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Bounding the bandwidths for graphs

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Abstract

Let G, H be finite graphs with $|V(H)| \geqslant |V(G)|$. The bandwidth of G with respect to H is defined to be $B_H(G) = \min_\pi \max_{uv \in E(G)} d_H(\pi(u), \pi(v))$, with the minimum taken over all injections π from $V(G)$ to $V(H)$, where $d_H(x, y)$ is the distance in H between two vertices $x, y \in V(H)$. This number is involved with the VLSI design and optimization, especially when the "host" graph H is a path P_n or a cycle C_n of length $n = |V(G)|$. In these two cases, $B_H(G)$ is known to be the ordinary bandwidth $B(G)$ and the cyclic bandwidth $B_c(G)$, respectively, and the corresponding decision problem is NP-complete. So estimations of $B(G)$, $B_c(G)$ and in general $B_H(G)$ are needed, especially in determining the bandwidths of some specific graphs. In this paper, we first propose a systematic method for obtaining lower bounds for the bandwidth $B_H(G)$. By using this method, we then get a number of lower bounds for $B(G)$ and $B_c(G)$ in terms of some distance- and degree-related parameters. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The bandwidth problem originated in the 1950s from the sparse matrix computation and received much attention since Harper [7] studied the bandwidth of the n-cube and Harary [5] publicized the problem at a conference in Prague. For the results and a large number of references on this subject, the reader is referred to the survey papers [2, 3]. Traditionally, there are two equivalent ways of defining the bandwidth of a finite, simple, undirected graph $G = (V(G), E(G))$. (A graph is *simple* if it has no loops and multiedges.) With the first definition, any bijection f from the vertex set $V(G)$ to the set $\{1, 2, \ldots, n\}$ is taken as a *labelling* of G, where $n = |V(G)|$ is the number of vertices of G. For such a labelling f, we denote

$$B(G, f) = \max_{uv \in E(G)} |f(u) - f(v)|.$$

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The *bandwidth* of G is then defined to be

$$B(G) = \min_f B(G, f),$$

where the minimum is taken over all possible labellings f of G.

In the second way of defining the bandwidth, $B(G)$ is viewed as the smallest value of the maximum "spans" of the edges of G when G is embedded on the *path* P_n of n vertices. It is this definition that enables us to generalize [3] the concept of bandwidth to a more general framework and makes the connections between bandwidth and VLSI optimization clear. Suppose we are given a *host graph* $H = (V(H), E(H))$ with at least n vertices. An *embedding* of G on H is an injection π from $V(G)$ to $V(H)$, and this can be viewed as a layout of G on H. Denote by $d_H(x, y)$ the *distance* in H between two vertices $x, y \in V(H)$ (that is, the length of a shortest path in H connecting x and y). Then

$$B_H(G, \pi) = \max_{uv \in E(G)} d_H(\pi(u), \pi(v))$$

is the longest distance in H between any two vertices of H hosting two adjacent vertices of G. The *bandwidth of G with respect to H* is defined [3] to be

$$B_H(G) = \min_{\pi} B_H(G, \pi)$$

with the minimum taking over all possible embeddings π.

Clearly, the bandwidth $B_{P_n}(G)$ of G with respect to P_n is exactly $B(G)$, which we call the *ordinary bandwidth* in the following. Other interesting candidates for the host graphs include the *cycle* C_n of n vertices and the *grid graph* $P_n \times P_n$ on the plane, and in both cases the corresponding bandwidths, known as the *cyclic bandwidth* $B_c(G)$ and the *two-dimensional bandwidth* $B_2(G)$, respectively, arise from the circuit layout models involved in VLSI design or optimization (see, [1, 8, 12]). These two kinds of bandwidth have been receiving increasing attention in recent years (see, e.g. [3, 8, 10–12]). Similar to the ordinary bandwidth, the cyclic bandwidth $B_c(G)$ can be defined [3, 10] equivalently as

$$B_c(G) = \min_f \max_{uv \in E(G)} d_c(f(u), f(v)),$$

where the minimum is taken over all bijection (*cyclic labelling*) from $V(G)$ to the additive group $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ of integers modulo n, and $d_c(f(u), f(v)) = \min\{|f(u) - f(v)|, n - |f(u) - f(v)|\}$.

The decision problems corresponding to the bandwidth [13] and the cyclic bandwidth [10] are known to be NP-complete. Therefore, it is unlikely to find the exact values of $B(G)$ and $B_c(G)$ for general graph G and hence the estimations in each case become important. Even in the case where the exact value of, say, the ordinary bandwidth is determinable, the estimation is also desired since if we can find a labelling f with $B(G, f)$ achieving a lower bound for $B(G)$, then that lower bound is in fact the bandwidth $B(G)$. With this strategy, Harper's lower bound [7]

$$B(G) \geqslant \max_{1\leqslant k\leqslant n} \min_{|S|=k} \max\{|N^-(S)|, |N^+(S)|\}, \tag{1}$$

Chvátal's *density lower bound* [4]

$$B(G) \geqslant \left\lceil \frac{n-1}{D(G)} \right\rceil, \tag{2}$$

and some other lower bounds (see, e.g. [14]) were frequently used in determining the bandwidths of some specific graphs, where $S \subseteq V(G)$, $N^+(S) = \{u \in V(G)\backslash S\colon\ uv \in E(G)$ for a vertex $v \in S\}$, $N^-(S) = N^+(V(G)\backslash S)$ and $D(G)$ is the diameter of G. (The *diameter* $D(G)$ is the maximum distance between two vertices of G if G is connected and is defined to be ∞ otherwise.) The same strategy is applicable to determining $B_c(G)$ (and in general $B_H(G)$) for some specific graphs G (see, e.g. [10, 11]).

The purpose of this paper is to provide a systematic method for obtaining lower bounds for the bandwidth $B_H(G)$ in terms of some graphical parameters. The basic idea (see the next section) is to relax the condition of embedding G on H with the aid of a graphical parameter possessing some kind of monotonic property. The method is genuinely simple and elementary. Nevertheless, it seems quite efficient when the parameters are chosen appropriately. We illustrate this by examining some distance- and degree-related parameters and thus yielding a number of lower bounds for the ordinary and cyclic bandwidths. In both cases, the method gives rise to new estimations, as well as improvements of some known results.

2. The parameter-relaxation method

We refer to Harary [6] for undefined terminology. *Graphs considered are finite, simple and undirected, and we always use $G = (V(G), E(G))$ to denote a graph with order $n = |V(G)|$ and size $m = |E(G)|$.* We write $G \subseteq H$ if G is isomorphic to a subgraph of a graph H, and denote by $G[S]$ the subgraph of G induced by $S \subseteq V(G)$. For a real number x, we use $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote the largest integer no more than x and the smallest integer no less than x, respectively. For an integer $k \geqslant 1$, the kth *power graph* of H, denoted by H^k, is the graph with the same vertex set as H in which two vertices u, v are adjacent if and only if $d_H(u, v) \leqslant k$. One can see that $G \subseteq H^k$ if and only if there exists an embedding π of G on H such that $B_H(G, \pi) \leqslant k$. Thus, we have the following basic result which is the starting point of our method.

Theorem 1. *Suppose G, H are graphs with $|V(H)| \geqslant |V(G)|$. Then*

$$B_H(G) = \min\{k\colon\ G \subseteq H^k\}. \tag{3}$$

This result is known in the literature for the ordinary and cyclic bandwidths (see, e.g. [3, 10]). The method provided in the paper, which refines the method used in [4], is based on the observation that there are a large number of graphical parameters which are either increasing or decreasing, where a graphical parameter φ is said to be *increasing* (*decreasing*, respectively) if $G_1 \subseteq G_2$ implies $\varphi(G_1) \leqslant \varphi(G_2)$ ($\varphi(G_1) \geqslant \varphi(G_2)$, respectively). A graphical parameter φ is said to be *spanning increasing* (*spanning decreasing*, respectively) if G_1 is isomorphic to a spanning subgraph of G_2 implies $\varphi(G_1) \leqslant \varphi(G_2)$ ($\varphi(G_1) \geqslant \varphi(G_2)$, respectively). It is clear that an/a increasing (decreasing, respectively) parameter is spanning increasing (spanning decreasing, respectively), but not conversely; and that φ is spanning increasing (spanning decreasing, respectively) if and only if the removal of one edge from a graph does not increase (decrease, respectively) the value of φ. Increasing parameters include the maximum degree Δ, the chromatic number χ (and some generalized chromatic numbers, see, e.g. [15]), the edge-chromatic number χ', the thickness θ, and so on. Also, the bandwidth numbers B, B_c and B_2 are increasing. Spanning decreasing parameters include the diameter D, the vertex independence number β, the edge-covering number α', the domination number γ, and so on. (The *domination number* of a graph G_1 is the minimum cardinality of a subset $S \subseteq V(G_1)$ such that each vertex in $V(G_1) \backslash S$ is adjacent to at least one vertex in S. Other parameters mentioned here can be found in [6].) For an/a increasing (decreasing, respectively) graphical parameter φ, it is clear that $\{k\colon\ G \subseteq H^k\} \subseteq \{k\colon\ \varphi(G) \leqslant \varphi(H^k)\}$ ($\{k\colon\ G \subseteq H^k\} \subseteq \{k\colon\ \varphi(G) \geqslant \varphi(H^k)\}$, respectively); and this is true also for spanning increasing (spanning decreasing, respectively) φ if H has the same order with G. Combining this observation with Theorem 1, we get the following result.

Theorem 2. *Suppose G, H are graphs with $|V(H)| \geqslant |V(G)|$ and φ is a graphical parameter.*
(a) *If φ is increasing, then*

$$B_H(G) \geqslant \min\{k\colon\ \varphi(G) \leqslant \varphi(H^k)\}. \tag{4}$$

(b) *If φ is decreasing, then*

$$B_H(G) \geqslant \min\{k\colon\ \varphi(G) \geqslant \varphi(H^k)\}. \tag{5}$$

Moreover, if $|V(H)| = |V(G)|$, then (4) *and* (5) *hold for spanning increasing and spanning decreasing parameters φ, respectively.*

For the ordinary and cyclic bandwidths, we can further give the local forms of (4) and (5). For this purpose we make the following convention: *In the remainder of the paper, we assume that P_n, C_n, G are defined on the same vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$, and we take P_n as the path $u_1 u_2 \ldots u_n$ and C_n as the cycle $u_1 u_2 \ldots u_n u_1$. For a non-empty subset $S = \{u_{i_1}, u_{i_2}, \ldots, u_{i_s}\}$ of $V(G)$, where $s = |S|$ and $i_1 < i_2 < \cdots < i_s$, we use P_s and C_s to denote the path $u_{i_1} u_{i_2} \ldots u_{i_s}$ and the cycle $u_{i_1} u_{i_2} \ldots u_{i_s} u_{i_1}$ with vertex set S, respectively.* For a parameter φ possibly defined only for the graphs with vertex sets contained in $V(G)$, we use the terminologies of spanning increasing and spanning decreasing in a similar way as above; and for such a parameter, (4) or (5) is valid also. With the convention above, we then have the following theorem.

Theorem 3. *Let* $\emptyset \neq S \subseteq V(G)$ *and set* $s = |S|$. *Let* φ *be a parameter defined for all graphs with vertex sets contained in* $V(G)$.
(a) *If* φ *is spanning increasing, then*

$$B(G) \geqslant \min\{k:\ \varphi(G[S]) \leqslant \varphi(P_s^k)\}, \tag{6}$$

$$B_c(G) \geqslant \min\{k:\ \varphi(G[S]) \leqslant \varphi(C_s^k)\}. \tag{7}$$

(b) *If* φ *is spanning decreasing, then*

$$B(G) \geqslant \min\{k:\ \varphi(G[S]) \geqslant \varphi(P_s^k)\}, \tag{8}$$

$$B_c(G) \geqslant \min\{k:\ \varphi(G[S]) \geqslant \varphi(C_s^k)\}. \tag{9}$$

Proof. Suppose $S = \{u_{i_1}, u_{i_2}, \ldots, u_{i_s}\}$ with $i_1 < i_2 < \cdots < i_s$, and let P_s be as above.

Claim 1. $(P_n^k)[S]$ *is a spanning subgraph of* P_s^k.

In fact, if $u_{i_a}, u_{i_b} \in S$ are adjacent in P_n^k, then $|i_b - i_a| \leqslant k$. Since $|b-a| \leqslant |i_b - i_a|$, this implies $|b-a| \leqslant k$. In other words, u_{i_a}, u_{i_b} are adjacent in the kth power graph P_s^k of P_s and hence Claim 1 follows.

Claim 2. *If* φ *is spanning increasing (spanning decreasing, respectively), then* $G \subseteq P_n^k$ *implies* $\varphi(G[S]) \leqslant \varphi(P_s^k)$ $(\varphi(G[S]) \geqslant \varphi(P_s^k)$, *respectively).*

In fact, if φ is spanning increasing (spanning decreasing, respectively), then $G \subseteq P_n^k$ $\Rightarrow G[S]$ is a spanning subgraph of $(P_n^k)[S] \Rightarrow G[S]$ is a spanning subgraph of $P_s^k \Rightarrow \varphi(G[S]) \leqslant \varphi(P_s^k)$ $(\varphi(G[S]) \geqslant \varphi(P_s^k)$, respectively), where the second implication is based on Claim 1.

Now (6) and (8) follow from Theorem 1 and Claim 2 immediately. Similarly, $(C_n^k)[S]$ is isomorphic to a spanning subgraph of the kth power graph C_s^k of the cycle $C_s = u_{i_1} u_{i_2} \ldots u_{i_s} u_{i_1}$, and hence (7) and (9) follow. □

Theorems 2 and 3 provide us a general approach for obtaining lower bounds for $B_H(G)$ and, in particular, $B(G)$ and $B_c(G)$. For a concrete parameter φ, $\varphi(H^k)$ is in general a function of k; and solving k from $\varphi(G) \leqslant \varphi(H^k)$ (if φ is increasing) or $\varphi(G) \geqslant \varphi(H^k)$ (if φ is decreasing) usually gives an inequality $k \geqslant g(\cdot)$ for some function g which involves $\varphi(G)$ (and possibly $|V(G)|, |V(H)|$, etc.). So from Theorem 2 we then get a lower bound $B_H(G) \geqslant \lceil g(\cdot) \rceil$. The derivation of "local-type" lower bounds for B and B_c from Theorem 3 is similar.

3. Lower bounds for the ordinary bandwidth

Theoretically, each spanning increasing or spanning decreasing parameter could induce two lower bounds for $B(G)$ via Theorems 2 and 3. For example, since the diameter D is spanning decreasing and $D(P_n^k) = \lceil (n-1)/k \rceil$, we get from (5) that $B(G) \geqslant \min\{k: D(G) \geqslant \lceil (n-1)/k \rceil\} \geqslant \min\{k: D(G) \geqslant (n-1)/k\} = \lceil (n-1)/D(G) \rceil$, which is precisely the density lower bound (2). Applying (8) to D in a similar way, we have $B(G) \geqslant \lceil (|S|-1)/D(G[S]) \rceil$ for any $\emptyset \neq S \subseteq V(G)$. Since $D(G_1) \geqslant D(G[S])$ for any subgraph $G_1 \subseteq G$ with vertex set S, this implies the following useful *local density lower bound* (see, e.g. [3, Theorem 3.3]):

$$B(G) \geqslant \max_{G_1 \subseteq G} \left\lceil \frac{|V(G_1)| - 1}{D(G_1)} \right\rceil . \tag{10}$$

Since the number of edges, the maximum degree Δ and the minimum degree δ are spanning increasing, and the independence number β and the chromatic number χ are spanning decreasing, we get the following known lower bounds (see, [2–4]) immediately from Theorem 2.

(i) $B(G) \geqslant n - \frac{1}{2}(1 + \sqrt{(2n-1)^2 - 8m})$.
(ii) $B(G) \geqslant \lceil \Delta(G)/2 \rceil$.
(iii) $B(G) \geqslant \delta(G)$.
(iv) $B(G) \geqslant \chi(G)$.
(v) $B(G) \geqslant \lceil n/\beta(G) \rceil - 1$.

We can top up this list by considering more spanning monotonic parameters. The aim of this section is, however, not to examine all such parameters possible and then derive the lower bounds for $B(G)$. We would rather focus on some typical parameters and see how the method gives rise to interesting results. We first consider a sequence of parameters relating to the distance. For $u \in S \subseteq V(G)$, define $e_S(u,G) = \max_{v \in S} d_G(u,v)$; and denote by $e_S^1(G) \leqslant e_S^2(G) \leqslant \cdots \leqslant e_S^s(G)$ the sequence of all such $e_S(u,G)$ in increasing order, where $s = |S|$. In particular, if $S = V(G)$, then we omit the subscript S. Thus, $e^1(G) \leqslant e^2(G) \leqslant \cdots \leqslant e^n(G)$ is the sequence of the eccentricities of the vertices of G in increasing order, where the *eccentricity* [6] $e(u,G)$ of a vertex $u \in V(G)$ is the maximum distance in G from u to any other vertex of G.

Lemma 1. *For $1 \leqslant j \leqslant n$, we have*

$$e^j(P_n^k) = \begin{cases} \left\lceil \dfrac{(n/2) + \lfloor (j-1)/2 \rfloor}{k} \right\rceil & \text{if } n \text{ is even,} \\[2ex] \left\lceil \dfrac{(n-1)/2 + \lfloor j/2 \rfloor}{k} \right\rceil & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If n is even, then $e(v_i, P_n^k) = e(v_{n+1-i}, P_n^k) = \lceil (n-i)/k \rceil$ for $1 \leqslant i \leqslant n/2$. This implies that $e^j(P_n^k) = \lceil (n/2) + \lfloor (j-1)/2 \rfloor / k \rceil$ for each j. The proof for odd integer n is similar and hence omitted. □

Theorem 4. *For any* $\emptyset \neq S \subseteq V(G)$, *we have*

$$B(G) \geqslant \begin{cases} \max_{1 \leqslant j \leqslant |S|} \left\lceil \dfrac{(|S|/2) + \lfloor (j-1)/2 \rfloor}{e_S^j(G)} \right\rceil & \text{if } |S| \text{ is even,} \\ \max_{1 \leqslant j \leqslant |S|} \left\lceil \dfrac{(|S|-1)/2 + \lfloor j/2 \rfloor}{e_S^j(G)} \right\rceil & \text{if } |S| \text{ is odd.} \end{cases} \quad (11)$$

Proof. Suppose that $S = \{u_{i_1}, u_{i_2}, \ldots, u_{i_s}\}$, where $s = |S|$ and $i_1 < i_2 < \cdots < i_s$, and that P_s is the path $u_{i_1} u_{i_2} \ldots u_{i_s}$. One can check that $e_S(u_{i_a}, P_n^k) \geqslant e(u_{i_a}, P_s^k)$ for each $u_{i_a} \in S$ and hence $e_S^j(P_n^k) \geqslant e^j(P_s^k)$ for each $1 \leqslant j \leqslant s$. Since e_S^j is spanning decreasing, by (5) we have $B(G) \geqslant \min\{k\colon\ e_S^j(G) \geqslant e_S^j(P_n^k)\} \geqslant \min\{k\colon\ e_S^j(G) \geqslant e^j(P_s^k)\}$. If s is even, then Lemma 1 gives $e^j(P_s^k) \geqslant ((s/2) + \lfloor (j-1)/2 \rfloor)/k$ and hence $B(G) \geqslant \min\{k\colon\ e_S^j(G) \geqslant ((s/2) + \lfloor (j-1)/2 \rfloor)/k\} = \lceil ((s/2) + \lfloor (j-1)/2 \rfloor)/e_S^j(G) \rceil$. Similarly, if s is odd, then $B(G) \geqslant \lceil ((|s|-1)/2 + \lfloor (j/2) \rfloor)/e_S^j(G) \rceil$. By the arbitrariness of j, the lower bound (11) follows. $\square$

Setting $S = V(G)$ in Theorem 4, we get the following lower bound for the bandwidth in terms of the eccentricities of the vertices of G.

Corollary 1.

$$B(G) \geqslant \begin{cases} \max_{1 \leqslant j \leqslant n} \left\lceil \dfrac{(n/2) + \lfloor (j-1)/2 \rfloor}{e^j(G)} \right\rceil & \text{if } n \text{ is even,} \\ \max_{1 \leqslant j \leqslant n} \left\lceil \dfrac{(n-1)/2 + \lfloor j/2 \rfloor}{e^j(G)} \right\rceil & \text{if } n \text{ is odd.} \end{cases} \quad (12)$$

Note that the maximum eccentricity $e^n(G)$ is exactly the diameter $D(G)$, and hence the nth term on the right-hand side of (12) is equal to $\lceil (n-1)/D(G) \rceil$ whether n is even or odd. So (12) is an improvement of the density lower bound (2). Similarly, denoting $D(S,G) = e_S^s(G) = \max_{u,v \in S} d_G(u,v)$ and considering the $|S|$th term on the right-hand side of (11), we get the following lower bound which improves the local density lower bound (10) as $D(S,G) \leqslant D(G[S])$.

Corollary 2 (Lin [9, Theorem 2]). *For each* k *with* $1 \leqslant k \leqslant n-1$, *denote by* $\mathscr{S}_k$ *the family of all maximal subsets* S *of* $V(G)$ *satisfying* $D(S,G) = k$. *Then we have*

$$B(G) \geqslant \max_{1 \leqslant k \leqslant n-1} \max_{S \in \mathscr{S}_k} \left\lceil \frac{|S|-1}{k} \right\rceil. \quad (13)$$

Now let us consider another sequence of parameters. Let $\ell \geqslant 1$ be an integer. For each $u \in V(G)$, we define the ℓ-*degree* of u, denoted by $d_\ell(u,G)$, to be the number of vertices of $V(G) \backslash \{u\}$ within distance ℓ from u. (Note that the 1-degree is the ordinary degree.) We denote the sequence of all such $d_\ell(u,G)$, for $u \in V(G)$, in increasing order by $d_\ell^1(G) \leqslant d_\ell^2(G) \leqslant \cdots \leqslant d_\ell^n(G)$.

Lemma 2. *If* $1 \leqslant k \leqslant (n-3)/2\ell$, *then*

$$d_\ell^j(P_n^k) = \begin{cases} k\ell + \lfloor \frac{j-1}{2} \rfloor & \text{if } 1 \leqslant j \leqslant 2k\ell + 2, \\ 2k\ell & \text{if } 2k\ell + 3 \leqslant j \leqslant n. \end{cases} \tag{14}$$

If $(n-1)/2\ell \leqslant k \leqslant n$, *then*

$$d_\ell^j(P_n^k) = \begin{cases} k\ell + \lfloor \frac{j-1}{2} \rfloor & \text{if } 1 \leqslant j \leqslant 2(n-k\ell) - 2, \\ n-1 & \text{if } 2(n-k\ell) - 1 \leqslant j \leqslant n. \end{cases} \tag{15}$$

Proof. Note that $d_{P_n^k}(u_i,u_j) \leqslant \ell$ if and only if $|i-j| \leqslant k\ell$. So we have $d_\ell(u_i,P_n^k) = \min\{i-1,k\ell\} + \min\{n-i,k\ell\}$. Thus, if $1 \leqslant k \leqslant (n-3)/2\ell$, then

$$d_\ell(u_i,P_n^k) = \begin{cases} k\ell + i - 1 & \text{if } 1 \leqslant i \leqslant k\ell + 1, \\ 2k\ell & \text{if } k\ell + 2 \leqslant i \leqslant n - k\ell - 1, \\ k\ell + n - i & \text{if } n - k\ell \leqslant i \leqslant n, \end{cases}$$

which implies (14). Similarly, if $(n-1)/2\ell \leqslant k \leqslant n$, then

$$d_\ell(u_i,P_n^k) = \begin{cases} k\ell + i - 1 & \text{if } 1 \leqslant i \leqslant n - k\ell - 1, \\ n-1 & \text{if } n - k\ell \leqslant i \leqslant k\ell + 1, \\ k\ell + n - i & \text{if } k\ell + 2 \leqslant i \leqslant n \end{cases}$$

and (15) follows. □

Theorem 5. *For any integer* $\ell \geqslant 1$, *we have*

$$B(G) \geqslant \max_{1 \leqslant j \leqslant n} \max \left\{ \left\lceil \frac{d_\ell^j(G) - \lfloor (j-1)/2 \rfloor}{\ell} \right\rceil, \left\lceil \frac{d_\ell^j(G)}{2\ell} \right\rceil \right\}. \tag{16}$$

Proof. From Lemma 2, one can see that

$$d_\ell^j(P_n^k) \leqslant k\ell + \left\lfloor \frac{j-1}{2} \right\rfloor \tag{17}$$

and

$$d_\ell^j(P_n^k) \leqslant 2k\ell \tag{18}$$

hold for $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant n$. Clearly, each d_ℓ^j is spanning increasing and hence from (4) and (17) we have

$$\begin{aligned} B(G) &\geqslant \min\{k\colon\ d_\ell^j(G) \leqslant d_\ell^j(P_n^k)\} \\ &\geqslant \min\left\{k\colon\ d_\ell^j(G) \leqslant k\ell + \left\lfloor \frac{j-1}{2} \right\rfloor\right\} \\ &= \left\lceil \frac{d_\ell^j(G) - \lfloor (j-1)/2 \rfloor}{\ell} \right\rceil. \end{aligned}$$

Similarly, from (4) and (18) one can get $B(G) \geqslant \lceil d_\ell^j(G)/2\ell \rceil$. Thus, we have

$$B(G) \geqslant \max \left\{ \left\lceil \frac{d_\ell^j(G) - \lfloor (j-1)/2 \rfloor}{\ell} \right\rceil, \left\lceil \frac{d_\ell^j(G)}{2\ell} \right\rceil \right\}$$

and (16) follows from the arbitrariness of j. □

Denote $\Delta_\ell(G) = d_\ell^n(G)$ (the maximum ℓ-degree) and $\delta_\ell(G) = d_\ell^1(G)$ (the minimum ℓ-degree). Then $\Delta_1(G) = \Delta(G)$, $\delta_1(G) = \delta(G)$ and $d_1^1(G) \leqslant d_1^2(G) \leqslant \cdots \leqslant d_1^n(G)$ is the ordinary degree sequence $d^1(G) \leqslant d^2(G) \leqslant \cdots \leqslant d^n(G)$ of G. So Theorem 5 implies the following consequences.

Corollary 3. (a) $B(G) \geqslant \max_{1 \leqslant \ell \leqslant n-1} \lceil \Delta_\ell(G)/2\ell \rceil$.
(b) $B(G) \geqslant \max_{1 \leqslant \ell \leqslant n-1} \lceil \delta_\ell(G)/\ell \rceil$.

Corollary 4 (Chvátal [4], see also Chinn et al. [2] and Chung [3]).

$$B(G) \geqslant \max_{1 \leqslant j \leqslant n} \max \left\{ d^j(G) - \left\lfloor \frac{j-1}{2} \right\rfloor, \left\lceil \frac{d^j(G)}{2} \right\rceil \right\}. \quad (19)$$

One can easily find examples where (a) ((b), respectively) in Corollary 3 is better than (ii) ((iii), respectively) mentioned at the beginning of this section. (The complete binary tree $T_{2,k}$ is such an example, see the end of next section.) The Petersen graph P can serve as an example for which the lower bound (16) is attainable and is better than (19). In fact, (19) gives $B(P) \geqslant 3$; whilst setting $\ell = 2$ in (16) we get $B(P) \geqslant 5$. This latter bound is tight as $B(P) = 5$.

Interestingly, by setting ℓ to be the diameter $D(G)$ in (16) we get the density lower bound (2) again since in such a case $d_\ell^j(G) = n-1$ for all $1 \leqslant j \leqslant n$.

4. Lower bounds for the cyclic bandwidth

In this section we use the parameter-relaxation method to derive lower bounds for the cyclic bandwidth B_c. As in the last section, we focus mainly on the eccentricities and the ℓ-degrees. Since C_n^k is the complete graph whenever $k > \lfloor n/2 \rfloor$, we have $B_c(G) \leqslant \lfloor n/2 \rfloor$ [10] from Theorem 1. *Therefore, we may assume that* $k \leqslant \lfloor n/2 \rfloor$ *in the following.*

We begin with some basic graphical parameters. It is not difficult to see that $|E(C_n^k)| = kn$, $\Delta(C_n^k) = 2k$, $B(C_n^k) = \min\{2k, n-1\} \leqslant 2k$, $\beta(C_n^k) = \lfloor n/(k+1) \rfloor \geqslant (n-k)/(k+1)$ and $\gamma(C_n^k) = \lceil n/(2k+1) \rceil \geqslant n/(2k+1)$. Since Δ, B and the number of edges are spanning increasing, and since β and γ are spanning decreasing, Theorem 2 implies

Theorem 6. (a) $B_c(G) \geqslant \lceil m/n \rceil$.
(b) $B_c(G) \geqslant \lceil \Delta(G)/2 \rceil$ [10].
(c) $B_c(G) \geqslant \lceil B(G)/2 \rceil$ [12].

(d) $B_c(G) \geqslant \lceil (n+1)/(\beta(G)+1) \rceil - 1$.
(e) $B_c(G) \geqslant \lceil \frac{1}{2}(n/\gamma(G) - 1) \rceil$.

In general, the density lower bound $B_c(G) \geqslant \lceil (n-1)/D(G) \rceil$ is no longer valid for the cyclic bandwidth, although it is valid whenever G is a tree [10]. However, we are able to prove the following theorem which implies a "density lower bound" ((22) below) and a "local density lower bound" ((21) below) in terms of the *radius* $r(G) = e^1(G)$. It is easy to see that $r(C_n^k) = D(C_n^k) = \lceil (\lfloor (n/2) \rfloor - 1)/k \rceil$ (so the eccentricities of the vertices of C_n^k are equal). Denote $r(S,G) = e_S^1(G)$ for $\emptyset \neq S \subseteq V(G)$. As r is spanning decreasing, an argument much similar to that in the proof of Theorem 4 leads to $B_c(G) \geqslant \lceil (\lfloor |S|/2 \rfloor - 1)/r(S,G) \rceil$. Thus, we have

Theorem 7. *For each integer k with $1 \leqslant k \leqslant n-1$, denote by $\mathscr{S}_k'$ the family of maximal subsets S of $V(G)$ satisfying $r(S,G) = k$. Then*

$$B_c(G) \geqslant \max_{1 \leqslant k \leqslant n-1} \max_{S \in \mathscr{S}_k'} \left\lceil \frac{\lfloor |S|/2 \rfloor - 1}{k} \right\rceil. \tag{20}$$

This result is a counterpart of Corollary 2 to the cyclic bandwidth. Note that $r(S,G) \leqslant r(G[S])$ for $S \subseteq V(G)$ and $r(G[S]) \leqslant r(G_1)$ for any subgraph $G_1 \subseteq G$ with vertex set S. Theorem 7 implies the following density and local density lower bounds for the cyclic bandwidth.

Corollary 5.

$$B_c(G) \geqslant \max_{G_1 \subseteq G} \left\lceil \frac{\lfloor |V(G_1)|/2 \rfloor - 1}{r(G_1)} \right\rceil. \tag{21}$$

In particular, we have

$$B_c(G) \geqslant \left\lceil \frac{\lfloor n/2 \rfloor - 1}{r(G)} \right\rceil. \tag{22}$$

We also found a lower bound for $B_c(G)$ in terms of $D(S,G)$. To this end we need the following lemma.

Lemma 3. *For any $\emptyset \neq S \subseteq V(G)$ and $1 \leqslant k \leqslant \lfloor n/2 \rfloor$, we have*

$$D(S, C_n^k) \geqslant \min \left\{ \left\lceil \frac{|S| - 1}{k} \right\rceil, \left\lceil \frac{\lceil n/|S| \rceil}{k} \right\rceil \right\}.$$

Proof. Suppose $S = \{u_{i_1}, u_{i_2}, \ldots, u_{i_s}\}$ with $1 = i_1 < i_2 < \cdots < i_s$, where $s = |S|$. If $i_s \leqslant \lfloor n/2 \rfloor$, then it is clear that $D(S, C_n^k) = d_{C_n^k}(u_{i_1}, u_{i_s}) = \lceil (i_s - 1)/k \rceil \geqslant \lceil (s-1)/k \rceil$. In the remaining case, the vertices in S separate the cycle C_n into s segments each with length no more than $\lfloor n/2 \rfloor - 1$. By the pigeonhole principle, at least one of them has length at least $\lceil n/s \rceil$. Thus, we have $D(S, C_n^k) \geqslant \lceil \lceil n/s \rceil / k \rceil$ and the proof is complete. $\square$

Theorem 8.

$$B_c(G) \geqslant \max\left\{ \max_{|S|(|S|-1)\leqslant n} \left\lceil \frac{|S|-1}{D(S,G)} \right\rceil, \max_{|S|(|S|-1)\geqslant n} \left\lceil \frac{\lceil n/|S| \rceil}{D(S,G)} \right\rceil \right\}. \tag{23}$$

Proof. Let $\emptyset \neq S \subseteq V(G)$. If $|S|(|S|-1)\leqslant n$, then $D(S,C_n^k)\geqslant\lceil(|S|-1)/k\rceil$ from Lemma 3. Note that $G\subseteq C_n^k$ implies $D(S,G)\geqslant D(S,C_n^k)$. Hence by (5) we get $B_c(G)\geqslant \min\{k:\ D(S,G)\geqslant\lceil(|S|-1)/k\rceil\}\geqslant \min\{k:\ D(S,G)\geqslant(|S|-1)/k\}=\lceil(|S|-1)/D(S,G)\rceil$. Similarly, if $|S|(|S|-1)\geqslant n$, then $D(S,C_n^k)\geqslant\lceil\lceil n/|S|\rceil/k\rceil$ and hence $B_c(G)\geqslant \lceil\lceil n/|S|\rceil/D(S,G)\rceil$. The result then follows from the arbitrariness of S. □

By the symmetry of C_n^k, the ℓ-degrees of the vertices of C_n^k are equal (that is, $\delta_\ell(C_n^k)=\Delta_\ell(C_n^k)$). This indicates that, when applying Theorem 2 to the ℓ-degrees, we can only get a lower bound of $B_c(G)$ in terms of $\Delta_\ell(G)$.

Lemma 4. *For any integer $\ell\geqslant 1$, we have*

$$\Delta_\ell(C_n^k)=\begin{cases} 2k\ell & \text{if } \ell\leqslant\lfloor\frac{\lfloor n/2\rfloor-1}{k}\rfloor,\\ n-1 & \text{if } \ell\geqslant\lceil\frac{\lfloor n/2\rfloor}{k}\rceil. \end{cases}$$

Proof. If $\ell\leqslant\lfloor(\lfloor n/2\rfloor-1)/k\rfloor$, then $k\ell+1\leqslant\lfloor n/2\rfloor$, and $u_2,u_3,\ldots,u_{k\ell+1}$ and $u_n,u_{n-1},\ldots,u_{n-k\ell+1}$ are the only vertices within distance ℓ from u_1 in C_n^k. Thus, we have $\Delta_\ell(C_n^k)=d_\ell(u_1,C_n^k)=2k\ell$. If $\ell\geqslant\lceil\lfloor n/2\rfloor/k\rceil$, then each vertex u_i is within distance ℓ from u_1 in C_n^k and hence $\Delta_\ell(C_n^k)=n-1$. □

Theorem 9. *For any integer $\ell\geqslant 1$, we have*

$$B_c(G)\geqslant \max_{1\leqslant\ell\leqslant n-1}\left\lceil\frac{\Delta_\ell(G)}{2\ell}\right\rceil. \tag{24}$$

Proof. From Lemma 4, one gets $\Delta_\ell(C_n^k)\leqslant 2k\ell$ for any integer $\ell\geqslant 1$ (this is true even when $\ell\geqslant\lceil\lfloor n/2\rfloor/k\rceil$). Applying Theorem 2, we then have $B_c(G)\geqslant\min\{k:\ \Delta_\ell(G)\leqslant\Delta_\ell(C_n^k)\}\geqslant\min\{k:\ \Delta_\ell(G)\leqslant 2k\ell\}=\lceil\Delta_\ell(G)/2\ell\rceil$. □

Theorem 9 improves Corollary 3(a) as $B(G)\geqslant B_c(G)$. The lower bound (24) is attainable and is indeed better than the existing bound $B_c(G)\geqslant\lceil\Delta(G)/2\rceil$ (see Theorem 6(b)) in some cases. For example, if G is the k-level complete binary tree $T_{2,k}$ with root v in which the ith level consists of 2^{i-1} vertices and each vertex in level $i<k$ has two "sons" in level $i+1$. Then $\Delta(G)=3$ and Theorem 6(b) gives $B_c(G)\geqslant 2$. It is easy to see that the right-hand side of (24) is $\lceil(2^{k-1}-1)/k-1\rceil=\lceil d_{k-1}(v)/2(k-1)\rceil$, which is exactly the bandwidth $B(T_{2,k})$ ([3, Theorem 3.7]). Since $B_c(T_{2,k})\leqslant B(T_{2,k})$, this value is also the cyclic bandwidth $B_c(T_{2,k})$ and hence the equalities in both Corollary 3(a) and (24) are attained.

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