Available at
www.ElsevierComputerScience.com
POWERED BY SCIENCE DIRECT
Theoretical Computer Science 310 (2004) 501–511
Theoretical Computer Science
www.elsevier.com/locate/tcs

Note

A channel assignment problem for optical networks modelled by Cayley graphs

Sanming Zhou[1]

Department of Mathematics and Statistics, The University of Melbourne, Parkville, VIC 3010, Australia

Received 21 May 2002; received in revised form 2 July 2003; accepted 11 July 2003
Communicated by D.-Z. Du

Abstract

A problem arising from a recent study of scalability of optical networks seeks to assign channels to the vertices of a network so that vertices distance 2 apart receive distinct channels. In this paper we introduce a general channel assignment scheme for Cayley graphs on abelian groups, and derive upper bounds for the minimum number of channels needed for such graphs. As application we give a systematic way of producing near-optimal channel assignments for connected graphs admitting a vertex-transitive abelian group of automorphisms. Hypercubes are examples of such graphs, and for them our near-optimal upper bound gives rise to the one obtained recently by Wan.

Keywords: Channel assignment; Optical network; Cayley graph; Hypercube

1. Introduction

Optical networking [12] has been widely recognized as a key technology in communication and computer networks due to its very promising applications in high-speed supercomputing, distributed computing, scientific visualisation, and so on. In an optical network, processors are interconnected by optical fibre links, each of which supports a given number of wavelengths. To enhance scalability Aly and Dowd [1,2] suggested to use a class of networks which efficiently combine space with time and/or wavelength division. In such a network, vertices are grouped into clusters with time and/or wavelength multiplexing, and the clusters are interconnected by fibre links. All clusters contain the same number, m_0, of vertices [14,15]. In the case where $m_0 > 1$, internal

[1] Supported by a Discovery Project Grant (DP0344803) from the Australian Research Council.
E-mail address: smzhou@ms.unimelb.edu.au (S. Zhou).

doi:10.1016/S0304-3975(03)00394-3

links in each cluster are provided to allow communications within the cluster. Taking the clusters as vertices we then get a new network, called the *cluster interconnection network* (CIN). For such a CIN, a key issue [14,15] is to assign channel sets to its vertices (clusters) so that no conflicts may happen at input couplers. Depending on whether $m_0 = 1$ or $m_0 > 1$, this channel assignment problem can be formulated [14] as the vertex colouring problems in the following definition.

Definition 1. Let $\Gamma = (V, E)$ be a simple undirected graph. An assignment of n colours to the vertices of Γ is called a $(2, n)$-*colouring* ($(\bar{2}, n)$-*colouring*, respectively) of Γ if vertices distance 2 apart (with distance at most 2, respectively) receive distinct colours. The minimum number of colours needed for such a colouring is denoted by $\chi_2(\Gamma)$ ($\chi_{\bar{2}}(\Gamma)$, respectively). A colouring scheme is *balanced* if each colour is used by the same number of vertices.

Equivalently, we can define a $(2, n)$-colouring of Γ as a partition $\{P_1, \ldots, P_n\}$ of the vertex set V such that any two non-adjacent vertices in the same part P_i are not joined by a length-two path of Γ, and define a $(\bar{2}, n)$-colouring of Γ as a $(2, n)$-colouring $\{P_1, \ldots, P_n\}$ such that each P_i is an independent set. (An *independent set* of Γ is a set of pairwise non-adjacent vertices of Γ.) In the following we will use these equivalent definitions.

The reader is referred to [14,15] for more background information on the colourings above. One can find in [15] an optimal $\bar{2}$-colouring for a special Cayley digraph called rotator digraph, and in [1] results on other CIN topologies. In [10] the concepts of $(2, n)$- and $(\bar{2}, n)$-colourings are generalised, and the problems of colouring hypercubes so that two vertices with distance exactly k, at most k, respectively, receive different colours are studied, where k is a positive integer.

Before proceeding to the main results in this paper, we would like to record a few connections between $\chi_2, \chi_{\bar{2}}$ and other invariants for graphs. First, as noticed in [10], from the definitions above we have

$$\chi_2(\Gamma) \leqslant \chi_{\bar{2}}(\Gamma) = \chi(\Gamma^2), \tag{1}$$

where χ denotes the chromatic number and Γ^2 is the square of Γ. (The *chromatic number* of a graph Γ is the minimum number of colours needed to colour the vertices of Γ such that adjacent vertices receive distinct colours. The *square* of Γ is the graph with the same vertex set as Γ in which two vertices are adjacent if and only if they are within distance 2 in Γ.)

Second, we point out that, the invariant $\chi_{\bar{2}}$ is equal to the radio colouring number, which was introduced by Harary [5] in studying the channel assignment problem for radio communication systems. An $L(2,1)$-*labelling* [4], or a *radio colouring* as used in [5], of a graph Γ is an assignment of labels—non-negative integers—to the vertices of Γ such that adjacent vertices receive labels that differ by at least 2, and vertices at distance 2 receive different labels. The minimum number of labels needed is called the *radio colouring number* of Γ by Harary [5]. One can show that this number is equal to $\chi(\Gamma^2)$ (see [17] for an explanation), and hence is the same as $\chi_{\bar{2}}(\Gamma)$.

The *vertex linear arboricity* $\mathrm{vla}(\Gamma)$ of Γ is the minimum number of parts into which V can be partitioned such that each part induces a forest whose connected components are paths. From this definition and the definition of χ_2 it follows that

$$\chi_2(\Gamma) \geqslant \mathrm{vla}(\Gamma).$$

If Γ is *triangle-free*, that is, Γ contains no cycle of length 3, then for each $v \in V$, any two neighbours of v are distance 2 apart and hence must be assigned different colours under any $(2,n)$-colouring of Γ. Thus, for triangle-free graphs Γ, we get

$$\chi_2(\Gamma) \geqslant \Delta(\Gamma) \tag{2}$$

by choosing v to be a vertex with maximum degree $\Delta(\Gamma)$.

Finally, we notice that, for triangle-free graphs Γ, $\chi_2(\Gamma)$ can be bounded below by the following three basic invariants: the *independence number* $\beta(\Gamma)$ (maximum size of an independent set of Γ), the *edge independence number* $\beta'(\Gamma)$ (maximum size of a set of edges no two of which have an end-vertex in common), and the *clique number* $\omega(\Gamma)$ (maximum size of a set of pairwise adjacent vertices of Γ). In fact, in a $(2,n)$-colouring $\{P_1,\dots,P_n\}$ of Γ, any two non-adjacent vertices in the same P_i are not joined by a path of Γ with length 2. A necessary (but not sufficient) condition for this to be true is that, for each P_i, all connected components of the subgraph $\Gamma[P_i]$ induced by P_i are complete graphs. In particular, if Γ is triangle-free, then the connected components of each $\Gamma[P_i]$ are isolated vertices or isolated edges. Hence, in this case, the partition $\{P_1,\dots,P_n\}$ is necessarily a $(2,n)$-colouring in the sense of [16], and consequently the 2-chromatic number of Γ defined in [16] provides a lower bound for $\chi_2(\Gamma)$. From this and [16, Theorem 4], we obtain

$$\chi_2(\Gamma) \geqslant \max\ \{\lceil \omega(\Gamma)/2\rceil, \lceil (|V| - 2\beta'(\Gamma))/\beta(\Gamma)\rceil, \lceil |V|^2/(|V|^2 - 2(|E| - \beta'(\Gamma)))\rceil\} \tag{3}$$

for any triangle-free graph Γ.

We will follow standard terminology and notation for graphs and groups, see for example [3,8,13], respectively.

2. Main results

Cayley graphs are recommended [6,11] strongly by computer scientists and mathematicians as good models for interconnection networks. Such graphs possess many desirable properties, including vertex-symmetry, maximal edge-fault tolerance and existence of uniform shortest path routings. As a matter of fact, a lot of networks currently being used are Cayley graphs. These include [6,11] hypercubes, butterflies, cube-connected cycles, star graphs and their generalisations, and many other networks of both theoretical and practical importance.

In this paper, we will introduce a general scheme for 2-colouring Cayley graphs Γ on abelian groups, and derive upper bounds for $\chi_2(\Gamma)$. These will be presented in

Theorem 1 and Corollary 1 below. The family of Cayley graphs on abelian groups is very large, with notable members including Hamming graphs, hypercubes, circulant graphs, etc., and our scheme applies to all of them. As application we give (Theorem 2) a systematic way of producing near-optimal 2-colourings for connected graphs admitting a vertex-transitive abelian group of automorphisms. Roughly speaking, any matrix over the field $\mathrm{GF}(2)=\{0,1\}$ with a certain property corresponds to such a 2-colouring, see the proof of Theorem 2. In particular, hypercubes are examples of such graphs, and for them our near-optimal upper bound gives rise to the one obtained recently in [14].

For any group G, a subset S of G is called a *Cayley set* of G if $1 \notin S$ and S is inverse-closed, namely $s \in S$ implies $s^{-1} \in S$, where 1 is the identity element of G. For such a set S, the *Cayley graph* of G with respect to S, denoted by $\Gamma(G,S)$, is the graph with vertices the elements of G in which $x, y \in G$ are adjacent if and only if $xy^{-1} \in S$. The conditions imposed on S ensure that $\Gamma(G,S)$ is a simple undirected graph. *Throughout the paper we will assume that G is a finite abelian group.* As usual, for a subgroup N of G, we use G/N to denote the quotient group of G by N, and $|G:N| := |G/N|$ the order of G/N. Thus, the elements of G/N are Nx for $x \in G$, where Nx is the coset of N in G containing x. For any subset X of G, denote $X/G = \{Nx\colon\ x \in X\}$. (Note that X/N is not necessarily a subgroup of G/N, and that $Ny \in X/N$ does not imply $y \in X$.) In particular, for a Cayley set S of G, we define

$$S/N := \{Ns\colon\ s \in S\}, \quad S^*/N := S/N - \{N\}. \tag{4}$$

Here and in the following "−" stands for set-theoretic subtraction. Since $(Ns)^{-1} = Ns^{-1}$ and S is closed under taking inverse, it follows that S^*/N is closed under taking inverse as well. Also, S^*/N does not contain the identity element N of G/N. So S^*/N is a Cayley set of the quotient group G/N, and thus we have the Cayley graph $\Gamma(G/N, S^*/N)$ defined on G/N. Denote

$$S^2 := \{xy\colon\ x, y \in S\}.$$

Since S is a Cayley set of G, we have $1 \notin S$ but $1 = ss^{-1} \in S^2$ as S is inverse-closed.

Theorem 1. *Let G be a finite abelian group and S a Cayley set of G. Let N be a subgroup of G such that $N \cap S^2 = \{1\}$. Then*

$$\chi_2(\Gamma(G,S)) \leqslant \chi_{\bar{2}}(\Gamma(G/N, S^*/N)). \tag{5}$$

Moreover, any $(\bar{2}, n)$-colouring $\{\mathscr{P}_i\}_{i=1}^n$ of $\Gamma(G/N, S^*/N)$ gives rise to a $(2,n)$-colouring of $\Gamma(G,S)$, namely $\{\bigcup_{Nx \in \mathscr{P}_i} Nx\}_{i=1}^n$, and the former is balanced if and only if the latter is balanced.

This shows a close relationship between 2-colourings of the original Cayley graph $\Gamma(G,S)$ and $\bar{2}$-colourings of the Cayley graph $\Gamma(G/N, S^*/N)$ on the quotient group G/N. Clearly, the colouring under which each coset (as a vertex of $\Gamma(G/N, S^*/N)$) receives a distinct colour is a balanced $\bar{2}$-colouring of $\Gamma(G/N, S^*/N)$. Applying Theorem 1 to this trivial case, we get the following corollary.

Corollary 1. *Let* G, S *and* N *be as in Theorem* 1. *Then*

$$\chi_2(\Gamma(G,S)) \leqslant |G:N|. \tag{6}$$

Moreover, the colouring under which two elements of G *receive the same colour if and only if they are in the same coset of* N *in* G *is a balanced* $(2,|G:N|)$-*colouring of* $\Gamma(G,S)$.

For any Cayley set S of G, if we choose $N=\{1\}$ to be the trivial subgroup of G, then of course the condition $N \cap S^2 = \{1\}$ is satisfied. This condition may be satisfied also by some non-trivial subgroups N of G, and we are interested in such N with $|G:N|$ as small as possible.

The *automorphism group* $\mathrm{Aut}(\Gamma)$ of a graph $\Gamma=(V,E)$ is the group of adjacency-preserving permutations on V. A subgroup G of $\mathrm{Aut}(\Gamma)$ is said to be *vertex-transitive* if for any $u,v \in V$ there exists $g \in G$ such that g permutes u to v. In this case we also say that Γ is *G-vertex-transitive*, or *vertex-transitive* if $G=\mathrm{Aut}(\Gamma)$. A G-vertex-transitive graph must be regular, that is, all the vertices have the same degree. Two graphs $\Gamma_i=(V_i,E_i), i=1,2$, are *isomorphic*, written $\Gamma_1 \cong \Gamma_2$, if there is a bijection from V_1 to V_2 such that $u,v \in V_1$ are adjacent in Γ_1 if and only if their images are adjacent in Γ_2.

Using Corollary 1 and techniques from linear algebra, we will prove the following:

Theorem 2. *Let* Γ *be a connected triangle-free graph. Suppose the automorphism group* $\mathrm{Aut}(\Gamma)$ *of* Γ *contains a vertex-transitive abelian subgroup. Then*

$$d \leqslant \chi_2(\Gamma) \leqslant 2^{\lceil \log_2 d \rceil}, \tag{7}$$

where d *is the degree of the vertices of* Γ. *Moreover, we can give balanced* $(2,2^{\lceil \log_2 d \rceil})$-*colourings* (*not unique*) *of* Γ *explicitly*.

Such colourings will be produced by using null spaces of certain matrices over GF(2), see the proof in Section 4. The reader is referred to [7–9] for existence and constructions of graphs satisfying the conditions of Theorem 2. Hypercubes Q_d are such graphs, and for them we can produce a number of balanced $(2,2^{\lceil \log_2 d \rceil})$-colourings of Q_d systematically and explicitly, see Corollary 2 below. In fact, we have $Q_d \cong \Gamma(V^+(d,2),S)$, so we may identify Q_d with $\Gamma(V^+(d,2),S)$, where $V^+(d,2)$ is the additive group of the d-dimensional linear space $V(d,2)$ of row vectors over GF(2), and $S=\{\mathbf{e}_1,\dots,\mathbf{e}_d\}$ is the standard basis of $V(d,2)$. (For each i, $\mathbf{e}_i$ is the vector of $V(d,2)$ with the i-coordinate 1 and all other coordinates 0.) For a $d \times n$ matrix A over GF(2), denote by N_A the additive group of the null space $\{\mathbf{x} \in V(d,2)\colon\ \mathbf{x}A=\mathbf{0}_n\}$ of A, where $\mathbf{0}_n$ is the zero vector (identity element) of $V(n,2)$. From a geometric point of view, the cosets of N_A in $V^+(d,2)$ are *flats* of $V(d,2)$ induced by N_A. Theorem 2 and its proof imply the following:

Corollary 2. *Let* Q_d *be the* d-*dimensional cube. Then* [14]

$$d \leqslant \chi_2(Q_d) \leqslant 2^{\lceil \log_2 d \rceil}. \tag{8}$$

Moreover, for any $d \times \lceil \log_2 d \rceil$ *matrix* A *over* GF(2) *with rank* $\lceil \log_2 d \rceil$ *and rows pairwise distinct, the colouring of* Q_d *under which two vectors of* $V(d,2)$ *receive the same colour if and only if they are in the same flat induced by* N_A *is a balanced* $(2, 2^{\lceil \log_2 d \rceil})$*-colouring of* Q_d.

Remark 1. (a) From the proof of Theorem 2, we will see that the upper bound in (7) remains true if Γ is not triangle-free (that is, $xy = z$ holds for some $x, y, z \in S$). But the lower bound $\chi_2(\Gamma) \geqslant d$ is not guaranteed in this case. For example, let $\Gamma = \Gamma(G,S)$ with $G = V^+(2,2)$ and $S = G - \{\mathbf{0}_2\}$. Then $\Gamma \cong K_4$ has triangles and admits G as a vertex-transitive abelian subgroup of automorphisms [3, Lemma 16.3]. For this graph $\chi_2(\Gamma) \geqslant d$ does not hold since Γ has degree $d = 3$ whilst $\chi_2(\Gamma) = 1$.

(b) For d a power of 2, the lower and upper bounds in (7) coincide, so we get $\chi_2(\Gamma) = d$ and similarly $\chi_2(Q_d) = d$. For other integers d, the upper bound $2^{\lceil \log_2 d \rceil}$ in (7) and (8) is near-optimal. In this latter case, from our proof of Theorem 2, and by using (5) instead of (6), it seems that the upper bound $2^{\lceil \log_2 d \rceil}$ can be improved. (If this is true, then the improved bound will not be generated by balanced 2-colourings.) On the contrary, Wan [14] conjectured that $\chi_2(Q_d) = 2^{\lceil \log_2 d \rceil}$ for all d. He also presented [14] a specific balanced $(2, 2^{\lceil \log_2 d \rceil})$-colouring of Q_d. Corollary 2 enables us to produce systematically a number of such near-optimal 2-colourings. The existence of the matrices A in this corollary is guaranteed by the fact that $V(\lceil \log_2 d \rceil, 2)$ contains $2^{\lceil \log_2 d \rceil} \geqslant d$ distinct vectors.

(c) Wan [14] also gives an upper bound for $\chi_{\bar{2}}(Q_d)$, namely $\chi_{\bar{2}}(Q_d) \leqslant 2^{\lceil \log_2 (d+1) \rceil}$. This bound follows from a general result [17, Theorem 3.1] on the radio colouring number.

3. Proof of Theorem 1

For any graph $\Gamma = (V, E)$ and partition $\mathscr{P}$ of V, the *quotient graph* $\Gamma_{\mathscr{P}}$ of Γ with respect to $\mathscr{P}$ is defined to be the graph with vertex set $\mathscr{P}$ in which $P, Q \in \mathscr{P}$ are adjacent if and only if there exist $u \in P$ and $v \in Q$ such that $\{u, v\} \in E$. We will use $\Gamma[P,Q]$ to denote the bipartite subgraph of Γ with vertex set $P \cup Q$ and all such edges $\{u, v\}$ of Γ between P and Q. In the case where each part of $\mathscr{P}$ is an independent set of Γ with k vertices, for some integer $k \geqslant 1$, and $\Gamma[P,Q]$ is a perfect matching of k edges, the graph Γ is called a *k-fold cover* [3] of the quotient $\Gamma_{\mathscr{P}}$. In particular, for a Cayley graph $\Gamma(G,S)$ and a subgroup N of G, G/N is a natural partition of G with cosets Nx as its parts. Hence, we have the quotient graph $(\Gamma(G,S))_{G/N}$ of $\Gamma(G,S)$ with respect to G/N.

One can see that the sets S/N and S^*/N defined in (4) satisfy

$$S/N = \{Nx \in G/N\colon\ Nx \cap S \neq \emptyset\}, \tag{9}$$

$$S^*/N = (S - N)/N. \tag{10}$$

In fact, since $s \in Ns \cap S$ for any $s \in S$, $S/N \subseteq \{Nx \in G/N\colon\ Nx \cap S \neq \emptyset\}$. Conversely, if $Nx \cap S \neq \emptyset$, say $s \in Nx \cap S$, then $Nx = Ns \in S/N$ and hence (9) is proved. For $Ns \in S/N$, where $s\in S$, we have $Ns\in S^*/N \Leftrightarrow Ns\neq N \Leftrightarrow s\in S-N$, and hence (10) is valid. In the case where $N\cap S^2=\{1\}$, we have the following Lemma 1, which will be needed in the proof of Theorem 1. Note that case (b) in this lemma cannot occur if the order of G is odd. An element x of a group G is called an *involution* of G if $x\neq 1$ but $x^2=1$. It is well known that [3, Proposition 16.2] any Cayley graph $\Gamma(G,S)$ is G-vertex-transitive with degree $|S|$.

Lemma 1. *Let G be a finite abelian group and S a Cayley set of G. Let N be a subgroup of G such that $N\cap S^2=\{1\}$. Then*

$$|Nx\cap S| = \begin{cases} 0 & \text{if } Nx\notin S/N, \\ 1 & \text{if } Nx\in S/N \end{cases} \tag{11}$$

and the mapping $s\mapsto Ns$ for $s\in S$ is a bijection from S to S/N. In particular, $|S/N|=|S|$. Moreover, $(\Gamma(G,S))_{G/N}\cong\Gamma(G/N,S^*/N)$, and the following (a) *and* (b) *hold.*

(a) *If $N\notin S/N$, then $S^*/N=S/N$, $N\cap S=\emptyset$, each coset Nx is an independent set of $\Gamma(G,S)$, and $\Gamma(G,S)$ is an $|N|$-fold cover of $\Gamma(G/N,S^*/N)$.*
(b) *If $N\in S/N$, then $S^*/N=S/N-\{N\}$, $N\cap S=\{s\}$ for an involution s of G, $|N|$ is even, each coset Nx induces a perfect matching of $|N|/2$ edges, and deleting from $\Gamma(G,S)$ all such matchings results in an $|N|$-fold cover of $\Gamma(G/N,S^*/N)$.*

Proof. By (9), $|Nx\cap S|=0$ if $Nx\notin S/N$ and $|Nx\cap S|\geqslant 1$ if $Nx\in S/N$. In the latter case, if $y,z\in Nx\cap S$, say $y=gx$, $z=hx$ for some $g,h\in N$, then $yz^{-1}=(gx)(hx)^{-1}=gh^{-1}$ and hence $yz^{-1}\in N\cap S^2$. But $N\cap S^2=\{1\}$ by our assumption, so $yz^{-1}=1$, that is, $y=z$. Thus, $|Nx\cap S|=1$ and (11) is proved. In particular, $|Ns\cap S|=1$ for $s\in S$. Hence the mapping $s\mapsto Ns$ is a bijection from S to S/N, and consequently $|S/N|=|S|$.

In the following we set $\Gamma:=\Gamma(G,S)$ and $\Gamma^*:=\Gamma(G/N,S^*/N)$. Since $(gx)(hx)^{-1}=gh^{-1}\in N$ for any $g,h\in N$ and $x\in G$, by the definition of a Cayley graph we have

(i) Two elements gx,hx in the same coset Nx are adjacent in Γ if and only if $gh^{-1}\in N\cap S$.

Since G is abelian, for distinct Nx,Ny and $g,h\in N$ we have: gx,hy are adjacent in $\Gamma \Leftrightarrow (gx)(hy)^{-1}\in S \Leftrightarrow (ugx)(uhy)^{-1}\in S$ for any $u\in N$. Note that ugx runs over Nx when u runs over N, and that $uhy\neq u'hy$ whenever $u\neq u'$. So all elements in Nx have the same number of neighbours in Ny, and similarly all elements in Ny have the same number of neighbours in Nx. Therefore, we have

(ii) $\Gamma[Nx,Ny]$ is a regular subgraph of Γ, for Nx,Ny adjacent in the quotient graph $\Gamma_{G/N}$.

In the case where $N\notin S/N$, we have $S^*/N=S/N$ and $N\cap S=\emptyset$ by (11). Thus, by (i) each coset Nx is an independent set of Γ. We have: $Nx,Ny\in G/N$ are adjacent in $\Gamma^* \Leftrightarrow Nx(Ny)^{-1}\in S/N \Leftrightarrow N(xy^{-1})=Ns$ for some $s\in S \Leftrightarrow xy^{-1}=gs$ for some $g\in N$ and $s\in S \Leftrightarrow x(gy)^{-1}=s$ for some $g\in N$ and $s\in S \Leftrightarrow x\in Nx$ and $gy\in Ny$ are adjacent in Γ for some $g\in N \Leftrightarrow Nx,Ny$ are adjacent in the quotient graph $\Gamma_{G/N}$. Here in the last step we used (ii) above. Hence we have $\Gamma^* \cong \Gamma_{G/N}$. Moreover, since Γ and Γ^* have the same degree, namely $|S| = |S^*/N|$, $\Gamma[Nx, Ny]$ is forced to be a perfect matching between Nx and Ny. In other words, Γ is an $|N|$-fold cover of Γ^*.

In the remaining case where $N \in S/N$, we have $S^*/N = S/N - \{N\}$ and, by (11), $N \cap S = \{s\}$ for some $s \in G$. Since $1 \notin S$, we have $s \neq 1$. Since $s^2 \in N \cap S^2$, we have $s^2 = 1$ by our assumption, and hence s is an involution of G. As $N \cap S = \{s\}$, it follows from (i) that $gx, hx \in Nx$ are adjacent if and only if $gh^{-1} = s$, where $g, h \in N$. Thus, each $gx \in Nx$ is adjacent to exactly one vertex in Nx, namely $s^{-1}gx$. So N must be of even order, and the subgraph of Γ induced on Nx is a perfect matching of $|N|/2$ edges. By a similar argument as above, one can show that $\Gamma^* \cong \Gamma_{G/N}$, and that deleting from Γ all such matchings results in an $|N|$-fold cover of Γ^*. □

Proof of Theorem 1. Let G, S, N be as in Theorem 1. Denote $\Gamma := \Gamma(G, S)$ and $\Gamma^* := \Gamma(G/N, S^*/N)$. We prove first that any two elements in the same coset Nx are not distance 2 apart in Γ. Suppose on the contrary that gx, hx are distinct elements in Nx with distance 2 in Γ, where $g, h \in N$. Then $g \neq h$ and gx, hx have a common neighbour, say $uy \in Ny$ for some $u \in N$ and $y \in G$. We have $Nx \neq Ny$ for otherwise the subgraph of Γ induced by Nx would contain the path gx, uy, hx of length 2, which contradicts Lemma 1. Since gx, uy are adjacent in Γ, it follows that $(gx)(uy)^{-1} \in S$. Similarly, since uy, hx are adjacent in Γ, we have $(uy)(hx)^{-1} \in S$. So $gh^{-1} = ((gx)(uy)^{-1})((uy)(hx)^{-1}) \in S^2$, and hence $gh^{-1} \in N \cap S^2$. But $N \cap S^2 = \{1\}$ by our assumption, so we have $gh^{-1} = 1$ and $g = h$. This final contradiction shows that the distance of any two elements in the same coset of N is not equal to 2.

Suppose $\{\mathscr{P}_1, \ldots, \mathscr{P}_n\}$ is a $(\bar{2}, n)$-colouring of Γ^* for some integer n, that is, a partition of G/N such that any two cosets in the same part $\mathscr{P}_i$ are distance at least 3 apart in Γ^*. For each i, define P_i to be the union of the cosets in $\mathscr{P}_i$, that is, $P_i := \bigcup_{Nx \in \mathscr{P}_i} Nx$. We assert that the distance in Γ between any two elements of P_i is not equal to 2. Suppose otherwise, and let $gx, hy \in P_i$ be elements which are distance 2 apart in Γ. Let gx, uz, hy be a length-two path of Γ, where $g, h, u \in N$. By the result in the previous paragraph, gx, hy must be in distinct cosets Nx, Ny, and Nx, Ny must be in $\mathscr{P}_i$ by the definition of P_i. If $uz \in Nx$, then the edge joining uz and hy connects Nx and Ny, and thus Nx, Ny are adjacent in $\Gamma_{G/N}$. In other words, they are adjacent in Γ^* since $\Gamma_{G/N} \cong \Gamma^*$ by Lemma 1. This contradicts the assumption that any two cosets in $\mathscr{P}_i$ are distance at least 3 apart in Γ^*. Hence $Nz \neq Nx$, and similarly $Nz \neq Ny$. Note that Nx and Nz are adjacent in $\Gamma_{G/N}$ since $\{gx, uz\}$ is an edge of Γ between Nx and Nz. Similarly, Nz and Ny are adjacent in $\Gamma_{G/N}$. Thus, Nx, Nz, Ny is a length-two path of $\Gamma_{G/N} \cong \Gamma^*$, which again contradicts the assumption above. Therefore, the distance in Γ between any two elements of P_i is not equal to 2, and so we can colour all of them with the same colour i without violating the regulation of $(2, n)$-colouring. In other words, $\{P_1, \ldots, P_n\}$ is a $(2, n)$-colouring of Γ, which is induced by the $(\bar{2}, n)$-colouring $\{\mathscr{P}_1, \ldots, \mathscr{P}_n\}$ of Γ^*. Moreover, since $|P_i| = |N||\mathscr{P}_i|$ for each i, all $\mathscr{P}_i$ have the same cardinality if and only if all P_i have the same cardinality. That is, $\{\mathscr{P}_1, \ldots, \mathscr{P}_n\}$ is a balanced $(\bar{2}, n)$-colouring of Γ^* if and only if $\{P_1, \ldots, P_n\}$ is a balanced $(2, n)$-colouring of Γ. For $n = \chi_{\bar{2}}(\Gamma^*)$, a $(\bar{2}, n)$-colouring of Γ^* exists, and it gives rise to a $(2, n)$-colouring of Γ. Hence (5) follows and the proof is complete. □

Proof of Corollary 1. The colouring under which the members of G/N all receive different colours is a $(\bar{2}, |G:N|)$-colouring of $\Gamma(G/N, S^*/N)$. From the proof above, this colouring induces the $(2, |G:N|)$-colouring of $\Gamma(G,S)$ under which two elements receive the same colour if and only if they are in the same coset Nx. This colouring is balanced since each colour is used by $|N|$ vertices. □

4. Proofs of Theorem 2 and Corollary 2

Now we prove Theorem 2 by using Corollary 1. In the proof we will use the following lemma. Although it is stated in [3] for the full automorphism group $\mathrm{Aut}(\Gamma)$, the result is valid for any transitive abelian subgroup of $\mathrm{Aut}(\Gamma)$ and the proof is the same. For a G-vertex-transitive graph $\Gamma = (V,E)$, if for any $u, v \in V$ there exists a unique $g \in G$ which maps u to v, then G is said to be *regular* on V.

Lemma 2 (Biggs [3, Proposition 16.5]). *Let $\Gamma = (V,E)$ be a graph such that* $\mathrm{Aut}(\Gamma)$ *contains a vertex-transitive abelian subgroup G. Then G is regular on V, and G is an elementary abelian 2-group.*

Recall that we use $V^+(n,2)$ to denote the additive group of the linear space $V(n,2)$. The operation of this group is addition of vectors. Henceforth, we will use $N + \mathbf{x}$ and $2S$ to replace $N\mathbf{x}$ and S^2, respectively, where N is a subgroup of $V^+(n,2)$ and $\mathbf{x} \in V(n,2)$. It is well known that $V^+(n,2)$ is isomorphic to the elementary abelian 2-group $\mathbb{Z}_2^n$ of order 2^n. From the definition of a Cayley graph, one can see that $\Gamma(G,S)$ is connected if and only if S is a *generating set* of G, that is, each element of G has the form $s_1^{n_1} \cdots s_t^{n_t}$ for some $s_1, \ldots, s_t \in S$ and integers $n_1, \ldots, n_t$, where $t \geqslant 1$.

Proof of Theorem 2. Suppose $\Gamma = (V,E)$ is a connected triangle-free graph with degree d such that $\mathrm{Aut}(\Gamma)$ has a vertex-transitive abelian subgroup G. Since Γ is triangle-free, we have $\chi_2(\Gamma) \geqslant d$ by (2). So we need to prove the upper bound in (7) only.

By Lemma 2, G is regular on V and G is an elementary abelian 2-group. Hence $|G| = 2^\ell$ and $G \cong \mathbb{Z}_2^\ell$ for a positive integer ℓ. In the following we will identify G with the group $V^+(\ell,2)$. Since G is regular on V, by Biggs [3, Lemma 16.3] Γ is isomorphic to a Cayley graph of G, that is, $\Gamma \cong \Gamma(G,S)$ for a Cayley set $S := \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ of G, where each $\mathbf{x}_i \in V(\ell,2) - \{\mathbf{0}_\ell\}$. Since Γ is connected, S must be a generating set of G. This is equivalent to saying that S contains a basis of the linear space $V(\ell,2)$. Hence $\ell \leqslant d$. Also, we have $d < 2^\ell$ as S is a proper subset of G. Set $n := \lceil \log_2 d \rceil$. Then $2^{n-1} < d \leqslant 2^n$. So $2^{n-1} < d < 2^\ell$, which implies $n \leqslant \ell$ and hence $n \leqslant d$.

We will show by explicit construction that there exists an $\ell \times n$ matrix A over GF(2) with rank n such that $\mathbf{x}_1 A, \ldots, \mathbf{x}_d A$ are pairwise distinct. Once this is achieved, then the null space

$$U_A := \{\mathbf{x} \in V(\ell,2):\ \mathbf{x}A = \mathbf{0}_n\}$$

of A is an $(\ell - n)$-dimensional subspace of $V(\ell,2)$, and thus $|G : N_A| = 2^n$ holds for the additive group N_A of U_A. Also, since $\mathbf{x}_1 A, \ldots, \mathbf{x}_d A$ are distinct, we have $(\mathbf{x}_i + \mathbf{x}_j)A \neq \mathbf{0}_n$

for $i \neq j$ and hence $N_A \cap (2S) = \{\mathbf{0}_n\}$. Thus, from Corollary 1 we get $\chi_2(\Gamma) \leqslant |G : N_A| = 2^n$, which is exactly the upper bound in (7). Also from Corollary 1, for each such A we can give explicitly a balanced $(2, 2^n)$-colouring of Γ, namely the one under which two vectors of $V(\ell, 2)$ receive the same colour if and only if they are in the same flat $N_A + \mathbf{x}$ induced by N_A.

Now it remains to construct explicitly matrices A with required properties. Since $n \leqslant d \leqslant 2^n$ and $V(n, 2)$ has 2^n vectors, we can choose d distinct vectors $\mathbf{c}_1, \ldots, \mathbf{c}_d$ in $V(n, 2)$ such that the $d \times n$ matrix C with the ith row $\mathbf{c}_i$ has rank n. Thus, the n columns of C are independent vectors of dimension d. Since $n \leqslant \ell \leqslant d$, we can add $\ell - n$ column vectors of dimension d to C to form a $d \times \ell$ matrix Y of rank ℓ. Thus $YB = C$, where $B = \binom{I_n}{0}$ with I_n the $n \times n$ identity matrix and 0 the all-zero matrix of dimension $(\ell - n) \times n$. Let X be the $d \times \ell$ matrix with the ith row $\mathbf{x}_i$, for $1 \leqslant i \leqslant d$. Then X has rank ℓ since S is a generating set of G. Since Y has also rank ℓ, there exists a non-singular $\ell \times \ell$ matrix D over GF(2) such that $Y = XD$. Now we set $A = DB$. Then the non-singularity of D implies that A has the same rank as B, namely n. Moreover, we have $XA = X(DB) = YB = C$, which implies $\mathbf{x}_i A = \mathbf{c}_i$ for each i. Thus, $\mathbf{x}_1 A, \ldots, \mathbf{x}_d A$ are pairwise distinct, and the matrix A satisfies all the requirements. □

Note that in the above proof of $\chi_2(\Gamma) \leqslant 2^n$ we do not need the condition that Γ is triangle-free. One can see that the matrix A is not unique, and each A gives rise to one balanced $(2, 2^n)$-colouring of Γ.

Proof of Corollary 2. We will use the proof of Theorem 2 and the notation there. The d-dimensional cube $Q_d = \Gamma(V^+(d, 2), S)$ admits $G = V^+(d, 2) \cong \mathbb{Z}_2^d$ as a vertex-transitive abelian subgroup of automorphisms, where $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ (see the paragraph before Corollary 2). So $\ell = d$, and we need to find $d \times n$ matrices A over GF(2) with rank $n = \lceil \log_2 d \rceil$ such that $\mathbf{e}_1 A, \ldots, \mathbf{e}_d A$ are pairwise distinct, that is, the rows of A are pairwise distinct. Such matrices exist since $V(n, 2)$ contains $2^n \geqslant d$ vectors. In fact, we can choose A to be any $d \times n$ matrix over GF(2) such that the first n rows are n linearly independent vectors of $V(n, 2)$ and the remaining $d - n$ rows are pairwise distinct and distinct from the first n rows.

For each $d \times n$ matrix A with rank n and rows pairwise distinct, from the proof of Theorem 2, the colouring such that two vectors of $V(d, 2)$ receive the same colour if and only if they are in the same flat $N_A + \mathbf{x}$ is a balanced $(2, 2^n)$-colouring of Q_d. This completes the proof. □

Acknowledgements

The author thanks Dr. Peng-Jun Wan for kindly providing his papers [14,15].

References

[1] K.A. Aly, Reconfigurable WDM interconnection of large-scale multiprocessor systems, Ph.D. Thesis, Department of Electrical and Computer Engineering, SUNY, Buffalo, 1993.

[2] K.A. Aly, P.W. Dowd, A class of scalable optical interconnection networks through discrete broadcast-selected multi-domain WDM, in: Proc. IEEE INFOCOM'94, Torando, Canada, 1994.
[3] N.L. Biggs, Algebraic Graph Theory, 2nd Edition, Cambridge University Press, Cambridge, 1993.
[4] J.R. Griggs, R.K. Yeh, Labelling graphs with a condition at distance two, SIAM J. Discrete Math. 5 (1992) 586–595.
[5] F. Harary, M. Plantholt, Graphs whose radio coloring number equals the number of nodes, in: P. Hansen, O. Marcotte (Eds.), Graph Colouring and Applications (Montreal, QC, 1997), CRM Proc., Lecture Notes, Vol. 23, American Mathematical Society, Providence, RI, 1999, pp. 99–100.
[6] M.C. Heydemann, Cayley graphs and interconnection networks, in: G. Hahn, G. Sabidussi (Eds.), Graph Symmetry (Montreal, PQ, 1996), NATO Advanced Science Institutes Series C, Mathematica and Physical Sciences, Vol. 497, Kluwer Academic Publishers, Dordrecht, 1997, pp. 167–224.
[7] W. Imrich, Graphs with transitive abelian automorphism group, in: P. Erdős, A.R. Rényi, V. Sós (Eds.), Combinatorica Theory and its Applications, Vol. 4, Colloquia Mathematica Societatis János Bolyai, North-Holland, Amsterdam, 1970, pp. 651–656.
[8] W. Imrich, S. Klavžar, Product Graphs, Wiley, New York, 2000.
[9] W. Imrich, M.E. Watkins, On automorphism groups of Cayley graphs, Period. Math. Hungar. 7 (1976) 243–258.
[10] D.S. Kim, D.-Z. Du, P.M. Pardalos, A coloring problem on the n-cube, Discrete Appl. Math. 103 (2000) 307–311.
[11] S. Lakshmivarahan, J.S. Jwo, S.K. Dhall, Symmetry in interconnection networks based on Cayley graphs of permutation groups: a survey, Parallel Comput. 19 (1993) 361–407.
[12] R. Ramaswami, K. Sivarajan, Optical Networks, Morgan Kauffman Publishers, Los Altos, CA, 1998.
[13] J.S. Rose, A Course on Group Theory, Cambridge University Press, Cambridge, 1978.
[14] P.-J. Wan, Near-optimal conflict-free channel set assignments for an optical cluster-based hypercube network, J. Combin. Optim. 1 (1997) 179–186.
[15] P.-J. Wan, Conflict-free channel set assignment for an optical cluster interconnection network based on rotator digraphs, Theoret. Comput. Sci. 207 (1998) 193–201.
[16] S. Zhou, Locally restricted colorings of graphs, J. Combin. Math. Combin. Comput. 43 (2002) 147–157.
[17] S. Zhou, Labelling Cayley graphs on abelian groups, submitted.