,X} = \{H\}$ occurs if and only if $\{H, HX\}$ is a partition of G. (Note that $H \cap X = \emptyset$ implies $H \cap HX = \emptyset$.) In this extreme case we have $\langle \mathcal{G}_{H,X} \rangle = \{H\}$ and hence (27) becomes

$$\lambda_{j,k}(\Gamma(G, X)) \leq (|G : H| - 1)j.$$

4. Proof of Theorem 2.5. To prove Theorem 2.5 we need the following well known result.

LEMMA 4.1 (see [1, Proposition 16.5]). *Let Γ be a graph whose automorphism group contains a vertex-transitive Abelian subgroup G. Then G is regular on $V(\Gamma)$, and G is an elementary Abelian 2-group.*

(Note that in [1] this proposition is stated for the full automorphism group $\mathrm{Aut}(\Gamma)$ of Γ. However, it is valid for a transitive Abelian subgroup of $\mathrm{Aut}(\Gamma)$ as well, and the proof is the same.)

In the following we will use $V(d, 2)$ to denote the d-dimensional linear space of row vectors over the field $\mathrm{GF}(2) = \{0, 1\}$ of characteristic 2, and $V^+(d, 2)$ to denote the additive group of $V(d, 2)$. For this group the operation is addition of row vectors, and hence we will use $H + x$ in place of Hx. Denote by $\mathbf{0}_d$ the zero vector of $V(d, 2)$. Then it is the identity element of $V^+(d, 2)$. It is well known that $V^+(d, 2)$ is isomorphic to the elementary Abelian 2-group $\mathbb{Z}_2^d$.

As we will soon see, any connected graph Γ with $\mathrm{Aut}(\Gamma)$ containing a vertex-transitive Abelian subgroup G is isomorphic to a Cayley graph on G. To prove Theorem 2.5 by using Theorem 2.2, we need to identify a subgroup of G such that it avoids the relevant Cayley set and produces the upper bounds (10) and (11). This is equivalent to identifying a subspace of $V(d, 2)$ with certain properties, and hence is a matrix problem essentially. The existence of such a subspace is guaranteed by the following lemma.

LEMMA 4.2. *Let d, ℓ, n be positive integers such that $n \leq \ell \leq d$ and $2^{n-1} \leq d < 2^n$. Let $t := \min\{2^n - d - 1, n\}$. Then for any d nonzero, pairwise distinct vectors $\mathbf{x}_1, \ldots, \mathbf{x}_d$ of $V(\ell, 2)$ which generate $V(\ell, 2)$, there exists an $\ell \times n$ matrix M over* $\mathrm{GF}(2)$ *such that*

(a) *M has rank n;*

(b) *$\mathbf{x}_1 M, \ldots, \mathbf{x}_d M$ are nonzero and pairwise distinct; and*

(c) *$V(n, 2) - \{\mathbf{x}_1 M, \ldots, \mathbf{x}_d M\}$ contains t independent vectors.*

Proof. Since $t \leq n$, we can choose t independent vectors $\mathbf{d}_1, \ldots, \mathbf{d}_t$ of $V(n, 2)$. Since $V(n, 2)$ has $2^n - 1$ nonzero vectors and $t + d \leq 2^n - 1$ by the definition of t, we can choose d distinct nonzero vectors, say $\mathbf{c}_1, \ldots, \mathbf{c}_d$, from $V(n, 2) - \{\mathbf{d}_1, \ldots, \mathbf{d}_t\}$. Moreover, we may require that the $d \times n$ matrix C with the ith row $\mathbf{c}_i$ has rank n, so that its columns are independent. For example, if $1 \le t < n$, then we can set $\mathbf{d}_i$, for $1 \le i \le t$, to be the vector with the jth entry 0 if $j < i$ and 1 if $j \ge i$; if $t = n$, then we can set $\mathbf{d}_n$ to be $(1, 0, \ldots, 0, 1)$ and define other $\mathbf{d}_i$'s in the same way. (In the case where $t = 0$ we leave $\mathbf{d}_t$ undefined.) Set $\mathbf{c}_1 = (1, 0, \ldots, 0), \ldots, \mathbf{c}_n = (0, 0, \ldots, 1)$ to be the standard basis of $V(n, 2)$, and choose distinct nonzero vectors $\mathbf{c}_{n+1}, \ldots, \mathbf{c}_d$ from $V(n, 2) - \{\mathbf{c}_1, \ldots, \mathbf{c}_n, \mathbf{d}_1, \ldots, \mathbf{d}_t\}$. Then $\mathbf{c}_1, \ldots, \mathbf{c}_n, \mathbf{c}_{n+1}, \ldots, \mathbf{c}_d, \mathbf{d}_1, \ldots, \mathbf{d}_t$ satisfy all the conditions above. Moreover, the matrix C has the form

$$C = \begin{pmatrix} I_n \\ J \end{pmatrix},$$

where I_n is the identity matrix of order n over GF(2) and J is the $(d - n) \times n$ matrix of rows $\mathbf{c}_{n+1}, \ldots, \mathbf{c}_d$. Since $\ell \le d$ and the columns of C are independent vectors of dimension d, we can add $\ell - n$ column vectors of dimension d to C to form a $d \times \ell$ matrix Y of rank ℓ. Thus, the columns of Y are independent, and the rows $\mathbf{y}_1, \ldots, \mathbf{y}_d$ of Y are extensions of $\mathbf{c}_1, \ldots, \mathbf{c}_d$, respectively, that is,

$$\mathbf{y}_i = (\mathbf{c}_i \mid \overbrace{*, \ldots, *}^{\ell-n})$$

for each i. Set

$$B = \begin{pmatrix} I_n \\ 0 \end{pmatrix},$$

where 0 is the $(\ell - n) \times n$ matrix with all entries zero. Then B is an $\ell \times n$ matrix of rank n, and it satisfies $YB = C$. Let A be the $d \times \ell$ matrix with the ith row $\mathbf{x}_i$, for $1 \le i \le d$. Then A has rank ℓ by our assumption. Since Y has also rank ℓ, from linear algebra there exists a nonsingular $\ell \times \ell$ matrix N over GF(2) such that $Y = AN$. Now we set $M = NB$. Then the nonsingularity of N ensures that M has the same rank as B, that is, M has rank n. Also, we have $AM = A(NB) = YB = C$, which implies $\mathbf{x}_i M = \mathbf{c}_i$ for each i. Thus, $\mathbf{x}_1 M, \ldots, \mathbf{x}_d M$ are nonzero and pairwise distinct. Moreover, $\mathbf{d}_1, \ldots, \mathbf{d}_t$ are t independent vectors in $V(n, 2) - \{\mathbf{x}_1 M, \ldots, \mathbf{x}_d M\}$. □

Proof of Theorem 2.5. Let Γ be a connected graph such that $\mathrm{Aut}(\Gamma)$ contains a vertex-transitive Abelian subgroup G. By Lemma 4.1, G is regular on $V(\Gamma)$, and G is an elementary Abelian 2-group. Hence $|G| = 2^\ell$ and $G \cong \mathbb{Z}_2^\ell$ for a positive integer ℓ (see, e.g., [25, 7.40]). In the following we will identify G with the group $V^+(\ell, 2)$. Since G is regular on $V(\Gamma)$, by [1, Lemma 16.3] Γ is isomorphic to a Cayley graph of G, namely $\Gamma \cong \Gamma(G, X)$ for a Cayley set

$$X := \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$$

of G, where $d := |X|$ is the degree of vertices of Γ and each $\mathbf{x}_i \in V(\ell, 2)$. Moreover, X must be a generating set of G as Γ is connected. Hence $\ell \le d$. Also, we have $d < 2^\ell$ as X is a proper subset of G. Let $n := 1 + \lfloor \log_2 d \rfloor$ and $t := \min\{2^n - d - 1, n\}$. Then $2^{n-1} \le d < 2^n$ and hence $2^{n-1} \le d < 2^\ell$, which implies $n \le \ell$. From Lemma 4.2 there exists an $\ell \times n$ matrix M over GF(2) with properties (a)–(c) in that lemma. Since M has rank n by property (a) there, its null space

$$U := \{\mathbf{x} \in V(\ell, 2) : \mathbf{x}M = \mathbf{0}_n\}$$

is an $(\ell - n)$-dimensional subspace of $V(\ell, 2)$. Let $H := U^+$ be the additive group of U. Then $|G : H| = 2^n$. From the definition (23) of $\mathcal{G}_{H,X}$ one can check that

$$\mathcal{G}_{H,X} = \{H + \mathbf{z} : \mathbf{z} \in V(\ell, 2),\ \mathbf{z}M \ne \mathbf{x}_q M \text{ for all } q = 1, \ldots, d\}. \tag{29}$$

By property (b) in Lemma 4.2, $\mathbf{x}_1 M, \ldots, \mathbf{x}_d M$ are nonzero and pairwise distinct. This is equivalent to saying that H avoids X. Thus, from (5) we have $\mu(\Gamma) \leq |G : H| = 2^n$ as claimed in (11). By property (c) in Lemma 4.2, $V(n,2) - \{\mathbf{x}_1 M, \ldots, \mathbf{x}_d M\}$ contains t independent vectors, say $\mathbf{d}_1, \ldots, \mathbf{d}_t$. Since M has rank n, there exist $\mathbf{y}_1, \ldots, \mathbf{y}_t \in V(\ell, 2)$ such that $\mathbf{y}_i M = \mathbf{d}_i$ for each $i = 1, \ldots, t$. Since no $\mathbf{d}_i$ is the same as any $\mathbf{x}_q M$, by (29) we know that all $H + \mathbf{y}_i \in \mathcal{G}_{H,X}$. On the other hand, since $\mathbf{d}_1, \ldots, \mathbf{d}_t$ are independent, $H + \mathbf{y}_1, \ldots, H + \mathbf{y}_t$ are independent in the quotient linear space $V(\ell, 2)/U$. Therefore,

$$|\langle \mathcal{G}_{H,X} \rangle| \geq |\langle H + \mathbf{y}_1, \ldots, H + \mathbf{y}_t \rangle| = 2^t.$$

By (27) and noting $|G : H| = 2^n$ we then have

$$\begin{aligned} \lambda_{j,k}(\Gamma) &\leq 2^n \left(\max\{k, \lceil j/2 \rceil\} + \frac{\min\{j-k, \lfloor j/2 \rfloor\}}{|\langle \mathcal{G}_{H,X} \rangle|} \right) - j \\ &\leq 2^n \max\{k, \lceil j/2 \rceil\} + 2^{n-t} \min\{j-k, \lfloor j/2 \rfloor\} - j \end{aligned}$$

as claimed in (10). □

The major part of the proof above was to show that the group G contains a subgroup H which avoids X and is such that $|\langle \mathcal{G}_{H,X} \rangle| \geq 2^t$. This was achieved by identifying a matrix M over GF(2) with properties (a)–(c) in Lemma 4.2. From [17, Corollary 4.14], the graph Γ in Theorem 2.5 contains the ℓ-cube Q_ℓ as a spanning subgraph, where ℓ is as in the proof above.

In the case where $\Gamma = Q_d$, we have $\ell = d$, $G = \mathbb{Z}_2^d$, and $Q_d \cong \Gamma(\mathbb{Z}_2^d, X)$, where $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ is the standard basis of $V(d, 2)$. Thus, in the proof of Lemma 4.2, we have $A = I_d$, $Y = N$, and $M = C$, and hence the ith row of M is $\mathbf{x}_i M = \mathbf{c}_i$, for $i = 1, 2, \ldots, d$. Therefore, by Lemma 4.2, in this case we can choose M to be any $d \times n$ matrix over GF(2) with rank n such that its rows are nonzero and pairwise distinct, and the subspace of $V(n, 2)$ spanned by those vectors which are not equal to any row of M has dimension at least t. For each choice of M, the additive group of the null space of M avoids X, and following the proof of Theorem 2.2 we then get an $L(j,k)$-labelling of Q_d which uses 2^n labels and has span $2^n \max\{k, \lceil j/2 \rceil\} + 2^{n-t} \min\{j-k, \lfloor j/2 \rfloor\} - j$.

5. Proof of Theorem 2.9. First, we have the following simple lower bounds for $\lambda_{j,k}(H_{n_1,n_2,\ldots,n_d})$ and $\mu(H_{n_1,n_2,\ldots,n_d})$.

LEMMA 5.1. *Let $n_1 \geq n_2 \geq \cdots \geq n_d$ (≥ 2) be a sequence of $d \geq 2$ integers. Then, for any $j \geq k \geq 1$, we have*

$$\lambda_{j,k}(H_{n_1,n_2,\ldots,n_d}) \geq (n_1 n_2 - 1)k \tag{30}$$

$$\mu(H_{n_1,n_2,\ldots,n_d}) \geq n_1 n_2. \tag{31}$$

Proof. Note that $H_{n_1,n_2,\ldots,n_d}$ contains a subgraph isomorphic to H_{n_1,n_2}. Since H_{n_1,n_2} has diameter 2, under any $L(j,k)$-labelling of $H_{n_1,n_2,\ldots,n_d}$, the $n_1 n_2$ vertices of H_{n_1,n_2} must be assigned labels with a mutual difference of at least k. From this both bounds follow immediately. □

Note that, if the equality in (30) occurs, then the equality in (31) occurs as well.

In the proof of Theorem 2.9 we will borrow some ideas from the proof of [9, Theorem 3.1]. However, we do not need a counting argument as used there. We will also use the monotonicity of $\lambda_{j,k}$ and μ: for any subgraph Σ of a graph Γ, we have

$$\lambda_{j,k}(\Sigma) \leq \lambda_{j,k}(\Gamma), \quad \mu(\Sigma) \leq \mu(\Gamma).$$

These hold because any $L(j,k)$-labelling of Γ is also an $L(j,k)$-labelling of Σ as $j \geq k$.

Proof of Theorem 2.9. It suffices to prove

$$\lambda_{j,k}(H_{n_1,n_2,\ldots,n_d}) \leq (n_1 n_2 - 1)\max\{k, \lceil j/2 \rceil\} \tag{32}$$

$$\mu(H_{n_1,n_2,\ldots,n_d}) \leq n_1 n_2 \tag{33}$$

for any sequence $n_1 \geq n_2 \geq \cdots \geq n_d$ (≥ 2) such that $n_1 > d \geq 2$, n_2 divides n_1, n_i divides n_2 for $i = 3,\ldots,d$, and each prime factor of n_i, for $i = 1,\ldots,d$, is no less than d. In fact, once this is achieved, then for any sequence $n_1, n_2, \ldots, n_d$ satisfying the conditions of Theorem 2.9 we will have

$$(n_1 n_2 - 1)k \leq \lambda_{j,k}(H_{n_1,n_2,\ldots,n_d}) \leq \lambda_{j,k}(H_{n_1,n_2,\ldots,n_2}) \leq (n_1 n_2 - 1)\max\{k, \lceil j/2 \rceil\}$$

and hence (17) and (19) follow. (Note that $\max\{k, \lceil j/2 \rceil\} = k$ whenever $2k \geq j$.) Here the first inequality is just (30), the second one is due to the fact that $H_{n_1,n_2,\ldots,n_d}$ is isomorphic to a subgraph of $H_{n_1,n_2,\ldots,n_2}$ and that $\lambda_{j,k}$ is monotonic, and the last one is a special case (where $n_2 = n_3 = \cdots = n_d$) of (32). The truth of (18) can be proved in a similar way using (31) and (33).

So from now on we suppose that the sequence $n_1 \geq n_2 \geq \cdots \geq n_d \geq 2$ satisfies the conditions in the previous paragraph. Denote $\Gamma := H_{n_1,n_2,\ldots,n_d}$. Then Γ is isomorphic to the Cayley graph $\Gamma(G, X)$, where

$$G := \langle g_1 \rangle \times \langle g_2 \rangle \times \cdots \times \langle g_d \rangle \tag{34}$$

is the direct product of cyclic groups $\langle g_i \rangle$ of order n_i $(i = 1, 2, \ldots, d)$ and

$$X := \{(x_1, x_2, \ldots, x_d) : \text{there is exactly one } i \text{ such that } x_i \neq 1\} \tag{35}$$

which is clearly a Cayley set of G. Note that the identity element of G is $1_G = (1, 1, \ldots, 1)$, where the 1 in the ith position is the identity element of $\langle g_i \rangle$. We will prove the existence of a subgroup H of G such that H avoids X, $|G : H| = n_1 n_2$, and $\mathcal{G}_{H,X}$ generates G/H (which is equivalent to saying that $G - HX$ generates G in view of (25)). Once this is achieved, we then have $\lambda_{j,k}(\Gamma) \leq (n_1 n_2 - 1)\max\{k, \lceil j/2 \rceil\}$ by (8) and $\mu(\Gamma) \leq n_1 n_2$ by (5), and hence (32) and (33) follow.

Since n_2 is a divisor of n_1 and n_i is a divisor of n_2 for $i = 3, \ldots, d$, $\langle g_2 \rangle$ is isomorphic to a subgroup of $\langle g_1 \rangle$, and $\langle g_i \rangle$ is isomorphic to a subgroup of $\langle g_2 \rangle$ for $i = 3, \ldots, d$. For simplicity of notation, we will take $\langle g_2 \rangle$ as a subgroup of $\langle g_1 \rangle$, and take each such $\langle g_i \rangle$ as a subgroup of $\langle g_2 \rangle$. Thus, for $u = (u_1, u_2, \ldots, u_d) \in G$, we have $\prod_{i=1}^d u_i \in \langle g_1 \rangle$, $\prod_{i=2}^d u_i^{i-1} \in \langle g_2 \rangle$, and

$$\psi : u \mapsto \left(\prod_{i=1}^d u_i, \prod_{i=2}^d u_i^{i-1} \right)$$

defines a mapping from G to $\langle g_1 \rangle \times \langle g_2 \rangle$. It is not difficult to check that ψ is a homomorphism from G to $\langle g_1 \rangle \times \langle g_2 \rangle$. Moreover, ψ is surjective since for any $(u_1, u_2) \in \langle g_1 \rangle \times \langle g_2 \rangle$ we have $\psi(u_1 u_2^{-1}, u_2, 1, \ldots, 1) = (u_1, u_2)$. Define $H := \mathrm{Ker}(\psi)$ to be the kernel of ψ, that is,

$$H = \{u \in G : \psi(u) = (1, 1)\}.$$

Then H is a subgroup of G and, by the homomorphism theorem, $G/H \cong \langle g_1 \rangle \times \langle g_2 \rangle$ via the bijection defined by $Hu \leftrightarrow \psi(u)$ for $u \in G$. In particular, H has index $|G : H| = n_1 n_2$ in G. Moreover, we have

Claim 1. H avoids X.

Proof of Claim 1. For any $x = (1, \ldots, x_i, \ldots, 1) \in X$ and $y = (1, \ldots, y_q, \ldots, 1) \in X$, we have $x_i \neq 1$ and $y_q \neq 1$. So $\psi(x) = (x_i, x_i^{i-1}) \neq (1, 1)$, and hence $H \cap X = \emptyset$. Clearly, we have $\psi(xy) = (x_i y_q, x_i^{i-1} y_q^{q-1})$. Thus, if $i = q$, then $\psi(xy) = (1, 1)$ if and only if $xy = (1, 1, \ldots, 1) = 1_G$. If $i \neq q$, say $i < q$, then $\psi(xy) = (1, 1)$ implies $y_q^{q-i} = 1$, which happens only when $d \geq 3$ and the order $o(y_q)$ of y_q is a divisor of $q - i$. In particular, we have $o(y_q) \leq d - 1$ in this case. However, since $o(y_q) > 1$ is a divisor of n_q, we have $o(y_q) \geq d$ by our assumption. This contradiction shows that the product of any two elements of X is not in $H - \{1_G\}$, that is, $H \cap X^2 = \{1_G\}$ and hence claim 1 follows. □

To verify that $\mathcal{G}_{H,X}$ is a generating set of G/H, we prove first the following result, which will be used also in explicitly $L(j, k)$-labelling the vertices of Γ.

Claim 2. There exist $Hv, Hw \in \mathcal{G}_{H,X}$ with orders n_1, n_2, respectively, such that

$$G/H = \langle Hv, Hw \rangle.$$

Proof of Claim 2. To prove this we first assume that $n_1 \neq n_2$. In this case we set $v := (g_1 g_2^{-1}, g_2, 1, \ldots, 1)$ and $w := (g_2^{-1}, g_2, 1, \ldots, 1)$. Then $\psi(v) = (g_1, g_2)$ and $\psi(w) = (1, g_2)$. Clearly, (g_1, g_2) and $(1, g_2)$ generate $\langle g_1 \rangle \times \langle g_2 \rangle$, and they have orders n_1, n_2, respectively. Since $G/H \cong \langle g_1 \rangle \times \langle g_2 \rangle$ via the bijection $Hu \leftrightarrow \psi(u)$ for $u \in G$, it follows that $G/H = \langle Hv, Hw \rangle$ and the orders of Hv, Hw in G/H are n_1, n_2, respectively. Note that, for any $u \in G$, $Hu \cap X \neq \emptyset \Leftrightarrow \psi(u) = \psi(x)$ for some $x \in X$ $\Leftrightarrow \psi(u) = (x_i, x_i^{i-1})$ for some $x_i \neq 1$. In particular, if $Hv \cap X \neq \emptyset$, then $g_1 = x_i$ and $g_2 = x_i^{i-1}$ for some $x_i \neq 1$, which implies $g_2 = g_1^{i-1}$ and hence $i \geq 2$. On the other hand, since $x_i \in \langle g_i \rangle$, it follows from $g_1 = x_i$ that $\langle g_i \rangle = \langle g_1 \rangle$ and hence $n_1 = \cdots = n_i$. In particular, since $i \geq 2$, we have $n_1 = n_2$, which contradicts our assumption. Thus, we must have $Hv \cap X = \emptyset$. Similarly, $Hw \cap X = \emptyset$ for otherwise we would have $(1, g_2) = (x_i, x_i^{i-1})$ for some $x_i \neq 1$, which implies $g_2 = 1$, a contradiction. Therefore, $Hv, Hw \in \mathcal{G}_{H,X}$ and all conditions in claim 2 are satisfied.

In the remaining case we have $n_1 = n_2$, so that g_2 has the same order as g_1. Thus, since $\langle g_2 \rangle$ is a subgroup of $\langle g_1 \rangle$ by our assumption, we have $\langle g_2 \rangle = \langle g_1 \rangle$ and hence $g_1 = g_2^r$ for an integer r, $1 \leq r \leq n_1$, which is coprime to n_1. Set $v := (g_1 g_2^r, g_2^{-r}, 1, \ldots, 1)$ and $w := (g_2^{-1}, g_2, 1, \ldots, 1)$. Then $\psi(v) = (g_1, g_2^{-r}) = (g_1, g_1^{-1})$ and $\psi(w) = (1, g_2)$. By a similar argument as above, one can see that $G/H = \langle Hv, Hw \rangle$ and the orders of Hv, Hw in G/H are n_1, n_2, respectively. Also, $Hw \cap X = \emptyset$ as seen above. If $Hv \cap X \neq \emptyset$, then $g_1 = x_i, g_1^{-1} = x_i^{i-1}$ for some $x_i \neq 1$, and hence $g_1^i = 1$. This implies that n_1 divides i, which is impossible since $1 \leq i \leq d < n_1$. Thus, we must have $Hv \cap X = \emptyset$, and Hv, Hw satisfy the conditions in claim 2. This completes the proof of claim 2. □

Now H avoids X by claim 1, and $\mathcal{G}_{H,X}$ is a generating set of G/H by claim 2. Thus, by (8) we have $\lambda_{j,k}(\Gamma) \leq (n_1 n_2 - 1) \max\{k, \lceil j/2 \rceil\}$ as claimed in (32), and by (5) we have $\mu(\Gamma) \leq n_1 n_2$ as claimed in (33). From our discussion in the first paragraph of this proof, the truth of (17) and (18) follows. Moreover, we can give explicitly an $L(j, k)$-labelling of Γ having span $(n_1 n_2 - 1) \max\{k, \lceil j/2 \rceil\}$ and using $n_1 n_2$ labels. In fact, claim 2 implies that $G/H = \{H(v^i w^\ell) : 0 \leq i < n_1, 0 \leq \ell < n_2\}$. Hence the cosets in G/H can be ordered in the following way to form a sequence. For $1 \leq t \leq n_1 n_2$, there exists a unique pair (i, ℓ) of integers with $1 \leq i \leq n_2$ and

$1 \le \ell \le n_1$ such that $t = (i-1)n_1 + \ell$. We then define the tth term Hu_t of the sequence to be $H(v^{\ell-i}w^{i-1})$. It can be checked that, for any two consecutive cosets Hu_t, Hu_{t+1} in the sequence, $H(u_t u_{t+1}^{-1})$ is either Hv^{-1} or Hw^{-1}. Since $Hv \cap X = Hw \cap X = \emptyset$, we have $Hv^{-1} \cap X = Hw^{-1} \cap X = \emptyset$ and hence $H(u_t u_{t+1}^{-1}) \cap X = \emptyset$. From the proof of Corollary 2.4, the labelling under which all elements of Hu_t are labelled by $(t-1)\max\{k, \lceil j/2 \rceil\}$ is an $L(j,k)$-labelling of Γ. This labelling has span $(n_1n_2-1)\max\{k, \lceil j/2 \rceil\}$ and uses n_1n_2 labels, and hence is optimal for μ. In the case where $2k \ge j$, we have $\max\{k, \lceil j/2 \rceil\} = k$ and hence (17) together with (30) gives $\lambda_{j,k}(\Gamma) = (n_1n_2-1)k$, as stated in (19). Moreover, in this case the $L(j,k)$-labelling above is optimal for $\lambda_{j,k}$ as well. □

Proof of Corollary 2.10. The truth of (20) and (21) follows from (19) and (18), respectively. In addition, in the present case where $2k = j = 2$, the labelling given in the last paragraph of the proof of Theorem 2.9 is a no-hole $L(2,1)$-labelling, and it is optimal for λ and μ simultaneously. □

Remark 5.2. (a) The conditions that $n_1 > d$ and each prime factor of n_1 is no less than d cannot be removed from Theorem 2.9 simultaneously for otherwise the result will not be guaranteed. In fact, for the d-cube Q_d with $d \ge 3$, both conditions are not satisfied; we have $\lambda(Q_d) \ge d+3$ [19], whilst the right-hand side of (20) is 3. This suggests that hypercubes deserve a different treatment, and this has been done in the previous section.

(b) Unlike [9, Theorem 3.1], Theorem 2.9 and Corollary 2.10 apply even when there are only two complete graph factors (that is, $d = 2$) in the Cartesian product, as long as n_2 divides n_1 and $n_1 > 2$. For such pairs (n_1, n_2), the λ-number of H_{n_1,n_2} is one less than the number of vertices, and each label is used exactly once in any $L(2,1)$-labelling optimal for λ. Harary [14] has asked for a characterization of graphs with this property.

(c) For any graph Γ, we have $\lambda(\Gamma) \ge \mu(\Gamma) - 1$ by definition, and the equality occurs if and only if there exists a no-hole $L(2,1)$-labelling which is optimal for both λ and μ. The Hamming graphs in Corollary 2.10 constitute a family of infinitely many graphs for which $\lambda(\Gamma) = \mu(\Gamma) - 1$ holds.

6. Concluding remarks. In this paper we introduced a general approach to $L(j,k)$-labelling Cayley graphs on Abelian groups. Then we used this approach to study the $L(j,k)$-labelling problem for Hamming graphs and those graphs whose automorphism groups contain a vertex-transitive Abelian subgroup. The results we obtained for these two families of graphs implied the known results [29, Theorem 3.7] and [9, Theorem 3.1] as special cases. It is expected that the approach would be useful in studying labelling problems for other families of Cayley graphs on Abelian groups.

Based on Theorem 2.9 we may ask naturally the following questions.

QUESTION 6.1. (a) Let j and k be integers with $2k \ge j \ge k \ge 1$. Is

$$\lambda_{j,k}(H_{n_1,n_2,\ldots,n_d}) = (n_1n_2-1)k$$

true for any sequence $n_1 \ge n_2 \ge \cdots \ge n_d$ of $d \ge 2$ integers which are no less than 2 but not all equal to 2?

(b) In particular, is $\lambda(H_{n_1,n_2,\ldots,n_d}) = n_1n_2 - 1$ true for the same sequence?

In other words, we would like to know whether (19) is valid for any Hamming graph other than a hypercube provided that $2k \ge j$. The result in [10, Theorem 4.2] shows that the answer to (b) is affirmative for H_{n_1,n_2} with $2 \le n_2 \le n_1$ and $(n_1, n_2) \ne (2,2)$. In general, a recent result of the author with Chang and Lu [5] shows that, if n_1 is substantially larger than n_2 and d, then the answer to (b) above is affirmative. As we have seen in the proof of Theorem 2.9, if we could find a subgroup H of the group G (defined in (34)) such that H avoids the Cayley set X (defined in (35)), $|G : H| = n_1 n_2$ and $\mathcal{G}_{H,X}$ is a generating set of G/H, then the answer to both (a) and (b) of Question 6.1 is positive. However, we suspect that in general the answers to these questions are negative.

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