Imprimitive Symmetric Graphs

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To My Parents

Abstract

A finite graph Γ is said to be G-symmetric if G is a group of automorphisms of Γ acting transitively on the ordered pairs of adjacent vertices of Γ. In most cases, the group G acts imprimitively on the vertices of Γ, that is, the vertex set of Γ admits a nontrivial G-invariant partition $\mathcal{B}$. The purpose of this thesis is to study such graphs, called imprimitive G-symmetric graphs.

In the first part of the thesis, we discuss in detail the geometric approach, introduced by Gardiner and Praeger in 1995, for studying imprimitive symmetric graphs which we use throughout. According to this approach, three configurations can be associated with $(\Gamma, \mathcal{B})$, namely the quotient graph $\Gamma_{\mathcal{B}}$ of Γ with respect to $\mathcal{B}$, the bipartite subgraph $\Gamma[B, C]$ of Γ induced by two adjacent blocks B, C of $\mathcal{B}$, and a certain 1-design $\mathcal{D}(B)$ induced on B (possibly with repeated blocks). The approach involves an analysis of these configurations and addresses the problem of reconstructing Γ from the triple $(\Gamma_{\mathcal{B}}, \Gamma[B, C], \mathcal{D}(B))$.

In the second part, we study the case where the block size k of $\mathcal{D}(B)$ is one less than the block size v of $\mathcal{B}$. We first assume that $\mathcal{D}(B)$ contains no repeated blocks, and prove that, under the assumption $k = v - 1 \geq 2$, this occurs precisely when $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. In this case, we find a very natural and simple construction of Γ from $\Gamma_{\mathcal{B}}$ and the induced action of G on $\mathcal{B}$, and prove that up to isomorphism it produces all such graphs Γ. If in addition $\Gamma_{\mathcal{B}}$ is a complete graph, then we classify all the possibilities for (Γ, G). We show that $\Gamma[B, C] \cong K_{v-1,v-1}$ if and only if $\Gamma_{\mathcal{B}}$ is $(G, 3)$-arc transitive, and that $\Gamma[B, C]$ is a matching of $v - 1$ edges and $\Gamma_{\mathcal{B}}$ is not a complete graph if and only if $\Gamma_{\mathcal{B}}$ is a certain near n-gonal graph for some even integer $n \geq 4$. In the general case where $\mathcal{D}(B)$ may contain repeated blocks, we give a construction of such graphs from G-point- and G-block-transitive 1-designs, and prove further that up to isomorphism it gives rise to all such graphs. By using this, we then classify such graphs arising from the classical projective and affine geometries.

In the last part, we will investigate the influence of certain "local" actions induced by the setwise stabilizer G_B on the structure of Γ, with emphasis on the case where Γ is G-locally quasiprimitive. In particular, we will study the case where the actions of G_B on B and on the neighbourhood of B in $\Gamma_{\mathcal{B}}$ are permutationally isomorphic.

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Publications Used in This Thesis

The majority of this thesis is based on certain parts of the following publications. As a coauthor or the sole author, I was involved actively in the research, planning and writing of these papers.

1. A. Gardiner, Cheryl E. Praeger and Sanming Zhou, Cross ratio graphs, *Proc. London Math. Soc.*, to appear. (Reference [46])

2. Cai Heng Li, Cheryl E. Praeger and Sanming Zhou, A class of finite symmetric graphs with 2-arc transitive quotients, *Math. Proc. Cambridge Philos. Soc.* **129** (2000), no. 1, 19-34. (Reference [53])

3. Cai Heng Li, Cheryl E. Praeger, Akshay Venkatesh and Sanming Zhou, Finite locally quasiprimitive graphs, *Discrete Math.*, to appear. (Reference [54])

4. Sanming Zhou, Almost covers of 2-arc transitive graphs, submitted to *J. London Math. Soc.* (Reference [97])

5. Sanming Zhou, Imprimitive symmetric graphs, 3-arc graphs and 1-designs, *Discrete Math.*, to appear. (Reference [98])

6. Sanming Zhou, Constructing a class of symmetric graphs, submitted to *European J. Combinatorics.* (Reference [99])

7. Sanming Zhou, Symmetric graphs and flag graphs, preprint. (Reference [100])

8. Sanming Zhou, Classifying a family of symmetric graphs, *Bull. Austral. Math. Soc.* **63** (2001), 329-335. (Reference [101])

List of Key Symbols

Groups and geometries

G, H, K	Groups
$G.H$	Semidirect product of G by H
$H \leq G$	H is a subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
$N_G(H)$	Normalizer of H in G, where $H \leq G$
$\mathrm{Core}_G(H)$	Core of H in G, where $H \leq G$
$\mathrm{soc}(G)$	Socle of G
α^G	G-orbit containing α
G_α	Stabilizer of α in G
G_Δ	Setwise stabilizer of Δ in G
$G_{(\Delta)}$	Pointwise stabilizer of Δ in G
$\mathrm{fix}_\Omega(T)$	Fixed point set
S_n	Symmetric group of degree n
A_n	Alternating group of degree n
$\mathbb{Z}_n$	Additive group of integers modulo n
$\mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}$	Mathieu groups
$\mathrm{PGL}(n, q)$	Projective general linear group
$\mathrm{PSL}(n, q)$	Projective special linear group
$\mathrm{P\Gamma L}(n, q)$	Semilinear projective group
$\mathrm{AGL}(n, q)$	Affine group
$\mathrm{A\Gamma L}(n, q)$	Semilinear affine group
$\mathrm{GF}(q)$	Finite field with q elements
$\mathrm{Sq}(q)$	Set of squares of $\mathrm{GF}(q)$
$V(n, q)$	n-Dimensional linear space over $\mathrm{GF}(q)$
$\mathrm{PG}(n, q)$	Projective geometry
$\mathrm{AG}(n, q)$	Affine geometry
$c(u, w; y, z)$	Cross-ratio of the 4-tuple (u, w, y, z)

Graphs

Γ, Σ, Ξ	Graphs
$V(\Gamma)$	Vertex set of Γ
$\mathrm{Arc}(\Gamma)$	Arc set of Γ
$\mathrm{Arc}_s(\Gamma)$	Set of s-arcs of Γ
$\Gamma(\alpha)$	Neighbourhood of α in Γ
$\Gamma[X]$	Subgraph of Γ induced by X
$\mathrm{Aut}(\Gamma)$	Full automorphism group of Γ
$\mathrm{val}(\Gamma)$	Valency of (a regular graph) Γ
$\mathrm{girth}(\Gamma)$	Girth of Γ
$\mathrm{diam}(\Gamma)$	Diameter of Γ
$\overline{\Gamma}$	Complement of Γ
$n \cdot \Gamma$	Union of n vertex-disjoint copies of Γ
K_n	Complete graph with n vertices
$K_{m,m}$	Complete bipartite graph with m vertices in each part
K_m^n	Complete n-partite graph with m vertices in each part
C_n	Cycle with length n
P_n	Path with length n
$\mathcal{B}$	G-invariant partition of $V(\Gamma)$, Γ a G-symmetric graph
$\Gamma_{\mathcal{B}}$	Quotient graph of Γ with respect to $\mathcal{B}$
$\Gamma[B, C]$	Bipartite subgraph induced by adjacent blocks $B, C \in \mathcal{B}$
$\mathcal{D}(B)$	1-Design induced on $B \in \mathcal{B}$
$\Gamma(B)$	Union of $\Gamma(\alpha)$, for $\alpha \in B$
$\Gamma_{\mathcal{B}}(B)$	Neighbourhood of B in $\Gamma_{\mathcal{B}}$
$\Gamma_{\mathcal{B}}(i, B)$	Set of blocks of $\mathcal{B}$ with distance no more than i from B
$\Gamma_{\mathcal{B}}(\alpha)$	Set of blocks of $\mathcal{D}(B)$ incident with α, where $\alpha \in B$
v	Block size of $\mathcal{B}$
k	Block size of $\mathcal{D}(B)$
r	Number of blocks in $\Gamma_{\mathcal{B}}(\alpha)$
b	Valency of $\Gamma_{\mathcal{B}}$
s	Valency of $\Gamma[B, C]$
$G_{[\alpha]}$	Subgroup of G_α fixing each $C \in \Gamma_{\mathcal{B}}(\alpha)$ setwise

(Continued)

$G_{[B]}$	Subgroup of G_B fixing setwise each block in $\Gamma_{\mathcal{B}}(B)$
$G_{[i,B]}$	Subgroup of G fixing setwise each block in $\Gamma_{\mathcal{B}}(i, B)$
$\mathrm{CR}(v; x, n)$	Untwisted cross-ratio graph
$\mathrm{TCR}(v; x, n)$	Twisted cross-ratio graph
$\Xi(\Sigma, \Delta)$	3-Arc graph of Σ with respect to Δ
$\mathrm{F}(\mathcal{D}, \Theta, \Psi)$	Flag graph of (a 1-design) $\mathcal{D}$ with respect to (Θ, Ψ)
$\mathcal{B}_N$	Normal partition induced by N

Contents

Chapter 1

Introduction

Things have roots and branches; affairs have scopes and beginnings.
To know what precedes and what follows will lead one near the WAY.
Confucius (551-479 B.C.), The Great Learning

1.1 Introduction

The study of symmetric graphs has long been one of the main themes in Algebraic Graph Theory. By definition a graph Γ is G-symmetric if Γ admits G as a group of automorphisms such that G is transitive on the ordered pairs of adjacent vertices of Γ. Roughly speaking, in most G-symmetric graphs Γ, the group G acts imprimitively on the vertices of Γ, that is, G is transitive on the vertex set $V(\Gamma)$ of Γ and $V(\Gamma)$ admits a nontrivial G-invariant partition $\mathcal{B}$. In this case Γ is said to be an imprimitive G-symmetric graph.

This thesis is dedicated to a study of imprimitive symmetric graphs, using a geometric approach which was first introduced by Gardiner and Praeger in 1995 for locally primitive symmetric graphs. According to this approach, the following three configurations can be associated with the triple $(\Gamma, G, \mathcal{B})$ above:

(i) the quotient graph $\Gamma_{\mathcal{B}}$ of Γ with respect to $\mathcal{B}$;

(ii) the bipartite subgraph $\Gamma[B, C]$ of Γ induced on $B \cup C$ with isolated vertices deleted, where B, C are blocks of $\mathcal{B}$ adjacent in $\Gamma_{\mathcal{B}}$; and

(iii) the 1-design $\mathcal{D}(B) = (B, \Gamma_{\mathcal{B}}(B), \mathrm{I})$ induced on a block $B \in \mathcal{B}$ such that $\alpha \mathrm{I} C$ for $\alpha \in B$ and $C \in \Gamma_{\mathcal{B}}(B)$ if and only if α is adjacent to some vertex of C, where $\Gamma_{\mathcal{B}}(B)$ is the neighbourhood of B in $\Gamma_{\mathcal{B}}$.

The graph Γ is thus "decomposed" into the "product" of these configurations, and the approach involves an analysis of them. Clearly the triple $(\Gamma_{\mathcal{B}}, \Gamma[B, C], \mathcal{D}(B))$ reflects the structure of Γ. In some cases, it even determines Γ uniquely (up to isomorphism), and this happens in particular when $\Gamma[B, C]$ is a complete bipartite graph between B and C. In this case Γ is the lexicographic product of $\Gamma_{\mathcal{B}}$ by an empty graph on $|B|$ vertices. However, in most cases the triple above does not determine the graph Γ. We will see a simple example of this in Section 4.2, see Remark 4.2.1. This suggests the following natural question.

Question 1 To what extent does the triple $(\Gamma_{\mathcal{B}}, \Gamma[B, C], \mathcal{D}(B))$ determine the graph Γ ?

As is widely recognized, the class of imprimitive symmetric graphs is very large. Because of this it might be more fruitful to consider some special classes of imprimitive symmetric graphs. With respect to this we propose the following problem.

Problem 1 For certain classes of triples $(\Sigma, \Pi, \mathcal{D})$, characterize or classify all possible $(\Gamma, G, \mathcal{B})$ such that $(\Gamma_{\mathcal{B}}, \Gamma[B, C], \mathcal{D}(B)) = (\Sigma, \Pi, \mathcal{D})$.

The effectiveness of the approach relies not only on a thorough understanding of the three configurations above but also on the feasibility of reconstructing Γ from the triple $(\Gamma_{\mathcal{B}}, \Gamma[B, C], \mathcal{D}(B))$. Therefore, refining Question 1, one may naturally ask the following question.

Question 2 Under what circumstances can we reconstruct the graph Γ from the triple $(\Gamma_{\mathcal{B}}, \Gamma[B, C], \mathcal{D}(B))$?

The study in this thesis will be more or less centered around these rather general problems. We will first discuss in detail the approach above for general imprimitive symmetric graphs. By using this we will then study an interesting and enlightening case where, for adjacent blocks B, C of $\mathcal{B}$, each part of the bipartition of $\Gamma[B, C]$ has size $|B| - 1$, that is, there exists a unique vertex in B which is not adjacent to any vertex in C. We will give a construction of such graphs Γ and, in particular, we will show that Γ can be reconstructed from $\Gamma_{\mathcal{B}}$ and the induced action of G on $\mathcal{B}$. We will also characterize or classify certain subclasses of such graphs. Finally, we will analyse the induced actions of the setwise stabilizer G_B on the block B and on the neighbourhood $\Gamma_{\mathcal{B}}(B)$ of B in $\Gamma_{\mathcal{B}}$, and study the influence of these actions on the structure of the graph Γ. A more detailed introduction to the main results in this thesis will be given in Section 1.3.

We now leave this discussion for a while and have an excursion to see some sample results in the world of symmetric graphs.

1.2 Literature review

Let Γ be a finite, undirected graph and let s be a positive integer. An s-arc of Γ is a sequence of $s+1$ vertices of Γ, not necessarily all distinct, such that any two consecutive vertices are adjacent and any three consecutive vertices are distinct. If Γ admits a group G of automorphisms such that G is transitive on the s-arcs of Γ, then Γ is said to be (G, s)-arc transitive. In most cases, such a graph is also G-vertex-transitive, and we assume this throughout without mentioning explicitly. Under this assumption the (G, s)-arc transitivity of Γ implies the $(G, s-1)$-arc transitivity of Γ, for $s > 1$. Usually a 1-arc is called an arc and a $(G, 1)$-arc transitive graph is called a G-symmetric graph.

Investigations of symmetric graphs can be found in the literature as early as in the 1940's when Tutte [82] proved that, for a G-symmetric cubic graph Γ, the order of the stabilizer G_α in G of a vertex α is at most 48. Based on this he proved in the same paper that there is no finite s-arc transitive cubic graph if $s > 5$. This fundamental result stimulated greatly the study of symmetric graphs and highly arc-transitive graphs, and its far-reaching influence in this area can be felt even after several decades. For example, by refining the ideas used in [82, 83], Sims [77, 78] generalized this result considerably. He proved in particular that, for a G-symmetric cubic graph Γ with G primitive on the vertices, the order of G_α is a divisor of 48. In a series of papers (see [11] and [27]-[31]), Djoković (partly with Bouwer) extended Tutte's work in several directions. In particular he showed [28] that, if Γ has valency $p+1$, for a prime p, and if the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ contains a subgroup acting regularly on the s-arcs of Γ, then $s \leq 5$ or $s = 7$. (Moreover [29], p must be a Mersenne prime if p is odd and $s \geq 2$.) Almost immediately, Gardiner [37] pointed out that requiring the existence of such a regular subgroup is a redundancy, and he proved that the same bound for s is valid if Γ is a graph with valency $p+1$, p a prime, such that $\mathrm{Aut}(\Gamma)$ is transitive on the s-arcs but not on the $(s+1)$-arcs of Γ. That the case $s = 7$ actually occurs was shown by the graph [4] derived from the families of points and lines on certain quadric surfaces in finite geometries. In [38] Gardiner proved further that, if Γ is (G, s)-arc but not $(G, s+1)$-arc transitive such that G_α is doubly primitive on the neighbourhood $\Gamma(\alpha)$ of α in Γ, and that the pointwise stabilizer $G_{(\Gamma(\alpha)\cup\{\alpha\})} \neq 1$, then we have $s \leq 5$ or $s = 7$ as well. By analysing the stabilizers of adjacent vertices, Goldschmidt [47] obtained an important extension of Tutte's result which inspired a lot of subsequent work on groups and geometries. In particular, he proved that if G is a group of automorphisms of a cubic graph such that G is transitive on the edges and the stabilizer in G of a vertex is finite, then the order of the subgroup of G fixing each of two adjacent vertices divides 2^7. A significantly simplified proof of this result was given by Weiss in [94]. In the 1970's and early 1980's, Weiss [88, 89, 90, 91, 92] obtained a number of results regarding the structure of the stabilizer of a vertex for a finite s-arc transitive (not necessarily cubic) graph, $s \geq 1$.

Also inspired by Tutte's fundamental result above, a lot of work has been done in constructing symmetric graphs and highly arc-transitive graphs. Tutte himself gave the first example of a connected 5-arc transitive cubic graph, and Conway constructed (but did not publish, see [6, pp.145]) infinitely many such graphs as covers of a given one. After that, a number of new 5-arc transitive cubic graphs were constructed [7, 8, 9, 17]. In [18], Conder found infinitely many new 5-arc transitive cubic graphs by showing that, for all but finitely many positive integers n, both the alternating and symmetric groups of degree n may be represented as full automorphism groups of 5-arc transitive cubic graphs. In [61], all cubic symmetric graphs with small girth (up to 6) were determined. An infinite family of 4-arc transitive cubic graphs each with girth 12 was constructed in [19], and a classification of 4- and 5-arc transitive cubic graphs with girth less than or equal to 13 was given (with some exceptions) in [62]. In [57] Lorimer determined all cubic symmetric graphs of order (that is, the number of vertices) at most 120 which are neither bipartite graphs nor Cayley graphs. A complete classification of cubic symmetric graphs with order at most 240 was recently given in [22] by using the result (see e.g. [21]) that the group of a cubic symmetric graph is a homomorphic image of one of seven finitely presented groups. There are also a few constructions and characterizations of symmetric graphs concerning the valency: Lorimer [56, 59] studied symmetric graphs with prime valency, and Praeger and Xu characterized [73] connected symmetric graphs of twice prime valency whose automorphism groups have abelian normal p-subgroups which are not semiregular on vertices. This latter work motivated the investigation of symmetric graphs of valency 4 conducted in [41, 42] by Gardiner and Praeger.

From a group-theoretic point of view, a symmetric graph can be defined as an orbital graph of a transitive permutation group (see e.g. [70, Section 2]). In [76] Sabidussi introduced a way of identifying the self-paired orbital involved, and developed a group-theoretic method for constructing an isomorphic copy of the given symmetric graph (see also [56, 58, 60]). More precisely, a graph is G-symmetric if and only if it is isomorphic to a certain kind of "coset graph" with vertices the right cosets in G of a certain subgroup of G. This group-theoretic approach has proved to be very useful in constructing and classifying some classes of symmetric graphs. It also indicates the strong connection between groups and symmetric graphs. In particular the classification of finite simple groups has had a great impact on research into symmetric graphs (see e.g. [13, 65]). A number of important results have been proved by using this powerful mathematical tool. The first one of them is the celebrated theorem of Weiss [93] which asserts that, apart from the cycles, there are no s-arc transitive graphs for $s > 7$. As mentioned earlier, 7-arc transitive graphs do exist; and Conder and Walker [20] prove recently that there are infinitely many such graphs. In fact, they proved that, for all but finitely many positive integers n, there are two connected graphs which admit, respectively, the alternating and symmetric groups of degree n as 7-arc transitive groups of automorphisms. Before the classification of finite simple groups, Chao [15] classified symmetric graphs with prime order. By using the classification of finite simple groups, Cheng and Oxley [16] determined all symmetric graphs with twice prime orders, and Wang and Xu [87] classified all symmetric graphs with triple prime orders. In [72], Praeger, Wang and Xu classified all symmetric graphs of order a product of two distinct primes by using the classification [74] of all vertex-primitive graphs of order a product of two distinct primes. All G-symmetric graphs with order $6p$ such that $p \geq 5$ is a prime and G is solvable were classified in [85].

Naturally, a G-symmetric graph Γ can be called a primitive or imprimitive G-symmetric graph according to whether G is primitive or imprimitive on the vertices of Γ. By using the result of [77], Wong [96] determined all primitive cubic symmetric graphs. As a consequence of the work of Wang [86], which relies on the classification of finite simple groups, all primitive symmetric graphs of valency 4 are known. In general, for studying primitive symmetric graphs we need to understand the possible structures of finite primitive groups. The information needed is contained in the O'Nan-Scott Theorem (see [55] or [80]), which categorizes finite primitive groups into several types. This theorem has been proved to be very useful in studying finite primitive groups and their applications, and in particular in studying primitive symmetric graphs. Similar to the primitive case, a G-symmetric graph Γ is said to be quasiprimitive if G is quasiprimitive on the vertices of Γ. (A permutation group is quasiprimitive if each of its nontrivial normal subgroups is transitive. Any primitive group is quasiprimitive, but not conversely.) Considering the local action, a G-symmetric graph Γ is said to be G-locally primitive (G-locally quasiprimitive, respectively) if in its induced action G_α is primitive (quasiprimitive, respectively) on $\Gamma(\alpha)$. Since a G-vertex-transitive graph Γ is $(G, 2)$-arc transitive if and only if G_α is 2-transitive on $\Gamma(\alpha)$ and since 2-transitive groups are primitive, it is clear that any $(G, 2)$-arc transitive graph is G-locally primitive, and in turn any G-locally primitive graph is G-locally quasiprimitive.

As a result of the classification of finite simple groups, all the finite 2-transitive groups are known (see e.g. [13, 51]). Because of this, an extensive study of 2-arc transitive graphs has been conducted during the past two decades. It was proved in [14] that, under certain conditions, a 2-arc transitive graph must be the incidence graph of a (known) symmetric design. In [49], Ivanov investigated 2- but not 3-arc transitive graphs. In [50], Ivanov and Praeger classified all primitive affine 2-arc transitive graphs and all bi-primitive affine 2-arc transitive graphs. (A 2-arc transitive graph Γ is said to be affine if there is a vector space N and a subgroup $G \leq \mathrm{Aut}(\Gamma)$ such that $N \leq G \leq \mathrm{AGL}(N)$ with N acting regularly on the vertices of Γ and G acting 2-arc transitively on Γ, where $\mathrm{AGL}(N)$ is the group of all affine transformations of N and N is identified with the subgroup of translations.) In [33, 34] Fang and Praeger constructed and classified some classes of 2-arc transitive graphs admitting a Suzuki group or a Ree group (see also Fang's PhD Thesis [35]). Recently, Hassani, Nochefranca and Praeger [48] studied 2-arc transitive graphs admitting a two-dimensional projective linear group. Examples of 2-arc transitive graphs of girth 5 containing Petersen subgraphs were constructed in [64] via a certain kind of flag-transitive geometry. A construction is given in [3] for all the pairs (Γ, G) such that Γ is $(G, 2)$-arc transitive and G has a minimal normal subgroup which is nonabelian and regular on the vertices of Γ. A classification of 2-arc transitive circulants was given in [1]. In [68] Praeger gave an O'Nan-Scott type Theorem for finite quasiprimitive groups, and this has been the impetus for a lot of work on quasiprimitive symmetric graphs and locally quasiprimitive graphs conducted by Praeger and her colleagues (see for example [52, 54, 71]). In [68] Praeger also proved that every finite, non-bipartite, 2-arc transitive graph is a cover of a quasiprimitive 2-arc transitive graph; and moreover among the possible types of quasiprimitive groups only four of them (namely, affine type, almost simple type, product type, twisted wreath type) can appear as a quasiprimitive, 2-arc transitive group of automorphisms of a connected graph. For bipartite 2-arc transitive graphs, a useful reduction was given in [69], also by Praeger.

Imprimitive symmetric graphs have been studied in various ways in the literature. Classical examples of such graphs include the "covering graphs" constructed in [6, Chapter 19] and some highly arc-transitive graphs constructed in [7, 8, 17, 18, 21, 62]. There have been a few characterizations of some special classes of imprimitive symmetric graphs [36, 41, 42, 73, 81]. Nevertheless, unlike the primitive case, it seems that there is no powerful mathematical tool available for dealing with imprimitive symmetric graphs. In this sense the main difficulty in studying symmetric graphs lies in the imprimitive case. Recently, Gardiner and Praeger [43] proposed a geometric approach to studying imprimitive symmetric graphs and discussed in detail the case where Γ is G-locally primitive. Further [44, 45], they indicated an extension of their approach for the whole class of imprimitive symmetric graphs. Recall that a G-symmetric graph Γ is imprimitive if and only if its vertex set admits a nontrivial G-invariant partition $\mathcal{B}$. So in this case we have a natural quotient graph $\Gamma_{\mathcal{B}}$ of Γ with respect to $\mathcal{B}$. According to the approach of Gardiner and Praeger, such a graph Γ can be "decomposed" into the "product" of this quotient and two other configurations, namely the "inter-block" bipartite graph $\Gamma[B,C]$ and the 1-design $\mathcal{D}(B)$ which we defined in the previous section. It was suggested [43, 44, 45] that these three configurations may have a strong influence on the structure of Γ.

1.3 Main results and the structure of the thesis

This thesis can be divided into the following three parts.

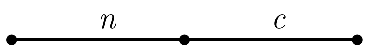
PART I. The first part consists of this introductory chapter and the next two chapters. In Chapter 2, we will introduce notation, terminology and preliminary results for permutation groups, designs and graphs that will be used throughout. In Chapter 3, we will discuss in detail the geometric approach of Gardiner and Praeger [43] for studying imprimitive symmetric graphs, and thus set the framework for the whole thesis. As mentioned in Section 1.1, the triple $(\Gamma_{\mathcal{B}}, \Gamma[B,C], \mathcal{D}(B))$ will be associated with any G-symmetric graph Γ admitting a nontrivial G-invariant partition $\mathcal{B}$. Here $\Gamma[B,C]$ is the induced bipartite subgraph of Γ with bipartition $\{\Gamma(C)\cap B, \Gamma(B)\cap C\}$, where $\Gamma(B)$ is the set of vertices of Γ adjacent to at least one vertex of B. In most cases we will identify $\mathcal{D}(B) = (B, \Gamma_{\mathcal{B}}(B), \mathrm{I})$ with the 1-design with point set B and blocks $\Gamma(C)\cap B$ (with possible repetition), so $\mathcal{D}(B)$ has block size $k = |\Gamma(C)\cap B|$, where C runs over all the blocks in the neighbourhood $\Gamma_{\mathcal{B}}(B)$ of B in $\Gamma_{\mathcal{B}}$. The study of Γ will involve a detailed analysis of these three configurations, as well as addressing the problem of the reconstruction of Γ from the triple above.

PART II. The heart of the thesis is the second part, which contains Chapters 4 through 9. Since the class of imprimitive symmetric graphs is very large, it is unrealistic to discuss all the cases in the thesis. In this part we concentrate on the case where the block size k of the 1-design $\mathcal{D}(B)$ is one less than the block size v of the partition $\mathcal{B}$. As we will see later, this case is rather enlightening and unexpectedly rich in both theory and examples. Chapter 4 is devoted to a general analysis of this case and thus provides a basis for subsequent study in this part. As fundamental properties for this case, we will prove that the induced action of G on $\mathcal{B}$ is faithful and the induced action of G_B on B is 2-transitive, where G_B is the setwise stabilizer of B in G. We will also study (Section 4.4) an extreme case for which all of Γ, $\Gamma_{\mathcal{B}}$ and $\Gamma[B,C]$ can be determined explicitly. (Chapter 4 is based on certain parts of [53] and [101].)

In Chapter 5, we study the case where in addition $\mathcal{D}(B)$ contains no repeated blocks. Not only is this a natural assumption geometrically, but also we will prove that, under the assumption $k = v - 1 \geq 2$, $\mathcal{D}(B)$ contains no repeated blocks if and only if the quotient $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive (Theorem 5.1.2). In this case, the valency of $\Gamma_{\mathcal{B}}$ is equal to v, the vertices of Γ can be labelled in a natural way by the arcs of $\Gamma_{\mathcal{B}}$, and each vertex of Γ has a unique mate. We will give (Section 5.2) a very natural and simple construction of a class of graphs Γ such that $k = v - 1 \geq 2$ and $\mathcal{D}(B)$ contains no repeated blocks. Moreover, we will show that every graph Γ satisfying these conditions can be constructed by using this construction (see Theorem 5.2.3). The construction bears some similarity to the "covering graph" construction of Biggs [6, pp.149-154]. The ingredients for our construction are a $(G, 2)$-arc transitive graph Σ and a self-paired G-orbit Δ on 3-arcs of Σ. Given these, we define the 3-arc graph of Σ with respect to Δ to be the graph with vertices the arcs of Σ in which two vertices represented respectively by arcs $(\sigma, \tau), (\sigma', \tau')$ of Σ are adjacent if and only if $(\tau, \sigma, \sigma', \tau')$ is a 3-arc in Δ. The possibilities for $\Gamma[B, C]$ depend on the pair $(\Gamma_{\mathcal{B}}, G)$, and vice versa. For example, under the assumptions above, we will prove (Theorem 5.3.1) that the extreme case $\Gamma[B, C] \cong K_{v-1,v-1}$ (where $\Gamma[B, C]$ contains the maximum possible number of edges) occurs if and only if $\Gamma_{\mathcal{B}}$ is $(G, 3)$-arc transitive. (This chapter is based on publication [53].)

The 3-arc graph construction above enables us to classify in Chapter 6 all the pairs (Γ, G) with Γ a G-symmetric graph such that $k = v - 1 \geq 2$, $\mathcal{D}(B)$ contains no repeated blocks and $\Gamma_{\mathcal{B}}$ is a complete graph. From this construction the classification of such graphs Γ is equivalent to classifying all 3-arc graphs of $(G, 2)$-arc transitive complete graphs Σ of valency v. In this case G is 3-transitive on the vertices of Σ. Thus the classification of such (Γ, G) relies on the classification of 3-transitive permutation groups, and hence depends on the classification of finite simple groups. The examples of such graphs Γ arising from 3-transitive projective groups are the so-called cross-ratio graphs, which can be defined in terms of cross ratios of quadruples of points of the projective line $\mathrm{PG}(1, v)$. Other examples of such graphs Γ include two graphs arising from each of the 3-transitive affine groups, and two graphs arising from each of the Mathieu groups M_{11} (degree 12) and M_{22} (degree 22). The classification of all possible (Γ, G) will be given in Theorem 6.6.1. (This chapter is based on certain parts of [46] and [99].)

Continuing our discussion for the case where $k = v - 1 \geq 2$ and $\mathcal{D}(B)$ contains no repeated blocks, we will study in Chapter 7 the second extreme case for $\Gamma[B, C]$, namely the case where $\Gamma[B, C] \cong (v-1) \cdot K_2$ is a matching of $v - 1$ edges (and thus $\Gamma[B, C]$ contains the minimum possible number of edges). In this case, we call Γ an almost cover of $\Gamma_{\mathcal{B}}$. If in addition $\Gamma_{\mathcal{B}}$ is a complete graph, then using the result in Chapter 6 we get a classification (Theorem 7.2.1) of all the possibilities for such (Γ, G). In the general case where $\Gamma_{\mathcal{B}}$ is connected but not complete, we find a surprising connection between such graphs Γ and an interesting class of graphs, namely near-polygonal graphs, which are associated with the Buekenhout geometries [12, 75] of the following diagram:



More precisely, in this case we will prove (Theorem 7.3.1) that, for some even integer $n \geq 4$, $\Gamma_{\mathcal{B}}$ must be a near n-gonal graph with respect to a G-orbit on n-cycles of $\Gamma_{\mathcal{B}}$; and moreover we will show that any $(G, 2)$-arc transitive near n-gonal graph (where n is even) with respect to a G-orbit on n-cycles can occur as such a quotient $\Gamma_{\mathcal{B}}$. (A near n-gonal graph [75] is a connected graph Σ of girth at least four together with a set $\mathcal{E}$ of n-cycles of Σ such that each 2-arc of Σ is contained in a unique member of $\mathcal{E}$. In this case we also say that Σ is a near n-gonal graph with respect to $\mathcal{E}$.) It was known in [43] that any G-locally primitive graph Γ admitting a G-invariant partition $\mathcal{B}$ with $k = v - 1 \geq 2$ is an almost cover of $\Gamma_{\mathcal{B}}$. So the results above apply in particular to this case, and we get an amended form (see Corollary 7.4.1) of [43, Theorem 5.4]. We conclude Chapter 7 by giving necessary and sufficient conditions for a $(G, 2)$-arc transitive graph of girth at least four to be near-polygonal. In view of the results above, these are needed in constructing almost covers of $(G, 2)$-arc transitive graphs. (This chapter is based on paper [97].)

In Chapter 8 we study a large class of symmetric graphs, namely the class of G-symmetric graphs such that the dual 1-design of $\mathcal{D}(B)$ contains no repeated blocks. Our study in this chapter reveals a very close connection between such graphs and certain point- and block-transitive 1-designs. More precisely, we will give a construction of such graphs Γ from some G-point-transitive and G-block-transitive 1-designs $\mathcal{D}$, and prove that, up to isomorphism, it produces all of them (see Theorem 8.2.1).

Each of the constructed graphs, called the G-flag graphs of $\mathcal{D}$, has vertex set a certain G-orbit Θ on the flags of $\mathcal{D}$ which satisfies some natural conditions. In particular, any G-symmetric graph Γ with $k = 1$ satisfies the condition above and thus is isomorphic to a G-flag graph. We will characterize such a graph Γ as a G-flag graph with Θ satisfying some additional condition (see Theorem 8.3.1). This chapter is preparatory for the next chapter, and this is the reason why we include it in the second part. However, we should point out that this chapter is of interest for its own sake, and the construction above seems to be useful in classifying or characterizing some interesting classes of symmetric graphs. (This chapter is based on paper [100].)

In Chapter 9 we return to the general case of $k = v-1 \geq 2$ without assuming the non-repetition of the blocks of $\mathcal{D}(B)$. Based on the similar idea as in the previous chapter, we will give a construction of such graphs Γ from some G-point-transitive and G-block-transitive 1-designs $\mathcal{D}$, and prove that, up to isomorphism, it produces all such graphs (see Theorem 9.2.1). In the particular case where the design $\mathcal{D}$ involved is the trivial design with block size 2, the construction gives rise to the 3-arc graphs introduced in Chapter 5. (However, the 3-arc graph construction is interesting and useful for its own sake.) In the case where $\mathcal{D}$ is a certain G-doubly transitive design, the constructed graph Γ has complete quotient $\Gamma_{\mathcal{B}}$. Using this construction we will classify (Theorems 9.4.1 and 9.5.1) all the G-symmetric graphs Γ such that $k = v - 1 \geq 2$ and $\Gamma_{\mathcal{B}}$ is complete when the design $\mathcal{D}$ involved is either the projective geometry $\mathrm{PG}(n, q)$ or the affine geometry $\mathrm{AG}(n, q)$. (This chapter is based on [99] and part of [46].)

Part III. The third part of the thesis consists of the last two chapters. The main purpose of this part is to investigate certain local actions induced by G_B, and to study their influence on the structure of Γ. In Chapter 10 we will study the induced actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$. We will see that the relationships between the kernels of these two actions affect significantly the structure of Γ, especially in the case where Γ is G-locally quasiprimitive. In the last chapter, Chapter 11, we will study a specific case where the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent. Geometrically, this requires that the group of automorphisms of $\mathcal{D}(B)$ induced by G_B acts in the same way on the points and blocks of $\mathcal{D}(B)$. In this case $\mathcal{D}(B)$ plays a more active role in influencing the structure of Γ and $\Gamma_{\mathcal{B}}$, and we will show that a labelling method similar to that used in Chapter 5 applies. Based on this labelling we will prove among other things that Γ is such a graph if and only if it is isomorphic to a 3-arc graph of some G-symmetric but not necessarily $(G, 2)$-arc transitive graph. (Chapter 11 is based on publication [98].)

Chapter 2

Notation, definitions and preliminaries

If names are not rectified, then language will not be in accord with truth; if language is not in accord with truth, then things cannot be accomplished.

Confucius (551-479 B.C.), LUN YÜ [THE ANALECTS] 13:3

This chapter is a collection of basic definitions and preliminary results relating to permutation groups, incidence structures, designs and graphs that will be used in subsequent chapters.

2.1 Permutation groups

A bijection from a finite set Ω to itself is said to be a *permutation* of Ω, and the *symmetric group* on Ω, denoted by $\mathrm{Sym}(\Omega)$ as usual, is the group of all permutations of Ω equipped with the ordinary composition of mappings. Any subgroup G of $\mathrm{Sym}(\Omega)$ is said to be a *permutation group* on Ω. If the size $|\Omega|$ of Ω is equal to n, then we say that G is a permutation group of *degree* n.

Let G be a finite group. Suppose that, for each $\alpha \in \Omega$ and $g \in G$, there corresponds a member of Ω, denoted by α^g. We say that this correspondence defines an *action* of G on Ω, or G *acts on* Ω, if for any $\alpha \in \Omega$ and $g, h \in G$ the following (i)-(ii) hold:

(i) $\alpha^1 = \alpha$, where 1 is the identity of the group G;

(ii) $(\alpha^g)^h = \alpha^{gh}$.

In other words, an action of G on Ω is a mapping $(\alpha, g) \mapsto \alpha^g$ from $\Omega \times G$ to Ω which satisfies the conditions (i), (ii) above. In such a case, the *degree* of the action of G on Ω is defined to be $|\Omega|$. We say that an element g of G *fixes* a point α of Ω if $\alpha^g = \alpha$. The *kernel* of the action of G on Ω is defined to be the subgroup of all elements of G which fix each point of Ω. If this kernel is equal to the identity subgroup of G, then G is said to act *faithfully* on Ω.

Example 2.1.1 Every permutation group G on Ω acts naturally on Ω, where α^g is the image of α under g, for $\alpha \in \Omega$ and $g \in G$. Clearly, such an action of G on Ω is faithful. Except where stated otherwise, we will always assume that this is the action we are dealing with whenever we have a permutation group.

Closely related to group actions is the concept of permutation representation. By definition a *permutation representation* of a group G on a finite set Ω is a group homomorphism $\varphi : G \to \mathrm{Sym}(\Omega)$. For such a permutation representation φ, the image $(G)\varphi$ of G is a permutation group on Ω, denoted by G^Ω. Thus, G acts on Ω via the natural action of $(G)\varphi$ on Ω. That is, for $\alpha \in \Omega$ and $g \in G$ we define $\alpha^g := \alpha^{(g)\varphi}$, the image of α under the permutation $(g)\varphi$ of Ω. It is clear that the kernel of this action of G on Ω is exactly the kernel $\mathrm{Ker}(\varphi)$ of the group homomorphism φ, and if $\mathrm{Ker}(\varphi) = 1$ then we say that the permutation representation φ is *faithful.* Conversely, if G acts on a finite set Ω, then each $g \in G$ induces a permutation $\hat{g}$ of Ω defined by $\hat{g} : \alpha \mapsto \alpha^g$ for $\alpha \in \Omega$. Hence such an action determines a permutation representation φ of G on Ω, defined by $\varphi : g \mapsto \hat{g}$ for $g \in G$, whose kernel is exactly the kernel of the action of G on Ω.

Example 2.1.2 (*Right multiplication*) Suppose G is a group and $H \leq G$ is a subgroup of G. Let $[G : H]$ be the set of right cosets of H in G. Then $(Ha)^g = Hag$, for $Ha \in [G : H]$ and $g \in G$, defines an action of G on $[G : H]$. One can see that the kernel of this action is $\bigcap_{g \in G} H^g$, which is called the *core* of H in G and is denoted by $\mathrm{Core}_G(H)$, where $H^g := g^{-1}Hg$.

Naturally, an action of G on Ω induces an equivalence relation $\sim_G$ on Ω defined by

$$\alpha \sim_G \beta \text{ if and only if } \alpha^g = \beta \text{ for some } g \in G.$$

The equivalence classes of $\sim_G$ are said to be G*-orbits* on Ω. So any two G-orbits are either identical or disjoint, and the G-orbit containing a given point α of Ω is

$$\alpha^G := \{\alpha^g : g \in G\}.$$

We say that G is *transitive* on Ω if there is only one G-orbit on Ω, and that G is *intransitive* on Ω otherwise. For a subset Δ of Ω, we define

$$\Delta^g := \{\alpha^g : \alpha \in \Delta\}.$$

In particular, if $\Delta^g = \Delta$ for each $g \in G$, then Δ is said to be G*-invariant*; in this case G induces an action on Δ. We call the subgroups

$$G_\Delta := \{g \in G : \Delta^g = \Delta\}$$

and

$$G_{(\Delta)} := \{g \in G : \alpha^g = \alpha \text{ for each } \alpha \in \Delta\}$$

the *setwise stabilizer* and the *pointwise stabilizer* of Δ in G, respectively. In particular, for $\alpha, \beta, \gamma \in \Omega$, the subgroup $G_\alpha := G_{\{\alpha\}}$ of G is called the *stabilizer* of α in G, and we set $G_{\alpha\beta} := (G_\alpha)_\beta, G_{\alpha\beta\gamma} := (G_{\alpha\beta})_\gamma$, etc. If G_α acts trivially on Ω, that is, $G_\alpha = \mathrm{Core}_G(G_\alpha)$ for all $\alpha \in \Omega$, then G is said to be *semiregular* on Ω. If G acts transitively and semiregularly on Ω, then we say that G is *regular* on Ω. The following results can be found in standard books on permutation groups (see e.g. [26, 95]).

Lemma 2.1.1 *Suppose that G is a group acting on a finite set Ω. Let $\alpha \in \Omega$ and $g, h \in G$. Then*

(a) $G_{\alpha^g} = g^{-1}G_\alpha g$.

(b) $\alpha^g = \alpha^h$ *if and only if* $G_\alpha g = G_\alpha h$.

(c) $|\alpha^G| \cdot |G_\alpha| = |G|$. *In particular, G is transitive on Ω if and only if $|\Omega| = |G : G_\alpha|$; and in this case G acts regularly on Ω if and only if $|\Omega| = |G : \mathrm{Core}_G(G_\alpha)|$.*

Now suppose G_1, G_2 are groups acting on finite sets Ω_1, Ω_2, respectively. If there exist a bijection $\rho : \Omega_1 \to \Omega_2$ and a group homomorphism $\psi : G_1 \to G_2$ such that

$$\rho(\alpha^g) = (\rho(\alpha))^{\psi(g)}$$

for all $\alpha \in \Omega_1$ and $g \in G_1$, then the action of G_1 on Ω_1 is said to be *permutationally isomorphic* to the action of G_2 on Ω_2. In particular, if a group G acts on both Ω_1 and Ω_2, then we say that the actions of G on Ω_1 and Ω_2 are *permutationally equivalent* if there exists a bijection $\rho : \Omega_1 \to \Omega_2$ such that

$$\rho(\alpha^g) = (\rho(\alpha))^g$$

for all $\alpha \in \Omega_1$ and $g \in G$.

For a positive integer k, we use $\Omega^{(k)}$ to denote the set of k-tuples of distinct members of Ω. Let G act on Ω. Then G induces a natural action on $\Omega^{(k)}$ defined by

$$(\alpha_1, \alpha_2, \ldots, \alpha_k)^g := (\alpha_1^g, \alpha_2^g, \ldots, \alpha_k^g)$$

for $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in \Omega^{(k)}$ and $g \in G$. If, under this action, G is transitive on $\Omega^{(k)}$, then G is said to be *k-transitive* on Ω; if G is regular on $\Omega^{(k)}$, then G is said to be *sharply k-transitive* on Ω. As a consequence of the finite simple group classification, all 2-transitive permutation groups are known up to permutation isomorphism. The following classification theorem is from [13]. For a group G, the *socle* $\mathrm{soc}(G)$ of G is defined to be the product of all minimal normal subgroups of G.

Theorem 2.1.1 ([13, pp.8]) *Let G be a finite 2-transitive permutation group of degree n. Then either $\mathrm{soc}(G)$ is an elementary abelian group, or $\mathrm{soc}(G)$ is a nonabelian simple group and one of the cases listed in Table 1 (next page) occurs, where k is the maximum degree of transitivity of the group G.*

As mentioned in [13], the socle of G is k-transitive in all cases in Table 1 except where (i) $G = S_n$ is n-transitive while $\mathrm{soc}(G) = A_n$ is $(n-2)$-transitive, (ii) $G \leq \mathrm{P\Gamma L}(2, q)$ (with q odd) is 3-transitive while $\mathrm{soc}(G) = \mathrm{PSL}(2, q)$ is 2-transitive, and (iii) $G = \mathrm{Ree}(3)$ is 2-transitive of degree 28 while $\mathrm{soc}(G) = \mathrm{PSL}(2, 8)$ is 1-transitive. Note that the 2-transitive groups with abelian socles are not included in Table 1. For such groups, one can consult [51]. In particular, any group G with $\mathrm{AGL}(d, q) \leq G \leq \mathrm{A\Gamma L}(d, q)$ is 2-transitive on the d-dimensional vector space over $\mathrm{GF}(q)$; and moreover G is 3-transitive if and only if $G = \mathrm{AGL}(d, 2)$, or $G = \mathbb{Z}_2^4.A_7 < \mathrm{AGL}(4, 2)$.

soc(G)	n	k	Remarks
$A_n, n \geq 5$	n	n	Two representations if $n = 6$
$\mathrm{PSL}(d,q), d \geq 2$	$(q^d-1)/(q-1)$	3 *if* $d = 2$	$(d,q) \neq (2,2),(2,3)$
		2 *if* $d > 2$	Two representations if $d > 2$
$\mathrm{PSU}(3,q)$	q^3+1	2	$q > 2$
$\mathrm{Suz}(q)$	q^2+1	2	$q = 2^{2a+1} > 2$
$\mathrm{Ree}(q)$	q^3+1	2	$q = 3^{2a+1} > 3$
$\mathrm{PSp}(2d,2)$	$2^{2d-1}+2^{d-1}$	2	$d > 2$
$\mathrm{PSp}(2d,2)$	$2^{2d-1}-2^{d-1}$	2	$d > 2$
$\mathrm{PSL}(2,11)$	11	2	Two representations
$\mathrm{PSL}(2,8)$	28	2	
A_7	15	2	Two representations
M_{11}	11	4	
M_{11}	12	3	
M_{12}	12	5	Two representations
M_{22}	22	3	
M_{23}	23	4	
M_{24}	24	5	
HS	176	2	Two representations
Co_3	276	2	

TABLE 1 Socles of 2-transitive groups

Finally, an element of a finite group G is called a 2-*element* if its order in G is a power of 2. In particular, an element of G with order 2 is called an *involution* of G. We refer to [26, 95] for terminology and notation on permutation groups not defined above.

2.2 Partitions, blocks and primitivity

A *partition* of a finite set Ω is a set $\mathcal{B}$ of subsets of Ω such that $\bigcup_{B\in\mathcal{B}} B = \Omega$ and $B \cap C = \emptyset$ for distinct $B, C \in \mathcal{B}$. We call each member of $\mathcal{B}$ a *block* of $\mathcal{B}$, and if $\mathcal{B}$ has n blocks then we say that it is an *n-partition* of Ω. Clearly, $\{\{\alpha\} : \alpha \in \Omega\}$ and $\{\Omega\}$ are partitions of Ω, which we call the *trivial partitions* of Ω. For two partitions $\mathcal{B}_1, \mathcal{B}_2$ of Ω, we say that $\mathcal{B}_1$ is a *refinement* of $\mathcal{B}_2$ if each block of $\mathcal{B}_2$ is a union of some blocks of $\mathcal{B}_1$; and we say that $\mathcal{B}_1$ is a *genuine refinement* of $\mathcal{B}_2$ if in addition $\mathcal{B}_1 \neq \{\{\alpha\} : \alpha \in \Omega\}$ and $\mathcal{B}_1 \neq \mathcal{B}_2$.

Let $\mathcal{B}$ be a partition of Ω and let G act on Ω. If $B^g \in \mathcal{B}$ for any $B \in \mathcal{B}$ and $g \in G$, then $\mathcal{B}$ is said to be a *G-invariant partition* of Ω. In such a case, G permutes blockwise the blocks of $\mathcal{B}$ and thus induces a natural (possibly unfaithful) action on $\mathcal{B}$. Obviously, the trivial partitions $\{\{\alpha\} : \alpha \in \Omega\}$, $\{\Omega\}$ of Ω are G-invariant. Suppose G is transitive on Ω. If the trivial partitions are the only G-invariant partitions of Ω, then G is said to be *primitive* on Ω; otherwise G is said to be *imprimitive* on Ω. In general, if G is k-transitive on Ω, for some $k > 1$, such that the pointwise stabilizer in G of any $k-1$ distinct points of Ω is primitive on the remaining points, then G is said to be *k-primitive* on Ω.

Note that each block B of a G-invariant partition of Ω is a *block of imprimitivity* for G in Ω in the sense that, for each $g \in G$, either $B^g = B$ or $B^g \cap B = \emptyset$. Conversely, for a transitive group G acting on Ω, any block B of imprimitivity for G in Ω induces a G-invariant partition of Ω, namely $\{B^g : g \in G\}$; and in this case each block of this partition is also a block of imprimitivity for G in Ω. Thus, a partition $\mathcal{B}$ of Ω is G-invariant if and only if each block of $\mathcal{B}$ is a block of imprimitivity for G in Ω. Hence G is primitive on Ω if and only if the only blocks of imprimitivity for G in Ω are Ω and $\{\alpha\}$, for $\alpha \in \Omega$. Clearly, we have $G_\alpha \le G_B \le G$ for $\alpha \in B$. Conversely, for any subgroup H of G with $G_\alpha \le H \le G$, the H-orbit $B := \alpha^H$ containing α is a block of imprimitivity for G in Ω, and hence B induces a G-invariant partition of Ω. Further, if $G_\alpha \le H_1 \le H_2 < G$, then $\alpha^{H_1} \subseteq \alpha^{H_2}$ and the partition corresponding to H_1 refines the partition corresponding to H_2. So the lattice of G-invariant partitions of Ω (with partial order the refinement of partitions) is isomorphic to the lattice of subgroups H of G containing G_α. Therefore, G is primitive on Ω if and only if G_α is a maximal subgroup of G.

In the following we will write $G_{B,C} := (G_B)_C, G_{B,C,D} := (G_{B,C})_D$ for $B, C, D \in \mathcal{B}$. For any subset T of G, we define

$$\mathrm{fix}_\Omega(T) := \{\alpha \in \Omega : \alpha^g = \alpha \,\text{for all}\, g \in T\},$$

the *fixed point set* of T in Ω. The following lemma will be used in our later discussion.

Lemma 2.2.1 ([26, pp.19]) *If a group G acts transitively on a finite set Ω, then, for each $\alpha \in \Omega$, $\mathrm{fix}_\Omega(G_\alpha)$ is a block of imprimitivity for G in Ω.*

We conclude this section by giving the definition of quasiprimitivity, which relies on the following result.

Lemma 2.2.2 (see e.g. [70, Lemma 10.1]) *Let a group G act on a finite set Ω, and let N be a normal subgroup of G. Then the set of N-orbits on Ω is a G-invariant partition of Ω.*

We will denote this partition by $\mathcal{B}_N$ and called it the *G-normal partition* of Ω induced by N. Clearly, the trivial partitions $\{\{\alpha\} : \alpha \in \Omega\}$ and $\{\Omega\}$ are G-normal partitions induced by the identity subgroup and G itself, respectively. If these are the only G-normal partitions of Ω, then G is said to be *quasiprimitive* on Ω. In other words, G is quasiprimitive on Ω if and only if every non-indentity normal subgroup of G is transitive on Ω. Thus, G is quasiprimitive on Ω implies in particular that it is transitive on Ω. It follows from the definition that G is primitive on Ω implies that it is quasiprimitive on Ω. Note that the converse of this is not true (see e.g. [70, Section 10]).

2.3 Incidence structures and designs

We refer to [5, 10] for terminology and notation on design theory. An *incidence structure* is a triple $\mathcal{D} = (V, \mathbf{B}, \mathrm{I})$, where V, $\mathbf{B}$ are disjoint finite sets and I is a binary relation between V and $\mathbf{B}$, that is, $\mathrm{I} \subseteq V \times \mathbf{B}$. The members of V, $\mathbf{B}$ and I are called the *points*, *blocks* and *flags* of $\mathcal{D}$, respectively. If (α, X) is a flag of $\mathcal{D}$, then we simply write $\alpha \mathrm{I} X$ and say that α, X are *incident* with each other. The *trace* of a block X (a point α, respectively) of $\mathcal{D}$ is the subset $\{\alpha \in V : \alpha \mathrm{I} X\}$ of V (the subset $\{X \in \mathbf{B} : \alpha \mathrm{I} X\}$ of $\mathbf{B}$, respectively). If two blocks have the same trace, then they are said to be *repeated blocks* of $\mathcal{D}$. As usual in the literature, in the case where $\mathcal{D}$ contains no repeated blocks we may identify each block with its trace and thus identify $\mathbf{B}$ with a set of subsets of V. If the traces of all blocks of $\mathcal{D}$ have the same cardinality k (which we call the *block size* of $\mathcal{D}$) and if the traces of all points of $\mathcal{D}$ have the same cardinality r, then $\mathcal{D}$ is said to be a *1-(v, k, r) design*, where $v := |V|$. In such a case, we have $vr = bk$ by counting the number of flags of $\mathcal{D}$ in two different ways, where we set $b := |\mathbf{B}|$. A 1-(v, k, r) design $\mathcal{D}$ is said to be a *t-(v, k, λ) design*, for some integers $t \geq 2$ and $\lambda \geq 1$, if any t distinct points are incident with λ blocks simultaneously. The *dual* of a 1-(v, k, r) design $\mathcal{D} = (V, \mathbf{B}, \mathrm{I})$ is the 1-(b, r, k) design $\mathcal{D}^* := (\mathbf{B}, V, \mathrm{I}^*)$ with $X \mathrm{I}^* \alpha$ if and only if $\alpha \mathrm{I} X$. For two 1-designs $\mathcal{D} = (V, \mathbf{B}, \mathrm{I})$ and $\mathcal{D}' = (V', \mathbf{B}', \mathrm{I}')$, an *isomorphism* from $\mathcal{D}$ to $\mathcal{D}'$ is a bijection $\psi : V \cup \mathbf{B} \to V' \cup \mathbf{B}'$

such that $\psi(V) = V', \psi(\mathbf{B}) = \mathbf{B}'$, and $\alpha \mathrm{I} X$ if and only if $\psi(\alpha)\mathrm{I}'\psi(X)$. If there exists an isomorphism ψ from $\mathcal{D}$ to $\mathcal{D}^*$, then $\mathcal{D}$ is said to be *self-dual*; and if moreover $\psi^2 = 1$ then ψ is called a *polarity* of $\mathcal{D}$. An isomorphism from $\mathcal{D}$ to itself is said to be an *automorphism* of $\mathcal{D}$, and all such automorphisms form the (full) *automorphism group* of $\mathcal{D}$, denoted by $\mathrm{Aut}(\mathcal{D})$. In general, if G is a group acting on the points and the blocks of $\mathcal{D}$ respectively such that the incidence relation of $\mathcal{D}$ is preserved by these actions, that is, $\alpha \mathrm{I} X$ if and only if $\alpha^g \mathrm{I} X^g$ for $\alpha \in V$, $X \in \mathbf{B}$ and $g \in G$, then we say that $\mathcal{D}$ *admits* G as a group of automorphisms. In this case G induces an action on the flags of $\mathcal{D}$. If G is transitive on the points (blocks, flags, respectively) of $\mathcal{D}$, then $\mathcal{D}$ is said to be *G-point-transitive* (*G-block-transitive*, *G-flag-transitive*, respectively). As a convention, when we say $\mathcal{D}$ is G-transitive, we mean it is G-point-transitive. Similar convention applies to G-doubly transitive 1-designs. For a point α of $\mathcal{D}$, we set $\mathbf{B}_\alpha := \{X \setminus \{\alpha\} : X \in \mathbf{B}, \alpha \mathrm{I} X\}$ and let I_α be the incidence relation between $V \setminus \{\alpha\}$ and $\mathbf{B}_\alpha$ induced by I. If $\mathcal{D}$ is a 2-(v, k, λ) design, then $\mathcal{D}_\alpha := (V \setminus \{\alpha\}, \mathbf{B}_\alpha, \mathrm{I}_\alpha)$ is a 1-$(v-1, k-1, \lambda-1)$ design, and in this case $\mathcal{D}$ is said to be an *extension* of $\mathcal{D}_\alpha$.

A *linear space* [5] is an incidence structure of points and blocks (called *lines*) in which any two distinct points are incident with exactly one line, any point is incident with at least two lines, and any line with at least two points. A linear space with each line incident with exactly two points is called a *trivial linear space*.

2.4 Graphs

All the graphs in this thesis will refer to finite, undirected and simple graphs. Such a graph Γ can be defined as an incidence structure (V, E, I) with no repeated blocks such that each block is incident with exactly 2 points. The members of V, E are called the *vertices* and *edges* of Γ, respectively. As usual, we use $V(\Gamma)$ and $E(\Gamma)$ to denote respectively the vertex set V and the edge set E of Γ, and thus we write $\Gamma = (V(\Gamma), E(\Gamma))$. Two vertices α, β of Γ are said to be *adjacent* if there exists an edge e of Γ which is incident with both α and β. In such a case, we say that e *joins* α and β and we may identify e with the unordered pair $\{\alpha, \beta\}$. So we may identify $E(\Gamma)$ with the set of all such unordered pairs of vertices of Γ. For $\alpha \in V(\Gamma)$, we use $\Gamma(\alpha)$ to denote the *neighbourhood* of α in Γ, that is, the set of vertices of Γ

adjacent to α. The *valency* of α in Γ is defined to be the size of $\Gamma(\alpha)$. A vertex of Γ with valency 0 is called an *isolated vertex* of Γ. If all vertices of Γ have the same valency, then Γ is said to be *regular*. In this case, this common valency is called the valency of Γ and is denoted by $\mathrm{val}(\Gamma)$. A *complete graph* is a graph in which any two distinct vertices are adjacent, whilst an *empty graph* is a graph in which any two vertices are not adjacent. A *subgraph* of Γ is a graph $\Sigma = (V(\Sigma), E(\Sigma))$ with $V(\Sigma) \subseteq V(\Gamma), E(\Sigma) \subseteq E(\Gamma)$. For a subset X of $V(\Gamma)$, we use $\Gamma[X]$ to denote the subgraph of Γ *induced* by X, that is, the graph with vertex set X in which $\alpha, \beta \in X$ are adjacent if and only if they are adjacent in Γ. In particular, if $\Gamma[X]$ is a complete graph, then X is said to be a *clique* of Γ; and if $\Gamma[X]$ is an empty graph, then X is said to be an *independent set* of Γ. The graph Γ is said to be an *n-partite graph* if $V(\Gamma)$ admits an n-partition with each block an independent set of Γ. If in addition any two vertices in distinct parts of this n-partition are adjacent, then Γ is said to be a *complete n-partite graph*. In particular, a 2-partite graph is called a *bipartite graph*. For two graphs $\Gamma = (V(\Gamma), E(\Gamma))$ and $\Sigma = (V(\Sigma), E(\Sigma))$ (with or without common vertices), the *union* of Γ and Σ, denoted by $\Gamma \cup \Sigma$, is the graph with vertex set $V(\Gamma) \cup V(\Sigma)$ and edge set $E(\Gamma) \cup E(\Sigma)$. The union of finitely many graphs is defined similarly. In particular, we will use $n \cdot \Gamma$ to denote the union of n vertex-disjoint copies of Γ. The *lexicographic product* $\Gamma[\Sigma]$ of Γ by Σ is defined to be the graph with vertex set $V(\Gamma) \times V(\Sigma)$ in which (α, β) and (σ, τ) are adjacent if and only if either α, σ are adjacent in Γ, or $\alpha = \sigma$ and β, τ are adjacent in Σ. We will use $\overline{\Gamma}$ to denote the *complement* of a graph Γ, that is, the graph with the same vertices as Γ in which α, β are adjacent if and only if they are not adjacent in Γ.

A *path* of a graph Γ of *length* n is a sequence $\alpha_0, \alpha_1, \ldots, \alpha_n$ of $n+1$ distinct vertices such that α_{i-1}, α_i are adjacent for $i = 1, 2, \ldots, n$. Such a path is said to connect α_0 and α_n. Define a binary relation $\sim_\Gamma$ on $V(\Gamma)$ such that $\alpha \sim_\Gamma \beta$ if and only if there exists a path of Γ connecting α and β. Then it is an equivalence relation on $V(\Gamma)$, and we call the subgraphs of Γ induced by the equivalence classes of $\sim_\Gamma$ the *connected components* of Γ. The graph Γ is said to be *connected* if it has only one connected component, and *disconnected* otherwise. The *distance* in Γ between two given vertices α, β, denoted by $d_\Gamma(\alpha, \beta)$ (or simply $d(\alpha, \beta)$ if no ambiguity exists), is the shortest length of a path of Γ connecting α and β if they are in the same connected component of Γ, and is defined to be ∞ otherwise. (As a convention, we define $d(\alpha, \alpha) = 0$.) The *diameter* of Γ, denoted by $\mathrm{diam}(\Gamma)$, is the largest distance in Γ between any two vertices of Γ. For $n \geq 3$, an *n-cycle* (or a cycle of length n) of Γ is an $(n+1)$-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha_0)$ of vertices of Γ such that $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are pairwise distinct and α_{i-1}, α_i are adjacent for $i = 1, 2, \ldots, n$ (subscripts modulo n). The *girth* of Γ, denoted by $\mathrm{girth}(\Gamma)$, is the shortest length of a cycle of Γ if Γ contains cycles, and is defined to be ∞ otherwise.

Let Γ, Σ be two graphs. A mapping $\psi : V(\Gamma) \to V(\Sigma)$ is called a (graph) *homomorphism* from Γ to Σ if ψ maps adjacent vertices of Γ to adjacent vertices of Σ. If in addition ψ is one-to-one, then it is called a (graph) *monomorphism*; and if in addition ψ is a bijection with ψ^{-1} a homomorphism from Σ to Γ, then ψ is said to be a (graph) *isomorphism* from Γ to Σ. In particular, an isomorphism from Γ to itself is said to be an *automorphism* of Γ. All such automorphisms of Γ form a subgroup of $\mathrm{Sym}(V(\Gamma))$, called the *full automorphism group* of Γ and denoted by $\mathrm{Aut}(\Gamma)$. Any subgroup of $\mathrm{Aut}(\Gamma)$ is called an automorphism group of Γ. In general, if G is a group acting on $V(\Gamma)$ such that, for any $g \in G$, two vertices α, β of Γ are adjacent in Γ implies that α^g, β^g are adjacent in Γ, then we say that Γ *admits* G as a group of automorphisms.

We will use $K_n, P_n, C_n, K_{m,m}, K_m^n$ to denote, respectively, the complete graph on n vertices, the path of length n, the cycle of length n, the complete bipartite graph with m vertices in each part of its bipartition, and the complete n-partite graph with m vertices in each part of its n-partition. The graph $n \cdot K_2$ with n edges is called a *matching*.

Chapter 3

Imprimitive symmetric graphs: A geometric approach

From TAO proceeds the one; one produces two; this makes three. From these three proceed all things. All things thus bear the imprint of the negative **yin** *behind and embrace the positive* **yang** *in front, and through the blending of the vital force (***ch'i***) they achieve harmony.*

Lao Tzu (6th or 4th Cent. B.C. ?), TAO TE CHING 42

The geometric approach we will use in this thesis was first introduced by Gardiner and Praeger in [43]. Although their paper was written in the context of G-locally primitive graphs, the same approach is well suitable for studying general imprimitive G-symmetric graphs, and the theory was extended to such graphs in [44, 45, 53]. In this chapter we will introduce this approach and prove some basic results involved, and thus set the framework for the whole thesis. Most results in Section 3.2 were known explicitly or implicitly in [43, 44, 45, 66]. We start with some definitions relating to symmetric graphs.

3.1 Symmetric and highly arc-transitive graphs

Let Γ be a graph and s a positive integer. An *s-arc* of Γ is an $(s+1)$-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of vertices of Γ such that α_{i-1} is adjacent to α_i for $1 \le i \le s$ and $\alpha_{i-1} \ne \alpha_{i+1}$ for $1 \le i \le s-1$. We will use $\mathrm{Arc}_s(\Gamma)$ to denote the set of s-arcs of Γ. Usually a 1-arc is called an *arc*, and instead of $\mathrm{Arc}_1(\Gamma)$ we use $\mathrm{Arc}(\Gamma)$ to denote the set of arcs of Γ.

Suppose that Γ admits a group G as a group of automorphisms. Then, in a natural way, G induces an action on $\mathrm{Arc}_s(\Gamma)$ defined by $(\alpha_0, \alpha_1, \ldots, \alpha_s)^g := (\alpha_0^g, \alpha_1^g, \ldots, \alpha_s^g)$, for $(\alpha_0, \alpha_1, \ldots, \alpha_s) \in \mathrm{Arc}_s(\Gamma)$ and $g \in G$. If, under this induced action, G is transitive on $\mathrm{Arc}_s(\Gamma)$, then Γ is said to be (G, s)-*arc transitive*. As usual in the literature, a $(G, 1)$-arc transitive graph is called a G-*symmetric graph*, or simply a symmetric graph if the group G is not important in the context. Likewise, if G is transitive on $V(\Gamma)$, then Γ is said to be G-*vertex-transitive*. Clearly, any G-vertex-transitive graph is regular.

Perhaps it is the right time to say a few words about the definition of a (G, s)-arc transitive graph. In most cases (but not always), such a graph Γ is also G-vertex-transitive; and this is the case researchers are interested in. In fact, if Γ is a G-symmetric graph with no isolated vertices, then it must be G-vertex-transitive since each vertex of Γ can be taken as the initial vertex of an arc. In general, it follows from [70, Theorem 9.3] that, if Γ is a (G, s)-arc transitive graph with each connected component containing at least one s-arc, then either Γ is G-vertex-transitive and is (G, i)-arc transitive for each i with $1 \leq i \leq s$, or the connected components of Γ are isomorphic trees. Therefore, as usual in the literature, we will be concerned with G-vertex-transitive, (G, s)-arc transitive graphs only.

Convention 3.1.1 By a (G, s)-arc transitive (G-symmetric, respectively) graph, we will always refer to a G-vertex-transtive, (G, s)-arc transitive (G-symmetric, respectively) graph with valency at least one.

As we see above, under the assumption of G-vertex-transitivity, (G, s)-arc transitivity implies $(G, s-1)$-arc transitivity. Here, for $s = 1$, we may interprete $(G, 0)$-arc transitivity as G-vertex-transitivity. Conversely, if Γ is $(G, s-1)$-arc transitive and, for some fixed $(s-1)$-arc $(\alpha_0, \alpha_1, \ldots, \alpha_{s-1})$ of Γ, the stabilizer $G_{\alpha_0 \alpha_1 \ldots \alpha_{s-1}}$ is transitive on $\Gamma(\alpha_{s-1}) \setminus \{\alpha_{s-2}\}$, then Γ is (G, s)-arc transitive. In particular we have part (b) of the following lemma. Part (a) of this lemma follows from the definition of a G-symmetric graph.

Lemma 3.1.1 *Let Γ be a G-vertex-transitive graph, and let $\alpha \in V(\Gamma)$. Then the following* (a)-(b) *hold.*

(a) Γ *is G-symmetric if and only if G_α is transitive on $\Gamma(\alpha)$.*

(b) Γ *is $(G, 2)$-arc transitive if and only if G_α is 2-transitive on $\Gamma(\alpha)$.*

We should warn that, for $s \geq 3$, the similar assertion "Γ is (G, s)-arc transitive if and only if G_α is s-transitive on $\Gamma(\alpha)$" is not valid. In view of (a) above, if Γ is G-symmetric such that G_α is primitive on $\Gamma(\alpha)$, then Γ is said to be *G-locally primitive*. Similarly, if Γ is G-symmetric such that G_α is quasiprimitive on $\Gamma(\alpha)$, then Γ is said to be *G-locally quasiprimitive*. In general, for a given property $\mathcal{P}$, if the action of G_α on $\Gamma(\alpha)$ has the property $\mathcal{P}$, then following [66] we say that Γ is *G-locally $\mathcal{P}$*. Since any 2-transitive group is primitive, it follows from (b) above that any $(G, 2)$-arc transitive graph is G-locally primitive. Similarly, any G-locally primitive graph is G-locally quasiprimitive.

Since the objects studied in this thesis are imprimitive symmetric graphs, we now give a formal definition for such graphs.

Definition 3.1.1 Suppose Γ is a G-symmetric graph. If G acts imprimitively on $V(\Gamma)$, then Γ is said to be an *imprimitive G-symmetric graph.*

Finally, if a graph Γ admits G as a group of automorphisms, then G induces a natural action on the edges of Γ defined by $\{\alpha, \beta\}^g := \{\alpha^g, \beta^g\}$, for $\{\alpha, \beta\} \in E(\Gamma)$ and $g \in G$. If, under this action, G is transitive on $E(\Gamma)$, then Γ is said to be *G-edge-transitive.*

3.2 The geometric approach

Suppose Γ is an imprimitive G-symmetric graph. Then it follows from the definition that $V(\Gamma)$ admits a nontrivial G-invariant partition $\mathcal{B}$. For a vertex α of Γ, we will always use $B(\alpha)$ to denote the (unique) block of $\mathcal{B}$ containing α. Since $\mathcal{B}$ is G-invariant, we have

$$B(\alpha^g) = (B(\alpha))^g \tag{3.1}$$

for any $\alpha \in V(\Gamma)$ and $g \in G$. A standard approach to studying such a graph Γ is to analyse the *quotient graph* of Γ with respect to $\mathcal{B}$, denoted by $\Gamma_{\mathcal{B}}$, which is defined to be the graph with vertex set $\mathcal{B}$ in which two blocks $B, C \in \mathcal{B}$ are *adjacent* if and only if there exists at least one edge of Γ joining a vertex of B and a vertex of C. To extract useful information about Γ from this quotient graph, we require naturally that $\Gamma_{\mathcal{B}}$ is a nonempty graph. In this case we have the following lemma (see [6, Proposition 22.1] or [66, Lemma 1.1(c)] for the "only if" part).

Lemma 3.2.1 *Suppose that Γ is a nonempty G-symmetric graph whose vertex set admits a nontrivial G-invariant partition $\mathcal{B}$. Then $\Gamma_{\mathcal{B}}$ is a nonempty graph if and only if each block of $\mathcal{B}$ is an independent set of Γ.*

Proof Since Γ is nonempty, there exists an arc (α, β) of Γ. The G-symmetry of Γ implies that each arc of Γ has the form (α^g, β^g) for some $g \in G$. So from (3.1) we have: $\Gamma_{\mathcal{B}}$ is a nonempty graph $\Leftrightarrow B(\alpha) \neq B(\beta)$ for some arc (α, β) of $\Gamma \Leftrightarrow B(\alpha^g) \neq B(\beta^g)$ for all arcs (α^g, β^g) of Γ (where $g \in G$) $\Leftrightarrow$ each block of $\mathcal{B}$ is an independent set of Γ. $\square$

In other words, $\Gamma_{\mathcal{B}}$ is an empty graph if and only if each block of $\mathcal{B}$ consists of connected components of Γ. In order to avoid this somewhat trivial case, we make the following convention throughout this thesis.

Convention 3.2.1 For the pair $(\Gamma, \mathcal{B})$ above, we always assume that $\Gamma_{\mathcal{B}}$ is a nonempty graph. Thus each block of $\mathcal{B}$ is an independent set of Γ.

The following lemma shows that $\Gamma_{\mathcal{B}}$ inherits the G-symmetry from Γ.

Lemma 3.2.2 ([66, Lemma 1.1(a)]**)** *Suppose that Γ is a G-symmetric graph and $\mathcal{B}$ is a nontrivial G-invariant partition of $V(\Gamma)$. Then $\Gamma_{\mathcal{B}}$ is G-symmetric under the induced action of G on $\mathcal{B}$.*

Proof Since G is transitive on $V(\Gamma)$ and $\mathcal{B}$ is a G-invariant partition of $V(\Gamma)$, it follows that G is transitive on $\mathcal{B}$, that is, $\Gamma_{\mathcal{B}}$ is G-vertex-transitive. Let $(B, C), (D, E)$ be two arcs of $\Gamma_{\mathcal{B}}$. Then there exist $\alpha \in B, \beta \in C, \gamma \in D, \delta \in E$ such that $(\alpha, \beta), (\gamma, \delta) \in \mathrm{Arc}(\Gamma)$. By the G-symmetry of Γ, there exists $g \in G$ such that $(\alpha, \beta)^g = (\gamma, \delta)$. From (3.1) above, this implies $B^g = (B(\alpha))^g = B(\alpha^g) = B(\gamma) = D$, and similarly $C^g = E$. Hence $(B, C)^g = (D, E)$ and $\Gamma_{\mathcal{B}}$ is G-symmetric. $\square$

We remark that, if Γ is connected, then $\Gamma_{\mathcal{B}}$ is connected as well ([66, Lemma 1.1(b)]). Since the connected components of a symmetric graph are all symmetric and are pairwise isomorphic, without loss of generality we may even require that $\Gamma_{\mathcal{B}}$ is connected. Nevertheless we will not assume this in most parts of this thesis. (Whenever we need the connectedness of $\Gamma_{\mathcal{B}}$ we will state this explicitly.)

The induced action of G on $\mathcal{B}$ is not necessarily faithful. If the kernel of this action is K, then $B^{Kg} := B^g$, for $B \in \mathcal{B}$ and $Kg \in G/K$, defines a faithful action of G/K on $\mathcal{B}$. Moreover, under this action $\Gamma_{\mathcal{B}}$ is (G/K)-symmetric.

The quotient graph $\Gamma_{\mathcal{B}}$ conveys a lot of information about the graph Γ. Nevertheless, it does not determine Γ completely since it does not tell us how adjacent blocks of $\mathcal{B}$ are joined by edges of Γ. To compensate for this shortage, we need to consider the "inter-block" subgraph $\Gamma[B, C]$ of Γ induced by $(\Gamma(C)\cap B)\cup(\Gamma(B)\cap C)$, where B, C are adjacent blocks of $\mathcal{B}$, where for any block $D \in \mathcal{B}$ we set

$$\Gamma(D) := \bigcup_{\alpha \in D} \Gamma(\alpha).$$

Since each block of $\mathcal{B}$ is an independent set of Γ, this subgraph $\Gamma[B, C]$ is a bipartite graph with bipartition $\{\Gamma(C) \cap B, \Gamma(B) \cap C\}$. Denote by $\Gamma_{\mathcal{B}}(B)$ the neighbourhood of B in $\Gamma_{\mathcal{B}}$. To depict genuinely the structure of Γ we also need a "cross-sectional" geometry, namely the incidence structure $\mathcal{D}(B) := (B, \Gamma_{\mathcal{B}}(B), \mathrm{I})$ in which a point $\alpha \in B$ and a block $C \in \Gamma_{\mathcal{B}}(B)$ are incident if and only if α is adjacent in Γ to at least one vertex of C. Clearly, the trace of the block C of $\mathcal{D}(B)$ is $\Gamma(C) \cap B$. We denote by $\Gamma_{\mathcal{B}}(\alpha)$ the trace of α in $\mathcal{D}(B)$, that is,

$$\Gamma_{\mathcal{B}}(\alpha) := \{C \in \Gamma_{\mathcal{B}}(B) : \alpha \in \Gamma(C)\}.$$

We will show in the following that, up to isomorphism, $\Gamma[B, C]$ and $\mathcal{D}(B)$ are respectively independent of the choice of adjacent blocks B, C and the block B, and that $\mathcal{D}(B)$ is in fact a 1-design (see Lemmas 3.2.3(a) and 3.2.5(a) below). Thus, with any imprimitive G-symmetric graph Γ and nontrivial G-invariant partition $\mathcal{B}$ of $V(\Gamma)$ we have associated three configurations, namely the quotient graph $\Gamma_{\mathcal{B}}$, the bipartite graph $\Gamma[B, C]$, and the 1-design $\mathcal{D}(B)$. In a very informal way we can say that the graph Γ is "decomposed" into the "product" of these three configurations which, according to Gardiner and Praeger [43], might have a strong influence on the structure of Γ. The usefulness of this geometric approach to studying imprimitive symmetric graphs lies in the following two aspects:

(i) a detailed analysis of the three configurations above; and

(ii) an attempt at reconstructing Γ from the triple $(\Gamma_{\mathcal{B}}, \Gamma[B, C], \mathcal{D}(B))$.

In the subsequent chapters we will follow this approach and study some specific classes of imprimitive symmetric graphs.

Now let us prove some basic properties regarding $\Gamma[B,C]$ and $\mathcal{D}(B)$.

Lemma 3.2.3 *Suppose the triple* $(\Gamma, G, \mathcal{B})$ *is as in Lemma 3.2.2. Then the following* (a)-(c) *hold for adjacent blocks* B, C *of* $\mathcal{B}$.

(a) *The bipartite graph* $\Gamma[B,C]$ *is, up to isomorphism, independent of the choice of adjacent blocks* B, C *of* $\mathcal{B}$.

(b) ([66, Lemma 1.4(b)]) $\Gamma[B,C]$ *is* $(G_{B\cup C})$*-symmetric and* $(G_{B,C})$*-edge-transitive.*

(c) $\Gamma(C) \cap B$ *and* $\Gamma(B) \cap C$ *are two* $(G_{B,C})$*-orbits on* $V(\Gamma)$.

(d) $|G_{B\cup C} : G_{B,C}| = 2$*, and hence* $G_{B,C} \trianglelefteq G_{B\cup C}$.

Proof Let $B, C; D, E$ be two pairs of adjacent blocks of $\mathcal{B}$. Then $(B,C), (D,E)$ are arcs of $\Gamma_{\mathcal{B}}$. Hence by Lemma 3.2.2 there exists $g \in G$ such that $(B,C)^g = (D,E)$. The restriction $\hat{g}$ of g on $B \cup C$ is a bijection from $B \cup C$ to $D \cup E$. Since g preserves the adjacency of Γ, $\hat{g}$ is an isomorphism form $\Gamma[B,C]$ to $\Gamma[D,E]$, and thus (a) is proved.

Now let us prove (b). Since Γ is G-symmetric, for any two arcs $(\alpha,\beta), (\gamma,\delta)$ of $\Gamma[B,C]$, there exists $g \in G$ such that $(\alpha,\beta)^g = (\gamma,\delta)$. Since either α, γ are in the same block of B, C and β, δ in the other, or α, δ are in the same block of B, C and β, γ in the other, it follows from (3.1) that $(B,C)^g = (B,C)$ or (C,B) respectively. So $g \in G_{B\cup C}$ and hence $\Gamma[B,C]$ is $(G_{B\cup C})$-symmetric. For any two edges $\{\alpha,\beta\}, \{\gamma,\delta\}$ of $\Gamma[B,C]$, we may assume without loss of generality that $\alpha, \gamma \in B$ and $\beta, \delta \in C$. Then there exists $h \in G$ such that $(\alpha,\beta)^h = (\gamma,\delta)$. Again by (3.1) we have $h \in G_{B,C}$. Thus $\Gamma[B,C]$ is $(G_{B,C})$-edge-transitive, and hence (b) is proved. Since each vertex in $\Gamma(C) \cap B$ is incident with an edge of $\Gamma[B,C]$ and since $G_{B,C}$ is transitive on the edges of $\Gamma[B,C]$, we conclude that $G_{B,C}$ is transitive on $\Gamma(C) \cap B$. Similarly, $G_{B,C}$ is transitive on $\Gamma(B) \cap C$. Since $\Gamma(C) \cap B$ and $\Gamma(B) \cap C$ are $(G_{B,C})$-invariant, part (c) follows.

Finally, for any $g, h \in G_{B\cup C} \setminus G_{B,C}$ and $x \in G_{B,C}$, one can check that $g^{-1}xh \in G_{B,C}$ and hence $g^{-1}(G_{B,C})h = G_{B,C}$. Therefore, $|G_{B\cup C} : G_{B,C}| = 2$ and thus $G_{B,C} \trianglelefteq G_{B\cup C}$. $\square$

Lemma 3.2.4 *Suppose the triple* $(\Gamma, G, \mathcal{B})$ *is as in Lemma 3.2.2, and let* $\alpha \in V(\Gamma)$. *Then the following* (a)-(b) *hold.*

(a) ([66, Lemma 1.4(a)]) $\Gamma(\alpha)$ *admits a* G_α*-invariant partition, namely* $\{\Gamma(\alpha) \cap C : C \in \Gamma_{\mathcal{B}}(\alpha)\}$. *Moreover,* G_α *is transitive on the blocks of this partition.*

(b) *If* $\Gamma[B, C]$ *is a matching, then the blocks of this partition are singletons and the actions of* G_α *on* $\Gamma(\alpha)$ *and* $\Gamma_{\mathcal{B}}(\alpha)$ *are permutationally equivalent.*

Proof Set $\mathcal{P} := \{\Gamma(\alpha) \cap C : C \in \Gamma_{\mathcal{B}}(\alpha)\}$. Then clearly $\mathcal{P}$ is a partition of $\Gamma(\alpha)$. For any block $\Gamma(\alpha) \cap C$ of $\mathcal{P}$ and $g \in G_\alpha$, one can easily check that $(\Gamma(\alpha) \cap C)^g = \Gamma(\alpha) \cap C^g \in \mathcal{P}$ and hence $\mathcal{P}$ is G_α-invariant. Let $\Gamma(\alpha) \cap D$ be a second block of $\mathcal{P}$. Then there exist $\beta \in C, \gamma \in D$ which are adjacent to α. So there exists $h \in G$ such that $(\alpha, \beta)^h = (\alpha, \gamma)$. Thus $h \in G_\alpha$ and (3.1) implies $C^h = D$. Therefore, we have $(\Gamma(\alpha) \cap C)^h = \Gamma(\alpha) \cap C^h = \Gamma(\alpha) \cap D$, which implies that G_α is transitive on $\mathcal{P}$ and hence (a) is proved.

If $\Gamma[B, C]$ is a matching for adjacent blocks B, C of $\mathcal{B}$, then (a) implies that each $C \in \Gamma_{\mathcal{B}}(\alpha)$ contains a unique vertex adjacent to α and hence $\rho : \beta \mapsto B(\beta)$, for $\beta \in \Gamma(\alpha)$, defines a bijection from $\Gamma(\alpha)$ to $\Gamma_{\mathcal{B}}(\alpha)$. From (3.1), we then have $\rho(\beta^g) = B(\beta^g) = (B(\beta))^g = (\rho(\beta))^g$ for $g \in G_\alpha$, and hence (b) follows. □

From Lemma 3.2.3(a), we know that

$$k := |\Gamma(B) \cap C|$$

is independent of the choice of adjacent blocks B, C of $\mathcal{B}$. Since G is transitive on $V(\Gamma)$, the value

$$r := |\Gamma_{\mathcal{B}}(\alpha)|$$

is independent of the choice of $\alpha \in V(\Gamma)$. Consequently, the incidence structure $\mathcal{D}(B)$ is a 1-(v, k, r) design, where

$$v := |B|$$

denotes the block size of $\mathcal{B}$. Similarly, the G-vertex-transitivity of Γ and Lemma 3.2.4(a) together imply that

$$s := |\Gamma(\alpha) \cap C|$$

is independent of the choice of the flag (α, C) of $\mathcal{D}(B)$ (where $\alpha \in B$ and $C \in \Gamma_{\mathcal{B}}(\alpha)$). Clearly, we have $\mathrm{val}(\Gamma) = rs$ and $\mathrm{val}(\Gamma[B, C]) = s$. Let

$$b := \mathrm{val}(\Gamma_{\mathcal{B}})$$

denote the valency of $\Gamma_{\mathcal{B}}$. Then b is equal to the number of blocks of $\mathcal{D}(B)$. Throughout this thesis we will reserve these symbols v, r, b, k, s for the above-defined parameters with respect to the G-invariant partition $\mathcal{B}$ of $V(\Gamma)$. In particular, if $k = v$, $s = 1$, then the bipartite graph $\Gamma[B, C]$ is a perfect matching between B and C, and in this case the graph Γ is called a *cover* of $\Gamma_{\mathcal{B}}$. In general, if $k = v$, then following [54], Γ is said to be a *multicover* of $\Gamma_{\mathcal{B}}$. Similarly, if $k = v - 1$ and $s = 1$, that is, $\Gamma[B, C] \cong (v - 1) \cdot K_2$, then we say that Γ is an *almost cover* of $\Gamma_{\mathcal{B}}$ and that $\Gamma_{\mathcal{B}}$ is *almost covered* by Γ. Lemma 3.2.4 has the following consequence for G-locally primitive graphs.

Corollary 3.2.1 ([43, Lemma 3.1(b)]) *Suppose that Γ is a G-locally primitive graph and $\mathcal{B}$ is a nontrivial G-invariant partition of $V(\Gamma)$. Then either*

(a) $\Gamma[B, C] \cong k \cdot K_2$ *is a matching of k edges, for adjacent blocks B, C of $\mathcal{B}$; or*

(b) Γ *is a bipartite graph with each part of the bipartition of a connected component contained in some block of $\mathcal{B}$, the traces of any two distinct blocks of $\mathcal{D}(B)$ are disjoint (thus $\mathcal{D}(B)$ contains no repeated blocks), and $v = bk$.*

Proof Let $B \in \mathcal{B}$ and $\alpha \in B$. By Lemma 3.2.4(a), $\{\Gamma(\alpha) \cap C : C \in \Gamma_{\mathcal{B}}(\alpha)\}$ is a G_α-invariant partition of $\Gamma(\alpha)$. Since Γ is G-locally primitive, for $C \in \Gamma_{\mathcal{B}}(\alpha)$ we have either (i) $|\Gamma(\alpha) \cap C| = 1$; or (ii) $\Gamma(\alpha) \cap C = \Gamma(\alpha)$. In the first case, we have $\Gamma[B, C] \cong k \cdot K_2$ and (a) occurs. In the second case, we have $\Gamma(\alpha) \subseteq C$. This, together with Lemma 3.2.3, implies that the connected component of Γ containing α is a bipartite graph with one part of its bipartition contained in $\Gamma(C) \cap B$ and the other contained in $\Gamma(B) \cap C$. Thus Γ is a bipartite graph. Moreover, we have $r = 1$ and the traces of any two blocks of $\mathcal{D}(B)$ are disjoint. Hence $v = bk$ and (b) occurs. $\square$

Lemma 3.2.5 *Suppose the triple $(\Gamma, G, \mathcal{B})$ is as in Lemma 3.2.2. Then the following (a)-(c) hold for $B \in \mathcal{B}$.*

(a) *$\mathcal{D}(B)$ is a 1-(v,k,r) design; moreover, up to isomorphism, it is independent of the choice of $B \in \mathcal{B}$.*

(b) *G_B induces a group of automorphisms of $\mathcal{D}(B)$ which is transitive on the points, the blocks and the flags of $\mathcal{D}(B)$.*

(c) *G induces a transitive action on the set of triples $(\mathcal{D}(B), \alpha, C)$, where $B \in \mathcal{B}$ and (α, C) is a flag of $\mathcal{D}(B)$.*

Proof (a) That $\mathcal{D}(B)$ is a 1-(v,k,r) design has been shown earlier. For two blocks $B, D \in \mathcal{B}$, there exists $g \in G$ such that $B^g = D$. So we have $(\Gamma_{\mathcal{B}}(B))^g = \Gamma_{\mathcal{B}}(D)$ and hence g induces a bijection from $B \cup \Gamma_{\mathcal{B}}(B)$ to $D \cup \Gamma_{\mathcal{B}}(D)$. For $\alpha \in B$ and $C \in \Gamma_{\mathcal{B}}(B)$, we have: (α, C) is a flag of $\mathcal{D}(B) \Leftrightarrow \alpha \in \Gamma(C) \cap B \Leftrightarrow \alpha^g \in \Gamma(C^g) \cap B^g$ $\Leftrightarrow \alpha^g \in \Gamma(C^g) \cap D \Leftrightarrow (\alpha^g, C^g)$ is a flag of $\mathcal{D}(D)$. Therefore, g induces an isomorphism from $\mathcal{D}(B)$ to $\mathcal{D}(D)$.

(b) Similarly, each $g \in G_B$ induces a bijection from $B \cup \Gamma_{\mathcal{B}}(B)$ to itself such that $B^g = B$ and $(\Gamma_{\mathcal{B}}(B))^g = \Gamma_{\mathcal{B}}(B)$. We have: (α, C) is a flag of $\mathcal{D}(B) \Leftrightarrow \alpha \in \Gamma(C) \cap B$ $\Leftrightarrow \alpha^g \in \Gamma(C^g) \cap B \Leftrightarrow (\alpha^g, C^g)$ is a flag of $\mathcal{D}(B)$. So g induces an automorphism of $\mathcal{D}(B)$. Therefore, G_B induces a group of automorphisms of $\mathcal{D}(B)$.

Since Γ is G-vertex-transitive, G_B is transitive on the point set B of $\mathcal{D}(B)$. Since $\Gamma_{\mathcal{B}}$ is G-symmetric (Lemma 3.2.2), by Lemma 3.1.1(a) G_B is transitive on the block set $\Gamma_{\mathcal{B}}(B)$ of $\mathcal{D}(B)$. Suppose (α, C), (β, D) are two flags of $\mathcal{D}(B)$. Then α is adjacent to a vertex $\gamma \in C$ and β is adjacent to a vertex $\delta \in D$. By the G-symmetry of Γ, there exists $g \in G$ such that $(\alpha, \gamma)^g = (\beta, \delta)$. This implies $g \in G_B$ and $(\alpha, C)^g = (\beta, D)$, and hence G_B is transitive on the flags of $\mathcal{D}(B)$.

(c) Clearly, $(\mathcal{D}(B), \alpha, C)^g := (\mathcal{D}(B^g), \alpha^g, C^g)$, for $g \in G$, defines an action of G on the set of triples $(\mathcal{D}(B), \alpha, C)$ such that $B \in \mathcal{B}$ and (α, C) is a flag of $\mathcal{D}(B)$. Since G is transitive on $\mathcal{B}$ and G_B is transitive on the flags of $\mathcal{D}(B)$, as shown in (b), we see that G is transitive on the set of such triples. $\square$

By Lemma 3.2.5(b), the number of times a block C of $\mathcal{D}(B)$ is repeated is independent of the choice of B, C. We call this number the *multiplicity* of $\mathcal{D}(B)$. *For most of the time, we will view $\mathcal{D}(B)$ as the 1-(v,k,r) design with point set B and blocks the subsets $\Gamma(C) \cap B$ of B (for $C \in \Gamma_{\mathcal{B}}(B)$) each repeated m times, where m is the multiplicity of $\mathcal{D}(B)$.*

We conclude this section by examining certain "local actions" induced by the action of G on $V(\Gamma)$. For $B \in \mathcal{B}$, the pointwise stabilizer $G_{(B)}$ is, by definition, the kernel of the action of G_B on B. We will use $G_{[B]}$ to denote the kernel of the action of G_B on $\Gamma_{\mathcal{B}}(B)$. Thus, $G_{[B]}$ is the subgroup of G_B fixing each $C \in \Gamma_{\mathcal{B}}(B)$ blockwise. For $\alpha \in V(\Gamma)$, the stabilizer G_α of α in G induces a natural action on $\Gamma_{\mathcal{B}}(\alpha)$. We denote by $G_{[\alpha]}$ the kernel of this action, that is, $G_{[\alpha]}$ is the subgroup of G_α fixing each $B \in \Gamma_{\mathcal{B}}(\alpha)$ setwise. One can see that $G_{[\alpha]}$ is also the subgroup of G_α fixing each $\Gamma(\alpha) \cap B$ setwise, and hence it induces a natural action on $\Gamma(\alpha) \cap B$. A detailed study of the influence of these "local actions" on the structure of Γ will be conducted in Chapters 10 and 11. For the moment we content ourselves with the following lemma the proof of which is straightforward. For a vertex $\alpha \in V(\Gamma)$ and a block $B \in \mathcal{B}$, we set $G_{\alpha,B} := (G_\alpha)_B$.

Lemma 3.2.6 *Suppose the triple* $(\Gamma, G, \mathcal{B})$ *is as in Lemma 3.2.2.*

(a) *For* $B, C \in \mathcal{B}$*, the following* (i)-(ii) *hold.*

  (i) *The action of* G_B *on* B *and the action of* G_C *on* C *are permutationally isomorphic; and the action of* G_B *on* $\Gamma_{\mathcal{B}}(B)$ *and the action of* G_C *on* $\Gamma_{\mathcal{B}}(C)$ *are permutationally isomorphic.*

  (ii) *The action of* $G_{(B)}$ *on* $\Gamma_{\mathcal{B}}(B)$ *and the action of* $G_{(C)}$ *on* $\Gamma_{\mathcal{B}}(C)$ *are permutationally isomorphic.*

(b) *For any* $\alpha, \beta \in V(\Gamma)$*, the following* (i)-(iii) *hold.*

  (i) *The action of* $G_{(B(\alpha))}$ *on* $\Gamma(\alpha)$ *and the action of* $G_{(B(\beta))}$ *on* $\Gamma(\beta)$ *are permutationally isomorphic. In particular, if* $\alpha, \beta \in B$*, then the actions of* $G_{(B)}$ *on* $\Gamma(\alpha)$ *and* $\Gamma(\beta)$ *are permutationally isomorphic.*

  (ii) *The action of* $G_{(B(\alpha))}$ *on* $\Gamma_{\mathcal{B}}(\alpha)$ *and the action of* $G_{(B(\beta))}$ *on* $\Gamma_{\mathcal{B}}(\beta)$ *are permutationally isomorphic. In particular, if* $\alpha, \beta \in B$*, then the actions of* $G_{(B)}$ *on* $\Gamma_{\mathcal{B}}(\alpha)$ *and* $\Gamma_{\mathcal{B}}(\beta)$ *are permutationally isomorphic.*

  (iii) *The action of* G_α *on* $\Gamma_{\mathcal{B}}(\alpha)$ *and the action of* G_β *on* $\Gamma_{\mathcal{B}}(\beta)$ *are permutationally isomorphic.*

(c) *For any* $\alpha \in V(\Gamma)$ *and* $B, C \in \Gamma_{\mathcal{B}}(\alpha)$*, the following* (i)-(ii) *hold.*

(i) *The action of $G_{\alpha,B}$ on $\Gamma(\alpha) \cap B$ and the action of $G_{\alpha,C}$ on $\Gamma(\alpha) \cap C$ are permutationally isomorphic.*

(ii) *The actions of $G_{[\alpha]}$ on $\Gamma(\alpha) \cap B$ and on $\Gamma(\alpha) \cap C$ are permutationally isomorphic.*

Proof (a) Since G is transitive on $\mathcal{B}$, there exists $g \in G$ such that $B^g = C$. So g induces a bijection ρ from B to C. By Lemma 2.1.1(a), $x \mapsto g^{-1}xg$ for $x \in G_B$ defines an isomorphism from G_B to G_C. Since, for $\gamma \in B$, $\rho(\gamma^x) = (\gamma^x)^g = (\gamma^g)^{g^{-1}xg} = (\rho(\gamma))^{g^{-1}xg}$, the first assertion in (i) follows. We have $(\Gamma_{\mathcal{B}}(B))^g = \Gamma_{\mathcal{B}}(C)$ and g induces a bijection $\lambda : D \mapsto D^g$ from $\Gamma_{\mathcal{B}}(B)$ to $\Gamma_{\mathcal{B}}(C)$. Since $\lambda(D^x) = (D^x)^g = (D^g)^{g^{-1}xg} = (\lambda(D))^{g^{-1}xg}$, the second assertion in (i) then follows.

Similarly, for any $x \in G_{(B)}$, we have $g^{-1}xg \in G_{(C)}$ and $x \mapsto g^{-1}xg$ defines an isomorphism from $G_{(B)}$ to $G_{(C)}$. One can see that the action of $G_{(B)}$ on $\Gamma_{\mathcal{B}}(B)$ is permutationally isomorphic to the action of $G_{(C)}$ on $\Gamma_{\mathcal{B}}(C)$ with respect to λ.

(b) Since G is transitive on $V(\Gamma)$, there exists $g \in G$ such that $\alpha^g = \beta$. Hence g induces a bijection $\rho : \Gamma(\alpha) \to \Gamma(\beta)$ defined by $\rho : \gamma \mapsto \gamma^g$ for $\gamma \in \Gamma(\alpha)$. For each $x \in G_{(B(\alpha))}$, one can see that $g^{-1}xg \in G_{(B(\beta))}$ and hence $x \mapsto g^{-1}xg$ defines an isomorphism from $G_{(B(\alpha))}$ to $G_{(B(\beta))}$. It is clear that, for any $\gamma \in \Gamma(\alpha)$ and $x \in G_{(B)}$, we have $\rho(\gamma^x) = (\gamma^x)^g = (\gamma^g)^{g^{-1}xg} = (\rho(\gamma))^{g^{-1}xg}$, and hence (i) follows. Similarly, g induces a bijection $\lambda : \Gamma_{\mathcal{B}}(\alpha) \to \Gamma_{\mathcal{B}}(\beta)$ defined by $\lambda : D \mapsto D^g$ for $D \in \Gamma_{\mathcal{B}}(\alpha)$. We have $\lambda(D^x) = (\lambda(D))^{g^{-1}xg}$ for $D \in \Gamma_{\mathcal{B}}(\alpha)$ and $x \in G_{(B)}$, and hence (ii) follows. By Lemma 2.1.1(a), we have $G_\beta = g^{-1}G_\alpha g$, and $x \mapsto g^{-1}xg$, for $x \in G_\alpha$, defines an isomorphism from G_α to G_β. Since $\lambda(D^x) = (D^x)^g = (D^g)^{g^{-1}xg} = (\lambda(D))^{g^{-1}xg}$ for $D \in \Gamma_{\mathcal{B}}(\alpha)$ and $x \in G_\alpha$, the assertion in (iii) then follows.

(c) Since $B, C \in \Gamma_{\mathcal{B}}(\alpha)$, α is adjacent to a vertex γ in B and a vertex δ in C. So there exists $g \in G$ such that $(\alpha, \gamma)^g = (\alpha, \delta)$. This implies that $g \in G_\alpha$ and $B^g = C$, and hence $(\Gamma(\alpha) \cap B)^g = \Gamma(\alpha) \cap C$. Thus g induces a bijection ρ from $\Gamma(\alpha) \cap B$ to $\Gamma(\alpha) \cap C$ defined by $\rho : \gamma \mapsto \gamma^g$ for $\gamma \in \Gamma(\alpha) \cap B$. By Lemma 2.1.1(a), we have $G_{\alpha,C} = (G_\alpha)_C = (G_\alpha)_{B^g} = g^{-1}(G_\alpha)_B g = g^{-1}(G_{\alpha,B})g$, and hence $x \mapsto g^{-1}xg$, for $x \in G_{\alpha,B}$, defines an isomorphism from $G_{\alpha,B}$ to $G_{\alpha,C}$. Since $\rho(\gamma^x) = (\gamma^x)^g = (\gamma^g)^{g^{-1}xg} = (\rho(\gamma))^{g^{-1}xg}$, the action of $G_{\alpha,B}$ on $\Gamma(\alpha) \cap B$ and the action of $G_{\alpha,C}$ on $\Gamma(\alpha) \cap C$ are permutationally isomorphic with respect to ρ. For each $x \in G_{[\alpha]}$, we have $g^{-1}xg \in G_{[\alpha]}$, and $x \mapsto g^{-1}xg$ defines an automorphism of

$G_{[\alpha]}$. By a similar argument as above one can see that the actions of $G_{[\alpha]}$ on $\Gamma(\alpha)\cap B$ and $\Gamma(\alpha)\cap C$ are permutationally isomorphic with respect to ρ. $\square$

3.3 Refining the given partition

The main result of this section is the following theorem, which shows that in some cases we can get a (nontrivial) refinement $\mathcal{B}^*$ of the given G-invariant partition $\mathcal{B}$. For $\mathcal{B}^*$, the parameters v^*, r^*, k^*, s^* have analogous meanings to the parameters v, r, k, s respectively for $\mathcal{B}$.

Theorem 3.3.1 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$. Then Γ admits a second G-invariant partition $\mathcal{B}^*$, which is a refinement of $\mathcal{B}$ (possibly $\mathcal{B}^* = \mathcal{B}$), such that the block size v^* of $\mathcal{B}^*$ is a common divisor of v and k, and that $s = cs^*, r^* = cr$ for some integer $c \geq 1$. The block of $\mathcal{B}^*$ containing $\alpha \in B$ is $B^* := \bigcap_{C\in\Gamma_{\mathcal{B}}(\alpha)}(\Gamma(C)\cap B)$. In particular, if G_α is primitive on $\Gamma_{\mathcal{B}}(\alpha)$, then either*

(a) *$\mathcal{D}(B)$ has no repeated blocks, or*

(b) *k divides v, $\mathcal{B}^* = \{(\Gamma(C)\cap B)^g : g \in G\}$ (where $C \in \Gamma_{\mathcal{B}}(B)$), Γ is a multicover of $\Gamma_{\mathcal{B}^*}$ with $v^* = k^* = k, s^* = s, r^* = r$, and $\Gamma[B, C] \cong \Gamma[B^*, C^*]$, where $B^* \in \mathcal{B}^*$ and $C^* \in \Gamma_{\mathcal{B}^*}(B^*)$.*

Proof Let $B \in \mathcal{B}$. For a fixed vertex $\alpha \in B$, we define a binary relation " $\sim_\alpha$ " on $\Gamma_{\mathcal{B}}(\alpha)$ by

$$E \sim_\alpha F \Leftrightarrow \Gamma(E)\cap B = \Gamma(F)\cap B$$

for $E, F \in \Gamma_{\mathcal{B}}(\alpha)$. Then clearly " $\sim_\alpha$ " is an equivalence relation on $\Gamma_{\mathcal{B}}(\alpha)$. Let n be the number of equivalence classes of " $\sim_\alpha$ ", and let $\mathbf{R}(\alpha) := \{R_i(\alpha) : 1 \leq i \leq n\}$ denote the partition of $\Gamma_{\mathcal{B}}(\alpha)$ induced by " $\sim_\alpha$ ". Then, for $g \in G_\alpha$ and $E, F \in \Gamma_{\mathcal{B}}(\alpha)$, we have: $E \sim_\alpha F \Leftrightarrow \Gamma(E)\cap B = \Gamma(F)\cap B \Leftrightarrow \Gamma(E^g)\cap B = \Gamma(F^g)\cap B \Leftrightarrow E^g \sim_\alpha F^g$. This implies that $\mathbf{R}(\alpha)$ is a G_α-invariant partition of $\Gamma_{\mathcal{B}}(\alpha)$. The block size of $\mathbf{R}(\alpha)$ is equal to the multiplicity m of $\mathcal{D}(B)$. Hence $mn = r$, and in particular n is a divisor of r. Also, since $E \in \Gamma_{\mathcal{B}}(\alpha)$ implies $E^g \in \Gamma_{\mathcal{B}}(\alpha^g)$ and $\Gamma(E^g)\cap B = (\Gamma(E)\cap B)^g$ for $g \in G_B$, from the definition of " $\sim_\alpha$ " we have $\mathbf{R}(\alpha^g) = (\mathbf{R}(\alpha))^g = \{(R_i(\alpha))^g : 1 \leq i \leq n\}$. Hence the size n of $\mathbf{R}(\alpha)$ is independent of the choice of $\alpha \in B$.

Now we define an equivalence relation " $\sim$ " on B by

$$\alpha \sim \beta \Leftrightarrow \Gamma_{\mathcal{B}}(\alpha) = \Gamma_{\mathcal{B}}(\beta) \text{ and } \mathbf{R}(\alpha) = \mathbf{R}(\beta)$$

for $\alpha, \beta \in B$. Since, for $g \in G_B$, $\alpha \sim \beta$ implies $\Gamma_{\mathcal{B}}(\alpha^g) = \Gamma_{\mathcal{B}}(\beta^g)$ and $\mathbf{R}(\alpha^g) = \mathbf{R}(\beta^g)$, the partition of B induced by " $\sim$ " is G_B-invariant. Let α be a fixed vertex in B, and let B^* be the block of this partition containing α. Then $G_\alpha \le G_{B^*} \le G_B$, and so (see the third paragraph of Section 2.2) $\mathcal{B}^* := \{B^{*g} : g \in G\}$ is a G-invariant partition of $V(\Gamma)$ refining $\mathcal{B}$. Thus $v^* = |B^*| = v/|G_B : G_{B^*}|$ divides v. We claim that $B^* = \bigcap_{C \in \Gamma_{\mathcal{B}}(\alpha)} (\Gamma(C) \cap B)$. In fact, let $C_{\alpha 1}, \ldots, C_{\alpha n}$ be representatives from $R_1(\alpha), \ldots, R_n(\alpha)$, respectively. If $\beta \in \bigcap_{i=1}^n (\Gamma(C_{\alpha i}) \cap B)$, then for each i, $\beta \in \Gamma(C_{\alpha i}) \cap B = \Gamma(C) \cap B$ for each $C \in R_i(\alpha)$, and hence $\Gamma_{\mathcal{B}}(\beta) = \Gamma_{\mathcal{B}}(\alpha)$ and $C_{\alpha 1}, \ldots, C_{\alpha n}$ are representatives from $R_1(\beta), \ldots, R_n(\beta)$. This implies $\mathbf{R}(\alpha) = \mathbf{R}(\beta)$ and hence $\beta \in B^*$. Conversely, for any $\beta \in B^*$, we have $\mathbf{R}(\alpha) = \mathbf{R}(\beta)$ and hence $\{R_1(\beta), \ldots, R_n(\beta)\} = \{R_1(\alpha), \ldots, R_n(\alpha)\}$. So $\beta \in \bigcap_{i=1}^n (\Gamma(C_{\alpha i}) \cap B)$. Thus we have proved that $B^* = \bigcap_{i=1}^n (\Gamma(C_{\alpha i}) \cap B)$, that is, $B^* = \bigcap_{C \in \Gamma_{\mathcal{B}}(\alpha)} (\Gamma(C) \cap B)$. Let $\gamma \in \Gamma(D) \cap B$, where $D \in \Gamma_{\mathcal{B}}(\alpha)$. Then there exists $g \in G_B$ such that $\gamma = \alpha^g$. So $\gamma \in B^{*g} = \bigcap_{C \in \Gamma_{\mathcal{B}}(\alpha)} (\Gamma(C) \cap B)^g = \bigcap_{C \in \Gamma_{\mathcal{B}}(\gamma)} (\Gamma(C) \cap B)$. Note that $\gamma \in \Gamma(D) \cap B$ implies $D \in \Gamma_{\mathcal{B}}(\gamma)$. So we have $B^{*g} \subseteq \Gamma(D) \cap B$. Therefore, $\Gamma(D) \cap B$ is a union of blocks of $\mathcal{B}^*$. Hence v^* is a divisor of k. Let c denote the number of blocks B^{*g} of $\mathcal{B}^*$ contained in D such that $\Gamma(\alpha) \cap B^{*g} \neq \emptyset$. Since G_B is transitive on the flags of $\mathcal{D}(B)$ (Lemma 3.2.5(b)), c is independent of the choice of D. Clearly, we have $s = cs^*$. Since $\mathrm{val}(\Gamma) = rs = r^*s^*$, this implies $r^* = cr$.

Now we suppose that G_α is primitive on $\Gamma_{\mathcal{B}}(\alpha)$. If $\mathcal{D}(B)$ contains repeated blocks, say C, D, let $\alpha \in \Gamma(C) \cap B = \Gamma(D) \cap B$. Then each part $R_i(\alpha)$ of the G_α-invariant partition $\mathbf{R}(\alpha)$ of $\Gamma_{\mathcal{B}}(\alpha)$ has size at least 2. So the primitivity of G_α on $\Gamma_{\mathcal{B}}(\alpha)$ implies that $\mathbf{R}(\alpha)$ must be the trivial partition $\{\Gamma_{\mathcal{B}}(\alpha)\}$. Hence $\Gamma(C) \cap B$ is the same for all $C \in \Gamma_{\mathcal{B}}(\alpha)$. Thus the equivalence relation " $\sim$ " on B defined above becomes: $\alpha \sim \beta$ if and only if $\Gamma(E) \cap B = \Gamma(F) \cap B$ for any $E \in \Gamma_{\mathcal{B}}(\alpha), F \in \Gamma_{\mathcal{B}}(\beta)$. This implies that the block of $\mathcal{B}^*$ containing α is $\Gamma(C) \cap B$, and hence $\mathcal{B}^* = \{(\Gamma(C) \cap B)^g : g \in G\}$. Thus $k = v^*$ and so k is a divisor of v, and Γ is a multicover of $\Gamma_{\mathcal{B}^*}$ (that is, $k^* = v^*$). In this case, we have $s^* = s, r^* = r$ and $\Gamma[B, C] \cong \Gamma[B^*, C^*]$ for adjacent blocks $B^* := \Gamma(C) \cap B$, $C^* := \Gamma(B) \cap C$ of $\mathcal{B}^*$. □

Theorem 3.3.1 implies the following known result.

Corollary 3.3.1 ([43, Lemma 3.3(c)]) *Suppose that Γ is a G-locally primitive graph. Suppose further that $V(\Gamma)$ admits a nontrivial G-invariant partition $\mathcal{B}$ such that the block size k of $\mathcal{D}(B)$ satisfies $2 \leq k \leq v-1$. Then either*

(a) *$\mathcal{D}(B)$ contains no repeated blocks, or*

(b) *k divides v, and $V(\Gamma)$ admits a second nontrivial G-invariant partition $\mathcal{B}^*$, which is a refinement of $\mathcal{B}$, such that Γ is a cover of $\Gamma_{\mathcal{B}^*}$.*

Proof Since Γ is G-locally primitive, Corollary 3.2.1 applies. If the second possibility in Corollary 3.2.1 occurs, then (a) above holds; otherwise we have $s = 1$ and hence G_α is primitive on $\Gamma_{\mathcal{B}}(\alpha)$ by the G-local primitivity of Γ and Lemma 3.2.4. In this latter case, from the second half of Theorem 3.3.1, either (a) or (b) above holds. □

Remark 3.3.1 We should emphasize that $\mathcal{B}^*$ could be a trivial partition of $V(\Gamma)$ or identical with $\mathcal{B}$ in some cases. For example, if $k = 1$, then $v^* = 1$ and of course $\mathcal{B}^*$ is trivial. However, there are other cases for which $\mathcal{B}^*$ is a genuine refinement of $\mathcal{B}$, and these are the cases we are interested in. Corollary 3.3.1 above shows that this is the case whenever Γ is G-locally primitive such that $2 \leq k \leq v-1$ and $\mathcal{D}(B)$ contains repeated blocks.

A nontrivial G-invariant partition $\mathcal{B}$ of $V(\Gamma)$ is said to be *minimal* if there is no genuine refinement of $\mathcal{B}$ which is also a G-invariant partition of $V(\Gamma)$. Any imprimitive G-symmetric graph Γ admits at least one minimal nontrivial G-invariant partition. Applying Theorem 3.3.1, the minimality of such a partition $\mathcal{B}$ implies that either $v^* = 1$ or $v^* = v$. In the first case, we have $\bigcap_{C \in \Gamma_{\mathcal{B}}(\alpha)}(\Gamma(C) \cap B) \neq \bigcap_{C \in \Gamma_{\mathcal{B}}(\beta)}(\Gamma(C) \cap B)$ for distinct $\alpha, \beta \in B$, and in this case we say that Γ is *vertex-distinct* with respect to $\mathcal{B}$. In the second case, since we have proved that v^* is a divisor of k, we must have $v^* = v = k$ and thus Γ is a multicover of $\Gamma_{\mathcal{B}}$. So Theorem 3.3.1 has the following consequence.

Corollary 3.3.2 *Suppose that Γ is an imprimitive G-symmetric graph, and let $\mathcal{B}$ be a minimal nontrivial G-invariant partition of $V(\Gamma)$. Then either Γ is vertex-distinct with respect to $\mathcal{B}$, or Γ is a multicover of $\Gamma_{\mathcal{B}}$.*

Chapter 4

The case $k = v - 1$: A general analysis

There is never a case when the root is in disorder and yet the branches are in order.

Confucius (551-479 B.C.), The Great Learning

In this chapter and the four chapters hereafter we will concentrate on the case where $k = v - 1$. This requirement is equivalent to the following: For distinct blocks $B, C \in \mathcal{B}$, either there are no edges between B and C, or there is a unique vertex $\alpha \in B$ such that $\Gamma(\alpha) \cap C = \emptyset$. Thus in this case the design $\mathcal{D}(B)$ is degenerate, with each $(v-1)$-element subset of B occurring as a (possibly repeated) block of $\mathcal{D}(B)$.

This chapter is devoted to a general analysis for the case $k = v - 1$. The results obtained here will be used in the next four chapters and Chapter 11.

4.1 Notation and preliminary results

Let us first introduce some special notation for the case where $k = v - 1$. Suppose Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - 1$. Let $B = B(\alpha)$ for $\alpha \in V(\Gamma)$, and let

$$\mathcal{B}(\alpha) := \{C \in \mathcal{B} : \Gamma(C) \cap B = B \setminus \{\alpha\}\}. \tag{4.1}$$

Thus $\mathcal{B}(\alpha)$ is the set of blocks which are adjacent to $B(\alpha)$ in $\Gamma_{\mathcal{B}}$ but contain no vertex adjacent to α in Γ. It is easy to check that $(\mathcal{B}(\alpha))^g = \{C^g : C \in \mathcal{B}(\alpha)\} = \mathcal{B}(\alpha^g)$ for any $\alpha \in V(\Gamma)$ and $g \in G$. If $B(\alpha) \in \mathcal{B}(\beta)$ and $B(\beta) \in \mathcal{B}(\alpha)$ hold simultaneously, then we say that α, β are *mates* and that α is the *mate of* β *in* $B(\alpha)$. Clearly, if β is the mate of α in $B(\beta)$, then α is the mate of β in $B(\alpha)$. Let

$$A(\alpha) := \{(B, C) : C \in \mathcal{B}(\alpha)\}, \tag{4.2}$$

the set of arcs of $\Gamma_{\mathcal{B}}$ from B to an element of $\Gamma_{\mathcal{B}}(B)$ containing no vertices adjacent to α. Then we can view $A(\alpha)$ as a label attached to the vertex α. Denote

$$\mathbf{A}(B) := \{A(\alpha) : \alpha \in B\} \tag{4.3}$$

for a block $B \in \mathcal{B}$, and set

$$\mathbf{A} := \{A(\alpha) : \alpha \in V(\Gamma)\}. \tag{4.4}$$

Lemma 4.1.1 *Suppose* Γ *is a* G*-symmetric graph which admits a* G*-invariant partition* $\mathcal{B}$ *of* $V(\Gamma)$ *such that* $k = v - 1 \geq 1$. *Then* $\mathbf{A}$ *is a* G*-invariant partition of* $\mathrm{Arc}(\Gamma_{\mathcal{B}})$, *and hence* G *induces an action on* $\mathbf{A}$. *The action of* G *on* $V(\Gamma)$ *and this action of* G *on* $\mathbf{A}$ *are permutationally equivalent with respect to the bijection* $\lambda : \alpha \mapsto A(\alpha)$. *In particular, we have* $(A(\alpha))^g = A(\alpha^g)$ *for* $\alpha \in V(\Gamma), g \in G$.

Proof Let α, β be distinct vertices of Γ. If $B(\alpha) \neq B(\beta)$, then the arcs in $A(\alpha)$ and $A(\beta)$ have different initial vertices; if $B(\alpha) = B(\beta)$, then $\mathcal{B}(\alpha) \cap \mathcal{B}(\beta) = \emptyset$ as $k = v - 1$. In both cases, we get $A(\alpha) \cap A(\beta) = \emptyset$ and hence $\mathbf{A}$ is a partition of $\mathrm{Arc}(\Gamma_{\mathcal{B}})$. It is straightforward to show that this partition is a G-invariant partition of $\mathrm{Arc}(\Gamma_{\mathcal{B}})$, and hence G induces an action on $\mathbf{A}$. Furthermore, the argument above shows that $\lambda : \alpha \mapsto A(\alpha)$ is a bijection from $V(\Gamma)$ to $\mathbf{A}$. For any $\alpha \in V(\Gamma)$ and $g \in G$, since $\mathcal{B}(\alpha^g) = (\mathcal{B}(\alpha))^g$, we have $\lambda(\alpha^g) = A(\alpha^g) = (A(\alpha))^g = (\lambda(\alpha))^g$. Therefore, the actions of G on $V(\Gamma)$ and $\mathbf{A}$ are permutationally equivalent with respect to λ. $\Box$

Next we define a graph Γ' associated with $(\Gamma, \mathcal{B})$ in the case where $k = v - 1$.

Definition 4.1.1 Let Γ' be the graph with vertex set $V(\Gamma)$ in which two vertices α, β are adjacent if and only if they are mates (see Figure 1). In other words, α, β are adjacent in Γ' if and only if $B(\alpha), B(\beta)$ are adjacent in $\Gamma_{\mathcal{B}}$, α is the only vertex in $B(\alpha)$ not adjacent to any vertex in $B(\beta)$, and β is the only vertex in $B(\beta)$ not adjacent to any vertex in $B(\alpha)$.

Note that $(\alpha,\beta) \mapsto (B(\alpha), B(\beta))$ establishes a bijection from the set of arcs of Γ' to the set of arcs of $\Gamma_{\mathcal{B}}$.

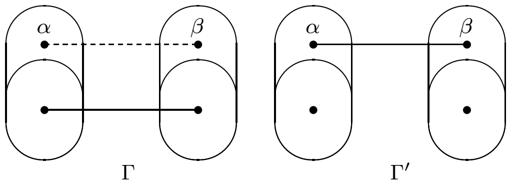


FIGURE 1 The definition of Γ'

Theorem 4.1.1 *Suppose Γ is a G-symmetric graph which admits a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - 1 \geq 1$. Then the graph Γ' defined above is G-symmetric.*

Proof Let $(\alpha,\beta), (\gamma,\delta)$ be distinct arcs of Γ'. Then $(B(\alpha), B(\beta)), (B(\gamma), B(\delta))$ are distinct arcs of $\Gamma_{\mathcal{B}}$. Since $\Gamma_{\mathcal{B}}$ is G-symmetric, there exists $g \in G$ such that $(B(\alpha), B(\beta))^g = (B(\gamma), B(\delta))$, that is, $(B(\alpha^g), B(\beta^g)) = (B(\gamma), B(\delta))$. Since α is the only vertex in $B(\alpha)$ not adjacent to any vertex in $B(\beta)$, we know that α^g is the only vertex in $B(\alpha^g) = B(\gamma)$ not adjacent to any vertex in $B(\beta^g) = B(\delta)$, and γ is the only vertex in $B(\gamma)$ not adjacent to any vertex in $B(\delta)$. So we must have $\alpha^g = \gamma$. Similarly, $\beta^g = \delta$. Hence $(\alpha,\beta)^g = (\gamma,\delta)$ and Γ' is a G-symmetric graph. $\square$

We say that the graph Γ is *vertex-distinguishable* with respect to $\mathcal{B}$ if, for any two adjacent blocks B, C of $\mathcal{B}$ and distinct vertices $\alpha, \beta \in \Gamma(B) \cap C$, we have $\Gamma(\alpha) \cap B \neq \Gamma(\beta) \cap B$. We conclude this section by proving the following lemma which exemplifies graphs of this kind. This lemma will be used in Theorem 4.3.1(d) and in the proof of Corollary 4.3.1.

Lemma 4.1.2 *Suppose Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$. Then Γ is vertex-distinguishable with respect to $\mathcal{B}$ if, for adjacent blocks B, C of $\mathcal{B}$, one of the following conditions holds:*

(a) $\Gamma[B, C]$ *is a matching;*

(b) $\Gamma[B, C]$ *is a complete bipartite graph minus a perfect matching between the vertices of $\Gamma(C) \cap B$ and $\Gamma(B) \cap C$;*

(c) $G_{B,C}$ *acts primitively on $\Gamma(B) \cap C$ and $\Gamma[B, C] \not\cong K_{k,k}$.*

Proof Clearly, the result is true whenever (a) or (b) occurs. Suppose that the condition (c) is satisfied. If there exist distinct $\alpha, \beta \in \Gamma(B) \cap C$ such that $\Gamma(\alpha) \cap B = \Gamma(\beta) \cap B$, then $\{\gamma \in C : \Gamma(\gamma) \cap B = \Gamma(\alpha) \cap B\}$ is a block of imprimitivity for $G_{B,C}$ in $\Gamma(B) \cap C$ and has size at least 2. Since this action is primitive, it follows that $\Gamma(\gamma) \cap B = \Gamma(\alpha) \cap B$ for all $\gamma \in \Gamma(B) \cap C$. This implies that $\Gamma[B, C] \cong K_{k,k}$, a contradiction. Thus, Γ is vertex-distinguishable with respect to $\mathcal{B}$. $\square$

4.2 The case where $k = 1$ and $v = 2$

We will distinguish the following two cases:

I. $k = v - 1 = 1$; and

II. $k = v - 1 \geq 2$.

In this section we discuss Case I, which can occur in a nontrivial way (see the examples in [43, Section 5] and see also Theorem 5.1.3 and the remarks following it). The characterization of Γ in Case I varies in difficulty according to the nature of $\Gamma_{\mathcal{B}}$. For example, if $\Gamma_{\mathcal{B}} = C_n$, then $r = 1$ and Γ is uniquely determined (see [43, Theorem 4.1(a)]), namely $\Gamma = n \cdot K_2$, while if $\Gamma_{\mathcal{B}}$ is a complete graph, then it seems rather difficult to determine or describe Γ (see [43, Section 4]). In Section 8.3 we will give a general construction of imprimitive G-symmetric graphs with $k = 1$ and $v \geq 2$. Here we prove some properties which hold only for the case where $k = v - 1 = 1$.

Suppose then that $k = v - 1 = 1$. For each vertex α, let $B(\alpha) = \{\alpha, \alpha'\}$ denote the block of $\mathcal{B}$ containing α, so $B(\alpha) = B(\alpha')$. The adjacency relation for the graph Γ' defined in Definition 4.1.1 becomes: α and β are adjacent in Γ' if and only if α' and β' are adjacent in Γ. Besides Γ', we can associate with Γ two other graphs $\Gamma^\star$ and $\Gamma^\#$ (see Figure 2) defined as follows.

Definition 4.2.1 (a) Let $\Gamma^\star$ be the graph with vertex set $V(\Gamma)$ in which $\{\alpha, \beta\}$ is an edge if and only if either $\{\alpha, \beta\}$ or $\{\alpha', \beta'\}$ is an edge of Γ;

(b) Let $\Gamma^\#$ be the graph with vertex set $V(\Gamma)$ such that $\{\alpha, \beta'\}$ and $\{\alpha', \beta\}$ are edges of $\Gamma^\#$ if and only if either $\{\alpha, \beta\}$ or $\{\alpha', \beta'\}$ is an edge of Γ.

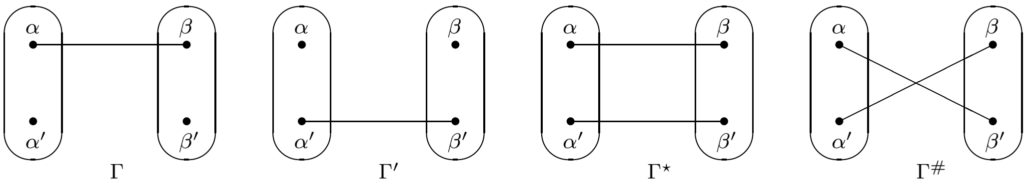


FIGURE 2 The definitions of $\Gamma', \Gamma^\star$ and $\Gamma^\#$

The graph $\Gamma^\star$ was defined in [43, Section 5] for a G-locally primitive graph Γ. The following result is analogous to [43, Lemma 5.1] without assuming G-local primitivity. It shows that the quotient graph $\Gamma_{\mathcal{B}}$ may be covered by two (possibly non-isomorphic) symmetric graphs. Let z be the involution which interchanges the two vertices in each block of $\mathcal{B}$.

Theorem 4.2.1 *Suppose that Γ is a G-symmetric graph and $\mathcal{B}$ is a nontrivial G-invariant partition of $V(\Gamma)$ with block size $v = k + 1 = 2$. Then*

(a) $\Gamma' \cong \Gamma$, *and Γ' is G-symmetric; and*

(b) *both $\Gamma^\star$ and $\Gamma^\#$ are $(G \times \langle z \rangle)$-symmetric, and $\mathcal{B}$ is a $(G \times \langle z \rangle)$-invariant partition of $V(\Gamma)$. Also, $\Gamma^\star_{\mathcal{B}} = \Gamma^\#_{\mathcal{B}} = \Gamma_{\mathcal{B}}$ and both $\Gamma^\star$ and $\Gamma^\#$ are covers of $\Gamma_{\mathcal{B}}$. Furthermore, if G is faithful on $V(\Gamma)$, then it is faithful on $\mathcal{B}$ as well.*

Proof By Theorem 4.1.1, Γ' is G-symmetric, and the mapping $z : \alpha \mapsto \alpha'$, for $\alpha \in V(\Gamma)$, is an isomorphism from Γ to Γ'. Thus (a) is proved.

Clearly, $\langle G, z \rangle \cong G \times \mathbb{Z}_2$. Since the edge set of $\Gamma^\star$ is the union of the sets of edges of Γ and Γ' it follows from (a) that $G \times \langle z \rangle \leq \mathrm{Aut}(\Gamma^\star)$ and that $G \times \langle z \rangle$ is transitive on the arcs of $\Gamma^\star$. Also, $\mathcal{B}$ is a $(G \times \langle z \rangle)$-invariant partition of $V(\Gamma)$, and $\Gamma^\star$ is a cover of $\Gamma^\star_{\mathcal{B}} = \Gamma_{\mathcal{B}}$. Moreover, $\Gamma_{\mathcal{B}} = \Gamma^\#_{\mathcal{B}}$, and $\Gamma^\#$ is a cover of $\Gamma_{\mathcal{B}}$. For two adjacent blocks $B = \{\alpha, \alpha'\}$ and $C = \{\beta, \beta'\}$ of $\Gamma_{\mathcal{B}}$, suppose that (α, β) is an arc of Γ. Then (α, β') and (β, α') are arcs of $\Gamma^\#$ which are interchanged by z. It is also easy to check that G preserves the edge set of $\Gamma^\#$. It follows that $G \times \langle z \rangle$ is transitive on the arcs of $\Gamma^\#$.

Let $B(\alpha) = \{\alpha, \alpha'\}$ be a block of $\mathcal{B}$. If $g \in G$ is any element which maps α to α', then g interchanges α and α'. Hence g interchanges $\Gamma_{\mathcal{B}}(\alpha)$ and $\Gamma_{\mathcal{B}}(\alpha')$. Note that $\Gamma_{\mathcal{B}}(\alpha)$ and $\Gamma_{\mathcal{B}}(\alpha')$ are disjoint since $k = 1$. Thus, g acts nontrivially on $\mathcal{B}$. It follows that, if G is faithful on $V(\Gamma)$, then G is also faithful on $\mathcal{B}$. $\square$

Remark 4.2.1 The graphs $\Gamma^\star$, $\Gamma^\#$ defined in Definition 4.2.1 may, or may not, be isomorphic to each other. For example, under the conditions of Theorem 4.2.1, if $\Gamma_{\mathcal{B}} = C_4$, then both $\Gamma^\star$ and $\Gamma^\#$ are $2 \cdot C_4$; while if $\Gamma_{\mathcal{B}} = C_3$, then $\Gamma^\star = C_6$ whilst $\Gamma^\# = 2 \cdot C_3$. So $\Gamma^\star$ and $\Gamma^\#$ may be non-isomorphic covers of $\Gamma_{\mathcal{B}}$.

4.3 A general discussion: $k = v - 1 \geq 2$

In the remaining sections of this chapter we investigate the general case where $v = k + 1 \geq 3$. Note that if, in addition, Γ is G-locally primitive, then $\mathcal{D}(B)$ contains no repeated blocks (by Corollary 3.3.1, noting that k does not divide v here). This however is not true in general, that is, the multiplicity of $\mathcal{D}(B)$ can be greater than one for general symmetric graphs with $v = k + 1 \geq 3$.

Theorem 4.3.1 *Suppose that Γ is a G-symmetric graph and $\mathcal{B}$ is a nontrivial G-invariant partition of $V(\Gamma)$ with block size $v = k + 1 \geq 3$. Let B be a block of $\mathcal{B}$ and $\alpha \in B$. Then the following* (a)-(d) *hold.*

(a) *$\mathcal{D}(B)$ has v distinct blocks and the multiplicity m of $\mathcal{D}(B)$ is equal to $|\mathcal{B}(\alpha)|$, so $b = mv$, $r = m(v-1)$, and $\mathcal{D}(B)$ is a* 2-$(v, v-1, m(v-2))$*-design.*

(b) *G_α has two orbits on $\Gamma_{\mathcal{B}}(B)$, namely, $\mathcal{B}(\alpha)$ and $\Gamma_{\mathcal{B}}(B) \setminus \mathcal{B}(\alpha)$.*

(c) *$G_{[B]} \leq G_{(B)}$ and equality holds whenever $\mathcal{D}(B)$ contains no repeated blocks. Moreover, if G is faithful on $V(\Gamma)$, then it is also faithful on $\mathcal{B}$.*

(d) *If G is faithful on $V(\Gamma)$, $\mathcal{D}(B)$ contains no repeated blocks, $\Gamma_{\mathcal{B}}$ is connected and Γ is vertex-distinguishable with respect to $\mathcal{B}$, then G_B acts faithfully on B and $\Gamma_{\mathcal{B}}(B)$.*

Proof (a) Since G_B is transitive on B, each $(v-1)$-subset of B is the trace of a block of $\mathcal{D}(B)$ and hence $\mathcal{D}(B)$ has v distinct blocks each repeated m times. So we have $m = |\mathcal{B}(\alpha)|$ and $b = mv$. This, together with $vr = bk = b(v-1)$, gives $r = m(v-1)$. In particular, $\mathcal{D}(B)$ is a 2-$(v, v-1, m(v-2))$-design.

(b) Clearly, $\mathcal{B}(\alpha)$ is G_α-invariant. Let $C, D \in \mathcal{B}(\alpha)$. Since $\Gamma_{\mathcal{B}}$ is G-symmetric, there exists $g \in G$ with $B^g = B, C^g = D$. Now $\alpha^g = \alpha$ for otherwise α is adjacent to no vertex in C but α^g is adjacent to at least one vertex in $C^g = D$. Thus, $g \in G_\alpha$ and hence G_α is transitive on $\mathcal{B}(\alpha)$. Now let $C, D \in \Gamma_{\mathcal{B}}(B) \setminus \mathcal{B}(\alpha)$. Then $\alpha \in \Gamma(C) \cap \Gamma(D) \cap B$. So there exist $\beta \in C, \gamma \in D$ which are adjacent to α. Since

Γ is G-symmetric, there exists $g \in G$ with $(\alpha, \beta)^g = (\alpha, \gamma)$. Thus, $g \in G_\alpha$ and $C^g = D$. So $\Gamma_{\mathcal{B}}(B) \setminus \mathcal{B}(\alpha)$ is a G_α-orbit.

(c) If $g \in G_{[B]}$ then, for each $\beta \in B$, g fixes setwise each block $C \in \mathcal{B}(\beta)$ and hence fixes setwise $\Gamma(C) \cap B$. Therefore, g fixes $B \setminus (\Gamma(C) \cap B) = \{\beta\}$. Thus, we have $G_{[B]} \leq G_{(B)}$. Moreover, if $g \in G$ fixes setwise each block of $\mathcal{B}$, then it lies in $G_{[B]}$ for each B, and hence fixes each vertex of Γ. This implies $g = 1$ provided that G is faithful on $V(\Gamma)$. So, if G is faithful on $V(\Gamma)$, then it is faithful on $\mathcal{B}$. Suppose that $\mathcal{D}(B)$ contains no repeated blocks and $g \in G_{(B)}$. Then for each $\alpha \in B$, g fixes the unique block in $\mathcal{B}(\alpha)$, and hence g fixes each block of $\Gamma_{\mathcal{B}}(B)$ setwise. So $g \in G_{[B]}$ and thus $G_{[B]} = G_{(B)}$.

(d) From (c) and the assumption that $\mathcal{D}(B)$ contains no repeated blocks, we have $G_{(B)} = G_{[B]}$. Let $g \in G_{(B)} = G_{[B]}$. Then for $C \in \Gamma_{\mathcal{B}}(B)$, g fixes the unique vertex in $C \setminus (\Gamma(B) \cap C)$, and for each $\beta \in \Gamma(B) \cap C$, we have $\beta^g \in \Gamma(B) \cap C$ and $\Gamma(\beta) \cap B = \Gamma(\beta^g) \cap B$ (since g fixes B pointwise). Since Γ is vertex-distinguishable with respect to $\mathcal{B}$, we get $\beta^g = \beta$. Thus $g \in G_{(C)}$ and hence $G_{(B)} \leq G_{(C)}$. By a similar argument $G_{(C)} \leq G_{(B)}$, so $G_{(B)} = G_{(C)}$. Since $\Gamma_{\mathcal{B}}$ is connected, this equality is true for any two blocks B, C (not necessarily adjacent), and hence $G_{(B)} = 1 = G_{[B]}$ since G is assumed to be faithful on $V(\Gamma)$. Thus, G_B is faithful on B and on $\Gamma_{\mathcal{B}}(B)$. $\square$

By Lemma 4.1.1 the induced action of G on $\mathbf{A}$ can be defined by $(A(\alpha))^g = A(\alpha^g)$, for $\alpha \in V(\Gamma), g \in G$, and this action is permutationally equivalent to the action of G on $V(\Gamma)$. Clearly, $\mathbf{A}(B)$ is a G_B-invariant subset of $\mathbf{A}$. Thus, G_B induces an action on $\mathbf{A}(B)$. Also, one can see that $\mathbf{B}(B) := \{\mathcal{B}(\alpha) : \alpha \in B\}$ is a G_B-invariant partition of $\Gamma_{\mathcal{B}}(B)$, and hence G_B induces an action on $\mathbf{B}(B)$ defined by $(\mathcal{B}(\alpha))^g = \mathcal{B}(\alpha^g)$ for $\alpha \in B$ and $g \in G_B$. The following theorem will play an important role in our later discussion. It shows in particular that the actions of G_B on B, $\mathbf{A}(B)$ and $\mathbf{B}(B)$ are permutationally equivalent and doubly transitive.

Theorem 4.3.2 *Suppose that* Γ *is a* G*-symmetric graph admitting a nontrivial* G*-invariant partition* $\mathcal{B}$ *with block size* $v = k+1 \geq 3$*. Let* $B \in \mathcal{B}$*,* $\alpha \in B$ *and* $C \in \mathcal{B}(\alpha)$*. Then the following* (a)-(c) *hold.*

(a) *The action of* G_B *on* B *is permutationally equivalent to the actions of* G_B *on* $\mathbf{A}(B)$ *and* $\mathbf{B}(B)$ *with respect to the bijections defined by* $\alpha \mapsto A(\alpha)$*,* $\alpha \mapsto \mathcal{B}(\alpha)$*,* *for $\alpha \in B$, respectively.*

(b) *$G_{B,C}$ is transitive on $B \setminus \{\alpha\}$, $\mathbf{A}(B) \setminus \{A(\alpha)\}$ and $\mathbf{B}(B) \setminus \{\mathcal{B}(\alpha)\}$. In particular, the actions of G_B on B, $\mathbf{A}(B)$ and $\mathbf{B}(B)$ are doubly transitive.*

(c) *$G_{B,C} = G_{\alpha,C} = G_{\delta,B} = G_{\alpha,\delta}$, where δ is the mate of α in C.*

Proof (a) Clearly, the actions of G_B on $\mathbf{B}(B)$ and $\mathbf{A}(B)$ are permutationally equivalent with respect to the natural bijection $\mathcal{B}(\alpha) \mapsto A(\alpha)$, for $\alpha \in B$. The permutation equivalence of the actions of G_B on B and $\mathbf{A}(B)$ follows immediately from Lemma 4.1.1.

(b) First, since G_B is transitive on B, from (a) above G_B is transitive on $\mathbf{A}(B)$ and $\mathbf{B}(B)$. Second, since $v = k + 1 \geq 3$, for distinct vertices $\beta, \gamma \in B \setminus \{\alpha\}$ there exist $\varepsilon, \eta \in C \setminus \{\delta\}$ which are adjacent in Γ to β, γ respectively, where δ is the mate of α in C. By the G-symmetry of Γ, there exists $g \in G$ such that $(\beta, \varepsilon)^g = (\gamma, \eta)$. This implies $g \in G_{B,C}$ and $(A(\beta))^g = A(\gamma)$. Hence $G_{B,C}$ is transitive on $\mathbf{A}(B) \setminus \{A(\alpha)\}$. Since by (a) above the actions of $G_{B,C}$ on $B \setminus \{\alpha\}$, $\mathbf{A}(B) \setminus \{A(\alpha)\}$ and on $\mathbf{B}(B) \setminus \{\mathcal{B}(\alpha)\}$ are permutationally equivalent, it follows that $G_{B,C}$ is also transitive on $B \setminus \{\alpha\}$ and $\mathbf{B}(B) \setminus \{\mathcal{B}(\alpha)\}$. Note that, since $(B, C) \in A(\alpha)$ and $\mathbf{A}$ is a G-invariant partition of $\mathrm{Arc}(\Gamma_{\mathcal{B}})$ (Lemma 4.1.1), $G_{B,C}$ is a subgroup of the setwise stabilizer $(G_B)_{A(\alpha)}$ of $A(\alpha)$ in G_B. So $(G_B)_{A(\alpha)}$ is transitive on $\mathbf{A}(B) \setminus \{A(\alpha)\}$. Therefore, we conclude that G_B is doubly transitive on $\mathbf{A}(B)$ and hence doubly transitive on B and $\mathbf{B}(B)$.

(c) Clearly, we have $G_{\alpha,C} \leq G_{B,C}$ since an element of G fixing α must fix B setwise. Conversely, if $g \in G$ fixes B and C setwise, then it must fix the unique vertex α of B not adjacent to any vertex of C. So we have $G_{B,C} \leq G_{\alpha,C}$ and hence $G_{B,C} = G_{\alpha,C}$. Similarly, one can show $G_{B,C} = G_{\delta,B}$ and $G_{B,C} = G_{\alpha,\delta}$. $\square$

Now let us consider the case where, in addition to our assumption $k = v - 1 \geq 2$, Γ is G-locally primitive and $\Gamma_{\mathcal{B}}$ is connected. In such a case, since k does not divide v, from Corollary 3.3.1 we know that (i) $\mathcal{D}(B)$ contains no repeated blocks, and hence $b = v \geq 3$ and $r = v - 1$ by Theorem 4.3.1(a). Also, Corollary 3.2.1 implies that (ii) $\Gamma[B, C] \cong (v - 1) \cdot K_2$ is a matching. From (ii) and Lemma 4.1.2 we know that Γ is vertex-distinguishable with respect to $\mathcal{B}$, and hence G_B is faithful on B provided that G is faithful on $V(\Gamma)$ (Theorem 4.3.1(d)). Also from (i) and (ii) we know that, for each $\alpha \in B$, there exists a bijection from $B \setminus \{\alpha\}$ to $\Gamma(\alpha)$, namely each $\beta \in B \setminus \{\alpha\}$ corresponds to the unique neighbour of α in the unique block of $\mathcal{B}(\beta)$. So G_α is primitive on $B \setminus \{\alpha\}$ as G_α is primitive on $\Gamma(\alpha)$. Therefore, G_B is doubly primitive on B, and hence is doubly primitive on $\Gamma_{\mathcal{B}}(B)$ by Theorem 4.3.2(a). So we deduce the following result which was proved in [43, Theorem 5.3].

Corollary 4.3.1 *Suppose that* Γ *is a* G*-locally primitive graph admitting a nontrivial* G*-invariant partition* $\mathcal{B}$ *such that* $v = k+1 \geq 3$ *and* $\Gamma_{\mathcal{B}}$ *is connected. Then* $b = v$, $r = v - 1$, *and the actions of* G_B *on* B *and* $\Gamma_{\mathcal{B}}(B)$ *are permutationally equivalent, and doubly primitive. Moreover, these actions are faithful if in addition* G *is faithful on* $V(\Gamma)$.

Now we consider the graph Γ' defined in Definition 4.1.1. Each maximal clique of Γ' has at most $m+1$ vertices since the valency of Γ' is m, where $m = |\mathcal{B}(\alpha)|$. The following result shows that if each maximal clique of Γ' does contain $m+1$ vertices, or equivalently if $\Gamma' \cong \ell \cdot K_{m+1}$ for some ℓ, then we obtain a second G-invariant partition of $V(\Gamma)$. This condition holds in particular when $m = 1$, and the following result will be used in this case in the next chapter.

Theorem 4.3.3 *Suppose that* Γ *is a* G*-symmetric graph admitting a nontrivial* G*-invariant partition* $\mathcal{B}$ *with blocks of size* $v = k + 1 \geq 3$. *Let* $\alpha \in V(\Gamma)$. *Then* $\mathcal{P} = \{(\{\alpha\} \cup \Gamma'(\alpha))^g : g \in G\}$ *is a* G*-invariant partition of* $V(\Gamma)$ *if and only if* $V(\Gamma)$ *is a disjoint union of* $(m + 1)$*-cliques of* Γ', *where* $m = |\mathcal{B}(\alpha)|$.

Proof Set $\Gamma'(\alpha) = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ and $B' = \{\alpha\} \cup \Gamma'(\alpha)$, and suppose that $V(\Gamma)$ is a disjoint union of $(m + 1)$-cliques of Γ'. Then B' is the unique $(m + 1)$-clique of Γ' containing α. Since G permutes the connected components of Γ', it follows that $\mathcal{P}$ is a G-invariant partition of $V(\Gamma)$.

Conversely, suppose $\mathcal{P}$ is a G-invariant partition of $V(\Gamma)$. For any i, $1 \leq i \leq m$, let $g \in G$ be such that $\alpha^g = \alpha_i$. Then $B'^g = B'$ since α_i is in both B' and B'^g, and hence $\Gamma'(\alpha_i) = \Gamma'(\alpha^g) = (\Gamma'(\alpha))^g = (B' \setminus \{\alpha\})^g = B' \setminus \{\alpha_i\}$. Therefore, B' is a clique of Γ' with the maximum possible size $m + 1$. In other words, $V(\Gamma)$ is a disjoint union of $(m + 1)$-cliques of Γ'. $\square$

4.4 Analysing an extreme case

Let $n := |\mathcal{B}(\alpha) \cap \mathcal{B}(\beta)|$ for adjacent vertices α, β of Γ. Since Γ is G-symmetric, this parameter n is independent of the choice of such α and β. Clearly, we have $0 \le n \le m$ with the extreme case $n = 0$ occurring only if $\mathrm{girth}(\Gamma_{\mathcal{B}}) \ge 4$, where $m = |\mathcal{B}(\alpha)|$ is the multiplicity of $\mathcal{D}(B)$. We will study this extreme case in the next chapter under the additional assumption $m = 1$ (see Theorems 5.1.2 and 5.1.3). In this section we study the second extreme case where $n = m$, that is, the case where $\mathcal{B}(\alpha) = \mathcal{B}(\beta)$ for adjacent vertices α, β of Γ. Our study below shows that in this case all of Γ, $\Gamma_{\mathcal{B}}$ and $\Gamma[B, C]$ can be determined explicitly. We first give examples of symmetric graphs with this property.

Example 4.4.1 (a) Let X be a 3-transitive group acting on a finite set I of degree $v + 1 \ge 4$. Let Γ be the graph with vertex set $V := I^{(2)}$ in which $(i, h), (i', h')$ are adjacent if and only if $i \ne i'$ and $h = h'$. Then the 3-transitivity of X implies that Γ is X-symmetric and admits the X-invariant partition $\mathcal{B} := \{\mathbf{i} : i \in I\}$, where $\mathbf{i}$ consists of members of V with first coordinate i. Clearly, we have $k = v - 1 \ge 2$, $\Gamma \cong (v+1) \cdot K_v$, $\Gamma_{\mathcal{B}} \cong K_{v+1}$, $\Gamma[B, C] \cong (v-1) \cdot K_2$ for adjacent blocks B, C of $\mathcal{B}$, and $\mathcal{D}(B)$ contains no repeated blocks. Also, for adjacent vertices $\alpha = (i, h), \alpha' = (i', h)$ of Γ, we have $\mathcal{B}(\alpha) = \mathcal{B}(\alpha') = \{\mathbf{h}\}$.

(b) Now let us consider the case where the multiplicity $m \ge 2$. Let X and I be as in (a) above and let Y be a 2-transitive group acting on a finite set J of degree m. Then $G := X \times Y$ is transitive on $V := I^{(2)} \times J$ in its action defined by $(i, h, j)^{(x,y)} := (i^x, h^x, j^y)$ for $(i, h, j) \in V$ and $(x, y) \in G$. Define the graph Γ with vertex set V in which $(i, h, j), (i', h', j')$ are adjacent if and only if $i \ne i'$ and $h = h'$. Then $\Gamma \cong (v + 1) \cdot K_m^v$, and the assumptions on X, Y imply that Γ is G-symmetric. Clearly, Γ admits $\mathcal{B} := \{[i, j] : i \in I, j \in J\}$ as a G-invariant partition, where $[i, j] := \{(i, h, j) : h \in I \setminus \{i\}\}$. We have $\Gamma_{\mathcal{B}} \cong K_m^{v+1}$ with $[i, j], [i', j']$ adjacent if and only if $i \ne i'$. Also, we have $\Gamma[B, C] \cong (v-1) \cdot K_2$ for adjacent blocks B, C of $\mathcal{B}$ (hence $k = v - 1 \ge 2$), and the multiplicity of $\mathcal{D}(B)$ is equal to m. Moreover, for adjacent vertices $\alpha = (i, h, j)$, $\alpha' = (i', h, j')$ of Γ, we have $\mathcal{B}(\alpha) = \mathcal{B}(\alpha') = \{[h, \ell] : \ell \in J\}$.

In the following theorem, we will show that the graphs Γ in Example 4.4.1 are the only G-symmetric graphs with $\Gamma_{\mathcal{B}}$ connected such that $k = v - 1 \ge 2$ and

$\mathcal{B}(\alpha) = \mathcal{B}(\beta)$ for adjacent vertices α, β of Γ, and that $\Gamma_{\mathcal{B}}$, $\Gamma[B, C]$ are as shown in this example. First, we have the following simple observation, which shows that the fact $\Gamma[B, C] \cong (v-1) \cdot K_2$ in Example 4.4.1 is not a coincidence.

Lemma 4.4.1 *Suppose that Γ is a G-symmetric graph admitting a nontrvial G-invariant partition $\mathcal{B}$ such that $k = v - 1 \geq 2$. Let m be the multiplicity of $\mathcal{D}(B)$, and let $n = |\mathcal{B}(\alpha) \cap \mathcal{B}(\beta)|$ for $(\alpha, \beta) \in \mathrm{Arc}(\Gamma)$, as defined above. Then we have*

$$sn \leq m,$$

where, recall that, $s = |\Gamma(\alpha) \cap C|$ (for a flag (α, C) of $\mathcal{D}(B)$) is the valency of $\Gamma[B, C]$. In particular, if $n > m/2$, then $\Gamma[B, C] \cong (v-1) \cdot K_2$.

Proof Let $B \in \mathcal{B}$ and $\alpha \in B$. Let $C \in \Gamma_{\mathcal{B}}(\alpha)$ and set $\Gamma(\alpha) \cap C = \{\beta_1, \ldots, \beta_s\}$. Then $\mathcal{B}(\alpha) \cap \mathcal{B}(\beta_i)$, for $i = 1, \ldots, s$, are pairwise disjoint and each of them contains n blocks of $\mathcal{B}(\alpha)$. So we have $sn \leq m$. In particular, if $n > m/2$, then we must have $s = 1$ and thus $\Gamma[B, C] \cong (v-1) \cdot K_2$. □

Theorem 4.4.1 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - 1 \geq 2$. Suppose further that $\Gamma_{\mathcal{B}}$ is connected and that $\mathcal{B}(\alpha) = \mathcal{B}(\beta)$ for adjacent vertices α, β of Γ. Let m be the multiplicity of $\mathcal{D}(B)$. Then $\Gamma \cong (v+1) \cdot K_m^v$, $\Gamma_{\mathcal{B}} \cong K_m^{v+1}$, $\Gamma[B, C] \cong (v-1) \cdot K_2$ for adjacent blocks B, C, and the induced action of G on the natural $(v+1)$-partition $\mathbf{B}$ of $\Gamma_{\mathcal{B}}$ is 3-transitive (thus $(\Gamma_{\mathcal{B}})_{\mathbf{B}} \cong K_{v+1}$ is $(G, 2)$-arc transitive). Moreover, the vertices of Γ can be labelled by ordered triples of integers such that the following* (a)-(c) *hold (where we set $I := \{0, 1, \ldots, v\}$ and $J := \{1, 2, \ldots, m\}$):*

(a) *$V(\Gamma) = I^{(2)} \times J$, and two vertices $(i, h, j), (i', h', j') \in V(\Gamma)$ are adjacent in Γ if and only if $i \neq i'$ and $h = h'$.*

(b) *$\mathcal{B} = \{[i, j] : i \in I, j \in J\}$, where $[i, j] := \{(i, h, j) : h \in I \setminus \{i\}\}$, and $[i, j], [i', j']$ are adjacent blocks if and only if $i \neq i'$.*

(c) *$\mathbf{B} = \{\mathbf{i} : i \in I\}$, where $\mathbf{i} = \{[i, j] : j \in J\}$.*

Conversely, the graph Γ defined in (a) together with the group $G = X \times Y$ satisfies all conditions of the theorem, where X is a group acting 3-transitively on I, Y is a group acting 2-transitively on J whenever $m \geq 2$, and the action of G on $V(\Gamma)$ is as defined in Example 4.4.1.

Proof By our assumption we have $n = m > m/2$. Thus Lemma 4.4.1 implies

(1) $\Gamma[D, E] \cong (v-1) \cdot K_2$ for adjacent blocks D, E of $\mathcal{B}$.

Let B be a block of $\mathcal{B}$ and let $\alpha_1, \alpha_2, \ldots, \alpha_v$ be vertices of B. For each $\alpha_i \in B$, we label (in an arbitrary way) the m blocks in $\mathcal{B}(\alpha_i)$ by $[i, j]$, $j \in J$. Also, we label the unique mate β_{ij} of α_i in the block $[i, j]$ by $(i, 0, j)$, $j \in J$. For each block $[i, j]$ and for each $h \in I \setminus \{0\}$ distinct from i, (1) implies that $[i, j]$ contains a unique vertex adjacent to α_h. We label such a vertex in $[i, j]$ by (i, h, j). In view of (1) one can see that each vertex in $[i, j]$ receives a unique label, and that the labels of distinct vertices in $[i, j]$ have distinct second coordinates. Therefore, for each $i \in I \setminus \{0\}$ and $j \in J$, we may identify the block $[i, j]$ with the set $\{(i, h, j) : h \in I \setminus \{i\}\}$. By our assumption, for $i, h \in I \setminus \{0\}$ with $i \neq h$ and $j \in J$, we have

(2) $\mathcal{B}((i, h, j)) = \mathcal{B}(\alpha_h) = \{[h, 1], [h, 2], \ldots, [h, m]\}$.

In particular, this implies that

(3) $[i, j], [i', j']$ are adjacent blocks, for distinct $i, i' \in I \setminus \{0\}$ and any $j, j' \in J$.

Moreover, if two vertices $(i, h, j), (i', h', j')$ are adjacent, where $i, i', h, h' \in I \setminus \{0\}$ with $i \neq h$, $i' \neq h'$ and $j, j' \in J$, then by (2) and our assumption we must have $\mathcal{B}(\alpha_h) = \mathcal{B}((i, h, j)) = \mathcal{B}((i', h', j')) = \mathcal{B}(\alpha_{h'})$, which is true only when $h = h'$. This, together with (1) and (3), implies the following assertion.

(4) For distinct $i, i' \in I \setminus \{0\}$ and any $j, j' \in J$, two labelled vertices (i, h, j), (i', h', j') of Γ are adjacent if and only if $h = h'$. In other words, for adjacent blocks $D = [i, j], E = [i', j']$ of $\mathcal{B}$, the bipartite subgraph $\Gamma[D, E]$ of Γ is the $(v-1)$-matching with edges joining (i, h, j) and (i', h, j'), for $h \in I \setminus \{i, i'\}$.

Therefore, (i, i', j) and (i', i, j') are mates and hence, for the graph Γ' defined in Definition 4.1.1, we have

(5) $\Gamma'((i, h, j)) = \{(h, i, j') : j' \in J\}$.

Now let us examine a particular labelled vertex, say (i, h, j). From Theorem 4.3.1(a) and (1) above, the valency of Γ is $m(v-1)$, and hence the neighbourhood $\Gamma((i, h, j))$ of (i, h, j) contains $m(v-1)$ vertices. From (4) we have $\{(i', h, j') :$

$i' \in I \setminus \{0, h, i\}, j' \in J\} \subseteq \Gamma((i, h, j))$ and this contributes $m(v-2)$ neighbours of (i, h, j). Note that α_h is also a neighbour of (i, h, j). Apart from these, there are $m-1$ remaining neighbours of (i, h, j), which we denote by $\delta_2, \ldots, \delta_m$, respectively. By (1) these vertices $\delta_2, \ldots, \delta_m$ belong to distinct blocks, say $B_2, \ldots, B_m$, of $\mathcal{B}$. For each δ_t, we have $\mathcal{B}(\delta_t) = \mathcal{B}((i, h, j)) = \mathcal{B}(\alpha_h) = \{[h, 1], [h, 2], \ldots, [h, m]\}$ by (2) and our assumption. In particular, this implies that all the blocks $[h, \ell]$, for $\ell \in J$, are adjacent to the block B_t. On the other hand, from (5) we have $\Gamma'((h, h', \ell)) = \{(h', h, t) : t \in J\}$ for each vertex $(h, h', \ell) \in [h, \ell] \setminus \{\beta_{h\ell}\}$. In other words, the m mates of each vertex in $[h, \ell] \setminus \{\beta_{h\ell}\}$ are in $\bigcup_{h' \in I \setminus \{0, h\}, t \in J} [h', t]$. So the only possibility is that $\beta_{h\ell}$ is the mate of δ_t in $[h, \ell]$, for each $\ell \in J$. Consequently, we have

(6) $\mathcal{B}(\beta_{h1}) = \cdots = \mathcal{B}(\beta_{hm}) = \{B, B_2, \ldots, B_m\}$, and hence none of $B, B_2, \ldots, B_m$ coincides with $[i, j]$ for any $i \in I \setminus \{0\}$, $j \in J$.

We know from (3) that the blocks $[i', j']$, for $i' \in I \setminus \{0, h\}$ and $j' \in J$, are all adjacent to $[h, \ell]$. Besides these $m(v-1)$ blocks, $B, B_2, \ldots, B_m$ are the only blocks of $\mathcal{B}$ adjacent to $[h, \ell]$ in $\Gamma_{\mathcal{B}}$ since $\Gamma_{\mathcal{B}}$ has valency mv (Theorem 4.3.1(a)). Therefore, if we apply the procedure above to another vertex (i', h, j'), we would get the same blocks $B_2, \ldots, B_m$. In other words, these blocks are independent of the choice of the vertex (i, h, j) (depending only on h), and hence they are adjacent to the block $[i, j]$ for any $i \in I \setminus \{0\}$ and $j \in J$. Moreover, since the mate δ_t of $\beta_{h\ell}$ in B_t is unique, the vertices $\delta_2, \ldots, \delta_m$ are also independent of the choice of (i, h, j) and thus they are common neighbours of all such vertices (i, h, j). Thus, since the valency of $\Gamma_{\mathcal{B}}$ is mv, $B, B_2, \ldots, B_m$ are the only unlabelled blocks of $\mathcal{B}$. From this and by a similar argument to that above, we see that for each $h \in I \setminus \{0\}$, all the vertices (i, h, j), $i \in I \setminus \{0, h\}$, $j \in J$, have a common neighbour in each B_t, which we now label by $(0, h, t)$. Since for distinct h, h' the vertices (i, h, j), (i, h', j) have different neighbours in B_t, the vertices of B_t receive pairwise distinct labels. Now let us label $B, B_2, \ldots, B_m$ with $[0, 1], [0, 2], \ldots, [0, m]$, respectively, and label each α_h with $(0, h, 1)$. Then all the vertices of Γ and all the blocks of $\mathcal{B}$ have been labelled. From the labelling above, the validity of (a) and (b) follows immediately.

Since the valency of Γ is $m(v-1)$, the argument above also shows that for each $h \in I$ the connected component of Γ containing the vertex α_h is the complete v-partite graph K_m^v with v-partition $\{\{(i, h, j) : j \in J\} : i \in I\}$, where we set $\alpha_0 = \beta_{11}$.

Hence we have $\Gamma \cong (v+1) \cdot K_m^v$. Also, $\Gamma_{\mathcal{B}}$ is the complete $(v+1)$-partite graph K_m^{v+1} with $(v+1)$-partition $\mathbf{B} := \{\mathbf{i} : i \in I\}$, where $\mathbf{i} := \mathcal{B}(\alpha_i) = \{[i,j] : j \in J\}$ for $i \in I$. Clearly, $(\Gamma_{\mathcal{B}})_{\mathbf{B}} \cong K_{v+1}$ and $\mathbf{B}$ is a G-invariant partition of $\mathcal{B}$. From Theorem 4.3.2(b), G_B is doubly transitive on $\{\mathcal{B}(\gamma) : \gamma \in B\}$. The setwise stabilizer in G of the block $\mathbf{0}$ contains G_B as a subgroup, and so is doubly transitive on the neighbourhood $\mathbf{B} \setminus \{\mathbf{0}\}$ of $\mathbf{0}$ in $(\Gamma_{\mathcal{B}})_{\mathbf{B}}$. Therefore, $(\Gamma_{\mathcal{B}})_{\mathbf{B}}$ is $(G,2)$-arc transitive and hence G is 3-transitive on $\mathbf{B}$.

Finally, for $G = X \times Y$ with X triply transitive on I and Y doubly transitive on J whenever $m \geq 2$, Example 4.4.1 shows that the graph Γ defined in (a) satisfies all the conditions in the theorem. $\Box$

Remark 4.4.1 In Theorem 4.4.1, G may or may not be faithful on $\mathbf{B}$. (This can be seen from Example 4.4.1, where the action of G on $\mathbf{B}$ is permutationally equivalent to the action of X on I which is not necessarily faithful.) Let K be the kernel of the action of G on $\mathbf{B}$, and set $H := G/K$. Then H is 3-transitive and faithful on $\mathbf{B}$ of degree $v+1$, and G is an extension of K by H. From the classification of finite highly transitive permutation groups (see Theorem 2.1.1 and the comments following it), H is one of the following: S_{v+1} ($v \geq 3$), A_{v+1} ($v \geq 4$), M_{v+1} ($v = 10, 11, 21, 22, 23$), M_{11} ($v = 11$), $\mathrm{AGL}(d,2)$ ($v = 2^d - 1$), $\mathbb{Z}_2^4.A_7$ ($v = 15$), and $\mathrm{PSL}(2,v) \leq H \leq \mathrm{P\Gamma}L(2,v)$ (v a prime power). Example 4.4.1 shows that the multiplicity m of $\mathcal{D}(B)$ can be any positive integer and H can be any group listed above.

Chapter 5

The case $k = v - 1 \geq 2$: $\mathcal{D}(B)$ contains no repeated blocks

What is most perfect seems to be incomplete; but its utility is unimpaired. What is most full seems to be empty; but its usefulness is inexhaustible. What is most straight seems to be crooked. The greatest skills seems to be clumsy. The greatest eloquence seems to stutter.

Lao Tzu (6th or 4th Cent. B.C. ?), TAO TE CHING 45

In this and the next two chapters we continue our study for the case where $k = v - 1 \geq 2$ under the additional assumption that $\mathcal{D}(B)$ contains no repeated blocks (that is, the multiplicity of $\mathcal{D}(B)$ is equal to 1). Not only is this a natural assumption geometrically, but also we will prove (Theorem 5.1.2) that it occurs if and only if $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. In this case each vertex of Γ can be labelled in a natural way by an arc of $\Gamma_{\mathcal{B}}$. Inspired by this labelling we then give a very simple and elegant method for constructing all such graphs. In particular, our construction shows that such a graph Γ can be reconstructed from the quotient $\Gamma_{\mathcal{B}}$ and the action of G on $\mathcal{B}$.

5.1 The case $\mathcal{D}(B)$ contains no repeated blocks

In the following we suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - 1 \geq 2$ and $\mathcal{D}(B)$ has no repeated blocks. Then the valency of the graph Γ' (defined in Definition 4.1.1) is 1 and thus each vertex $\alpha \in V(\Gamma)$ has a unique mate α', namely the unique vertex adjacent to α in Γ'. Hence the blocks of the partition $\mathcal{P}$ defined in Theorem 4.3.3 are the pairs $\{\alpha, \alpha'\}$, and the map $z : \alpha \mapsto \alpha'$ defines a G-invariant bijection on $V(\Gamma)$. So $A(\alpha)$ contains only one arc $(B(\alpha), B(\alpha'))$ and, by Theorem 4.3.1(b), $B(\alpha')$ is the unique block in $\Gamma_{\mathcal{B}}(B)$ fixed setwise by G_α. As in the G-locally primitive case [43], the mapping $\lambda : \alpha \mapsto A(\alpha)$ of Lemma 4.1.1 defines, for each $\alpha \in V(\Gamma)$, a unique *label* "$B(\alpha)B(\alpha')$" for α with the blocks of $\mathcal{B}$ containing α and α' as the first and the second coordinates, respectively. Set $B^* = B^z = \{\text{"}CB\text{"} : C \in \Gamma_{\mathcal{B}}(B)\}$ for $B \in \mathcal{B}$. Then it follows from the definition that no vertex in B^* is adjacent to any vertex in B, that is, $B^* \cap \Gamma(B) = \emptyset$. Thus no neighbour of $\alpha \in B$ has a label involving B as either coordinate.

Theorem 5.1.1 *Suppose that Γ is a G-symmetric graph, $\mathcal{B}$ is a nontrivial G-invariant partition of $V(\Gamma)$ with block size $v = k+1 \geq 3$ such that $\mathcal{D}(B)$ contains no repeated blocks. Then $\Gamma_{\mathcal{B}}$ has valency $b = v$. Let $z : \alpha \mapsto \alpha'$, $\alpha \in V(\Gamma)$, as defined above. Then*

(a) *the actions of G on $V(\Gamma)$ and on the set of arcs of $\Gamma_{\mathcal{B}}$ are permutationally equivalent, and each $\alpha \in V(\Gamma)$ can be uniquely labelled by a pair "BB'" of adjacent blocks of $\mathcal{B}$, where $B = B(\alpha)$ and B' is the unique block in $\Gamma_{\mathcal{B}}(B)$ fixed setwise by G_α;*

(b) *z centralises G and is an involution (that is, $z^2 = 1$), and $z \notin G$; also $\mathcal{P} = \{\{\alpha, \alpha'\} : \alpha \in V(\Gamma)\}$ is a $(G \times \langle z \rangle)$-invariant partition of $V(\Gamma)$;*

(c) *$\mathcal{B}^* := \{(B^*)^g : g \in G\}$ is a G-invariant partition of $V(\Gamma)$ with blocks of size v; and $G_{B^*} = G_B$ is doubly transitive on B and B^*.*

Proof Theorem 4.3.1(a) implies that $b = v$. Each $A(\alpha)$ can be identified with the arc $(B(\alpha), B(\alpha'))$ of $\Gamma_{\mathcal{B}}$ and each arc of $\Gamma_{\mathcal{B}}$ has this form. So it follows from Lemma 4.1.1 that the actions of G on $V(\Gamma)$ and on the set of arcs of $\Gamma_{\mathcal{B}}$ are permutationally equivalent. Clearly, z is an involution and leaves $\mathcal{P}$ invariant. This and Theorem 4.3.3 together imply that $\mathcal{P}$ is a $(G \times \langle z \rangle)$-invariant partition of $V(\Gamma)$. For each $g \in G$ and "BD" $\in V(\Gamma)$, we have "BD"^{zg} = "DB"^{g} = "D^gB^g" = "B^gD^g"^{z} = "BD"^{gz} and hence z centralises G. If $B^* \cap (B^*)^g \neq \emptyset$ for some $g \in G$ then, since $(B^*)^g = \{\text{"}C^gB^g\text{"} : \text{"}CB\text{"} \in B^*\} = (B^g)^*$, we have $B^* \cap (B^g)^* \neq \emptyset$, which implies $g \in G_B$ and consequently $(B^*)^g = B^*$. Thus, $\mathcal{B}^*$ is a G-invariant partition of $V(\Gamma)$

with block size v. Since z interchanges $\mathcal{B}$ and $\mathcal{B}^*$ whilst G leaves $\mathcal{B}$ invariant, it follows that $z \notin G$. Clearly, $G_{B^*} = G_B$ and the actions of G_B on B and B^* are permutationally equivalent with respect to $z : \alpha \mapsto \alpha'$. So by Theorem 4.3.2(b), G_B is doubly transitive on both B and B^*. □

Corollary 5.1.1 *Suppose that Γ is a G-symmetric graph with $\mathcal{B}$ a nontrivial G-invariant partition of block size $v = k + 1 \geq 3$ such that $\mathcal{D}(B)$ contains no repeated blocks. Then for any 2-arc (B, C, D) of $\Gamma_{\mathcal{B}}$, we have*

(a) *$G_{\text{“}CB\text{”}} = G_{\text{“}BC\text{”}} = G_{B,C}$ and hence $G_{\text{“}CB\text{”},D} = G_{B,C,D}$; and*

(b) *the actions of $G_{B,C,D}$ on $D \setminus \{\text{“}DC\text{”}\}$ and $\Gamma_{\mathcal{B}}(D) \setminus \{C\}$ are permutationally equivalent.*

Proof By Theorem 5.1.1(a) we have $G_{\text{“}CB\text{”}} = G_{\text{“}BC\text{”}} = G_{B,C}$, and hence $G_{\text{“}CB\text{”},D} = G_{B,C,D}$. Note that $G_{B,C,D}$ ($\leq G_D$) fixes “DC” and fixes C setwise. Also, Theorem 4.3.2 implies that the actions of G_D on D and $\Gamma_{\mathcal{B}}(D)$ are permutationally equivalent with respect to the bijection ρ : “DE” $\mapsto E$ for $E \in \Gamma_{\mathcal{B}}(D)$. Therefore, $G_{B,C,D}$ induces actions on $D \setminus \{\text{“}DC\text{”}\}$ and $\Gamma_{\mathcal{B}}(D) \setminus \{C\}$, respectively, and these two actions are permutationally equivalent with respect to the restriction of ρ to $D \setminus \{\text{“}DC\text{”}\}$. □

As advertised at the beginning of this chapter, we now prove that, if $k = v - 1 \geq 2$, then $\mathcal{D}(B)$ contains no repeated blocks if and only if $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive.

Theorem 5.1.2 *Suppose that Γ is a G-symmetric graph, and $\mathcal{B}$ is a nontrivial G-invariant partition of $V(\Gamma)$ with block size $v = k + 1 \geq 3$. Then $\mathcal{D}(B)$ contains no repeated blocks if and only if $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. Furthermore, in this case either*

(a) *adjacent vertices have labels involving four distinct blocks, or*

(b) *there exist two adjacent vertices of Γ which share the same second coordinate. In this case, $\Gamma[B, C] \cong (v - 1) \cdot K_2$, $\Gamma[B^*] \cong K_v$, $\Gamma \cong n(v + 1) \cdot K_v$ and $\Gamma_{\mathcal{B}} \cong n \cdot K_{v+1}$ for some integer $n \geq 1$, and the group induced by G on the connected component $\{B\} \cup \Gamma_{\mathcal{B}}(B)$ of $\Gamma_{\mathcal{B}}$ is 3-transitive. In particular, if $\Gamma_{\mathcal{B}}$ is connected, then $\Gamma \cong (v + 1) \cdot K_v$, $\Gamma_{\mathcal{B}} \cong K_{v+1}$ and G acts on $\mathcal{B}$ as a 3-transitive permutation group of degree $v + 1$.*

Proof Suppose $\mathcal{D}(B)$ has no repeated blocks. Then for each $\alpha \in B$, $\mathcal{B}(\alpha)$ can be identified with the unique block it contains, and thus $\mathbf{B}(B)$ can be identified with $\Gamma_{\mathcal{B}}(B)$. So Theorem 4.3.2(b) implies that G_B is doubly transitive on $\Gamma_{\mathcal{B}}(B)$. Since G is transitive on $\mathcal{B}$, it follows from Lemma 3.1.1(b) that $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive. Conversely suppose that $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive, and let α, β, γ be pairwise distinct vertices of B. (Note that $v \geq 3$.) If $\mathcal{D}(B)$ contains repeated blocks, then there are distinct blocks $C_1, C_2 \in \mathcal{B}(\alpha)$. Let $D \in \mathcal{B}(\beta)$ and $E \in \mathcal{B}(\gamma)$. By the $(G,2)$-arc transitivity of $\Gamma_{\mathcal{B}}$ there exists $g \in G_B$ with $(C_1, C_2)^g = (D, E)$. Note that, as mentioned before Theorem 4.3.2, $\mathbf{B}(B) = \{\mathcal{B}(\delta) : \delta \in B\}$ is a G_B-invariant partition of $\Gamma_{\mathcal{B}}(B)$. So $C_1^g = D$ implies $(\mathcal{B}(\alpha))^g = \mathcal{B}(\beta)$, whilst $C_2^g = E$ implies $(\mathcal{B}(\alpha))^g = \mathcal{B}(\gamma)$. This contradiction shows that $\mathcal{D}(B)$ contains no repeated blocks. Thus the first assertion is proved.

For the rest of the proof we assume that $\mathcal{D}(B)$ has no repeated blocks. If adjacent vertices of Γ have different second coordinates, then it follows from the definition of the labels that two adjacent vertices of Γ have labels involving four distinct blocks. Suppose there exist two adjacent vertices whose second coordinates are the same. Since G acts transitively on $\mathcal{B}$, we may assume without loss of generality that there are two adjacent vertices in B^*. Since G_{B^*} is doubly transitive on B^* (Theorem 5.1.1(c)), it follows that B^* induces a complete graph K_v. Since Γ is G-symmetric and since $\mathcal{B}^*$ is a G-invariant partition of $V(\Gamma)$ (Theorem 5.1.1(c)), it follows that each edge of Γ joins two vertices in the same block of $\mathcal{B}^*$. This means that each block of $\mathcal{B}^*$ induces a connected component K_v of Γ and hence $\Gamma = |\mathcal{B}^*| \cdot K_v$. This implies in particular that $\Gamma[B, C]$ is a matching of $v-1$ edges. Note that any two blocks in $\Gamma_{\mathcal{B}}(B)$ are adjacent in $\Gamma_{\mathcal{B}}$ and hence $\{B\} \cup \Gamma_{\mathcal{B}}(B)$ induces a complete subgraph K_{v+1} of $\Gamma_{\mathcal{B}}$. Since the valency of $\Gamma_{\mathcal{B}}$ is $b = v$, the subgraph induced by $\{B\} \cup \Gamma_{\mathcal{B}}(B)$ is a connected component of $\Gamma_{\mathcal{B}}$. This implies (i) $\Gamma_{\mathcal{B}} = n \cdot K_{v+1}$, and hence $\Gamma = n(v+1) \cdot K_v$, where n is the number of connected components of $\Gamma_{\mathcal{B}}$; and (ii) since G is transitive on $\mathcal{B}$ and G_B is doubly transitive on $\Gamma_{\mathcal{B}}(B)$, as shown above, it follows that the group induced on the connected component $\{B\} \cup \Gamma_{\mathcal{B}}(B)$ of $\Gamma_{\mathcal{B}}$ is 3-transitive. In particular, if $\Gamma_{\mathcal{B}}$ is connected, then $\Gamma_{\mathcal{B}} = K_{v+1}$, $\Gamma = (v+1) \cdot K_v$ and G is 3-transitive on $\mathcal{B} = \{B\} \cup \Gamma_{\mathcal{B}}(B)$ with degree $|\mathcal{B}| = v + 1$. $\square$

Remark 5.1.1 Under the assumption that $\mathcal{D}(B)$ contains no repeated blocks, two adjacent vertices α, β of Γ share the same second coordinate if and only if the size of $\mathcal{B}(\alpha) \cap \mathcal{B}(\beta)$ is equal to 1. Hence we can also prove the assertions in part (b) of Theorem 5.1.2 by applying Theorem 4.4.1 to each connected component of $\Gamma_{\mathcal{B}}$. If G is faithful on $V(\Gamma)$ and $\Gamma_{\mathcal{B}}$ is connected, then G acts faithfully (Theorem 4.3.1(c)) on $\mathcal{B}$ as one of the 3-transitive permutation groups listed in Remark 4.4.1.

According to Theorem 5.1.2, under the assumption that $\mathcal{D}(B)$ contains no repeated blocks, all possibilities for the graphs Γ, $\Gamma_{\mathcal{B}}$, $\Gamma[B, C]$ and the group G are known if there are two adjacent vertices of Γ sharing the same second coordinate. For the remaining case where the labels of any two adjacent vertices involve four distinct blocks, the following theorem gives some structural information about Γ and $\Gamma_{\mathcal{B}}$ provided the girth of $\Gamma_{\mathcal{B}}$ is sufficiently large.

Theorem 5.1.3 *Suppose that Γ is a G-symmetric graph, and $\mathcal{B}$ is a nontrivial G-invariant partition of $V(\Gamma)$ with block size $v = k + 1 \geq 3$ such that $\mathcal{D}(B)$ contains no repeated blocks. Suppose further that* $\mathrm{girth}(\Gamma_{\mathcal{B}}) \geq 5$. *Then*

(a) $\Gamma[\{\alpha, \alpha'\}, \{\beta, \beta'\}] \cong K_2$ *for adjacent blocks* $\{\alpha, \alpha'\}$ *and* $\{\beta, \beta'\}$ *of* $\mathcal{P}$.

(b) $\Gamma[B^*, C^*]$ *is a matching for adjacent blocks* B^*, C^* *of* $\mathcal{B}^*$, *and if in addition* $\mathrm{girth}(\Gamma_{\mathcal{B}}) \geq 7$ *then* $\Gamma[B^*, C^*] \cong K_2$.

(c) *The involution* $z : \alpha \mapsto \alpha'$ $(\alpha \in V(\Gamma))$ *defines a graph monomorphism from* Γ *to the complement* $\overline{\Gamma}$, *and* z *interchanges the two partitions* $\mathcal{B}$ *and* $\mathcal{B}^*$. *Moreover,* z *induces graph monomorphisms from* $\Gamma_{\mathcal{B}}$ *to* $\overline{\Gamma_{\mathcal{B}^*}}$, *and from* $\Gamma_{\mathcal{B}^*}$ *to* $\overline{\Gamma_{\mathcal{B}}}$, *defined by* $B \mapsto B^*$, *and* $B^* \mapsto B$, *respectively.*

Proof The assumption $\mathrm{girth}(\Gamma_{\mathcal{B}}) \geq 5$ implies that adjacent vertices of Γ have labels involving four distinct blocks. Suppose that $\{$"BD", "DB"$\}$ and $\{$"CE", "EC"$\}$ are blocks of $\mathcal{P}$ with "DB" and "EC" adjacent in Γ. (This is represented diagramatically in Figure 3, where the two dashed boxes represent B^* and C^* respectively.) Then B, C, D, E are pairwise distinct blocks by our assumption about the labels. Note that "BD" is not adjacent to "EC" and "DB" is not adjacent to "CE" for otherwise (B, D, E, B) or (C, D, E, C) would be a triangle of $\Gamma_{\mathcal{B}}$, contradicting the assumption that $\mathrm{girth}(\Gamma_{\mathcal{B}}) \geq 5$. Similarly, "$BD$" $=$ "DB"^{z} is not adjacent to "CE" $=$ "EC"^{z}, for otherwise (B, D, E, C, B) would be a 4-cycle of $\Gamma_{\mathcal{B}}$. Thus, $\Gamma[\{$"BD", "DB"$\}, \{$"CE", "EC"$\}] \cong K_2$ and (a) holds.

In particular, the non-adjacency of "BD" and "CE" implies that z is a graph monomorphism from Γ to $\overline{\Gamma}$. By the definition of z, two vertices α, β lie in the same block B of $\mathcal{B}$ if and only if α^z, β^z lie in the same block B^* of $\mathcal{B}^*$. Hence z induces the bijection $B \mapsto B^*$ from $\mathcal{B}$ to $\mathcal{B}^*$. Suppose B^*, C^* are adjacent blocks of $\mathcal{B}^*$, say "DB", "EC" are adjacent vertices of Γ, where $D \in \Gamma_{\mathcal{B}}(B), E \in \Gamma_{\mathcal{B}}(C)$ (see Figure 3). If B and C were adjacent in $\Gamma_{\mathcal{B}}$ then (B, D, E, C, B) would be a 4-cycle in $\Gamma_{\mathcal{B}}$, which is not the case. Thus B, C are not adjacent in $\Gamma_{\mathcal{B}}$, that is to say, if B, C are adjacent in $\Gamma_{\mathcal{B}}$, then B^*, C^* are not adjacent in $\Gamma_{\mathcal{B}^*}$. Therefore, the bijection $B \mapsto B^*$ induced by z is a graph monomorphism from $\Gamma_{\mathcal{B}}$ to $\overline{\Gamma_{\mathcal{B}^*}}$, and similarly the bijection $B^* \mapsto B$ is a graph monomorphism from $\Gamma_{\mathcal{B}^*}$ to $\overline{\Gamma_{\mathcal{B}}}$.

If "DB" were adjacent to a second vertex, say "E_1C", in C^*, then (D, E, C, E_1, D) would be a 4-cycle of $\Gamma_{\mathcal{B}}$, contradicting the assumption that $\mathrm{girth}(\Gamma_{\mathcal{B}}) \geq 5$. Therefore, $\Gamma[B^*, C^*]$ is a matching. Now suppose $\mathrm{girth}(\Gamma_{\mathcal{B}}) \geq 7$, and suppose that there is an edge $\{$"D_1B", "E_1C"$\}$ connecting B^* and C^*, distinct from $\{$"DB", "EC"$\}$. If $D_1 = D$ then $E_1 \neq E$ and (D, E, C, E_1, D) is a 4-cycle, and similarly if $E_1 = E$ then $D_1 \neq D$ and (E, D_1, B, D, E) is a 4-cycle. Hence $\{D, E\} \cap \{D_1, E_1\} = \emptyset$, but in this case $(B, D, E, C, E_1, D_1, B)$ is a 6-cycle. Hence $\Gamma[B^*, C^*] \cong K_2$. $\square$

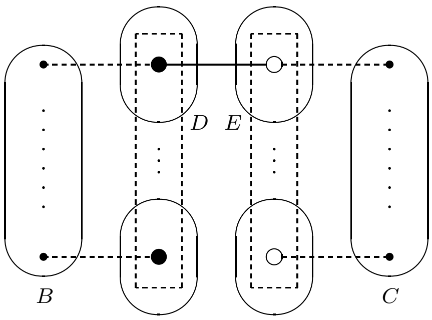


FIGURE 3 Blocks of $\mathcal{B}$, $\mathcal{B}^*$ and $\mathcal{P}$

It is worth noticing that, under the assumptions of Theorem 5.1.3, the G-invariant partition $\mathcal{P}$ satisfies all the assumptions of Section 4.2. Thus, from Theorem 4.2.1, we know that the graphs $\Gamma^\star = \Gamma \cup \Gamma'$ and $\Gamma^\#$ defined in Definition 4.2.1 with respect to $\mathcal{P}$ are both covers of $\Gamma_{\mathcal{P}}$.

Remark 5.1.2 From the group theoretic point of view (see, for example, [70, Theorem 2.1(b)]), Theorem 5.1.3(c) shows that z carries the arc set $\mathrm{Arc}(\Gamma)$ of Γ to a self-paired G-orbital on $V(\Gamma)$ disjoint from Γ_1 and hence $z(\mathrm{Arc}(\Gamma)) \subseteq \Gamma_i$ for some $i \geq 2$, where $\Gamma_i := \{(\alpha, \beta) : d_\Gamma(\alpha, \beta) = i\}$. This parameter i might have a strong influence on the structure of Γ. Essentially the same argument as that used in the proof of Theorem 5.1.3 shows that $i \geq \mathrm{girth}(\Gamma_{\mathcal{B}}) - 3$ (so in particular $i \geq 2$ if $\mathrm{girth}(\Gamma_{\mathcal{B}}) \geq 5$). However, we have been unable to determine the exact value of i.

One consequence of Theorem 5.1.3 is that the valencies of Γ and $\Gamma_{\mathcal{B}^*}$ are bounded as shown below. Recall that $\mathrm{val}(\Gamma)$ denotes the valency of a graph Γ.

Corollary 5.1.2 *Under the assumptions of Theorem 5.1.3,* $\mathrm{val}(\Gamma) \leq (|V(\Gamma)|-2)/4$, *and* $\Gamma_{\mathcal{B}^*}$ *has valency at most* $(|V(\Gamma)|/v) - v - 1$. *If in addition* $\mathrm{girth}(\Gamma_{\mathcal{B}}) \geq 7$, *then* $\mathrm{val}(\Gamma) \leq (|V(\Gamma)|/v^2) - (1/v) - 1$.

Proof By Theorem 5.1.3, each edge of Γ joining α and β corresponds to a unique 3-path $\alpha, \beta', \alpha', \beta$ of $\overline{\Gamma}$, and conversely each 3-path of $\overline{\Gamma}$ of this form corresponds to a unique edge of Γ. One can see that the 3-paths of $\overline{\Gamma}$ with this form corresponding to distinct edges of Γ are pairwise edge-disjoint, and that they have no common edges with Γ' (the latter being contained in $\overline{\Gamma}$). So $|E(\overline{\Gamma})| \geq 3|E(\Gamma)| + |V(\Gamma)|/2$, that is, $\mathrm{val}(\overline{\Gamma}) \geq 3 \cdot \mathrm{val}(\Gamma) + 1$. Thus, we have $\mathrm{val}(\Gamma) \leq (|V(\Gamma)| - 2)/4$. Now by Theorem 5.1.3(c), we have $\mathrm{val}(\Gamma_{\mathcal{B}}) + \mathrm{val}(\Gamma_{\mathcal{B}^*}) \leq |\mathcal{B}| - 1 = (|V(\Gamma)|/v) - 1$, which yields the second inequality since $\mathrm{val}(\Gamma_{\mathcal{B}}) = v$ by Theorem 5.1.1. Note that by Theorem 5.1.3(b), $\mathrm{val}(\Gamma_{\mathcal{B}^*}) = v \cdot \mathrm{val}(\Gamma)$ if $\mathrm{girth}(\Gamma_{\mathcal{B}}) \geq 7$, which implies the last inequality. □

5.2 The 3-arc graph construction

As mentioned in Section 3.2, a fundamental problem arising from the geometric approach used in the thesis is that of reconstructing Γ from the triple $(\Gamma_{\mathcal{B}}, \Gamma[B, C], \mathcal{D}(B))$. In this section we study this problem for the case where $k = v - 1 \geq 2$ and $\mathcal{D}(B)$ contains no repeated blocks. In this case $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive by Theorem 5.1.2, and we will show that the reconstruction can be achieved satisfactorily. We will give an explicit construction of such graphs from $(G, 2)$-arc transitive graphs of valency $v \geq 3$, and prove further that, up to isomorphism, it gives rise to all G-symmetric graphs Γ with properties above.

We present the construction in a general setting, starting with a regular graph Σ of valency $v \geq 3$. For a subset Δ of the set $\mathrm{Arc}_i(\Sigma)$ of i-arcs of Σ, the *paired subset* of Δ is defined by

$$\Delta^{\circ} := \{(\sigma_i, \sigma_{i-1}, \ldots, \sigma_1, \sigma_0) : (\sigma_0, \sigma_1, \ldots, \sigma_{i-1}, \sigma_i) \in \Delta\}$$

and Δ is said to be *self-paired* if $\Delta = \Delta^{\circ}$. The data needed for our construction are a regular graph Σ and a self-paired subset of $\mathrm{Arc}_3(\Sigma)$.

Definition 5.2.1 Let Σ be a regular graph of valency $v \geq 3$, and let Δ be a non-empty self-paired subset of $\mathrm{Arc}_3(\Sigma)$. Define $\Xi(\Sigma, \Delta)$ to be the graph with vertex set $\mathrm{Arc}(\Sigma)$ such that $(\sigma, \tau), (\sigma', \tau') \in \mathrm{Arc}(\Sigma)$ are joined by an edge in $\Xi(\Sigma, \Delta)$ if and only if $(\tau, \sigma, \sigma', \tau') \in \Delta$. We call $\Xi(\Sigma, \Delta)$ the 3-*arc graph* of Σ with respect to Δ.

The requirement that Δ is self-paired ensures that adjacency in $\Xi(\Sigma, \Delta)$ is well-defined (in the sense that (σ, τ) is joined to (σ', τ') if and only if (σ', τ') is joined to (σ, τ)). There are several natural partitions of the vertex set of $\Xi(\Sigma, \Delta)$, namely

(i) $\mathcal{P}(\Sigma) := \{\{(\sigma, \tau), (\tau, \sigma)\} : (\sigma, \tau) \in \mathrm{Arc}(\Sigma)\}$;

(ii) $\mathcal{B}(\Sigma) := \{B(\sigma) : \sigma \in V(\Sigma)\}$, where $B(\sigma) := \{(\sigma, \tau) : \tau \in \Sigma(\sigma)\}$;

(iii) $\mathcal{B}^*(\Sigma) := \{B^*(\sigma) : \sigma \in V(\Sigma)\}$, where $B^*(\sigma) := \{(\tau, \sigma) : \tau \in \Sigma(\sigma)\}$.

Now let G be a group of automorphisms of Σ. Then G induces natural actions on $\mathrm{Arc}(\Sigma)$ and $\mathrm{Arc}_3(\Sigma)$, and provided G leaves Δ invariant, G will preserve the adjacency relation for $\Xi(\Sigma, \Delta)$ and hence will induce an action as a group of automorphisms of $\Xi(\Sigma, \Delta)$. Moreover, the three partitions $\mathcal{P}(\Sigma)$, $\mathcal{B}(\Sigma)$ and $\mathcal{B}^*(\Sigma)$ are all G-invariant. We note the following relations between the G-actions on Σ and $\Xi(\Sigma, \Delta)$: the proofs are straightforward and are omitted.

Lemma 5.2.1 *Let Σ, Δ be as in Definition 5.2.1, and let G be a group of automorphisms of Σ which leaves Δ invariant. Then*

(a) *If G is faithful on the vertices of Σ, then it is also faithful on the vertices of $\Xi(\Sigma, \Delta)$.*

(b) *$\Xi(\Sigma, \Delta)$ is G-vertex-transitive if and only if Σ is G-symmetric.*

(c) $\Xi(\Sigma, \Delta)$ *is G-symmetric if and only if Σ is G-symmetric and Δ is a self-paired G-orbit on* $\mathrm{Arc}_3(\Sigma)$.

(d) *For $\sigma \in V(\Sigma)$, $G_\sigma = G_{B(\sigma)} = G_{B^*(\sigma)}$, and the actions of G_σ on $\Sigma(\sigma), B(\sigma)$ and $B^*(\sigma)$ are permutationally equivalent.*

Thus, if Σ is G-symmetric and Δ is a self-paired G-orbit on $\mathrm{Arc}_3(\Sigma)$, then $\Xi(\Sigma, \Delta)$ is an imprimitive G-symmetric graph relative to each of the partitions above. The following self-evident lemma tells us when a G-orbit on $\mathrm{Arc}_3(\Sigma)$ is self-paired, and this will be used in the next two chapters.

Lemma 5.2.2 *Suppose Σ is a G-symmetric graph. Then a G-orbit $\Delta = (\tau, \sigma, \sigma', \tau')^G$ on $\mathrm{Arc}_3(\Sigma)$ is self-paired if and only if there exists an element of G which reverses the 3-arc $(\tau, \sigma, \sigma', \tau')$, and this in turn is true if and only if there exists an element of G which interchanges the arcs (σ, τ) and (σ', τ').*

Bearing in mind the remarks at the beginning of this section, we now study 3-arc graphs $\Xi(\Sigma, \Delta)$ of a $(G, 2)$-arc transitive graph Σ with respect to self-paired G-orbits Δ on $\mathrm{Arc}_3(\Sigma)$, paying particular attention to the partition $\mathcal{B}(\Sigma)$. A 3-arc $(\tau, \sigma, \sigma', \tau')$ of Σ is said to be *proper* if $\tau \neq \tau'$, that is, $(\tau, \sigma, \sigma', \tau')$ is not a 3-cycle.

Theorem 5.2.1 *Suppose Σ is a $(G, 2)$-arc transitive graph with valency $v \geq 3$. Suppose Δ is a self-paired G-orbit of 3-arcs of Σ. Set $\Gamma := \Xi(\Sigma, \Delta)$. Then the following (a)-(d) hold.*

(a) *For adjacent blocks $B(\sigma), B(\sigma')$ of $\Gamma_{\mathcal{B}(\Sigma)}$, (σ, σ') is the unique element of $B(\sigma)$ which is not adjacent to an element of $B(\sigma')$ (that is, "$k = v - 1$"), and the valency of $\Gamma[B(\sigma), B(\sigma')]$ is equal to the size $|(\tau')^{G_{\tau\sigma\sigma'}}|$ of the $(G_{\tau\sigma\sigma'})$-orbit containing τ', where $(\tau, \sigma, \sigma', \tau') \in \Delta$. Hence* $\mathrm{val}(\Gamma) = (\mathrm{val}(\Sigma) - 1) \cdot |(\tau')^{G_{\tau\sigma\sigma'}}|$.

(b) *$\Gamma_{\mathcal{B}(\Sigma)} \cong \Sigma$, and $\mathcal{D}(B(\sigma))$ has no repeated blocks.*

(c) *If Δ contains a 3-cycle then Δ consists of all the 3-cycles of Σ, and both $\Xi(\Sigma, \Delta)$ and Σ are vertex disjoint unions of complete graphs, as specified in Theorem 5.1.2 (b). The connected components of $\Xi(\Sigma, \Delta)$ are the induced subgraphs on the blocks of $\mathcal{B}^*(\Sigma)$.*

(d) *On the other hand if Δ consists of proper 3-arcs then adjacent vertices of $\Xi(\Sigma, \Delta)$ involve four distinct vertices of Σ.*

Proof Since $B(\sigma), B(\sigma')$ are adjacent in $\Gamma_{\mathcal{B}(\Sigma)}$, there exist $(\sigma, \tau), (\sigma', \tau') \in \mathrm{Arc}(\Sigma)$ such that $(\tau, \sigma, \sigma', \tau') \in \Delta$. In particular $(\sigma, \sigma') \in \mathrm{Arc}(\Sigma)$. Conversely, if $(\sigma, \sigma') \in \mathrm{Arc}(\Sigma)$ then, since $\Delta \neq \emptyset$ and Σ is $(G, 2)$-arc transitive it follows that there exist τ, τ' such that $(\tau, \sigma, \sigma', \tau') \in \Delta$ and hence such that (σ, τ) and (σ', τ') are adjacent in Γ. Thus $B(\sigma)$ is adjacent to $B(\sigma')$ in $\Gamma_{\mathcal{B}(\Sigma)}$. This proves that $\Gamma_{\mathcal{B}(\Sigma)} \cong \Sigma$.

It follows from the definition of a 3-arc that (σ, σ') is not adjacent to any vertex of $B(\sigma')$. Let $(\sigma, \varepsilon) \in B(\sigma)$ with $\varepsilon \neq \sigma'$. Then some $g \in G$ maps the 2-arc (τ, σ, σ') to the 2-arc $(\varepsilon, \sigma, \sigma')$ of Σ, and hence g maps the edge $\{(\sigma, \tau), (\sigma', \tau')\}$ of Γ to $\{(\sigma, \varepsilon), (\sigma', (\tau')^g)\}$. Thus (σ, ε) is joined to some vertex of $B(\sigma') \setminus \{(\sigma', \sigma)\}$. It is now clear that the set of points of $\mathcal{D}(B(\sigma))$ incident with the block $B(\sigma')$ is $B(\sigma) \setminus \{(\sigma, \sigma')\}$. So $\mathcal{D}(B(\sigma))$ has no repeated blocks. Clearly the valency of $\Gamma[B(\sigma), B(\sigma')]$ is equal to $|(\tau')^{G_{\tau\sigma\sigma'}}|$, and from this the equality regarding the valency of Γ follows.

If Δ contains a 3-cycle then, since Σ is $(G, 2)$-arc transitive, the end vertices of every 2-arc of Σ are adjacent vertices of Σ, so Σ is a disjoint union of complete graphs. From the previous paragraph it follows that Δ contains all the 3-cycles of Σ, and that (σ, τ) is adjacent to (σ', τ') in $\Xi(\Sigma, \Delta)$ if and only if (σ, σ') is an arc of Σ and $\tau = \tau'$. Thus the connected components of $\Xi(\Sigma, \Delta)$ are the blocks $B^*(\tau)$ of $\mathcal{B}^*(\Sigma)$ and each is a complete graph. By Lemma 5.2.1, the conditions of Theorem 5.1.2 (b) hold, so $\Gamma \cong \Xi(\Sigma, \Delta)$ and $\Gamma_{\mathcal{B}(\Sigma)} \cong \Sigma$ are as given there. On the other hand, if Δ consists of proper 3-arcs then adjacent vertices (σ, τ) and (σ', τ') of $\Xi(\Sigma, \Delta)$ involve four distinct vertices of Σ. $\square$

Thus, under the assumptions of Theorem 5.2.1, we see that the graph $\Xi(\Sigma, \Delta)$ is a G-symmetric graph admitting the G-invariant partition $\mathcal{B}(\Sigma)$ such that $k = v-1 \geq 2$ and $\mathcal{D}(B(\sigma))$ contains no repeated blocks. We now show that every G-symmetric graph Γ with these properties for some G-invariant partition $\mathcal{B}$ is isomorphic to a 3-arc graph $\Xi(\Gamma_{\mathcal{B}}, \Delta)$ of the quotient graph $\Gamma_{\mathcal{B}}$, for a certain self-paired G-orbit Δ on $\mathrm{Arc}_3(\Gamma_{\mathcal{B}})$. Note that in this case we may identify $\mathcal{B}$ with $\mathcal{B}(\Gamma_{\mathcal{B}})$, and similarly identify $\mathcal{B}^*, \mathcal{P}$ defined in Theorem 5.1.1 with $\mathcal{B}^*(\Gamma_{\mathcal{B}}), \mathcal{P}(\Gamma_{\mathcal{B}})$ respectively.

Theorem 5.2.2 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ of block size $v = k+1 \geq 3$ such that $\mathcal{D}(B)$ contains no repeated blocks, so $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive and the vertices of Γ are labelled with the arcs of*

$\Gamma_{\mathcal{B}}$. *Then* $\Gamma \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ *for* Δ *the (self-paired)* G*-orbit in* $\mathrm{Arc}_3(\Gamma_{\mathcal{B}})$ *containing the* 3*-arc* (C, B, D, E)*, where* ("BC", "DE") *is an arc of* Γ. *In particular,* Δ *contains a* 3*-cycle if and only if* $\Gamma, \Gamma_{\mathcal{B}}$ *are as in Theorem 5.1.2 (b).*

Proof Let ("BC", "DE") be an arc of Γ. Then by the labelling defined before Theorem 5.1.1, it is clear that (C, B, D, E) is a 3-arc of $\Gamma_{\mathcal{B}}$. Let Δ be the G-orbit containing it. Since G is transitive on $\mathrm{Arc}(\Gamma)$, Δ is independent of the choice of arc ("BC", "DE"), and Δ is self-paired. Since every arc of Γ is of the form ("B^gC^g", "D^gE^g") for some $g \in G$, and since $(C^g, B^g, D^g, E^g) = (C, B, D, E)^g \in \Delta$, it follows from Definition 5.2.1 that $\Gamma \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$. Finally, by Theorem 5.2.1 (c) and (d), Δ contains a 3-cycle if and only if the second coordinates of labels for adjacent vertices of Γ are equal, and hence $\Gamma, \Gamma_{\mathcal{B}}$ are as in Theorem 5.1.2 (b). $\square$

Remark 5.2.1 (a) The structure of $\Xi(\Sigma, \Delta)$ for $(G, 2)$-arc transitive graphs Σ is of considerable interest. We will explore in Chapter 7 the family of these graphs for which Σ is a near-polygonal graph and Δ is the set of 3-arcs occurring in the distinguished "polygons" of Σ. This case is of particular interest in connection with Section 5 of [43].

(b) The construction of the graphs $\Xi(\Sigma, \Delta)$ bears some similarity to the covering graph construction of Biggs [6, pp.149-154]. The graphs $\Xi(\Sigma, \Delta)$ are "almost multicovers" of the 2-arc transitive graph Σ.

(c) Let Σ be a $(G, 2)$-arc transitive graph, and let σ, σ' be a pair of adjacent vertices of Σ. Then G contains an element g which interchanges σ and σ'. Let $\tau \in \Sigma(\sigma) \setminus \{\sigma'\}$. Then $\tau' := \tau^g \in \Sigma(\sigma') \setminus \{\sigma\}$, and $(\tau, \sigma, \sigma', \tau')$ is a 3-arc of Σ. Also $\tau^{g^2} \in \Sigma(\sigma) \setminus \{\sigma'\}$. If it is possible to choose g and τ such that $\tau^{g^2} = \tau$, then g maps the 3-arc $(\tau, \sigma, \sigma', \tau')$ to its reverse $(\tau', \sigma', \sigma, \tau)$, and hence the G-orbit Δ containing $(\tau, \sigma, \sigma', \tau')$ is self-paired. This is certainly possible if any one of the following conditions holds:

(i) σ and σ' are interchanged by an involution g;

(ii) the valency $|\Sigma(\sigma)|$ of Σ is even (since we may take g to be a 2-element, and $g^2 \in G_{\sigma\sigma'}$);

(iii) Σ is $(G, 3)$-arc transitive;

(iv) the actions of $G_{\sigma\sigma'}$ on $\Sigma(\sigma) \setminus \{\sigma'\}$ and $\Sigma(\sigma') \setminus \{\sigma\}$ are permutationally isomorphic, in the sense that $G_{\sigma\sigma'\tau}$ fixes a point $\varepsilon \in \Sigma(\sigma') \setminus \{\sigma\}$, and σ', τ are the only points of $\Sigma(\sigma)$ fixed by $G_{\sigma\sigma'\tau}$. (For if $h \in G_{\sigma\sigma'}$ maps τ' to ε, then gh interchanges σ and σ', and maps τ to ε, and hence normalises $G_{\sigma\sigma'\tau} = G_{\sigma\sigma'\varepsilon}$. Therefore gh interchanges τ and ε, and hence reverses the 3-arc $(\tau, \sigma, \sigma', \varepsilon)$.)

If any of these conditions holds, then Σ will occur as the quotient graph $\Gamma_{\mathcal{B}}$ for a graph Γ satisfying the hypotheses of Theorem 5.2.2.

To facilitate our later references, we summarize in the following the key information contained in Theorems 5.1.2, 5.2.1 and 5.2.2.

Theorem 5.2.3 *Let Γ be a G-symmetric graph, and $\mathcal{B}$ a nontrivial G-invariant partition of $V(\Gamma)$ with block size $v \geq 3$ such that $\mathcal{D}(B)$ has block size $v-1$. Then $\mathcal{D}(B)$ contains no repeated blocks if and only if $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive. In this case $\Gamma \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ for some self-paired G-orbit Δ of 3-arcs of $\Gamma_{\mathcal{B}}$. Conversely, for any self-paired G-orbit Δ of 3-arcs of a $(G,2)$-arc transitive graph Σ of valency $v \geq 3$, the graph $\Gamma = \Xi(\Sigma, \Delta)$, group G, and partition $\mathcal{B}(\Sigma)$ satisfy all the conditions above.*

5.3 Three-arc transitive quotient

From the discussion in the previous two sections, we see that even under the assumption that $k = v - 1 \geq 2$ and $\mathcal{D}(B)$ contains no repeated blocks, we are unable to determine the graph Γ completely. This suggests that more information on either the quotient graph $\Gamma_{\mathcal{B}}$ or the bipartite graph $\Gamma[B, C]$ may be needed in order to determine Γ. With regard to the quotient, since we have proved that $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive, we know that G_B is 2-transitive on $\Gamma_{\mathcal{B}}(B)$ by Lemma 3.1.1(b), and thus we may naturally investigate the following two extreme cases:

(i) $\Gamma_{\mathcal{B}}$ is $(G,3)$-arc transitive;

(ii) G_B is sharply 2-transitive on $\Gamma_{\mathcal{B}}(B)$.

With respect to the bipartite graph $\Gamma[B, C]$, we have also the following two extreme cases in which $\Gamma[B, C]$ contains the maximum and minimum possible numbers of edges, respectively:

(I) $\Gamma[B, C] \cong K_{v-1,v-1}$;

(II) $\Gamma[B, C] \cong (v-1) \cdot K_2$.

The purpose of this section is to study the extreme case (I). We find (see Theorem 5.3.1 below) with surprise that case (I) occurs if and only if the extreme case (i) for $\Gamma_{\mathcal{B}}$ occurs, which in turn occurs if and only if the self-paired G-orbit Δ needed in Theorem 5.2.2 for reconstructing Γ is equal to $\mathrm{Arc}_3(\Gamma_{\mathcal{B}})$. Therefore, in this case the graph Γ is uniquely determined by $\Gamma_{\mathcal{B}}$. In Chapter 7 we will study in detail the extreme case (II), and in particular we will prove (Proposition 7.1.1) that (ii) occurs only if (II) occurs.

Theorem 5.3.1 *Suppose that Γ is a G-symmetric graph, and $\mathcal{B}$ is a nontrivial G-invariant partition of $V(\Gamma)$ with block size $v = k+1 \geq 3$ such that $\mathcal{D}(B)$ contains no repeated blocks. Then the following conditions* (a)-(c) *are equivalent:*

(a) $\Gamma_{\mathcal{B}}$ *is* $(G, 3)$*-arc transitive;*

(b) $\Gamma[B, C] \cong K_{v-1,v-1}$*;*

(c) $\Gamma \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ *with* Δ *the set of all* 3*-arcs of* $\Gamma_{\mathcal{B}}$*.*

Thus in this case Γ *is uniquely determined by* $\Gamma_{\mathcal{B}}$*.*

Proof Since $\mathcal{D}(B)$ has no repeated blocks, $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive by Theorem 5.1.2. Suppose that ("BC", "DE") is an arc of Γ and let Δ be the G-orbit on $\mathrm{Arc}_3(\Gamma_{\mathcal{B}})$ containing the 3-arc (C, B, D, E). By Theorem 5.2.2, $\Gamma \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$. Now each 3-arc (C_1, B, D, E_1) of $\Gamma_{\mathcal{B}}$ corresponds to a unique ordered pair "BC_1", "DE_1" of vertices of Γ and vice versa, where $C_1 \in \Gamma_{\mathcal{B}}(B) \setminus \{D\}$ and $E_1 \in \Gamma_{\mathcal{B}}(D) \setminus \{B\}$. Thus we have the following: $\Gamma[B, D] \cong K_{v-1,v-1} \Leftrightarrow$ for any such C_1, E_1, "BC_1", "DE_1" are adjacent in $\Gamma \Leftrightarrow$ for any such C_1, E_1, there exists $g \in G$ with ("BC", "DE")^g = ("BC_1", "DE_1") $\Leftrightarrow$ for any such C_1, E_1, there exists $g \in G$ with $(C, B, D, E)^g = (C_1, B, D, E_1) \Leftrightarrow$ for any such C_1, E_1, the 3-arc (C_1, B, D, E_1) is in $\Delta \Leftrightarrow \Delta = \mathrm{Arc}_3(\Gamma_{\mathcal{B}}) \Leftrightarrow \Gamma_{\mathcal{B}}$ is $(G, 3)$-arc transitive. Thus (a), (b) and (c) are equivalent. $\square$

Chapter 6

Three-arc graphs of complete graphs: Classification

When things had been classified in organic categories, knowledge moved toward fulfillness.

Confucius (551-479 B.C.), THE GREAT LEARNING

The purpose of this chapter is to classify all G-symmetric graphs which admit a G-invariant partition $\mathcal{B}$ such that $k = v - 1 \geq 2$, $\mathcal{D}(B)$ contains no repeated blocks and $\Gamma_{\mathcal{B}} \cong K_{v+1}$ (note that $\mathrm{val}(\Gamma_{\mathcal{B}}) = v$ by Theorem 5.1.1), where $G \leq \mathrm{Aut}(\Gamma)$. By Theorem 5.2.3, this is equivalent to classifying 3-arc graphs of complete $(G, 2)$-arc transitive graphs $\Sigma := K_{v+1}$. In this case G must be 3-transitive on $V(\Sigma)$ (see Lemma 6.1.1 below), and by Theorem 4.3.1(c) and Lemma 5.2.1(a), G is also faithful on $V(\Sigma)$. Hence, by the classification of highly transitive groups (see Theorem 2.1.1 and the comments following it), G is one of the following groups of degree $v+1$ with the natural 3-transitive permutation representation on $V(\Sigma)$:

(i) S_{v+1} ($v \geq 3$);

(ii) A_{v+1} ($v \geq 4$);

(iii) $\mathrm{AGL}(d, 2)$ ($v = 2^d - 1 \geq 3$);

(iv) $\mathbb{Z}_2^4.A_7$ ($v = 15$);

(v) Mathieu groups M_{v+1} ($v = 10, 11, 21, 22, 23$) and M_{11} ($v = 11$); and

(vi) 3-transitive groups G satisfying $\mathrm{PGL}(2,v) \leq G \leq \mathrm{P\Gamma L}(2,v)$ ($v \geq 3$ is a prime power).

We will classify all the 3-arc graphs of Σ with respect to self-paired G-orbits on $\mathrm{Arc}_3(\Sigma)$. The feasibility of such a classification is due to the classification of 3-transitive permutation groups, as shown above, and hence relies on the classification of finite simple groups. To study 3-arc graphs of Σ arising from the groups G in (vi), we need a detailed description of the 3-transitive subgroups of $\mathrm{P\Gamma L}(2,v)$, and this will be given in Section 6.2. The 3-arc graphs obtained in this case are the so-called cross-ratio graphs, which were first introduced in [43] and studied systematically in [46]. Those arising from the other 3-transitive groups were classified in [45]. The 3-arc graphs arising from the groups G in (iii) and (iv) belong to a large class of symmetric graphs associated with the classical affine geometries which we will study in detail in Section 9.5. The results obtained in this chapter will be used in the next chapter.

6.1 Simple examples

Lemma 6.1.1 *Let Σ be a connected $(G,2)$-arc transitive graph with valency $v \geq 3$. Then* $\mathrm{girth}(\Sigma) = 3$ *if and only if $\Sigma \cong K_{v+1}$, which in turn is true if and only if G is 3-transitive on $V(\Sigma)$.*

Proof If $\Sigma \cong K_{v+1}$, then $\mathrm{girth}(\Sigma) = 3$ and G is 3-transitive on $V(\Sigma)$ since G_σ is 2-transitive on $\Sigma(\sigma) = V(\Sigma) \setminus \{\sigma\}$ and G is transitive on $V(\Sigma)$. Next suppose that G is 3-transitive on $V(\Sigma)$. Then, for each $\sigma \in V(\Sigma)$, G_σ is 2-transitive on $V(\Sigma)\setminus\{\sigma\}$ and hence $V(\Sigma) \setminus \{\sigma\}$ induces a complete graph K_v (note that $V(\Sigma) \setminus \{\sigma\}$ contains adjacent vertices). This implies $\Sigma \cong K_{v+1}$. Finally, if $\mathrm{girth}(\Sigma) = 3$, then $\Sigma(\sigma)$ induces a complete graph K_v by the 2-transitivity of G_σ on $\Sigma(\sigma)$. Hence $\Sigma \cong K_{v+1}$ by the connectedness of Σ. $\square$

In the remaining part of this chapter we will suppose $\Sigma := K_{v+1}$, G is one of the groups in (i)-(vi), and $\Gamma := \Xi(\Sigma, \Delta)$ with Δ a self-paired G-orbit on $\mathrm{Arc}_3(\Sigma)$. For simplicity we use $\sigma\tau$ to denote the arc (σ, τ) of Σ if there is no danger of confusion. We begin with the following two simple examples.

Example 6.1.1 *Unions of complete graphs.* If Δ contains a 3-cycle of Σ, then by Theorem 5.2.1(c), Δ is the set of all 3-cycles of $V(\Sigma)$, and in this case we have $\Gamma = \Xi(\Sigma, \Delta) \cong (v+1) \cdot K_v$, $\Gamma[B(\sigma), B(\sigma')] \cong (v-1) \cdot K_2$ for any two distinct vertices σ, σ' of Σ, and G can be any one of the groups listed in (i)-(vi) above.

Therefore, in the following discussion we may suppose that Δ consists of proper 3-arcs of Σ. Also by the 3-transitivity of G on $V(\Sigma)$, to seek self-paired G-orbits $\Delta := (\sigma', \sigma, \tau, \tau')^G$ on $\mathrm{Arc}_3(\Sigma)$, we can start from any chosen 2-arc (σ', σ, τ) of Σ. The next example determines all the 3-arc graphs of Σ (other than $(v+1) \cdot K_v$) arising from 4-transitive groups. For integers ℓ, n with $2 \le 2\ell < n$, the *Kneser graph* $K(n, \ell)$ is the graph with vertices all ℓ-subsets of a given n-set in which two such ℓ-subsets X, Y are adjacent if and only if $X \cap Y = \emptyset$.

Example 6.1.2 Let $\Sigma := K_{v+1}$, and let G be 4-transitive on $V(\Sigma)$. Then either $G = S_{v+1}$ $(v \ge 3)$, or $G = A_{v+1}$ $(v \ge 5)$, or $G = \mathrm{M}_{v+1}$ $(v = 10, 11, 22, 23)$. In each case, G is transitive on the set Δ of all proper 3-arcs of Σ, and hence Δ is the unique self-paired G-orbit on such 3-arcs. As mentioned in Section 5.2, $\mathcal{P} := \{\{\sigma\tau, \tau\sigma\} : \sigma, \tau \in V(\Sigma), \sigma \ne \tau\}$ is a G-invariant partition of the vertex set of $\Gamma = \Xi(\Sigma, \Delta)$. One can see that two blocks $P := \{\sigma\tau, \tau\sigma\}$, $Q := \{\delta\varepsilon, \varepsilon\delta\}$ of $\mathcal{P}$ are adjacent if and only if $\{\sigma, \tau\} \cap \{\delta, \varepsilon\} = \emptyset$, and in this case we have $\Gamma[P, Q] \cong K_{2,2}$. So $\Gamma_{\mathcal{P}}$ is isomorphic to the Kneser graph $K(v+1, 2)$, and $\sigma\tau, \delta\varepsilon$ are adjacent in Γ if and only if $\{\sigma, \tau\} \cap \{\delta, \varepsilon\} = \emptyset$. Thus Γ is isomorphic to $(K(v+1, 2))[\overline{K}_2]$, the lexicographic product of $K(v+1, 2)$ by the empty graph $\overline{K}_2$ on two vertices. One can see that, for distinct blocks B, C of $\mathcal{B}(\Sigma)$, $\Gamma[B, C]$ is isomorphic to $K_{v-1,v-1}$ minus a perfect matching.

So the only remaining groups are those in (iii), (iv), (vi), and the Mathieu groups M_{11} (with degree $v + 1 = 12$) and M_{22} (with degree $v + 1 = 22$). (Note that $S_4 \cong \mathrm{PGL}(2, 3)$ and $A_5 \cong \mathrm{PGL}(2, 4)$.) In the remaining sections of this chapter we will study the 3-arc graphs arising from such groups.

6.2 The 3-transitive subgroups of $\mathrm{P\Gamma L}(2, v)$

Let $v = p^e$ where p is a prime and $e \geq 1$. Then the multiplicative group $\mathrm{GF}(v)^{\#}$ of units of the finite field $\mathrm{GF}(v)$ is a cyclic group of order $p^e - 1$. As is well-known,

$$\mathrm{Sq}(v) := \{x^2 : x \in \mathrm{GF}(v)^{\#}\}$$

is a subgroup of $\mathrm{GF}(v)^{\#}$ with index one or two according as v is even or odd. We may identify the projective line $\mathrm{PG}(1, v)$ with $\mathrm{GF}(v) \cup \{\infty\}$ by identifying a point $[(y, z)]$ of the former with the element y/z of the latter, where ∞ satisfies the usual arithmetic rules such as $1/\infty = 0$, $\infty - y = \infty$, $y - \infty = -\infty$, $(\infty \cdot y)/(\infty \cdot z) = y/z$, $\infty^p = \infty$, etc. With respect to the bijection $[(y, z)] \mapsto y/z$, the action of $\mathrm{PGL}(2, v)$ on the points of $\mathrm{PG}(1, v)$ is permutationally equivalent (see e.g. [10, 63]) to the action of the group of Möbius transformations

$$t_{a,b,c,d} : z \mapsto \frac{az + b}{cz + d} \quad (a, b, c, d \in \mathrm{GF}(v), ad - bc \neq 0)$$

acting on $\mathrm{GF}(v) \cup \{\infty\}$. So we may identify $\mathrm{PGL}(2, v)$ with this group in the following. Then $\mathrm{PSL}(2, v) = \{t_{a,b,c,d} : ad - bc \in \mathrm{Sq}(v)\}$, and it follows from the definition that $\mathrm{P\Gamma L}(2, v) = \mathrm{PGL}(2, v).\langle\psi\rangle$, the semidirect product of $\mathrm{PGL}(2, v)$ by $\langle\psi\rangle$, where ψ is the Frobenius mapping defined by

$$\psi : z \mapsto z^p, z \in \mathrm{GF}(v) \cup \{\infty\}. \tag{6.1}$$

For an integer i with $0 \leq i < e$, we call $(i, t_{a,b,c,d})$ a *twisted pair* if either i is even and $ad - bc \in \mathrm{Sq}(v)$, or i is odd and $ad - bc \in \mathrm{GF}(v)^{\#} \setminus \mathrm{Sq}(v)$. The following theorem shows that the 3-transitive subgroups of $\mathrm{P\Gamma L}(2, v)$ fall into two categories.

Theorem 6.2.1 *Let $v = p^e$, where p is a prime and $e \geq 1$. Then a group G with $\mathrm{PSL}(2, v) \leq G \leq \mathrm{P\Gamma L}(2, v)$ is 3-transitive on $\mathrm{GF}(v) \cup \{\infty\}$ if and only if G is one of the following:*

(a) $G = \mathrm{PGL}(2, v).\langle\psi^n\rangle$ *for n a divisor of e;*

(b) $G = \mathrm{M}(n, v) := \{\psi^{in} t_{a,b,c,d} : (i, t_{a,b,c,d})$ *a twisted pair*$\}$, *where p is an odd prime, $e \geq 2$ is an even integer and n is a divisor of $e/2$.*

Proof Suppose first that $\mathrm{PGL}(2, v) \leq G \leq \mathrm{P\Gamma L}(2, v)$. Then, since $\mathrm{PGL}(2, v)$ is (sharply) 3-transitive on $\mathrm{GF}(v) \cup \{\infty\}$ (e.g. [10, Theorem 2.6.2]) and $\mathrm{P\Gamma L}(2, v) = \mathrm{PGL}(2, v).\langle\psi\rangle$, we have $G = \mathrm{PGL}(2, v).\langle\psi^n\rangle$ for some divisor n of e.

In the following we suppose that $\mathrm{PSL}(2,v) < G < \mathrm{P\Gamma L}(2,v)$, $G \not\geq \mathrm{PGL}(2,v)$ and that G is 3-transitive on $\mathrm{GF}(v) \cup \{\infty\}$. Since $\mathrm{PSL}(2,2^e) = \mathrm{PGL}(2,2^e)$, p must be an odd prime (and hence $\mathrm{Sq}(v)$ has index two in $\mathrm{GF}(v)^{\#}$). Let θ be a fixed element of $\mathrm{PGL}(2,v)$ which is not in $\mathrm{PSL}(2,v)$, and let $\bar{\psi}$, $\bar{\theta}$ be the left cosets of $\mathrm{PSL}(2,v)$ containing ψ, θ respectively. Then $\mathrm{PGL}(2,v)/\mathrm{PSL}(2,v) = \langle\bar{\theta}\rangle \cong \mathbb{Z}_2$ and $\mathrm{P\Gamma L}(2,v)/\mathrm{PSL}(2,v) = \langle\bar{\psi}\rangle \times \langle\bar{\theta}\rangle \cong \mathbb{Z}_e \times \mathbb{Z}_2$. Since $\mathrm{PGL}(2,v) \not\leq G$, we have $\overline{G} := G/\mathrm{PSL}(2,v) = \langle\bar{\psi}^n\bar{\theta}^t\rangle$ for a divisor n of e with $1 < n < e$ and $t = 0$ or 1. If $t = 0$, then $G = \mathrm{PSL}(2,v).\langle\bar{\psi}^n\rangle$ and hence $G_{0\infty} = \{\psi^{in}t_{a,0,0,1} : 0 \leq i < e, a \in \mathrm{Sq}(v)\}$. Thus G is not 3-transitive on $\mathrm{GF}(v) \cup \{\infty\}$ since $\mathrm{Sq}(v) \neq \mathrm{GF}(v)^{\#}$ and since each element $\psi^{in}t_{a,0,0,1}$ in $G_{0\infty}$ maps 1 to $a \in \mathrm{Sq}(v)$. This contradiction shows that $t = 1$ and hence $\overline{G} = \langle\bar{\psi}^n\bar{\theta}\rangle$. If e/n is odd, then $(\bar{\psi}^n\bar{\theta})^{e/n} = \bar{\theta}^{e/n} = \bar{\theta} \in \overline{G}$, which is not the case as $\mathrm{PGL}(2,v) \not\leq G$. Hence e is even and n divides $e/2$. Note that $(i, \theta^i t_{a,b,c,d})$ is a twisted pair for each i with $0 \leq i < e/n$ and for any $t_{a,b,c,d} \in \mathrm{PSL}(2,v)$. Therefore, we have $G = \{\psi^{in}\theta^i t_{a,b,c,d} : 0 \leq i < e/n, ad - bc \in \mathrm{Sq}(v)\} = \mathrm{M}(n,v)$.

To complete the proof, one can see that $\mathrm{M}(n,v)$ is 3-transitive on $\mathrm{GF}(v) \cup \{\infty\}$ for any $v = p^e$ with p an odd prime and $e \geq 2$ an even integer and for any divisor n of $e/2$. $\square$

Note that if $n = e$ then $\mathrm{PGL}(2,v).\langle\psi^n\rangle = \mathrm{PGL}(2,v)$. For p, e, n as in part (b) of Theorem 6.2.1, it follows from the definition that $\mathrm{M}(n,v) = \langle\mathrm{PSL}(2,v), \psi^n t_{a,0,0,1}\rangle$, where a is a primitive element of $\mathrm{GF}(v)$. This expression of $\mathrm{M}(n,v)$ is independent of the choice of the element a.

Corollary 6.2.1 *Let $v = p^e$, where p is a prime and $e \geq 1$. Let G be a 3-transitive subgroup of $\mathrm{P\Gamma L}(2,v)$, as specified in Theorem 6.2.1. Then $G_{\infty 01} = \langle\psi^n\rangle$ if $G = \mathrm{PGL}(2,v).\langle\psi^n\rangle$ (n a divisor of e); and $G_{\infty 01} = \langle\psi^{2n}\rangle$ if $G = \mathrm{M}(n,v)$ (for suitable p, e, n).*

Proof For $G = \mathrm{PGL}(2,v) \cdot \langle\psi^n\rangle$, we have $G_{\infty 0} = \{\psi^{in}t_{a,0,0,1} : i \geq 0, a \in \mathrm{GF}(v)^{\#}\}$. So we get $G_{\infty 01} = \langle\psi^n\rangle$. On the other hand, for $G = \mathrm{M}(n,v)$, by definition we have $G_{\infty 01} = \{\psi^{in} : i \text{ is even }\} = \langle\psi^{2n}\rangle$. $\square$

In particular, this implies that $(\mathrm{M}(e/2,v))_{\infty 01} = 1$ and hence $\mathrm{M}(e/2,v)$ is sharply 3-transitive on $\mathrm{GF}(v) \cup \{\infty\}$ (see e.g. [26, pp. 242-243] for details on this group).

6.3 Definitions of cross-ratio graphs

In this section, we use Σ to denote the complete graph K_{v+1} with vertex set $\mathrm{GF}(v) \cup \{\infty\}$, where $v = p^e$ with p a prime and $e \geq 1$ an integer. So Σ has arc set

$$\Omega(v) := \{yz : y, z \in \mathrm{GF}(v) \cup \{\infty\}, y \neq z\}.$$

Suppose G is a 3-transitive subgroup of $\mathrm{P\Gamma L}(2, v)$, so that Σ is $(G, 2)$-arc transitive. We will define cross-ratio graphs as 3-arc graphs of Σ with respect to self-paired G-orbits on $\mathrm{Arc}_3(\Sigma)$. In accordance with Theorem 6.2.1, we distinguish the following two cases.

We first consider 3-arc graphs of Σ arising from 3-transitive groups G given in Theorem 6.2.1(a). From the theory of finite fields, for each element $x \in \mathrm{GF}(v) \setminus \{0\}$, the subfield of $\mathrm{GF}(v)$ generated by x has the form $\mathrm{GF}(p^{n(x)})$, for some divisor $n(x)$ of e.

Lemma 6.3.1 *Let $x \in \mathrm{GF}(v) \setminus \{0, 1\}$. Let n be a divisor of $n(x)$ and $G := \mathrm{PGL}(2, v).\langle \psi^n \rangle$. Then $\Delta := (0, \infty, 1, x)^G$ is a self-paired G-orbit on $\mathrm{Arc}_3(\Sigma)$.*

Proof Since $1 \cdot (-1) - (-x) \cdot 1 = x - 1 \neq 0$, $t_{1,-x,1,-1}$ is an element of $\mathrm{PGL}(2, v)$. So $t_{1,-x,1,-1}$ is an element of G since $\mathrm{PGL}(2, v) \leq G$. Clearly, $t_{1,-x,1,-1}$ maps $(0, \infty, 1, x)$ to $(x, 1, \infty, 0)$. Hence, by Lemma 5.2.2, Δ is self-paired. $\square$

Definition 6.3.1 Let x, n, G and Δ be as in Lemma 6.3.1. Then the 3-arc graph $\Xi(\Sigma, \Delta)$ of Σ with respect to Δ is well-defined. We call this graph an *untwisted cross-ratio graph* and denote it by $\mathrm{CR}(v; x, n)$.

For 3-transitive subgroups of $\mathrm{P\Gamma L}(2, v)$ given in Theorem 6.2.1(b), we have the following lemma.

Lemma 6.3.2 *Let $v = p^e$ with p an odd prime and e an even integer, and let $x \in \mathrm{GF}(v) \setminus \{0, 1\}$ be such that $n(x)$ is even and $x - 1 \in \mathrm{Sq}(v)$. Let n be an even divisor of $n(x)$ and let $G := \mathrm{M}(n/2, v)$. Then $\Delta := (0, \infty, 1, x)^G$ is a self-paired G-orbit on $\mathrm{Arc}_3(\Sigma)$.*

Proof Since $x-1 \in \mathrm{Sq}(v)$, we have $t_{1,-x,1,-1} \in \mathrm{PSL}(2,v)$. Hence $t_{1,-x,1,-1} \in \mathrm{M}(n/2,v)$ as $\mathrm{PSL}(2,v) \leq \mathrm{M}(n/2,v)$. Since $t_{1,-x,1,-1}$ reverses $(0,\infty,1,x)$, the result follows immediately from Lemma 5.2.2. $\square$

Definition 6.3.2 Let p, e, x, n, G and Δ be as in Lemma 6.3.2. Then the 3-arc graph $\Xi(\Sigma,\Delta)$ of Σ with respect to Δ is well-defined. We call this graph a *twisted cross-ratio graph* and denote it by $\mathrm{TCR}(v;x,n)$.

In the following theorem, we show that the (untwisted and twisted) cross-ratio graphs can be defined equivalently in terms of cross-ratios. This approach for defining untwisted cross-ratio graphs was adopted in [46], and it justifies the terminology used. For distinct elements $u,w,y,z \in \mathrm{GF}(v) \cup \{\infty\}$, the *cross-ratio* is defined as

$$c(u,w;y,z) := \frac{(u-y)(w-z)}{(u-z)(w-y)}$$

(see e.g. [63, pp. 59]) with the usual convention for ∞ as mentioned in previous section. The cross-ratio can take all values in $\mathrm{GF}(v)$ except 0 and 1. For $x \in \mathrm{GF}(v) \setminus \{0,1\}$ and a divisor n of $n(x)$, the field automorphism ψ^n acts on $\mathrm{GF}(p^{n(x)})$ and there are exactly $n(x)/n$ images of x under the elements of $\langle\psi^n\rangle$, namely

$$B(x,n) := \{x^{\psi^{in}} : 0 \leq i < n(x)/n\}. \tag{6.2}$$

Thus $B(x,n)$ is the $\langle\psi^n\rangle$-orbit on $\mathrm{GF}(p^{n(x)})$ containing x.

Theorem 6.3.1 *Suppose $v=p^e$, where p is a prime and $e \geq 1$ an integer.*

(a) *Let $x \in \mathrm{GF}(v) \setminus \{0,1\}$. Let n be a divisor of $n(x)$ and $G := \mathrm{PGL}(2,v).\langle\psi^n\rangle$. Then $\mathrm{CR}(v;x,n)$ is the G-symmetric graph with vertex set $\Omega(v)$ in which uw and yz are adjacent if and only if $c(u,w;y,z) \in B(x,n)$.*

(b) *Let $x \in \mathrm{GF}(v) \setminus \{0,1\}$ be such that $n(x)$ is even and $x-1 \in \mathrm{Sq}(v)$. Let n be an even divisor of $n(x)$ and let $G := \mathrm{M}(n/2,v)$ (where p is odd and e is even). Then $\mathrm{TCR}(v;x,n)$ is the G-symmetric graph with vertex set $\Omega(v)$ in which yz and $\infty 0$ are adjacent if and only if $y \in \mathrm{GF}(v)^{\#}$ and*

$$z \in \begin{cases} B(x,n)y, & \text{if } y \in \mathrm{Sq}(v) \\ B(x,n)^{\psi^{n/2}}y, & \text{if } y \in \mathrm{GF}(v)^{\#} \setminus \mathrm{Sq}(v). \end{cases}$$

Proof The cross-ratio is invariant under the action of $\mathrm{PGL}(2, v)$ on 4-tuples of distinct elements of $\mathrm{GF}(v) \cup \{\infty\}$, and moreover $\mathrm{PGL}(2, v)$ is transitive on such 4-tuples with a fixed cross-ratio (see e.g. [63, pp. 59]). Under the action of the Frobenius mapping ψ, we have

$$c(u^\psi, w^\psi; y^\psi, z^\psi) = (c(u, w; y, z))^\psi.$$

(a) By the definition of $\mathrm{CR}(v; x, n)$ as a 3-arc graph and by Lemma 5.2.1(c), $\mathrm{CR}(v; x, n)$ is G-symmetric. Since $c(\infty, 0; 1, x) = x$, by the definition of a 3-arc graph we have: uw and yz are adjacent in $\mathrm{CR}(v; x, n) \Leftrightarrow (w, u, y, z) \in (0, \infty, 1, x)^G$ $\Leftrightarrow c(u, w, y, z) \in B(x, n)$.

(b) This can be proved in a similar manner, by using the properties of the cross-ratio mentioned above. □

Since $B(x, n(x)) = \{x\}$, part (a) of Theorem 6.3.1 implies that two vertices uw and yz are adjacent in $\mathrm{CR}(v; x, n(x))$ if and only if $c(u, w; y, z) = x$. Recall that $B(x, n)$ is the $\langle \psi^n \rangle$-orbit containing x, so in part (b) of Theorem 6.3.1 the sets $B(x, n)$ and $B(x, n)^{\psi^{n/2}}$ are disjoint and their union is $B(x, n/2)$.

From the discussion in Section 5.2, for $\Gamma = \mathrm{CR}(v; x, n)$ and $G = \mathrm{PGL}(2, v).\langle \psi^n \rangle$, or for $\Gamma = \mathrm{TCR}(v; x, n)$ and $G = \mathrm{M}(n/2, v)$, the vertices of Γ admit the following three G-invariant partitions:

$$\mathcal{P}(v) := \{\{yz, zy\} : y, z \in \mathrm{GF}(v) \cup \{\infty\}, y \neq z\};$$

$$\mathcal{B}(v) := \{B(y) : y \in \mathrm{GF}(v) \cup \{\infty\}\}, \text{ where } B(y) := \{yz : z \in \mathrm{GF}(v) \cup \{\infty\}, y \neq z\};$$

$$\mathcal{B}^*(v) := \{B^*(y) : y \in \mathrm{GF}(v) \cup \{\infty\}\}, \text{ where } B^*(y) := \{zy : z \in \mathrm{GF}(v) \cup \{\infty\}, y \neq z\}.$$

Moreover, yz is the unique vertex of $B(y)$ not adjacent to any vertex of $B(z)$ in Γ. Since $|B(x, n)| = n(x)/n$ and $\Gamma_{\mathcal{B}(v)} \cong K_{v+1}$ (Theorem 5.2.1(b)), this together with Theorem 6.3.1 implies the following consequence.

Corollary 6.3.1 *Let $\Gamma = \mathrm{CR}(v; x, n)$ or $\Gamma = \mathrm{TCR}(v; x, n)$ (for proper x and n). Then for distinct blocks B, C of $\mathcal{B}(v)$, the graph $\Gamma[B, C]$ has valency $n(x)/n$. Hence the valency of Γ is equal to $(q - 1)n(x)/n$.*

More results concerning cross-ratio graphs can be found in [46]. For example, all instances of isomorphism between the cross-ratio graphs are determined in [46, Theorem 7.2]. In particular, there are no isomorphisms between an untwisted cross-ratio graph and a twisted cross-ratio graph.

6.4 Characterizing cross-ratio graphs

In this section we will show that, for G a 3-transitive subgroup of $\mathrm{P\Gamma L}(2, v)$, the (twisted or untwisted) cross-ratio graphs are the only 3-arc graphs of $\Sigma = K_{v+1}$ with respect to self-paired G-orbits on proper 3-arcs of Σ. In fact, we have the following characterization for the cross-ratio graphs. Here we adopt the notation in the previous section.

Theorem 6.4.1 *Let $v = p^e \geq 3$, where p is a prime and $e \geq 1$. Suppose that Γ is a G-symmetric graph with vertex set $\Omega(v)$, where G is a 3-transitive subgroup of $\mathrm{P\Gamma L}(2, v)$ with the induced natural action on $\Omega(v)$. Then either*

(a) $\Gamma \cong (v+1) \cdot K_v$, *with connected components being either the blocks of $\mathcal{B}(v)$ or the blocks of $\mathcal{B}^*(v)$, or*

(b) $\Gamma \cong \binom{v+1}{2} \cdot K_2$, *with connected components the blocks of $\mathcal{P}(v)$, or*

(c) Γ *is isomorphic to* $\mathrm{CR}(v; x, n)$ *or* $\mathrm{TCR}(v; x, n)$ *for some x, n.*

Proof Let $(\infty 0, dx)$ be an arc of Γ. If $d = \infty$, then since G is 3-transitive on $\mathrm{GF}(v) \cup \{\infty\}$ and since the action of G on $\Omega(v)$ is induced by the action of G on $\mathrm{GF}(v) \cup \{\infty\}$, we know that two vertices uw, yz of Γ are adjacent if and only if $u = y$, that is, if and only if (a) holds with components the blocks $B(u)$ of $\mathcal{B}(v)$, for $u \in \mathrm{GF}(v) \cup \{\infty\}$. Similarly, if $x = 0$ then (a) holds with components the blocks $B^*(u)$ of $\mathcal{B}^*(v)$, for $u \in \mathrm{GF}(v) \cup \{\infty\}$; and if $dx = 0\infty$ then (b) holds. So suppose in the following that $d \neq \infty$, $x \neq 0$, and $dx \neq 0\infty$. Suppose that $d = 0$, so that $x \neq \infty$. Any element of G which maps $\infty 0$ to $0x$ must map $0x$ to xz, for some z, and hence there is no element of G which interchanges $\infty 0$ and $0x$, contradicting the arc-transitivity of Γ. Hence $d \neq 0$ and similarly $x \neq \infty$, so $\infty, 0, d, x$ are pairwise distinct. Since G is 3-transitive on $\mathrm{GF}(v) \cup \{\infty\}$, we may assume that $d = 1$.

By Theorem 6.2.1, for some divisor n of e, we have $G = \mathrm{PGL}(2, v).\langle\psi^n\rangle$, or $G = \mathrm{M}(n/2, v)$, where in the latter case p is odd and both e and n are even. By Corollary 6.2.1, $G_{\infty 01} = \langle\psi^n\rangle$, and since G is 3-transitive on $\mathrm{GF}(v) \cup \{\infty\}$, and transitive on arcs of Γ, it follows that $\langle\psi^n\rangle$ is transitive on the vertices of $\Gamma(\infty 0) \cap B(1)$. Thus this set consists of all pairs $1x'$, for $x' \in U(x) := \{x^{\psi^{in}} : \text{for some } i\}$. We can determine $\Gamma(\infty 0)$ since it is the orbit of $G_{\infty 0}$ containing $1x$. If $G = \mathrm{PGL}(2,v) \cdot \langle \psi^n \rangle$ then $\Gamma(\infty 0)$ consists of the pairs uw where $w \in U(x)u$. If $G = \mathrm{M}(n/2, v)$, then $\Gamma(\infty 0)$ consists of the pairs uw where $w \in U(x)u$ if u is a square, and where $w \in U(x)^{\psi^{n/2}}u$ if u is not a square.

The set $U(x)$ is contained in the subfield $\mathrm{GF}(p^{n(x)})$ generated by x, and so each element of $U(x)$ is left invariant by $\psi^{n(x)}$. Moreover $\psi^{n(x)}$ maps squares to squares. It follows that $\Gamma(\infty 0)$ is left invariant by $\langle G_{\infty 0}, \psi^{n(x)} \rangle$ and hence that $\langle G, \psi^{n(x)} \rangle$ leaves the set of arcs of Γ invariant, that is, $\langle G, \psi^{n(x)} \rangle$ is contained in $\mathrm{Aut}(\Gamma)$. Thus we may assume that $\psi^{n(x)} \in G$, and hence that n divides $n(x)$. This means that $U(x)$ is the set $B(x,n)$, as defined in (6.2). If $G = \mathrm{PGL}(2,v) \cdot \langle \psi^n \rangle$ then we have shown that the set of vertices adjacent to $\infty 0$ is the same for Γ and $\mathrm{CR}(v; x, n)$, and they admit the same arc-transitive group G. Hence in this case $\Gamma = \mathrm{CR}(v; x, n)$.

Suppose therefore that $G = \mathrm{M}(n/2, v)$. Since G is arc-transitive on Γ, some element $g = \psi^i t_{a,b,c,d}$ of G interchanges $\infty 0$ and $1x$. Since g interchanges ∞ and 1, and maps 0 to x, we have $g = \psi^i t_{1,-x,1,-1}$. Then, since g maps x to 0, we have $x^{\psi^i} = x$, and hence $n(x)$ divides i. Since n divides $n(x)$, this means that n divides i, and hence $\psi^i \in G$. Therefore $t_{1,-x,1,-1} \in G \cap \mathrm{PGL}(2,v) = \mathrm{PSL}(2,v)$, and so $x - 1 \in \mathrm{Sq}(v)$. Therefore the graph $\mathrm{TCR}(v; x, n)$ is defined, and we have shown that the set of vertices adjacent to $\infty 0$ is the same for Γ and $\mathrm{TCR}(v; x, n)$, and they admit the same arc-transitive group $\mathrm{M}(n/2, v)$. Hence in this case $\Gamma = \mathrm{TCR}(v; x, n)$.
$\square$

Theorem 6.4.1 and its proof imply the following corollary.

Corollary 6.4.1 *Let $v = p^e \geq 3$ with p a prime and $e \geq 1$.*

(a) *The graphs $\mathrm{CR}(v; x, n)$, $\mathrm{TCR}(v; x, n)$ and $(v+1) \cdot K_v$ (as in Example 6.1.1) are the only 3-arc graphs of $\Sigma = K_{v+1}$ with respect to self-paired G-orbits on $\mathrm{Arc}_3(\Sigma)$, where G is a 3-transitive subgroup of $\mathrm{P\Gamma L}(2, v)$.*

(b) *For $\Gamma = \mathrm{CR}(v; x, n)$ or $\Gamma = \mathrm{TCR}(v; x, n)$, the 3-transitive subgroup H such that Γ is H-symmetric is equal to $\mathrm{PGL}(2,v) \cdot \langle \psi^t \rangle$ or $\mathrm{M}(t/2, v)$ respectively, for some divisor t of e such that $\gcd(n(x), t) = n$.*

6.5 Affine 3-arc graphs

For an integer $d \geq 2$ and a prime power q, we use $V(d,q)$ to denote the d-dimensional linear space of row vectors over $\mathrm{GF}(q)$. We use $\mathbf{e}_i$ to denote the unit vector of $V(d,q)$ with i-th coordinate 1 and the remaining coordinates 0, for $i = 1, 2, \ldots, d$. The affine group $\mathrm{AGL}(d,q)$ consists of all affine transformations

$$t_{M,\mathbf{w}} : \mathbf{z} \mapsto \mathbf{z}M + \mathbf{w} \tag{6.3}$$

of $V(d,q)$, where M is a $d \times d$ invertible matrix over $\mathrm{GF}(q)$ and $\mathbf{w} \in V(d,q)$. Similarly the group $\mathrm{A\Gamma L}(d,q)$ consists all semilinear transformations $t_{M,\mathbf{w},\rho} : \mathbf{z} \mapsto \mathbf{z}^{\rho}M + \mathbf{w}$ of $V(d,q)$, where $\rho \in \mathrm{Aut}(\mathrm{GF}(q))$ and ρ acts componentwise on vectors of $V(d,q)$. The affine geometry $\mathrm{AG}(d,q)$ is the geometry with point set $V(d,q)$ and *n-flats* $(1 \leq n \leq d)$ of the form $U + \mathbf{w} := \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U\}$, where U is an n-dimensional subspace of $V(d,q)$ and $\mathbf{w} \in V(d,q)$. A 1-flat (2-flat, respectively) of $\mathrm{AG}(d,q)$ is usually called a *line* (*plane*, respectively) of $\mathrm{AG}(d,q)$. Three points of $\mathrm{AG}(d,q)$ are said to be *collinear* if they lie on a line, and four points of $\mathrm{AG}(d,q)$ are said to be *coplanar* if they lie on a plane.

In studying 3-arc graphs of $\Sigma = K_{v+1}$ arising from the groups in (iii) and (iv), we need the following basic result for $\mathrm{AG}(d,q)$, which we will use in Section 9.5 as well.

Lemma 6.5.1 *Suppose* $\mathrm{AGL}(d,q) \leq G \leq \mathrm{A\Gamma L}(d,q)$*, where $d \geq 2$ and q is a prime power. Then, for $1 \leq n \leq d$, G is transitive on ordered $(n+1)$-tuples of points of* $\mathrm{AG}(d,q)$ *not lying on any $(n-1)$-flat of* $\mathrm{AG}(d,q)$*.*

Proof Any given $n+1$ points $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ of $\mathrm{AG}(d,q)$ do not lie on the same $(n-1)$-flat if and only if $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0$ are independent vectors of $V(d,q)$. So in this case $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0$ can be taken as the first n vectors of an ordered base of $V(d,q)$. Hence there exists $t_{M,0} \in \mathrm{GL}(d,q)$ which maps $\mathbf{e}_1, \ldots, \mathbf{e}_n$ to $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0$, respectively. Thus $t_{M,\mathbf{x}_0} \in \mathrm{AGL}(d,q)$ maps $(\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n)$ to $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$. Since $(\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n)$ is a typical $(n+1)$-tuple not lying on any $(n-1)$-flat of $\mathrm{AG}(d,q)$, the result follows immediately. $\square$

We now determine the 3-arc graphs arising from the 3-transitive affine group (iii) in the introduction of this chapter. These graphs were classified in [45].

Example 6.5.1 *Affine* 3-*arc graphs.* Let $G := \mathrm{AGL}(d,2)$, $d \geq 2$, and let Σ be the complete graph with vertex set $V(d,2)$. Then the proper 3-arcs $(\tau, \sigma, \sigma', \tau')$ of Σ can be partitioned into the following two parts:

$$\Delta_1 := \Delta_1(d,2) = \{(\tau, \sigma, \sigma', \tau') : \tau, \sigma, \sigma', \tau' \text{ coplanar in } \mathrm{AG}(d,2)\},$$

$$\Delta_2 := \Delta_2(d,2) = \{(\tau, \sigma, \sigma', \tau') : \tau, \sigma, \sigma', \tau' \text{ non-coplanar in } \mathrm{AG}(d,2)\}.$$

Clearly, both Δ_1 and Δ_2 are self-paired. Note that each line of $\mathrm{AG}(d,2)$ contains exactly two points, and each plane of $\mathrm{AG}(d,2)$ contains exactly four points (see e.g. [84, Theorem 1.17]). Thus, for any proper 3-arc $(\tau, \sigma, \sigma', \tau')$ in Δ_1, the points τ, σ, σ' are non-collinear and moreover we have $\tau - \sigma = \tau' - \sigma'$. This together with Lemma 6.5.1 implies that G is transitive on Δ_1. Similarly, G is transitive on Δ_2. Since G preserves coplanarity, we conclude that Δ_1, Δ_2 are both self-paired G-orbits on proper 3-arcs of Σ, and they are the only such G-orbits. So we get two 3-arc graphs of Σ, namely $\Xi_i(d,2) := \Xi(\Sigma, \Delta_i)$ for $i = 1, 2$. (In defining the graph $\Xi_2(d,2)$ we require that $d \geq 3$ since $\Delta_2 \neq \emptyset$ if and only if $d \geq 3$.) It follows from the definition that $\Xi_1(d,2)$ is the graph with vertices the ordered pairs of distinct vectors of $V(d,2)$ in which $\mathbf{uw}, \mathbf{yz}$ are adjacent if and only if $\mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{z}$ are distinct and $\mathbf{u} - \mathbf{w} = \mathbf{y} - \mathbf{z}$. Also, $\Xi_2(d,2)$ is the graph with the same vertices in which $\mathbf{uw}, \mathbf{yz}$ are adjacent if and only if $\mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{z}$ are non-coplanar in $\mathrm{AG}(d,2)$.

Example 6.5.2 The group $G := \mathbb{Z}_2^4.A_7$ is a subgroup of $\mathrm{AGL}(4,2)$, where $\mathbb{Z}_2^4$ acts on $V(\Sigma) := V(4,2)$ by translations and, for $\tau := \mathbf{0}$, $G_\tau \cong A_7$ is a subgroup of $\mathrm{GL}(4,2) \cong A_8$ acting 2-transitively on $V(4,2) \setminus \{\tau\}$ in its natural action. Let σ, σ' be distinct points of $V(4,2) \setminus \{\tau\}$. Then from [24, pp.10] we have $G_{\sigma\tau} \cong \mathrm{PSL}(2,7)$, which is transitive on $V(4,2) \setminus \{\sigma, \tau\}$, and each involution in A_7 and also each element of order 3 in $\mathrm{PSL}(2,7)$ fixes exactly 3 nonzero vectors in $V(4,2)$. Hence in the action of $G_{\sigma\sigma'\tau} \cong A_4$ on $V(4,2) \setminus \{\sigma, \sigma', \tau, \sigma + \sigma' + \tau\}$ the stabilizer of any vector is trivial, that is, $G_{\sigma\sigma'\tau}$ has an orbit of length 12. Apart from this orbit, $G_{\sigma\sigma'\tau}$ has another orbit on $V(4,2) \setminus \{\sigma, \sigma', \tau\}$, namely $\{\sigma + \sigma' + \tau\}$. Since G is 3-transitive on $V(\Sigma)$, there are two G-orbits on proper 3-arcs of Σ. It is clear that these two G-orbits are $\Delta_1(4,2)$ and $\Delta_2(4,2)$. Therefore, we have exactly two 3-arc graphs of Σ, namely $\Xi_1(4,2)$ and $\Xi_2(4,2)$.

6.6 Mathieu graphs, and the classification theorem

In this last section we determine 3-arc graphs from the two Mathieu groups M_{11} (with degree $v+1=12$) and M_{22} (with degree $v+1=22$), and thus complete our classification. These graphs were classified in [45].

Example 6.6.1 The Mathieu group M_{11} with degree $v+1=12$ is the automorphism group of the unique 3-$(12,6,2)$ design $\mathcal{D}$. We assume that the point set of $\mathcal{D}$ is the same as the vertex set of $\Sigma := K_{12}$. For a 2-arc (σ',σ,τ) of Σ, let $X(\sigma',\sigma,\tau)$ denote the union of the two blocks of $\mathcal{D}$ containing σ',σ,τ. Then $(\mathrm{M}_{11})_{\sigma'\sigma\tau} \cong S_3$ has two orbits on $V(\Sigma)\setminus\{\sigma',\sigma,\tau\}$ (see [26, pp.231-232]), namely $V(\Sigma)\setminus X(\sigma',\sigma,\tau)$ and $X(\sigma',\sigma,\tau)\setminus\{\sigma',\sigma,\tau\}$. Let $\tau' \in V(\Sigma)\setminus\{\sigma',\sigma,\tau\}$. By the 3-transitivity of M_{11}, there exists $g \in \mathrm{M}_{11}$ such that $(\sigma,\tau,\tau')^g = (\tau,\sigma,\sigma')$. Set $(\sigma')^g = \delta$, so $(\sigma',\sigma,\tau,\tau')^g = (\delta,\tau,\sigma,\sigma')$. Since g is an automorphism of $\mathcal{D}$, the points $\sigma',\sigma,\tau,\tau'$ lie in the same block of $\mathcal{D}$ if and only if $\delta,\tau,\sigma,\sigma'$ lie in the same block of $\mathcal{D}$. This implies that, $\tau' \in V(\Sigma)\setminus X(\sigma',\sigma,\tau)$ ($\tau' \in X(\sigma',\sigma,\tau)\setminus\{\sigma',\sigma,\tau\}$, respectively) if and only if $\delta \in V(\Sigma)\setminus X(\sigma',\sigma,\tau)$ ($\delta \in X(\sigma',\sigma,\tau)\setminus\{\sigma',\sigma,\tau\}$, respectively). That is, δ and τ' are in the same $(\mathrm{M}_{11})_{\sigma'\sigma\tau}$-orbit on $V(\Sigma)\setminus\{\sigma',\sigma,\tau\}$. So there exists $h \in (\mathrm{M}_{11})_{\sigma'\sigma\tau}$ such that $\delta^h = \tau'$. This implies that gh reverses $(\sigma',\sigma,\tau,\tau')$ and hence Δ is self-paired (Lemma 5.2.2). So there are exactly two self-paired (M_{11})-orbits on proper 3-arcs of Σ, namely $\Delta_1 := (\sigma',\sigma,\tau,\tau')^{\mathrm{M}_{11}}$ for $\tau' \in V(\Sigma)\setminus X(\sigma',\sigma,\tau)$, and $\Delta_2 := (\sigma',\sigma,\tau,\tau')^{\mathrm{M}_{11}}$ for $\tau' \in X(\sigma',\sigma,\tau)\setminus\{\sigma',\sigma,\tau\}$. Thus we get two 3-arc graphs, namely $\Xi_i(\mathrm{M}_{11}) := \Xi(\Sigma,\Delta_i)$ for $i=1,2$. Note that $|V(\Sigma)\setminus X(\sigma',\sigma,\tau)| = 3$ and $|X(\sigma',\sigma,\tau)\setminus\{\sigma',\sigma,\tau\}| = 6$. So by Theorem 5.2.1(a) each vertex of $B(\sigma)$ other than $\sigma\tau$ is adjacent to three vertices of $B(\tau)$ in $\Xi_1(\mathrm{M}_{11})$, and adjacent to six vertices of $B(\tau)$ in $\Xi_2(\mathrm{M}_{11})$. One can see that $\alpha\alpha',\beta\beta'$ are adjacent in $\Xi_1(\mathrm{M}_{11})$ ($\Xi_2(\mathrm{M}_{11})$, respectively) if and only if $\alpha',\alpha,\beta,\beta'$ are distinct and $\beta' \in V(\Sigma)\setminus X(\alpha',\alpha,\beta)$ ($\beta' \in X(\alpha',\alpha,\beta)\setminus\{\alpha',\alpha,\beta\}$, respectively). Thus, $\Xi_1(\mathrm{M}_{11})$ and $\Xi_2(\mathrm{M}_{11})$ are the graphs defined in Proposition 5.1(e)(1) and (2) of [45], respectively.

Example 6.6.2 The Mathieu group M_{22} of degree $v+1=22$ is the automorphism group of the 3-$(22,6,1)$ Steiner system $\mathcal{D}$. We assume that the point set of $\mathcal{D}$

is the same as the vertex set of $\Sigma := K_{22}$. As in Example 6.6.1 above, we get two 3-arc graphs of Σ, namely the graph $\Xi_1(\mathrm{M}_{22})$ in which $\alpha\alpha', \beta\beta'$ are adjacent if and only if $\alpha', \alpha, \beta, \beta'$ are distinct and $\beta' \in V(\Sigma) \setminus X(\alpha', \alpha, \beta)$, and the graph $\Xi_2(\mathrm{M}_{22})$ in which $\alpha\alpha', \beta\beta'$ are adjacent if and only if $\alpha', \alpha, \beta, \beta'$ are distinct and $\beta' \in X(\alpha', \alpha, \beta) \setminus \{\alpha', \alpha, \beta\}$, where $X(\alpha', \alpha, \beta)$ denotes the unique block of $\mathcal{D}$ containing α', α, β. These two graphs are the graphs defined in Proposition 5.1(d)(1) and (2) of [45], respectively. Based on the same reason as in Example 6.6.1 one can see that each vertex of $B(\alpha)$ other than $\alpha\beta$ is adjacent to sixteen vertices of $B(\beta)$ in $\Xi_1(\mathrm{M}_{22})$, and adjacent to three vertices of $B(\beta)$ in $\Xi_2(\mathrm{M}_{22})$.

Applying Theorem 5.2.3, the discussion in this chapter gives rise to the following classification of all G-symmetric graphs Γ such that $v = k+1 \geq 3$, $\mathcal{D}(B)$ contains no repeated blocks and $\Gamma_{\mathcal{B}}$ is a complete graph. This classification was obtained in [45] by using a different approach. (By Theorem 4.3.2(b), in our case above G_B must be doubly transitive on B. So such graphs Γ are precisely those graphs studied in [45] with the additional properties that $\mathrm{val}(\Gamma_{\mathcal{B}}) = v$ and $v = k+1 \geq 3$. The objective of [45] is to classify G-symmetric graphs with complete quotients such that the induced action of G on each block of the G-invariant partition is doubly transitive.)

Theorem 6.6.1 *Suppose that Γ is a G-symmetric graph which admits a nontrivial G-invariant partition $\mathcal{B}$ of block size $v = k+1 \geq 3$ such that $\mathcal{D}(B)$ contains no repeated blocks and $\Gamma_{\mathcal{B}}$ is a complete graph, where $G \leq \mathrm{Aut}(\Gamma)$. Then $\Gamma_{\mathcal{B}} \cong K_{v+1}$, G is 3-transitive and faithful on $\mathcal{B}$, and either $\Gamma \cong (v+1) \cdot K_v$ with G an arbitrary 3-transitive permutation group of degree $v+1$, or one of the following* (a)-(f) *holds.*

(a) $\Gamma = (K(v+1,2))[\overline{K}_2]$, *and G is either S_{v+1} ($v \geq 3$), or A_{v+1} ($v \geq 5$), or M_{v+1} ($v = 10, 11, 22, 23$).*

(b) $(\Gamma, G) = (\mathrm{CR}(v; x, n), \mathrm{PGL}(2, v).\langle\psi^t\rangle)$, *where $v = p^e$ with p a prime and $e \geq 1$, $x \in \mathrm{GF}(v) \setminus \{0, 1\}$, n is a divisor of $n(x)$, and t is a divisor of e with $\gcd(n(x), t) = n$.*

(c) $(\Gamma, G) = (\mathrm{TCR}(v; x, n), \mathrm{M}(t/2, v))$, *where $v = p^e$ with p an odd prime and $e \geq 2$ an even integer, $x \in \mathrm{GF}(v) \setminus \{0, 1\}$ with $n(x)$ even and $x - 1$ a square of $\mathrm{GF}(v)$, n is an even divisor of $n(x)$, and t is a divisor of e with $\gcd(n(x), t) = n$.*

(d) $\Gamma = \Xi_1(d, 2)$ *or* $\Xi_2(d, 2)$ *(defined in Example 6.5.1), $v = 2^d - 1$, where $d \geq 2$, and either $G = \mathrm{AGL}(d, 2)$ or $d = 4$ and $G = \mathbb{Z}_2^4.A_7$.*

(e) $\Gamma = \Xi_1(\mathrm{M}_{11})$ *or* $\Xi_2(\mathrm{M}_{11})$ *(defined in Example 6.6.1),* $G = \mathrm{M}_{11}$*, and* $v = 11$*.*

(f) $\Gamma = \Xi_1(\mathrm{M}_{22})$ *or* $\Xi_2(\mathrm{M}_{22})$ *(defined in Example 6.6.2),* $G = \mathrm{M}_{22}$*, and* $v = 21$*.*

In possibility (b) above, if $v = 3$ then $\mathrm{PGL}(2,3) \cong S_4$ and we get only one cross-ratio graph $\mathrm{CR}(3;2,1) \cong 3 \cdot C_4$; if $v = 4$, then $\mathrm{PGL}(2,4) \cong A_5$ and we also have a unique cross-ratio graph $\mathrm{CR}(4;t,2) \cong \mathrm{CR}(4;t^2,2)$, which is isomorphic to the dodecahedron (see [43, Example 2.4(a)]), where we set $\mathrm{GF}(4) = \{0,1,t,t^2 = 1+t\}$.

Chapter 7

Almost covers of two-arc transitive graphs

Study it extensively, inquire into it accurately, think over it carefully, sift it clearly, and practice it earnestly.
Confucius (551-479 B.C.), THE DOCTRINE OF THE MEAN 20

In this chapter we continue our study of the case where $k = v - 1 \geq 2$ and $\mathcal{D}(B)$ contains no repeated blocks. We notice that the possibilities for $\Gamma[B, C]$ depend on the pair $(\Gamma_{\mathcal{B}}, G)$, and vice versa. For example, we have proved in Theorem 5.3.1 that the extreme case $\Gamma[B, C] \cong K_{v-1,v-1}$ occurs if and only if $\Gamma_{\mathcal{B}}$ is $(G, 3)$-arc transitive. In this chapter we investigate the other extreme case for $\Gamma[B, C]$, namely $\Gamma[B, C] \cong (v - 1) \cdot K_2$. In this case Γ is said to be an almost cover of $\Gamma_{\mathcal{B}}$ (see Section 3.2). By using Theorem 6.6.1, we first classify all such graphs Γ in the case where in addition $\Gamma_{\mathcal{B}} \cong K_{v+1}$ (Theorem 7.2.1). In the general case where $\Gamma_{\mathcal{B}} \not\cong K_{v+1}$ and $\Gamma_{\mathcal{B}}$ is connected, we find a surprising connection (Theorem 7.3.1) between such graphs Γ and an interesting class of graphs, namely near-polygonal graphs. For an integer $n \geq 4$, a *near n-gonal graph* [75] is a pair $(\Sigma, \mathcal{E})$ consisting of a connected graph Σ of girth at least 4, together with a set $\mathcal{E}$ of n-cycles of Σ, such that each 2-arc of Σ is contained in a unique member of $\mathcal{E}$. In this case we also say that Σ is a near n-gonal graph with respect to $\mathcal{E}$. The main results in this chapter may be summarized as follows.

Theorem 7.0.2 *Suppose Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ of block size $v \geq 3$ such that $\mathcal{D}(B)$ contains no repeated blocks, and*

$\Gamma_{\mathcal{B}}$ is connected and is almost covered by Γ, where $G \leq \mathrm{Aut}(\Gamma)$. Then the following (a)-(b) *hold.*

(a) *If $\Gamma_{\mathcal{B}} \cong K_{v+1}$, then all possibilities for Γ and G are known explicitly.*

(b) *If $\Gamma_{\mathcal{B}} \ncong K_{v+1}$, then for some even integer $n \geq 4$, $\Gamma_{\mathcal{B}}$ is a $(G,2)$-arc transitive near n-gonal graph with respect to a certain G-orbit on n-cycles of $\Gamma_{\mathcal{B}}$. Moreover, any $(G,2)$-arc transitive near n-gonal graph (where $n \geq 4$ is even) with respect to a G-orbit on n-cycles can appear as such a quotient $\Gamma_{\mathcal{B}}$.*

In Section 7.4, we will study the special case where Γ is a G-locally primitive almost cover of $\Gamma_{\mathcal{B}}$, and in the last section we will give criteria for testing when a $(G,2)$-arc transitive graph is near-polygonal. We will present and prove our results in this chapter in terms of 3-arc graphs. By Theorem 5.2.3, the graphs Γ in Theorem 7.0.2 are precisely 3-arc graphs $\Xi := \Xi(\Sigma, \Delta)$ which almost cover $\Xi_{\mathcal{B}(\Sigma)}$, where Σ is a $(G,2)$-arc transitive graph and Δ is a self-paired G-orbit on $\mathrm{Arc}_3(\Sigma)$. In this case we also say that Ξ *almost covers* Σ since $\Xi_{\mathcal{B}(\Sigma)} \cong \Sigma$ (Theorem 5.2.1(b)).

7.1 Preliminaries

As in the last Chapter, we will denote an arc (σ, τ) of a graph Σ by $\sigma\tau$ when this is convenient and unlikely to cause confusion. The following simple lemma follows from Theorem 5.2.1(a).

Lemma 7.1.1 *Let Σ be a connected $(G,2)$-arc transitive graph. Let Δ be a self-paired G-orbit on $\mathrm{Arc}_3(\Sigma)$, and let $(\tau, \sigma, \sigma', \tau') \in \Delta$. Then $\Xi(\Sigma, \Delta)$ almost covers Σ if and only if τ' is fixed by $G_{\tau\sigma\sigma'}$ (that is, $G_{\tau\sigma\sigma'} = G_{\tau\sigma\sigma'\tau'}$).*

Let $\Gamma = \Xi(\Sigma, \Delta)$ be a 3-arc graph of the $(G,2)$-arc transitive graph Σ. If Γ almost covers Σ, then for each $\tau \in \Sigma(\sigma) \setminus \{\sigma'\}$ there exists a unique $\tau' \in \Sigma(\sigma') \setminus \{\sigma\}$ such that $(\tau, \sigma, \sigma', \tau') \in \Delta$, and hence $\tau \mapsto \tau'$ defines a bijection from $\Sigma(\sigma) \setminus \{\sigma'\}$ to $\Sigma(\sigma') \setminus \{\sigma\}$. Note that this bijection depends on Δ. Since there will be no danger of confusion, we will denote it just by $\phi_{\sigma\sigma'}$. Recall that a G-vertex-transitive graph Σ is $(G,2)$-arc transitive if and only if G_σ is 2-transitive on $\Sigma(\sigma)$ for $\sigma \in V(\Sigma)$ (see Lemma 3.1.1(b)).

Lemma 7.1.2 *Let Σ be a connected $(G, 2)$-arc transitive graph, where $G \leq \mathrm{Aut}(\Sigma)$. Let Δ be a self-paired G-orbit on $\mathrm{Arc}_3(\Sigma)$ and let $\sigma\tau$ be an arc of Σ. Suppose that the 3-arc graph $\Gamma := \Xi(\Sigma, \Delta)$ almost covers Σ. Then the following* (a)-(d) *hold:*

(a) *The actions of G_σ on $B(\sigma)$ and $\Sigma(\sigma)$ are permutationally equivalent, doubly transitive and faithful.*

(b) *The actions of $G_{\sigma\tau}$ on $\Sigma(\sigma) \setminus \{\tau\}$ and on $\Gamma(\sigma\tau)$ are permutationally equivalent, where $\Gamma(\sigma\tau)$ is the neighbourhood of $\sigma\tau$ in Γ. In particular, Γ is G-locally primitive if and only if G_σ is 2-primitive on $\Sigma(\sigma)$; and $G_{\sigma\tau}$ is regular on $\Gamma(\sigma\tau)$ if and only if G_σ is sharply 2-transitive on $\Sigma(\sigma)$.*

(c) $\phi_{\sigma\tau}^{-1} = \phi_{\tau\sigma}$.

(d) $(\phi_{\sigma\tau}(\varepsilon))^g = \phi_{\sigma^g\tau^g}(\varepsilon^g)$ *for $\varepsilon \in \Sigma(\sigma)\setminus\{\tau\}$ and $g \in G$. In particular, the actions of $G_{\sigma\tau}$ on $\Sigma(\sigma) \setminus \{\tau\}$ and $\Sigma(\tau) \setminus \{\sigma\}$ are permutationally equivalent with respect to $\phi_{\sigma\tau}$.*

Proof (a) By Lemma 5.2.1(d), the actions of G_σ on $B(\sigma)$ and $\Sigma(\sigma)$ are permutationally equivalent with respect to the bijection $B(\sigma) \to \Sigma(\sigma)$ defined by $\sigma\tau \mapsto \tau$ for $\tau \in \Sigma(\sigma)$. Since Σ is $(G, 2)$-arc transitive, these actions are doubly transitive. The faithfulness follows from Lemma 5.2.1(a), Theorem 4.3.1(d) and Lemma 4.1.2(a).

(b) For each $\varepsilon \in \Sigma(\sigma)\setminus\{\tau\}$, let $\lambda(\varepsilon)$ denote the unique vertex in $B(\varepsilon)$ adjacent to $\sigma\tau$ in Γ. (The existence of $\lambda(\varepsilon)$ follows from Theorem 5.2.1(a).) Then λ establishes a bijection from $\Sigma(\sigma) \setminus \{\tau\}$ to $\Gamma(\sigma\tau)$. Clearly, $(\lambda(\varepsilon))^g \in \Gamma(\sigma\tau)$ for $g \in G_{\sigma\tau}$. Since $\lambda(\varepsilon) \in B(\varepsilon)$, we have $(\lambda(\varepsilon))^g \in (B(\varepsilon))^g = B(\varepsilon^g)$ and hence $\lambda(\varepsilon^g) = (\lambda(\varepsilon))^g$ by the definition of λ. Thus, the actions of $G_{\sigma\tau}$ on $\Sigma(\sigma) \setminus \{\tau\}$ and on $\Gamma(\sigma\tau)$ are permutationally equivalent with respect to λ. From this the last two assertions in (b) follow immediately.

(c) This is obvious from the definition of $\phi_{\sigma\tau}$.

(d) For $(\varepsilon, \sigma, \tau, \eta) \in \Delta$ and $g \in G$, since Δ is G-invariant we have $(\varepsilon^g, \sigma^g, \tau^g, \eta^g) \in \Delta$ and so $(\phi_{\sigma\tau}(\varepsilon))^g = \eta^g = \phi_{\sigma^g\tau^g}(\varepsilon^g)$ (by the definitions of $\phi_{\sigma\tau}$ and $\phi_{\sigma^g\tau^g}$). In particular, $(\phi_{\sigma\tau}(\varepsilon))^g = \phi_{\sigma\tau}(\varepsilon^g)$ for $g \in G_{\sigma\tau}$ and hence the assertion in the last sentence of (d) is true. □

The following result was advertised in Section 5.3. It shows that, if G_σ is sharply 2-transitive on $\Sigma(\sigma)$ (that is, G_σ holds the "weakest" 2-transitivity on $\Sigma(\sigma)$), then all the 3-arc graphs of Σ are forced to be almost covers of Σ.

Proposition 7.1.1 *Suppose that* Σ *a* $(G,2)$*-arc transitive graph of valency* $v \geq 3$ *such that* G_σ *is sharply* 2*-transitive on* $\Sigma(\sigma)$ *for* $\sigma \in V(\Sigma)$*. Then, for every self-paired* G*-orbit* Δ *on* $\mathrm{Arc}_3(\Sigma)$*, the* 3*-arc graph* $\Gamma := \Xi(\Sigma, \Delta)$ *is an almost cover of* Σ *and* $G_{\sigma\tau}$ *is regular on the neighbourhood* $\Gamma(\sigma\tau)$ *of* $\sigma\tau \in V(\Gamma)$ *in* Γ*.*

Proof Let $\sigma\tau$ be an arc of Σ. Then the sharp 2-transitivity of G_σ on $\Sigma(\sigma)$ implies that $G_{\sigma\tau}$ is regular on $\Sigma(\sigma) \setminus \{\tau\}$, and hence we have $|G_{\sigma\tau}| = |\Sigma(\sigma)| - 1$. Since $\Gamma(\sigma\tau)$ contains exactly s points of each block $B(\delta)$ for $\delta \in \Sigma(\sigma) \setminus \{\tau\}$, where s is the valency of the bipartite graph $\Gamma[B(\sigma), B(\delta)]$ as defined in Section 3.2, we then have $|\Gamma(\sigma\tau)| = s(|\Sigma(\sigma)| - 1) = s|G_{\sigma\tau}|$. On the other hand, since $G_{\sigma\tau}$ is transitive on $\Gamma(\sigma\tau)$, by the orbit-stabilizer property (see Lemma 2.1.1(c)), $|\Gamma(\sigma\tau)|$ is a divisor of $|G_{\sigma\tau}|$. So we have $s = 1$, that is, $\Gamma[B(\sigma), B(\tau)] = (v-1) \cdot K_2$, and hence Γ almost covers Σ. Since $G_{\sigma\tau}$ is regular on $\Sigma(\sigma) \setminus \{\tau\}$, from Lemma 7.1.2(b) we know that $G_{\sigma\tau}$ is also regular on $\Gamma(\sigma\tau)$. □

For a near n-gonal graph $(\Sigma, \mathcal{E})$, the cycles in $\mathcal{E}$ are called *basic cycles* of $(\Sigma, \mathcal{E})$. We use $C(\sigma, \tau, \varepsilon)$ to denote the unique basic cycle of $(\Sigma, \mathcal{E})$ containing a given 2-arc $(\sigma, \tau, \varepsilon)$ of Σ. We also use $\mathrm{Arc}_3(\Sigma, \mathcal{E})$ to denote the set of all 3-arcs of Σ which are contained in some basic cycle of $(\Sigma, \mathcal{E})$. Since the number of 2-arcs contained in a basic cycle of $(\Sigma, \mathcal{E})$ is $2n$ and since each 2-arc is contained in a unique basic cycle, we have $2n|\mathcal{E}| = |\mathrm{Arc}_2(\Sigma)| = v(v-1)|V(\Sigma)| = (v-1)|\mathrm{Arc}(\Sigma)|$, where $v = \mathrm{val}(\Sigma)$. So n and $|\mathcal{E}|$ are connected by

$$|\mathcal{E}| = (v-1)|\mathrm{Arc}(\Sigma)|/2n.$$

Any subgroup $G \leq \mathrm{Aut}(\Sigma)$ induces an action on n-cycles of Σ, and if $\mathcal{E}$ is G-invariant, then G induces an action on $\mathcal{E}$. A *circulant* is a Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, S)$ with vertex set the additive group $\mathbb{Z}_n$ of integers modulo n in which $x, y \in \mathbb{Z}_n$ are adjacent if and only if $x - y \in S$, where S is a subset of $\mathbb{Z}_n$ such that $0 \notin S$ and $-S := \{-x : x \in S\}$ is equal to S.

Lemma 7.1.3 *Suppose* $(\Sigma, \mathcal{E})$ *is a finite* $(G,2)$*-arc transitive near* n*-gonal graph. Then the following statements* (a)-(c) *are equivalent:*

(a) $\mathcal{E}$ *is* G*-invariant.*

(b) $\mathcal{E}$ *is a* G*-orbit on* n*-cycles of* Σ*.*

(c) $\mathrm{Arc}_3(\Sigma, \mathcal{E})$ *is a self-paired G-orbit on* $\mathrm{Arc}_3(\Sigma)$.

Moreover, if one of these occurs, then the following (d)-(e) *hold:*

(d) *Any element of G fixing a 2-arc $(\sigma, \tau, \varepsilon)$ of Σ must fix each vertex in $C(\sigma, \tau, \varepsilon)$.*

(e) *The subgraph of Σ induced by the vertex set of a basic cycle of $(\Sigma, \mathcal{E})$ is isomorphic to a circulant graph* $\mathrm{Cay}(\mathbb{Z}_n, S)$*, for some S with $1 \in S$. Moreover, each such basic cycle is chordless (that is,* $\mathrm{Cay}(\mathbb{Z}_n, S) \cong C_n$*) unless, for adjacent vertices σ, τ of Σ, either*

(i) *G_τ is sharply 2-transitive on $\Sigma(\tau)$ (and hence $|\Sigma(\tau)|$ is a prime power); or*

(ii) *$G_{\sigma\tau}$ is imprimitive on $\Sigma(\tau) \setminus \{\sigma\}$.*

Proof The equivalence of (a) and (b) is obvious since each 2-arc of Σ lies in a unique cycle of $\mathcal{E}$. If (a) holds, then $\mathrm{Arc}_3(\Sigma, \mathcal{E})$ is a G-orbit on $\mathrm{Arc}_3(\Sigma)$. Moreover, in this case $\mathrm{Arc}_3(\Sigma, \mathcal{E})$ is also self-paired. In fact, for $(\sigma, \tau, \varepsilon, \eta) \in \mathrm{Arc}_3(\Sigma, \mathcal{E})$ there exists $g \in G$ such that $(\sigma, \tau, \varepsilon)^g = (\eta, \varepsilon, \tau)$ as Σ is $(G, 2)$-arc transitive. Thus, $(C(\sigma, \tau, \varepsilon))^g = C(\eta, \varepsilon, \tau)$. But $C(\sigma, \tau, \varepsilon)$ is the unique basic cycle containing $(\sigma, \tau, \varepsilon)$, and it is also the unique basic cycle containing $(\eta, \varepsilon, \tau)$. So g fixes $C(\sigma, \tau, \varepsilon)$ and $\eta^g = \sigma$, implying $(\eta, \varepsilon, \tau, \sigma) = (\sigma, \tau, \varepsilon, \eta)^g \in \mathrm{Arc}_3(\Sigma, \mathcal{E})$. Hence $\mathrm{Arc}_3(\Sigma, \mathcal{E})$ is self-paired. Thus (a) implies (c). Conversely suppose that (c) holds. Let

$$C(\sigma_0, \sigma_1, \sigma_2) = (\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_0)$$

be the basic cycle of $(\Sigma, \mathcal{E})$ containing the 2-arc $(\sigma_0, \sigma_1, \sigma_2)$, and let $g \in G$. For each $i = 0, 1, \ldots, n-1$ (subscripts modulo n here and in the remaining part of the proof), it follows from (c) that both $(\sigma_{i-1}^g, \sigma_i^g, \sigma_{i+1}^g, \sigma_{i+2}^g)$ and $(\sigma_i^g, \sigma_{i+1}^g, \sigma_{i+2}^g, \sigma_{i+3}^g)$ lie in basic cycles, and they must lie in the same basic cycle since these two 3-arcs have the 2-arc $(\sigma_i^g, \sigma_{i+1}^g, \sigma_{i+2}^g)$ in common and since each 2-arc of Σ is contained in a unique basic cycle of $(\Sigma, \mathcal{E})$. Since this is true for all i, it follows that $(C(\sigma_0, \sigma_1, \sigma_2))^g$ must be a basic cycle of $(\Sigma, \mathcal{E})$ and hence (c) implies (a).

In the remainder of this proof, we suppose $\mathcal{E}$ is G-invariant, so both (b) and (c) hold. Thus the vertex sets of the basic cycles of $(\Sigma, \mathcal{E})$ induce mutually isomorphic subgraphs. If $g \in G$ fixes the 2-arc $(\sigma_0, \sigma_1, \sigma_2)$, then it fixes the basic cycle $C(\sigma_0, \sigma_1, \sigma_2)$ and, since g fixes each of σ_1, σ_2, it follows that g must fix σ_3. Inductively, one can see that g fixes each vertex in $C(\sigma_0, \sigma_1, \sigma_2)$ and thus (d) is proved. In proving (e), we set $V := \{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$, the vertex set of $C(\sigma_0, \sigma_1, \sigma_2)$, and denote by Σ_1 the subgraph of Σ induced by V. Since Σ is $(G,2)$-arc transitive, there exists $h \in G$ such that $(\sigma_{n-1}, \sigma_0, \sigma_1)^h = (\sigma_0, \sigma_1, \sigma_2)$. Since $\mathcal{E}$ is G-invariant it follows that h fixes V setwise and leaves $C(\sigma_0, \sigma_1, \sigma_2)$ invariant. The only element of $\mathrm{Aut}(\Sigma_1)$ which leaves $C(\sigma_0, \sigma_1, \sigma_2)$ invariant and maps $(\sigma_{n-1}, \sigma_0, \sigma_1)$ to $(\sigma_0, \sigma_1, \sigma_2)$ is the rotation $\rho : \sigma_i \mapsto \sigma_{i+1}$, for all i. Thus the permutation h^V of V induced by h is ρ, and by [6, Lemma 16.3], since $\langle \rho \rangle \cong \mathbb{Z}_n$ is regular on V, Σ_1 is isomorphic to a circulant $\mathrm{Cay}(\mathbb{Z}_n, S)$ for some S. Since σ_i is adjacent to σ_{i+1}, we have $1 \in S$ and the first part of (e) is proved. In proving the second part of (e), we assume that $C(\sigma_0, \sigma_1, \sigma_2)$ contains a chord. Since the group induced on $C(\sigma_0, \sigma_1, \sigma_2)$ contains ρ, it follows that σ_1 is adjacent to some vertex σ_i with $i \neq 0, 2$, that is to say, $\{\sigma_1, \sigma_i\}$ is a chord; and the set $X := \mathrm{fix}_{\Sigma(\sigma_1) \setminus \{\sigma_0\}}(G_{\sigma_0\sigma_1\sigma_2})$ contains both σ_2 and σ_i. On the other hand, the $(G,2)$-arc transitivity of Σ implies that $G_{\sigma_0\sigma_1}$ is transitive on $\Sigma(\sigma_1) \setminus \{\sigma_0\}$, and the stabilizer $G_{\sigma_0\sigma_1\sigma_2}$ (which fixes $C(\sigma_0, \sigma_1, \sigma_2)$ pointwise) fixes $|X| \geq 2$ points of $\Sigma(\sigma_1) \setminus \{\sigma_0\}$. By Lemma 2.2.1, X is a block of imprimitivity for $G_{\sigma_0\sigma_1}$ in $\Sigma(\sigma_1) \setminus \{\sigma_0\}$. Hence either $X = \Sigma(\sigma_1) \setminus \{\sigma_0\}$ or X induces a nontrivial $G_{\sigma_0\sigma_1}$-invariant partition of $\Sigma(\sigma_1) \setminus \{\sigma_0\}$. In the former case the possibility (i) in (e) occurs; whilst in the latter case the possibility (ii) in (e) occurs. Note that if (i) occurs then by [95, pp. 23] $|\Sigma(\sigma_1)|$ must be a prime power. $\square$

7.2 Almost covers of complete graphs

In this section we assume that $\Sigma := K_{v+1}$ is a $(G,2)$-arc transitive graph of valency $v \geq 3$, where $G \leq \mathrm{Aut}(\Sigma)$. So, by Lemma 6.1.1, G is 3-transitive on $V(\Sigma)$ and thus is one of the groups listed at the beginning of the previous chapter. In the following we will show that almost covers $\Xi(\Sigma, \Delta)$ of Σ exist (where Δ is a self-paired G-orbit on $\mathrm{Arc}_3(\Sigma)$), and the goal of this section is to determine all of them. Recall that in Theorem 6.6.1 we have classified all 3-arc graphs of Σ. So what we need to do here is to determine which of them are almost covers of Σ.

By Lemmas 5.2.2 and 7.1.1, a G-orbit $\Delta := (\tau, \sigma, \sigma', \tau')^G$ on $\mathrm{Arc}_3(\Sigma)$ is self-paired and $\Xi(\Sigma, \Delta)$ almost covers Σ if and only if $\sigma\tau, \sigma'\tau'$ can be reversed by an element of G and $G_{\tau\sigma\sigma'} = G_{\tau\sigma\sigma'\tau'}$. In the case where $(\tau, \sigma, \sigma', \tau')$ is a 3-cycle of Σ, we get a unique graph $\Xi(\Sigma, \Delta) \cong (v+1) \cdot K_v$ which almost covers Σ (see Example 6.1.1). Therefore, we may assume in the following that $(\tau, \sigma, \sigma', \tau')$ is proper. Then the requirement

$G_{\tau\sigma\sigma'} = G_{\tau\sigma\sigma'\tau'}$ implies that either $v+1 = |V(\Sigma)| = 4$, or G is 3- but not 4-transitive on $V(\Sigma)$. Hence the groups listed in Example 6.1.2 and the corresponding graph $(K(v+1,2))[\overline{K}_2]$ therein can be excluded. From the discussion in Examples 6.6.1 and 6.6.2, we can also exclude the Mathieu group M_{11} of degree 12 and the Mathieu group M_{22} of degree 22. Therefore, from the list at the beginning of Chapter 6, only $\mathrm{AGL}(d,2)$ $(v = 2^d - 1 \geq 3)$, $\mathbb{Z}_2^4.A_7$ $(v = 15)$ and the 3-transitive subgroups of $\mathrm{P\Gamma L}(2,v)$ (v is a prime power) can satisfy the conditions in Lemmas 5.2.2 and 7.1.1 for a proper 3-arc $(\tau, \sigma, \sigma', \tau')$ of Σ. The 3-transitive subgroups of $\mathrm{P\Gamma L}(2,v)$ were described in Theorem 6.2.1, and by Corollary 6.4.1(a) the 3-arc graphs arising from such groups are (twisted or untwisted) cross-ratio graphs. The following example tells us when a cross-ratio graph is an almost cover of Σ.

Example 7.2.1 *Cross-ratio graphs which almost cover* K_{v+1}. Let $v = p^e \geq 3$ be a prime power. For $\Gamma = \mathrm{CR}(v;x,n)$ and $G = \mathrm{PGL}(2,v).\langle\psi^n\rangle$ or for $\Gamma = \mathrm{TCR}(v;x,n)$ and $G = \mathrm{M}(n/2,v)$ (for proper x and n), the vertex set of Γ admits the G-invariant partition $\mathcal{B} := \mathcal{B}(v)$ (see Section 6.3 for definition) such that the block size of $\mathcal{D}(B)$ is equal to $v-1$, where $B \in \mathcal{B}$. By Corollary 6.3.1, for distinct blocks B, C of $\mathcal{B}$, the bipartite subgraph $\Gamma[B,C]$ has valency $n(x)/n$. So Γ almost covers Σ if and only if $n(x) = n$. Thus, by Corollary 6.4.1(a), the only 3-arc graphs $\Xi(\Sigma,\Delta)$ of $\Sigma = K_{v+1}$ which almost cover Σ are $\mathrm{CR}(v;x,n(x))$ and $\mathrm{TCR}(v;x,n(x))$, for $x \in \mathrm{GF}(v) \setminus \{0,1\}$, where Δ is a self-paired G-orbit on $\mathrm{Arc}_3(\Sigma)$. Note that $\gcd(n(x),t) = n(x)$ implies that $n(x)$ is a divisor of t. So by Corollary 6.4.1(b), the only 3-transitive subgroups H of $\mathrm{P\Gamma L}(2,v)$ such that $\mathrm{CR}(v;x,n(x))$ is H-symmetric have the form $H = \mathrm{PGL}(2,v).\langle\psi^t\rangle$, where t is a divisor of e and a multiple of $n(x)$. Similarly, the only 3-transitive subgroups H of $\mathrm{P\Gamma L}(2,v)$ such that $\mathrm{TCR}(v;x,n(x))$ is H-symmetric have the form $H = \mathrm{M}(t/2,v)$, where t is an even divisor of e and a multiple of $n(x)$.

The next example determines the 3-arc graphs arising from $\mathrm{AGL}(d,2)$ and $\mathbb{Z}_2^4.A_7$ which almost cover Σ.

Example 7.2.2 In Example 6.5.1 we have shown that, apart from the graph $(v+1)\cdot K_v$, there are only two 3-arc graphs of Σ arising from the group $G = \mathrm{AGL}(d,2)$ $(v = 2^d - 1 \geq 3)$ or the group $G = \mathbb{Z}_2^4.A_7$ (if $d = 4$), namely $\Xi_i(d,2)$ for $i = 1,2$. It follows from the definition that $\Xi_1(d,2)$ is, and $\Xi_2(d,2)$ is not, an almost cover of Σ.

The analysis above leads to the following classification theorem.

Theorem 7.2.1 *Suppose $\Sigma = K_{v+1}$ is a $(G, 2)$-arc transitive complete graph, where $v \geq 3$ and $G \leq \mathrm{Aut}(\Sigma)$. Suppose further that $\Gamma = \Xi(\Sigma, \Delta)$ almost covers Σ, where Δ is a self-paired G-orbit on $\mathrm{Arc}_3(\Sigma)$. Then either $\Gamma \cong (v+1) \cdot K_v$ with G an arbitrary 3-transitive permutation group of degree $v + 1$, or (Γ, G) is one of the following (where in* (a), (b), *$v = p^e$ with p a prime and $e \geq 1$):*

(a) *$(\mathrm{CR}(v; x, n(x)), \mathrm{PGL}(2, v).\langle\psi^t\rangle)$, where $x \in \mathrm{GF}(v) \setminus \{0, 1\}$, and t is a divisor of e and a multiple of $n(x)$;*

(b) *$(\mathrm{TCR}(v; x, n(x)), \mathrm{M}(t/2, v))$, where p is odd, e is even, $x \in \mathrm{GF}(v) \setminus \{0, 1\}$ with $n(x)$ even and $x - 1$ a square of $\mathrm{GF}(v)$, and t is an even divisor of e and a multiple of $n(x)$;*

(c) *$(\Xi_1(d, 2), \mathrm{AGL}(d, 2))$, where $v = 2^d - 1 \geq 3$; or*

(d) *$(\Xi_1(4, 2), \mathbb{Z}_2^4.A_7)$, where $v = 15$.*

7.3 Almost covers of non-complete graphs

Now we discuss the general case where Σ is a connected, non-complete, $(G, 2)$-arc transitive graph with valency $v \geq 3$. Then $\mathrm{girth}(\Sigma) \geq 4$ by Lemma 6.1.1. The main result in this case is the following theorem which, together with Theorems 5.2.3 and 7.2.1, yields a proof of Theorem 7.0.2 stated at the beginning of this chapter.

Theorem 7.3.1 *Suppose that Σ is a connected $(G, 2)$-arc transitive graph with valency $v \geq 3$ and that $\Sigma \ncong K_{v+1}$. Then Σ is almost covered by a 3-arc graph $\Xi(\Sigma, \Delta)$ of Σ if and only if, for some even integer $n \geq 4$, Σ is a near n-gonal graph with respect to a G-orbit $\mathcal{E}$ of n-cycles of Σ, and in this case we have $\Delta = \mathrm{Arc}_3(\Sigma, \mathcal{E})$, the set of all 3-arcs of Σ contained in the n-cycles in $\mathcal{E}$.*

Proof Suppose Σ is almost covered by a 3-arc graph $\Gamma := \Xi(\Sigma, \Delta)$ of Σ, where Δ is a self-paired G-orbit on $\mathrm{Arc}_3(\Sigma)$. Recall that, for adjacent vertices σ, σ' of Σ, we use $\phi_{\sigma\sigma'}$ to denote the the bijection from $\Sigma(\sigma) \setminus \{\sigma'\}$ to $\Sigma(\sigma') \setminus \{\sigma\}$ such that $\phi_{\sigma\sigma'}(\tau) = \tau'$ precisely when $(\tau, \sigma, \sigma', \tau') \in \Delta$. Let $(\sigma_0, \sigma_1, \sigma_2)$ be a 2-arc of Σ. Set $\sigma_3 := \phi_{\sigma_1\sigma_2}(\sigma_0)$, and inductively define $\sigma_{i+2} := \phi_{\sigma_i\sigma_{i+1}}(\sigma_{i-1})$ for $i \geq 1$. Then we get a sequence $\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{i-1}, \sigma_i, \sigma_{i+1}, \sigma_{i+2}, \ldots$ of vertices of Σ such that $(\sigma_{i-1}, \sigma_i, \sigma_{i+1}, \sigma_{i+2}) \in \Delta$ for each $i \geq 1$. Our assumption $\Sigma \not\cong K_{v+1}$ implies that girth$(\Sigma) \geq 4$ (Lemma 6.1.1) and hence all such 3-arcs $(\sigma_{i-1}, \sigma_i, \sigma_{i+1}, \sigma_{i+2})$ are proper, that is, any four consecutive vertices in this sequence are pairwise distinct. Since Σ has a finite number of vertices, the sequence must eventually contain repeated vertices. Let σ_n be the first vertex in the sequence that coincides with one of the preceding vertices. We claim that σ_n must coincide with σ_0. Suppose to the contrary that $\sigma_n = \sigma_\ell$ for some ℓ such that $1 \leq \ell < n$. Then since Σ is $(G, 2)$-arc transitive, there exists $g \in G$ such that $(\sigma_\ell, \sigma_{\ell+1}, \sigma_{\ell+2})^g = (\sigma_0, \sigma_1, \sigma_2)$. From Lemma 7.1.2(d), we have $\sigma_{\ell+3}^g = \phi_{\sigma_{\ell+1}^g \sigma_{\ell+2}^g}(\sigma_\ell^g) = \phi_{\sigma_1\sigma_2}(\sigma_0) = \sigma_3$. Inductively we have that $\sigma_{\ell+i}^g = \sigma_i$ for each $i \geq 0$. In particular, $\sigma_n^g = \sigma_{\ell+(n-\ell)}^g = \sigma_{n-\ell}$. But since $\sigma_n = \sigma_\ell$, we have $\sigma_{n-\ell} = \sigma_n^g = \sigma_\ell^g = \sigma_0$, contradicting the minimality of n. Therefore we must have $\sigma_n = \sigma_0$. Thus, each 2-arc $(\sigma_0, \sigma_1, \sigma_2)$ of Σ determines a unique (undirected) n-cycle $C(\sigma_0, \sigma_1, \sigma_2) := (\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_0)$ of Σ. Note again that $n \geq 4$ since girth$(\Sigma) \geq 4$.

Set $\tau := \phi_{\sigma_1\sigma_0}(\sigma_2)$, then we have $\sigma_2 = \phi_{\sigma_0\sigma_1}(\tau)$ by Lemma 7.1.2(c). We claim that τ must coincide with σ_{n-1}. For the 2-arc $(\tau, \sigma_0, \sigma_1)$, the construction in the previous paragraph will give the sequence $\tau, \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_n = \sigma_0$, and since the first repeated vertex is the same as the starting vertex τ, it follows that $\tau = \sigma_{n-1}$. Similarly, one can show that $\sigma_{n-2} = \phi_{\sigma_0\sigma_{n-1}}(\sigma_1)$ and hence $\sigma_1 = \phi_{\sigma_{n-1}\sigma_0}(\sigma_{n-2})$. Therefore, reading the subscripts modulo n (here and in the remainder of this section), we have $\sigma_{i+2} = \phi_{\sigma_i\sigma_{i+1}}(\sigma_{i-1})$ and hence $\sigma_{i-1} = \phi_{\sigma_{i+1}\sigma_i}(\sigma_{i+2})$ for each $i \geq 1$ (Lemma 7.1.2(c)). This implies that the 2-arcs $(\sigma_{i-1}, \sigma_i, \sigma_{i+1})$ and $(\sigma_{i+1}, \sigma_i, \sigma_{i-1})$ contained in $C(\sigma_0, \sigma_1, \sigma_2)$ (for $i \geq 1$) also determine the same n-cycle $C(\sigma_0, \sigma_1, \sigma_2)$. By definition of $C(\sigma_0, \sigma_1, \sigma_2)$ and by Lemma 7.1.2(d), we have $C(\sigma_0^g, \sigma_1^g, \sigma_2^g) = (C(\sigma_0, \sigma_1, \sigma_2))^g$ for $g \in G$ and hence $\mathcal{E} := \{C(\sigma, \tau, \varepsilon) : (\sigma, \tau, \varepsilon) \in \mathrm{Arc}_2(\Sigma)\}$ is G-invariant and each 2-arc lies in a unique cycle of $\mathcal{E}$. By the $(G, 2)$-arc transitivity of Σ, the length n of $C(\sigma, \tau, \varepsilon)$ is independent of the choice of $(\sigma, \tau, \varepsilon)$ and G is transitive on $\mathcal{E}$. Thus $\mathcal{E}$ is a G-orbit on n-cycles of Σ and Σ is a near n-gonal graph with respect to $\mathcal{E}$. Moreover, the argument above shows that $\Delta = \mathrm{Arc}_3(\Sigma, \mathcal{E})$. In particular, in the sequence $\sigma_0\sigma_1, \sigma_1\sigma_0, \sigma_2\sigma_3, \sigma_3\sigma_2, \ldots, \sigma_{2i-2}\sigma_{2i-1}, \sigma_{2i-1}\sigma_{2i-2}, \sigma_{2i}\sigma_{2i+1}, \sigma_{2i+1}\sigma_{2i}, \ldots$ of vertices of Γ, for each i, the $(2i-1)$-st vertex $\sigma_{2i-2}\sigma_{2i-1}$ and the $2i$-th vertex $\sigma_{2i-1}\sigma_{2i-2}$ are not adjacent, while the $2i$-th vertex and the $(2i+1)$-st vertex $\sigma_{2i}\sigma_{2i+1}$ are adjacent. By the definition of n, the n-th vertex of this sequence is $\sigma_{n-1}\sigma_{n-2}$, and it is adjacent to $\sigma_0\sigma_1$ ($= \sigma_n\sigma_{n+1}$) since $(\sigma_{i-1}, \sigma_i, \sigma_{i+1}, \sigma_{i+2}) \in \Delta$ for each i (subscripts modulo n). It follows that n must be an even integer.

To prove the "if" part of the theorem, suppose that $(\Sigma, \mathcal{E})$ is a $(G, 2)$-arc transitive near n-gonal graph with valency $v \geq 3$ and $\mathcal{E}$ is a G-orbit on n-cycles of Σ, for some even $n \geq 4$. Then by Lemma 7.1.3, $\Delta := \mathrm{Arc}_3(\Sigma, \mathcal{E})$ is a self-paired G-orbit on $\mathrm{Arc}_3(\Sigma)$. Let $\Gamma := \Xi(\Sigma, \Delta)$ and let $(\tau, \sigma, \sigma', \tau') \in \Delta$. Then $\sigma\tau \in B(\sigma)$ is adjacent to $\sigma'\tau' \in B(\sigma')$ in Γ. If $\sigma\tau$ is adjacent in Γ to a second vertex, say $\sigma'\varepsilon'$, of $B(\sigma')$, then $(\tau, \sigma, \sigma', \tau')$, $(\tau, \sigma, \sigma', \varepsilon')$ are distinct 3-arcs in Δ and hence the 2-arc (τ, σ, σ') is contained in two distinct basic cycles of $(\Sigma, \mathcal{E})$. This contradiction shows that $\Gamma[B(\sigma), B(\sigma')] \cong (v-1) \cdot K_2$ and hence Γ almost covers $\Gamma_{\mathcal{B}(\Sigma)}$. □

Remark 7.3.1 By Lemma 7.1.3(e), the vertex set of each basic cycle of $(\Sigma, \mathcal{E})$ in Theorem 7.3.1 induces a circulant subgraph of Σ, and these basic cycles are chordless unless either (e)(i) or (e)(ii) in that lemma occurs. This latter fact is interesting from a combinatorial point of view. The following example shows that the basic cycles of $(\Sigma, \mathcal{E})$ may contain chords. It also provides an example of such a graph Σ with the smallest valency (namely 3) and shows that the near n-gonal graph $(\Sigma, \mathcal{E})$ occurring in Theorem 7.3.1 is not necessarily an n-gonal graph. (A near n-gonal graph is said to be an *n-gonal graph* [75] if n is equal to the girth of the graph.) Moreover, it shows that the graph $\Xi(\Sigma, \Delta)$ may not be connected, even if Σ is connected and $(G, 2)$-arc transitive.

Example 7.3.1 Let Σ be the complete bipartite graph $K_{3,3}$ with vertex set $\{0, 1, 2, 3, 4, 5\}$ and bipartition $(\{0, 2, 4\}, \{1, 3, 5\})$. We will show that there exists a unique subgroup $G \leq \mathrm{Aut}(\Sigma)$ such that Σ is a $(G, 2)$-arc transitive near 6-gonal graph with respect to a G-orbit $\mathcal{E}$ of 6-cycles of Σ. By the definition of near polygonal graphs, one can easily check that

$$\mathcal{E}_1 := \{(0, 1, 2, 3, 4, 5, 0), (0, 5, 2, 1, 4, 3, 0), (0, 1, 4, 5, 2, 3, 0)\}$$

and

$$\mathcal{E}_2 := \{(0, 1, 2, 5, 4, 3, 0), (0, 3, 2, 1, 4, 5, 0), (0, 1, 4, 3, 2, 5, 0)\}$$

are the only possible sets $\mathcal{E}$ of 6-cycles of Σ such that $(\Sigma, \mathcal{E})$ is a near 6-gonal graph. On the other hand, we have $\mathrm{Aut}(\Sigma) = S_3 \,\mathrm{wr}\, S_2 \cong \langle (024), (02), (01)(23)(45) \rangle$ and again it is easily checked that (024) and (01)(23)(45) fix $\mathcal{E}_1$ and $\mathcal{E}_2$ setwise, whilst (02) interchanges $\mathcal{E}_1$ and $\mathcal{E}_2$. Thus $\mathrm{Aut}(\Sigma)$ interchanges $\mathcal{E}_1$ and $\mathcal{E}_2$ and so a subgroup G of $\mathrm{Aut}(\Sigma)$ with index 2 fixes $\mathcal{E}_1$ and $\mathcal{E}_2$ setwise. We have seen that G contains $H = \langle (024), (01)(23)(45) \rangle \cong A_3 \,\mathrm{wr}\, S_2$, but does not contain (02). Thus $|G : H| = 2$. The element (13) is the conjugate of (02) by (01)(23)(45), and hence $(13) \in \mathrm{Aut}(\Sigma)$ and (13) interchanges $\mathcal{E}_1$ and $\mathcal{E}_2$. Therefore (02)(13) fixes $\mathcal{E}_1$ and $\mathcal{E}_2$ setwise and does not lie in H, so $G = \langle H, (02)(13) \rangle$. It is easy to check that G is transitive on the 2-arcs of Σ, and hence $(\Sigma, \mathcal{E}_i)$ is a $(G, 2)$-arc transitive near 6-gonal graph for $i = 1$ and $i = 2$. If Σ is $(K, 2)$-arc transitive and K preserves the $\mathcal{E}_i$, then $K \leq G$ and $|K|$ is divisible by the number of 2-arcs, that is, by 36. Hence $K = G$. Finally, for $\Delta_i := \mathrm{Arc}_3(\Sigma, \mathcal{E}_i)$, $i = 1, 2$, we have $\Xi(\Sigma, \Delta_i) \cong 3 \cdot C_6$ (see Figure 4 below).

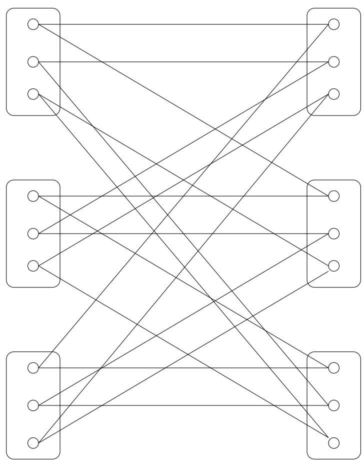


FIGURE 4 $\Gamma = 3 \cdot C_6, \Sigma = K_{3,3}$

The following proposition shows further that the graph Σ in Example 7.3.1 is the only connected trivalent non-complete graph which is $(G, 2)$-arc transitive and near n-gonal for an even integer n such that the basic cycles have chords.

Proposition 7.3.1 *Suppose Σ is a connected, $(G, 2)$-arc transitive, trivalent graph and $\Sigma \ncong K_4$. Suppose Δ is a self-paired G-orbit on* $\mathrm{Arc}_3(\Sigma)$ *such that $\Gamma := \Xi(\Sigma, \Delta)$ almost covers Σ. Then Σ is a near n-gonal graph with respect to some G-orbit $\mathcal{E}$ of n-cycles (and n is even). Moreover the cycles in $\mathcal{E}$ have chords if and only if $\Sigma \cong K_{3,3}$, $\Gamma \cong 3 \cdot C_6$, and $\mathcal{E} \cong \mathcal{E}_1$ or $\mathcal{E}_2$, where G, $\mathcal{E}_1$ and $\mathcal{E}_2$ are as in Example 7.3.1.*

Proof By Theorem 7.3.1, Σ is a near n-gonal graph with respect to some G-orbit $\mathcal{E}$ of n-cycles for an even integer $n \geq 4$. So we need only to prove that the cycles in $\mathcal{E}$ have chords if and only if $\Sigma, \Gamma, G, \mathcal{E}$ are as claimed. The "if" part was in fact proved in Example 7.3.1. We prove the "only if" part in the following.

Suppose $\{\sigma_0, \sigma_\ell\}$ is a chord of the basic cycle $C(\sigma_0, \sigma_1, \sigma_2) := (\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_0)$. Then $\{\sigma_i, \sigma_{i+\ell}\}$ is a chord of $C(\sigma_0, \sigma_1, \sigma_2)$ for each i (by Lemma 7.1.3(e)). Since Σ is trivalent and connected, the only possibility is $\ell = n/2$ and $\Sigma \cong \mathrm{Cay}(\mathbb{Z}_n, \{1, \ell, n-1\})$. Since $\Sigma \not\cong K_4$, we have $\ell \geq 3$. Now the unique n-cycle $C(\sigma_\ell, \sigma_0, \sigma_1)$ containing $(\sigma_\ell, \sigma_0, \sigma_1)$ must be the following sequence of vertices: $\sigma_\ell, \sigma_0, \sigma_1, \sigma_{\ell+1}, \sigma_{\ell+2}, \sigma_2, \sigma_3, \sigma_{\ell+3}, \sigma_{\ell+4}, \ldots$. If ℓ is even, this sequence does not even form an n-cycle since it never returns to the vertex σ_ℓ. (Once we arrive at $\sigma_{\ell-1}$, the next vertex in the sequence is σ_{n-1} and from σ_{n-1} the sequence returns to σ_0. For example, if $\ell = 4$, then the sequence is the 7-cycle $(\sigma_0, \sigma_1, \sigma_5, \sigma_6, \sigma_2, \sigma_3, \sigma_7, \sigma_0)$.) So ℓ is odd, and in this case the sequence does give an n-cycle. By the $(G, 2)$-arc transitivity of Σ, there exists $g \in G$ such that $(\sigma_{n-1}, \sigma_0, \sigma_1)^g = (\sigma_\ell, \sigma_0, \sigma_1)$. From Lemma 7.1.2(d), we have $(C(\sigma_{n-1}, \sigma_0, \sigma_1))^g = C(\sigma_\ell, \sigma_0, \sigma_1)$. Therefore, $\sigma_0^g = \sigma_0, \sigma_1^g = \sigma_1, \sigma_{n-1}^g = \sigma_\ell, \sigma_{n-3}^g = \sigma_{n-1}$. Since σ_0, σ_ℓ are adjacent, we know that σ_0^g and σ_ℓ^g are adjacent, and hence the only possibility for σ_ℓ^g is $\sigma_\ell^g = \sigma_{n-1}$ (note that $\sigma_\ell^g \neq \sigma_1^g = \sigma_1, \sigma_\ell^g \neq \sigma_{n-1}^g = \sigma_\ell$). But $\sigma_{n-3}^g = \sigma_{n-1}$ as mentioned above, so we get $\sigma_\ell^g = \sigma_{n-3}^g$ and hence $\sigma_\ell = \sigma_{n-3}$. Therefore, $n = 6$ and hence $\Sigma = \mathrm{Cay}(\mathbb{Z}_6, \{1, 3, 5\}) = K_{3,3}$. From the discussion in Example 7.3.1, we then have $\Gamma = 3 \cdot C_6$, $\mathcal{E}$ is either $\mathcal{E}_1$ or $\mathcal{E}_2$, and G is the group $\langle (024), (02)(13), (01)(23)(45) \rangle$. $\square$

7.4 Locally primitive almost covers

In this section, we examine an important special case which was the original motivation for the study in this chapter. Recall that if Γ is a G-locally primitive graph admitting a nontrivial G-invariant partition $\mathcal{B}$ of block size $v = k + 1 \geq 3$ such that $\Gamma_{\mathcal{B}}$ is connected, then $\mathcal{D}(B)$ contains no repeated blocks (Corollary 3.3.1) and $\Gamma_{\mathcal{B}}$ is almost covered by Γ (Corollary 3.2.1). By Theorem 5.2.3, $\Gamma = \mathrm{Arc}_\Delta(\Sigma)$ for some self-paired G-orbit Δ on 3-arcs of $\Sigma := \Gamma_{\mathcal{B}}$, and $\mathcal{B}$ is identical with $\mathcal{B}(\Sigma)$ (see the comments before Theorem 5.2.2). Since Γ is G-locally primitive, from Corollary 4.3.1, G_B is 2-primitive on B and $\Sigma(B)$ (this result also follows from Lemma

7.1.2(a) and (b)). If in addition girth$(\Sigma) = 3$ (that is, $\Sigma \cong K_{v+1}$, see Lemma 6.1.1), then G is 3-primitive on $\mathcal{B}$ and the argument in the proof of [43, Theorem 5.4] from (Line, Page) = (25, 534) to (12, 535) shows that we get the possibilities for (Γ, G) listed in part (a) and the second half of part (b) of [43, Theorem 5.4]. (It also comes from the classification of 3-primitive groups and the discussion in Example 7.2.1.) However, in the general case where girth$(\Sigma) \geq 4$, the argument in [43, lines 33-41, pp. 534] should be modified since the block D therein is not adjacent to C. In this case, as mentioned in Theorem 7.0.2(b), Σ is a near n-gonal graph with $n \geq 4$ and n even. Moreover, $G_B^{\Sigma(B)}$ is 2-primitive. Hence if basic cycles of Σ have chords, then by Lemma 7.1.3(e), G_B is sharply 2-primitive on $\Sigma(B)$. Hence G_B is also sharply 2-primitive on B, and so, v is a prime power and, for $\alpha \in B$, $G_\alpha^{B\setminus\{\alpha\}} = \mathbb{Z}_{v-1}$ with $v-1$ a prime. Hence either $v = 3$, or $v = 2^p$ for a prime p with $q = 2^p - 1$ a Mersenne prime. In the former case Proposition 7.3.1 implies that $\Sigma = K_{3,3}$, $\Gamma = 3 \cdot C_6$, and G and $\mathcal{E}$ are as in Example 7.3.1. In the latter case $K := \{g \in G_B^B : g = 1 \text{ or } g \text{ fixes no vertex in } B\}$ is a regular normal elementary abelian subgroup of G_B^B ([26, Theorem 3.4B, pp.88]) and so $G_B^B = (\mathbb{Z}_2)^p.\mathbb{Z}_q$. Theorems 5.2.3 and 7.3.1 and the argument above imply the following corollary, which is an amended form of [43, Theorem 5.4].

Corollary 7.4.1 *Suppose that Γ is a G-locally primitive graph admitting a nontrivial G-invariant partition $\mathcal{B}$ of block size $v = k+1 \geq 3$ such that $\Gamma_{\mathcal{B}}$ is connected. Then $\Gamma_{\mathcal{B}}$ is a $(G,2)$-arc transitive graph of valency v and is almost covered by Γ, the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent and 2-primitive, and the following* (a)-(b) *hold.*

(a) *If $\Gamma_{\mathcal{B}} \cong K_{v+1}$, then either*

  (i) $\Gamma \cong (v+1)\cdot K_v$ *and G is one of the following:* S_{v+1} *($v \geq 3$)*, A_{v+1} *($v \geq 4$)*, M_{v+1} *($v = 10, 11, 22, 23$)*, M_{11} *($v = 11$)*, $\mathrm{PGL}(2, 2^p)$ *($v = 2^p$ with $2^p - 1$ a Mersenne prime), or*

  (ii) $\Gamma \cong \mathrm{CR}(3; 2, 1) = 3 \cdot C_4$ *and* $G = \mathrm{PGL}(2,3)$ *($v = 3$), or*

  (iii) $\Gamma \cong \mathrm{CR}(2^p; x, n(x))$ *and* $G = \mathrm{PGL}(2, 2^p)$ *($v = 2^p$) for some* $x \in \mathrm{GF}(2^p) \setminus \{0, 1\}$ *with $2^p - 1$ a Mersenne prime.*

(b) *If* $\Gamma_{\mathcal{B}} \not\cong K_{v+1}$*, then for some even integer* $n \geq 4$*,* $\Gamma_{\mathcal{B}}$ *is a near* n*-gonal graph with respect to a certain* G*-orbit* $\mathcal{E}$ *on* n*-cycles of* $\Gamma_{\mathcal{B}}$*, and* $\Gamma \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ *for* $\Delta := \mathrm{Arc}_3(\Gamma_{\mathcal{B}}, \mathcal{E})$*. Moreover, each basic cycle of* $(\Gamma_{\mathcal{B}}, \mathcal{E})$ *is chordless unless* G_B^B *is sharply* 2*-primitive and either*

(i) $v = 3$*,* $\Gamma_{\mathcal{B}} \cong K_{3,3}$*,* $\Gamma \cong 3 \cdot C_6$*, and* G *and* $\mathcal{E}$ *are as in Example 7.3.1, or*

(ii) $G_B^B = (\mathbb{Z}_2)^p.\mathbb{Z}_q$ *and* $v = 2^p$ *with* p *a prime and* $q = 2^p - 1$ *a Mersenne prime.*

The smallest v in part (b)(ii) above is $v = 2^2 = 4$. In this case we have $G_B^B = (\mathbb{Z}_2)^2.\mathbb{Z}_3$ and a similar argument as in the proof of Proposition 7.3.1 shows that, if the basic cycles of $(\Gamma_{\mathcal{B}}, \mathcal{E})$ have chords, then the subgraph induced by the vertex set of each basic cycle is isomorphic to the circulant $\mathrm{Cay}(\mathbb{Z}_n, S)$ for $S = \{1, n/2, n-1\}$.

7.5 Two-arc transitive near-polygonal graphs

Let us review briefly the group-theoretic method for constructing 2-arc transitive graphs. Let G be a finite group. A subgroup H of G is said to be *core-free* if its core in G (see Example 2.1.2 for the definition) is equal to the identity, that is, $\bigcap_{g \in G} H^g = 1$. For such a subgroup H and for a 2-element g of G with $g \notin N_G(H)$ (where $N_G(H)$ is the normalizer of H in G), define $\Gamma(G, H, HgH) = (V^*, E^*)$ to be the graph such that

$$V^* := [G : H] = \{Hx : x \in G\},\ E^* := \{\{Hx, Hy\} : xy^{-1} \in HgH\}.$$

Sabidussi [76] (and others, see e.g. [56]) proved that $\Gamma(G, H, HgH)$ is a G-symmetric graph, and that any G-symmetric graph is isomorphic to $\Gamma(G, H, HgH)$ for a certain core-free subgroup H and 2-element g of G. Moreover, the graph $\Gamma(G, H, HgH)$ is connected if and only if $\langle H, g \rangle = G$. By refining this classic result, Fang and Praeger [33, Theorem 2.1] (see also [70, Theorem 11.1]) gave the following construction of $(G, 2)$-arc transitive graphs.

Theorem 7.5.1 ([33, Theorem 2.1]) *Let* G *be a finite group with a core-free subgroup* H *and a* 2*-element* g*. Then the graph* $\Gamma(G, H, HgH)$ *is a finite, connected,*

$(G, 2)$-arc transitive graph with G transitive on vertices (acting by right multiplication as defined in Example 2.1.2) if and only if

$$g \notin N_G(H), g^2 \in H, \langle H, g\rangle = G,$$

and the action of H on $[H : H \cap H^g]$ by right multiplication is doubly transitive.

Let $\mathcal{F}$ denote the class of G-symmetric graphs Γ such that $k = v - 1 \geq 2$, $\Gamma_{\mathcal{B}}$ is connected and $(G, 2)$-arc transitive, and $\Gamma_{\mathcal{B}} \ncong K_{v+1}$. Theorem 7.0.2(b) shows that the construction of the graphs in $\mathcal{F}$ can be reduced to that of $(G, 2)$-arc transitive near n-gonal graphs $(\Sigma, \mathcal{E})$ of valency $v \geq 3$ such that $\mathcal{E}$ is a G-orbit on n-cycles of Σ, for an even integer $n \geq 4$. In view of Theorem 7.5.1 above, in constructing the graphs in $\mathcal{F}$ we can start, at least theoretically, from $(G, 2)$-arc transitive graphs Σ. To make this approach effective, we need to know when such a graph Σ is a near-polygonal graph $(\Sigma, \mathcal{E})$ with $\mathcal{E}$ as above. The purpose of this section is to give the following necessary and sufficient conditions for a 2-arc transitive graph to be a near-polygonal graph.

Theorem 7.5.2 *Suppose that Σ is a connected $(G, 2)$-arc transitive graph with* $\mathrm{girth}(\Sigma) \geq 4$, *where $G \leq \mathrm{Aut}(\Sigma)$. Let $(\sigma, \tau, \varepsilon)$ be a 2-arc of Σ and set $H = G_{\sigma\tau\varepsilon}$. Then the following conditions* (a)-(c) *are equivalent:*

(a) *there exist an integer $n \geq 4$ and a G-orbit $\mathcal{E}$ on n-cycles of Σ such that $(\Sigma, \mathcal{E})$ is a near n-gonal graph;*

(b) *H fixes at least one vertex in $\Sigma(\varepsilon) \setminus \{\tau\}$;*

(c) *there exists $g \in N_G(H)$ such that $(\sigma, \tau)^g = (\tau, \varepsilon)$.*

Proof (a) $\Rightarrow$ (b) Suppose that $(\Sigma, \mathcal{E})$ is a near n-gonal graph for a G-orbit $\mathcal{E}$ on n-cycles of Σ, where $n \geq 4$. Let $C(\sigma, \tau, \varepsilon) = (\sigma, \tau, \varepsilon, \eta, \ldots, \sigma)$ be the basic cycle containing the 2-arc $(\sigma, \tau, \varepsilon)$. Then we have $\eta \in \Sigma(\varepsilon) \setminus \{\tau\}$. We claim that η is fixed by H. Suppose otherwise, say $\eta^g \neq \eta$ for some $g \in H$, then $(C(\sigma, \tau, \varepsilon))^g = (\sigma, \tau, \varepsilon, \eta^g, \ldots, \sigma)$ is a basic cycle containing $(\sigma, \tau, \varepsilon)$ which is different from $C(\sigma, \tau, \varepsilon)$. This contradicts with the uniqueness of the basic cycle containing a given 2-arc, and hence (b) holds.

(b) $\Rightarrow$ (c) Suppose H fixes $\eta \in \Sigma(\varepsilon) \setminus \{\tau\}$. Then we have $H \leq G_{\tau\varepsilon\eta}$. Since Σ is $(G, 2)$-arc transitive, there exists $g \in G$ such that $(\sigma, \tau, \varepsilon)^g = (\tau, \varepsilon, \eta)$ and hence $G_{\tau\varepsilon\eta} = H^g$. Therefore, $H^g = H$ and $g \in N_G(H)$.

(c) $\Rightarrow$ (a) Suppose that there exists $g \in N_G(H)$ such that $(\sigma, \tau)^g = (\tau, \varepsilon)$. Set $\eta := \varepsilon^g$. Then $\eta \in \Sigma(\varepsilon) \setminus \{\tau\}$, $(\sigma, \tau, \varepsilon)^g = (\tau, \varepsilon, \eta)$ and hence $G_{\tau\varepsilon\eta} = H^g = H$. Set $\sigma_0 = \sigma, \sigma_1 = \tau, \sigma_2 = \varepsilon$ and $\sigma_3 = \eta$, and set $\sigma_4 = \sigma_3^g$. Then $\sigma_4 \in \Sigma(\sigma_3) \setminus \{\sigma_2\}$ and $G_{\sigma_2\sigma_3\sigma_4} = (G_{\sigma_1\sigma_2\sigma_3})^g = H^g = H$. Now set $\sigma_5 = \sigma_4^g$, then similarly $\sigma_5 \in \Sigma(\sigma_4) \setminus \{\sigma_3\}$ and $G_{\sigma_3\sigma_4\sigma_5} = (G_{\sigma_2\sigma_3\sigma_4})^g = H^g = H$. Continuing this process, we get inductively a sequence $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \ldots$ of vertices of Σ with the following properties:

(1) $\sigma_i = \sigma_{i-1}^g$ for all $i \geq 1$, and hence $\sigma_{i+1} \in \Sigma(\sigma_i) \setminus \{\sigma_{i-1}\}$ for $i \geq 1$ and $\sigma_i = \sigma_0^{g^i}$ for $i \geq 0$; and

(2) $G_{\sigma_{i-1}\sigma_i\sigma_{i+1}} = H$ for all $i \geq 1$.

Since we have finitely many vertices in Σ, this sequence will eventually contain repeated terms. Suppose σ_n is the first vertex in this sequence which coincides with one of the preceding vertices. Without loss of generality we may suppose that σ_n coincides with σ_0 for if $\sigma_n = \sigma_i$ for some $i \geq 1$ then we can begin with σ_i and relabel the vertices in the sequence. Thus, we get an n-cycle $J := (\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots, \sigma_{n-1}, \sigma_0)$ (note that $n \geq 4$ as girth$(\Sigma) \geq 4$). Let $\mathcal{E}$ denote the G-orbit on n-cycles of Σ containing J. In the following we will prove that each 2-arc of Σ is contained in exactly one of the "basic cycles" in $\mathcal{E}$ and hence $(\Sigma, \mathcal{E})$ is indeed a near n-gonal graph.

By the $(G, 2)$-arc transitivity of Σ, it is clear that each 2-arc $(\sigma', \tau', \varepsilon')$ of Σ is contained in at least one member J^x of $\mathcal{E}$, where $x \in G$ is such that $(\sigma', \tau', \varepsilon') = (\sigma, \tau, \varepsilon)^x$. So it suffices to show that if two members of $\mathcal{E}$ have a 2-arc in common then they are identical; or, equivalently, if J^x and J have a 2-arc in common then they are identical.

Suppose then that J^x and J have a 2-arc in common for some $x \in G$. Note that, for each $i \geq 0$, g^i maps each vertex σ_j to σ_{j+i} and so $\langle g \rangle$ leaves J invariant (subscripts modulo n here and in the rest of this proof). So, replacing J^x by J^{xg^i} for some i if necessary, we may suppose without loss of generality that $(\sigma_0, \sigma_1, \sigma_2)$ is a common 2-arc of J^x and J. Then $(\sigma_0, \sigma_1, \sigma_2) \in J^x$ implies that $(\sigma_0, \sigma_1, \sigma_2) = (\sigma_{i-1}, \sigma_i, \sigma_{i+1})^x$ for some $1 \leq i \leq n$. Thus, $(\sigma_0, \sigma_1, \sigma_2) = (\sigma_0, \sigma_1, \sigma_2)^{g^{i-1}x}$ and hence $g^{i-1}x \in H$. From the properties (1)-(2) above, we then have $\sigma_{j+i-1}^x = \sigma_j^{g^{i-1}x} = \sigma_j$ for each vertex σ_j on J. That is, $\sigma_j^x = \sigma_{j-i+1}$ for each j and hence $J^x = J$. Thus, we have proved that each 2-arc of Σ is contained in exactly one member of $\mathcal{E}$, and so $(\Sigma, \mathcal{E})$ is a near n-gonal graph. $\square$

Chapter 8

Flag graphs: A general construction

He who learns but does not think is lost; he who thinks but does not learn is in great danger.

Confucius (551-479 B.C.), LUN YÜ [THE ANALECTS] 2:15

In this chapter we temporarily leave the case where $v = k + 1 \geq 3$. Instead we will give a natural construction of a large class of symmetric graphs, namely the class of G-symmetric graphs Γ such that the dual 1-design of $\mathcal{D}(B)$ contains no repeated blocks. We will prove that up to isomorphism this construction produces all such graphs, and in particular that Γ can be reconstructed from the quotient $\Gamma_{\mathcal{B}}$ and the action of G on $\mathcal{B}$. The study in this chapter reveals a close connection between symmetric graphs and 1-designs. In fact, the ingredients for our construction are a G-point-transitive and G-block-transitive 1-design $\mathcal{D}$, a G-orbit Θ on the flags of $\mathcal{D}$ satisfying some natural conditions, and a certain self-paired G-orbit Ψ on the ordered pairs of distinct flags of $\mathcal{D}$. Given these, the constructed graph, called a flag graph, is defined to have vertex set Θ and arc set Ψ. The construction was initially introduced in the course of our attempt to construct symmetric graphs with $v = k + 1 \geq 3$. However, for convenience of narration we will first present in this chapter the general construction. The utility of this construction is not fully explored in this thesis: We just apply it to characterizing two large classes of symmetric graphs, namely the classes of symmetric graphs with $k = 1$ and $v = k + 1 \geq 3$, respectively. It is hoped that some interesting symmetric graphs can be constructed and characterized by using this rather general approach.

8.1 Preliminary

Let Γ be a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$. We use $\mathcal{D}^*(B) := (\Gamma_{\mathcal{B}}(B), B, \mathrm{I}^*)$ to denote the dual 1-design of $\mathcal{D}(B) = (B, \Gamma_{\mathcal{B}}(B), \mathrm{I})$. So the "points" of $\mathcal{D}^*(B)$ are those blocks of $\mathcal{B}$ which are adjacent to B, and the "blocks" of $\mathcal{D}^*(B)$ are the points of B. Note that the trace of the "block" $\alpha \in B$ of $\mathcal{D}^*(B)$ is the subset $\Gamma_{\mathcal{B}}(\alpha)$ of $\Gamma_{\mathcal{B}}(B)$, and that $\mathcal{D}^*(B)$ has block size r, where $r = |\Gamma_{\mathcal{B}}(\alpha)|$ as in Section 3.2. By Lemma 3.2.5(b), G_B induces a point-, block- and flag-transitive group of automorphisms of $\mathcal{D}^*(B)$. As a vital observation, we notice that $\mathcal{D}^*(B)$ can be "expanded" to the following 1-design which has point set $\mathcal{B}$ and admits G as a point- and block-transitive group of automorphisms. For each $\alpha \in V(\Gamma)$, set

$$\mathcal{L}(\alpha) := \{B(\alpha)\} \cup \Gamma_{\mathcal{B}}(\alpha). \tag{8.1}$$

We should warn that, for distinct vertices α, β of Γ, it may happen that $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$. (This might be true even for distinct vertices α, β in the same block of $\mathcal{B}$.) Denote by $\mathbf{L}$ the set of all $\mathcal{L}(\alpha)$, $\alpha \in V(\Gamma)$, with repeated ones identified. Note that $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$ if and only if $\mathcal{L}(\alpha^g) = \mathcal{L}(\beta^g)$ for any $g \in G$. Therefore, $(\mathcal{L}(\alpha))^g := \mathcal{L}(\alpha^g)$, for $\alpha \in V(\Gamma)$ and $g \in G$, defines an action of G on $\mathbf{L}$. We define

$$\mathcal{D}(\Gamma, \mathcal{B}) := (\mathcal{B}, \mathbf{L})$$

to be the incidence structure where B is incident with $\mathcal{L}(\alpha)$ if and only if $B \in \mathcal{L}(\alpha)$.

Lemma 8.1.1 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$. Then*

(a) $\mathcal{D}(\Gamma, \mathcal{B})$ *is a* 1*-design with block size* $r+1$*; and*

(b) $\mathcal{D}(\Gamma, \mathcal{B})$ *admits G as a group of automorphisms, and G is transitive on the points and the blocks of* $\mathcal{D}(\Gamma, \mathcal{B})$.

Proof It is clear that G is transitive on $\mathcal{B}$ and on $\mathbf{L}$, and that G preserves the incidence relation of $\mathcal{D}(\Gamma, \mathcal{B})$. So G induces a group of automorphisms of $\mathcal{D}(\Gamma, \mathcal{B})$, and each $B \in \mathcal{B}$ is incident with the same number of elements of $\mathbf{L}$. Clearly, $\mathcal{D}(\Gamma, \mathcal{B})$ has block size $r+1$. $\square$

We notice that in a lot of cases (see Examples 8.1.1 and 8.1.2 below) the 1-design $\mathcal{D}^*(B)$ contains no repeated blocks, and the purpose of this chapter is to give a construction of all G-symmetric graphs with this property. (Recall that two "blocks" α, β of $\mathcal{D}^*(B)$ are said to be repeated if $\Gamma_{\mathcal{B}}(\alpha) = \Gamma_{\mathcal{B}}(\beta)$.) The feasibility of such a construction lies on the observation that in this case the flags $(B(\alpha), \mathcal{L}(\alpha))$ of $\mathcal{D}(\Gamma, \mathcal{B})$, for $\alpha \in V(\Gamma)$, are pairwise distinct, or equivalently, for each $B \in \mathcal{B}$ the members of

$$\mathbf{L}(B) := \{\mathcal{L}(\alpha) : \alpha \in B\}$$

are pairwise distinct. Therefore, in this case $V(\Gamma)$ can be identified with the set

$$\Theta(\Gamma, \mathcal{B}) := \{(B(\alpha), \mathcal{L}(\alpha)) : \alpha \in V(\Gamma)\}$$

of flags of $\mathcal{D}(\Gamma, \mathcal{B})$ by $\alpha \mapsto (B(\alpha), \mathcal{L}(\alpha))$. We denote by

$$\mathbf{E}(B) := \{\Gamma_{\mathcal{B}}(\alpha) : \alpha \in B\} \tag{8.2}$$

the set of distinct traces of the "blocks" of $\mathcal{D}^*(B)$. Denote by $G_{B,\Gamma_{\mathcal{B}}(\alpha)}$ and $G_{B,\mathcal{L}(\alpha)}$ the setwise stabilizers of $\Gamma_{\mathcal{B}}(\alpha)$ and $\mathcal{L}(\alpha)$ in G_B, respectively. Note that G_B preserves $\mathbf{L}(B)$ and hence induces an action on $\mathbf{L}(B)$.

Lemma 8.1.2 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$. Then $\Theta(\Gamma, \mathcal{B})$ is a G-orbit on the flags of $\mathcal{D}(\Gamma, \mathcal{B})$. The flags $(B(\alpha), \mathcal{L}(\alpha))$ for $\alpha \in V(\Gamma)$ are pairwise distinct if and only if $\mathcal{D}^*(B)$ contains no repeated blocks, and in this case the following* (a)-(d) *hold (where $B \in \mathcal{B}$ in* (c) *and* (d)*).*

(a) *The mapping $\rho : \alpha \mapsto (B(\alpha), \mathcal{L}(\alpha))$, for $\alpha \in V(\Gamma)$, defines a bijection from $V(\Gamma)$ to $\Theta(\Gamma, \mathcal{B})$.*

(b) *The actions of G on $V(\Gamma)$ and on $\Theta(\Gamma, \mathcal{B})$ are permutationally equivalent with respect to the bijection ρ in (a).*

(c) *The action of G_B on B is permutationally equivalent to the actions of G_B on $\mathbf{E}(B)$, $\mathbf{L}(B)$ with respect to the bijections defined by $\alpha \mapsto \Gamma_{\mathcal{B}}(\alpha)$, $\alpha \mapsto \mathcal{L}(\alpha)$, for $\alpha \in B$, respectively. Hence we have $G_{B,\Gamma_{\mathcal{B}}(\alpha)} = G_{B,\mathcal{L}(\alpha)} = G_\alpha$.*

(d) *$G_{B,\mathcal{L}(\alpha)}$ is transitive on $\Gamma_{\mathcal{B}}(\alpha)$, for $\alpha \in B$.*

Proof Since G is transitive on $V(\Gamma)$, it is easy to see that $\Theta(\Gamma, \mathcal{B})$ is a G-orbit on the flags of $\mathcal{D}(\Gamma, \mathcal{B})$. Clearly, the flags $(B(\alpha), \mathcal{L}(\alpha))$, $(B(\beta), \mathcal{L}(\beta))$ in $\Theta(\Gamma, \mathcal{B})$ corresponding to two distinct vertices α, β are identical if and only if $B(\alpha) = B(\beta)$ and $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$, that is, if and only if α, β are in the same block of $\mathcal{B}$ and $\Gamma_{\mathcal{B}}(\alpha) = \Gamma_{\mathcal{B}}(\beta)$. In other words, the flags $(B(\alpha), \mathcal{L}(\alpha))$ for $\alpha \in V(\Gamma)$ are pairwise distinct if and only if $\mathcal{D}^*(B)$ contains no repeated blocks. In this case it is easily checked that both (a) and (b) are true. The truth of (c) follows from a routine argument. From Lemma 3.2.5(c) it follows that G_α is transitive on $\Gamma_{\mathcal{B}}(\alpha)$, that is, $G_{B,\mathcal{L}(\alpha)}$ is transitive on $\Gamma_{\mathcal{B}}(\alpha)$. Thus (d) is proved. $\Box$

Remark 8.1.1 Under the assumption that $\mathcal{D}^*(B)$ contains no repeated blocks, $\mathcal{D}(\Gamma, \mathcal{B})$ is an extension of $\mathcal{D}^*(B)$ if and only if $\Gamma_{\mathcal{B}}$ is a complete graph. (See Section 2.3 for the definition of an extension of a design.) In this case $\mathcal{D}(\Gamma, \mathcal{B})$ is a 2-design with G acting doubly transitively on its points.

We conclude this section by giving examples of two large classes of symmetric graphs such that $\mathcal{D}^*(B)$ contains no repeated blocks.

Example 8.1.1 Suppose Γ is a G-symmetric graph such that $k = 1$, that is, $\Gamma[B, C] \cong K_2$ for adjacent blocks B, C of $\mathcal{B}$. Then clearly we have $\Gamma_{\mathcal{B}}(\alpha) \cap \Gamma_{\mathcal{B}}(\beta) = \emptyset$ for distinct vertices α, β in the same block of $\mathcal{B}$. In particular, $\mathcal{D}^*(B)$ contains no repeated blocks. Note that in this case the block size r of $\mathcal{D}^*(B)$ is equal to the valency $\mathrm{val}(\Gamma)$ of Γ. Moreover, we have $\mathrm{val}(\Gamma_{\mathcal{B}}) = vr$.

We remind the reader that symmetric graphs satisfying $k = 1$ have appeared in Sections 4.1, 4.2 and Theorem 5.1.3(a)(b).

Example 8.1.2 Suppose Γ is a G-symmetric graph such that $k < v$ and G_B is doubly transitive on B. Then $\mathcal{D}^*(B)$ must contain no repeated blocks. In fact, suppose otherwise, then since G_B is doubly transitive on the blocks of $\mathcal{D}^*(B)$ we would have $\Gamma_{\mathcal{B}}(\alpha) = \Gamma_{\mathcal{B}}(\beta)$ for all $\alpha, \beta \in B$. This implies $k = v$ and thus contradicts our assumption.

8.2 Flag graphs

For simplicity we assume without mentioning explicitly that the 1*-designs used for our constructions in this chapter have no repeated blocks.* Let $\mathcal{D} = (V, \mathbf{B})$ be a 1-design. As usual we may identify each block $L \in \mathbf{B}$ with the subset of V consisting of the points incident with L. Let Θ be a set of flags of $\mathcal{D}$, and Ψ a subset of the set $\Theta^{(2)}$ of ordered pairs of distinct flags in Θ. If Ψ is *self-paired*, that is, $((\sigma, L), (\tau, N)) \in \Psi$ implies $((\tau, N), (\sigma, L)) \in \Psi$, then we define the *flag graph* of $\mathcal{D}$ with respect to (Θ, Ψ), denoted by $\Gamma(\mathcal{D}, \Theta, \Psi)$, to be the graph with vertex set Θ in which two "vertices" $(\sigma, L), (\tau, N) \in \Theta$ are adjacent if and only if $((\sigma, L), (\tau, N)) \in \Psi$. The self-parity of Ψ guarantees that this graph is well-defined. For a given point σ of $\mathcal{D}$, we denote by $\Theta(\sigma)$ the set of flags in Θ with point entry σ. Let G be a group of automorphisms of $\mathcal{D}$. If Θ is a G-orbit on the flags of $\mathcal{D}$, then $\Theta(\sigma)$ is a G_σ-orbit on the flags of $\mathcal{D}$ with point entry σ. In this case, $\Gamma(\mathcal{D}, \Theta, \Psi)$ is G-vertex-transitive and its vertex set Θ admits

$$\mathcal{B}(\Theta) := \{\Theta(\sigma) : \sigma \in V\} \tag{8.3}$$

as a natural G-invariant partition. If furthermore Ψ is a G-orbit on $\Theta^{(2)}$ (under the induced action), then $\Gamma(\mathcal{D}, \Theta, \Psi)$ is G-symmetric. For a flag (σ, L) of $\mathcal{D}$, we use $G_{\sigma,L}$ to denote the subgroup of G fixing (σ, L), that is, the subgroup of G fixing σ and L setwise. For the purpose of this chapter, the G-orbit Θ will be required to satisfy some additional properties.

Definition 8.2.1 Let $\mathcal{D}$ be a G-point-transitive and G-block-transitive 1-design with block size at least 2. Let σ be a point of $\mathcal{D}$. A G-orbit Θ on the flags of $\mathcal{D}$ is said to be *feasible* if

(a) $|\Theta(\sigma)| \geq 2$; and

(b) $G_{\sigma,L}$ is transitive on $L \setminus \{\sigma\}$, for some (and hence all) $(\sigma, L) \in \Theta$.

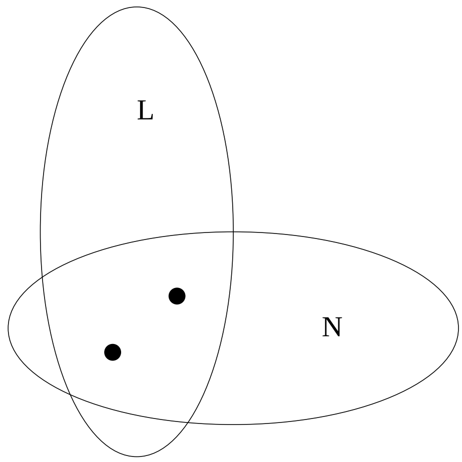


FIGURE 5 A compatible ordered pair of flags

Since G is transitive on the points of $\mathcal{D}$, the validity of (a), (b) above does not depend on the choice of the point σ. Let Θ be a feasible G-orbit on the flags of $\mathcal{D}$. We say that $((\sigma, L), (\tau, N)) \in \Theta^{(2)}$ is *compatible* with Θ if $\sigma \neq \tau$ and $\sigma, \tau \in L \cap N$ (see Figure 5). In the following we use $\mathrm{C}(\mathcal{D}, \Theta)$ to denote the set of those members of $\Theta^{(2)}$ which are compatible with Θ. One can easily see that $\mathrm{C}(\mathcal{D}, \Theta)$ is a G-invariant subset of $\Theta^{(2)}$. In this chapter we will consider only those flag graphs $\Gamma(\mathcal{D}, \Theta, \Psi)$ such that $\mathcal{D}$ and G are as in Definition 8.2.1, Θ is a feasible G-orbit on the flags of $\mathcal{D}$, and Ψ is a self-paired G-orbit on $\mathrm{C}(\mathcal{D}, \Theta)$ (that is, Ψ is a self-paired G-orbit on $\Theta^{(2)}$ whose members are all compatible with Θ). For such a Ψ, either $L = N$ for all $((\sigma, L), (\tau, N)) \in \Psi$, or $L \neq N$ for all $((\sigma, L), (\tau, N)) \in \Psi$. For convenience we will call such a graph the *G-flag graph* of $\mathcal{D}$ with respect to (Θ, Ψ). In the following we show that these graphs can represent all G-symmetric graphs admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $\mathcal{D}^*(B)$ contains no repeated blocks.

Theorem 8.2.1 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $\mathcal{D}^*(B)$ contains no repeated blocks. Let r be the block size of $\mathcal{D}^*(B)$, that is, $r = |\Gamma_{\mathcal{B}}(\alpha)|$. Then $\Gamma \cong \Gamma(\mathcal{D}, \Theta, \Psi)$ for a certain G-point-transitive and G-block-transitive 1-design $\mathcal{D}$ with block size $r+1$, a certain feasible G-orbit Θ on the flags of $\mathcal{D}$, and a certain self-paired G-orbit Ψ on $\mathrm{C}(\mathcal{D}, \Theta)$.*

Conversely, for any G-point-transitive and G-block-transitive 1-design $\mathcal{D}$ with no repeated blocks and with block size $r+1$, any feasible G-orbit Θ on the flags of $\mathcal{D}$, *and any self-paired G-orbit Ψ on* $\mathrm{C}(\mathcal{D}, \Theta)$*, the graph* $\Gamma := \Gamma(\mathcal{D}, \Theta, \Psi)$*, group* G*, partition* $\mathcal{B} := \mathcal{B}(\Theta)$ *and integer* r *satisfy all the conditions above.*

Proof Suppose that Γ, G, $\mathcal{B}$ and r are as in the first part of the theorem. Then by Lemma 8.1.1, $\mathcal{D} := \mathcal{D}(\Gamma, \mathcal{B})$ is a G-point-transitive and G-block-transitive 1-design with block size $r+1$. From Lemma 8.1.2, $\Theta := \Theta(\Gamma, \mathcal{B})$ is a G-orbit on the flags of $\mathcal{D}$, and the mapping $\rho : \gamma \mapsto (B(\gamma), \mathcal{L}(\gamma))$, for $\gamma \in V(\Gamma)$, is a bijection from $V(\Gamma)$ to Θ. In particular, we have $|\Theta(B)| = |B| \geq 2$. For $(B, \mathcal{L}) \in \Theta(B)$, say $\mathcal{L} = \mathcal{L}(\alpha)$ for some $\alpha \in B$, we have $\mathcal{L} \setminus \{B\} = \Gamma_{\mathcal{B}}(\alpha)$. So it follows from Lemma 8.1.2(d) that $G_{B,\mathcal{L}}$ is transitive on $\mathcal{L} \setminus \{B\}$. Therefore, Θ is a feasible G-orbit on the flags of $\mathcal{D}$.

Clearly, for each arc (α, β) of Γ, we have $B(\alpha) \neq B(\beta)$ and $B(\alpha), B(\beta) \in \mathcal{L}(\alpha) \cap \mathcal{L}(\beta)$. Therefore, setting

$$\Psi := \{((B(\alpha), \mathcal{L}(\alpha)), (B(\beta), \mathcal{L}(\beta))) : (\alpha, \beta) \in \mathrm{Arc}(\Gamma)\},$$

then $\Psi \subseteq \mathrm{C}(\mathcal{D}, \Theta)$ and Ψ is self-paired. By Lemma 8.1.2(b), the actions of G on $V(\Gamma)$ and Θ are permutationally equivalent with respect to the bijection ρ defined above. Since Γ is G-symmetric, this implies that $\Psi = ((B(\alpha), \mathcal{L}(\alpha)), (B(\beta), \mathcal{L}(\beta)))^G$, for a fixed arc (α, β) of Γ. Hence Ψ is a self-paired G-orbit on $\mathrm{C}(\mathcal{D}, \Theta)$. One can easily check that the bijection ρ defines an isomorphism from Γ to the G-flag graph $\Gamma(\mathcal{D}, \Theta, \Psi)$, and hence the first part of Theorem 8.2.1 is proved.

Suppose conversely that $\mathcal{D}, G, \Theta, \Psi$ and r are as in the second part of Theorem 8.2.1. Let $\Gamma := \Gamma(\mathcal{D}, \Theta, \Psi)$, and let $\mathcal{B} := \mathcal{B}(\Theta)$ be as defined in (8.3). Then it follows from the definition that Γ is a G-symmetric graph with vertex set Θ, and that $\mathcal{B}$ is a nontrivial G-invariant partition of Θ with block size $|\Theta(\sigma)| \geq 2$, where σ is a point of $\mathcal{D}$. To complete the proof, we need to show that the block size of $\mathcal{D}^*(\Theta(\sigma))$ is equal to r and that $\mathcal{D}^*(\Theta(\sigma))$ contains no repeated blocks.

Let $\Theta(\sigma), \Theta(\tau)$ be adjacent blocks of $\mathcal{B}$. Then there exist $(\sigma, L) \in \Theta(\sigma)$ and $(\tau, N) \in \Theta(\tau)$ such that $(\sigma, L), (\tau, N)$ are adjacent in Γ, that is, $((\sigma, L), (\tau, N)) \in \Psi$. So we have $\sigma \neq \tau$ and $\sigma, \tau \in L \cap N$ by the compatibility of the members of Ψ. Since Θ is feasible, it follows from (b) in Definition 8.2.1 that, for any $\tau_1 \in L \setminus \{\sigma\}$, there exists $g \in G_{\sigma,L}$ such that $\tau^g = \tau_1$. Setting $N_1 := N^g$, then we have $(\tau_1, N_1) = (\tau, N)^g \in \Theta$. Since g fixes σ, it fixes $\Theta(\sigma)$ setwise, and moreover $\sigma \in N$ implies $\sigma \in N_1$. Also, $\sigma \neq \tau$ implies that $\sigma = \sigma^g \neq \tau^g = \tau_1$. Thus we have $((\sigma, L), (\tau_1, N_1)) = ((\sigma, L), (\tau, N))^g \in$

Ψ, that is, (σ, L) and (τ_1, N_1) are adjacent in Γ. Hence $\Theta(\tau_1) \in \Gamma_{\mathcal{B}}((\sigma, L))$ provided that $\tau_1 \in L \setminus \{\sigma\}$. We now prove that the converse of this is true as well. In fact, suppose that $\Theta(\delta) \in \Gamma_{\mathcal{B}}((\sigma, L))$. Then there exists $(\delta, M) \in \Theta(\delta)$ such that (σ, L) and (δ, M) are adjacent in Γ. So $((\sigma, L), (\delta, M)) \in \Psi$ and hence there exists $h \in G$ such that $((\sigma, L), (\tau, N))^h = ((\sigma, L), (\delta, M))$. Thus we have $h \in G_{\sigma,L}$, $\tau^h = \delta$ and $N^h = M$. Since h fixes σ and fixes L setwise, and since $\tau \in L \setminus \{\sigma\}$, we have $\delta = \tau^h \in L \setminus \{\sigma\}$. So we have proved that $\Gamma_{\mathcal{B}}((\sigma, L)) = \{\Theta(\delta) : \delta \in L \setminus \{\sigma\}\}$, and thus $\mathcal{D}^*(\Theta(\sigma))$ has block size $|L \setminus \{\sigma\}| = r$. Moreover, since $\mathcal{D}$ contains no repeated blocks, we have $L \neq L_1$ for distinct $(\sigma, L), (\sigma, L_1) \in \Theta(\sigma)$. This together with the argument above implies that $\Gamma_{\mathcal{B}}((\sigma, L)) \neq \Gamma_{\mathcal{B}}((\sigma, L_1))$, and hence $\mathcal{D}^*(\Theta(\sigma))$ contains no repeated blocks. □

The special case where in addition $\Gamma_{\mathcal{B}}$ is a complete graph (that is, $\Gamma_{\mathcal{B}} \cong K_{b+1}$) is particularly interesting. Since $\Gamma_{\mathcal{B}}$ is G-symmetric, this case occurs if and only if G is doubly transitive on $\mathcal{B}$. So in this case $\mathcal{D}(\Gamma, \mathcal{B}) = (\mathcal{B}, \mathbf{L})$ is a G-doubly transitive and G-block-transitive 2-$(b+1, r+1, \lambda)$ design, for some integer $\lambda \geq 1$. Conversely, if $\mathcal{D}$ is a G-doubly transitive and G-block-transitive 2-$(b+1, r+1, \lambda)$ design, then for any G-flag graph $\Gamma := \Gamma(\mathcal{D}, \Theta, \Psi)$ of $\mathcal{D}$ we have $\Gamma_{\mathcal{B}(\Theta)} \cong K_{b+1}$. So Theorem 8.2.1 implies the following corollary.

Corollary 8.2.1 *Let $b \geq 2$ and $r \geq 1$ be integers, and let G be a group. Then the following* (a), (b) *are equivalent.*

(a) Γ *is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $\mathcal{D}^*(B)$ contains no repeated blocks and has block size r, and such that $\Gamma_{\mathcal{B}} \cong K_{b+1}$.*

(b) $\Gamma \cong \Gamma(\mathcal{D}, \Theta, \Psi)$, *for a G-doubly transitive and G-block-transitive* 2-$(b+1, r+1, \lambda)$ *design $\mathcal{D}$, a feasible G-orbit Θ on the flags of $\mathcal{D}$, and a self-paired G-orbit Ψ on* $\mathrm{C}(\mathcal{D}, \Theta)$.

Let us consider the graphs Γ in Example 8.1.2 with the additional property that $\Gamma_{\mathcal{B}} \cong K_{b+1}$. From the corollary above, they are all G-flag graphs of some G-doubly transitive 2-designs. So all the graphs appearing in [45, Theorems 1.1 and 1.2] are in fact G-flag graphs. Thus, from the general theory above, the close connection of such graphs with certain doubly transitive 2-designs shown in [45] is not a coincidence.

8.3 Symmetric graphs with $k = 1$

In this section we will study G-symmetric graphs Γ admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $k = 1$, that is, $\Gamma[B, C] \cong K_2$ for adjacent blocks B, C of $\mathcal{B}$. This seemingly trivial case is notoriously difficult to manage, even in the case where in addition $\Gamma_{\mathcal{B}}$ is a complete graph (see [43, Section 4]). The behaviour of such graphs seems to be quite wild, and to the best knowledge of the author there is no useful description of them up to now. In Example 8.1.1 we have shown that in this case $\mathcal{D}^*(B)$ has block size val(Γ) and contains no repeated blocks. Hence, from Theorem 8.2.1, Γ is isomorphic to a G-flag graph of $\mathcal{D} := \mathcal{D}(\Gamma, \mathcal{B})$. In the following we will further characterize Γ as a G-flag graph $\Gamma(\mathcal{D}, \Theta, \Psi)$ with Θ satisfying some additional condition. Using the notation in Section 8.1, we see that in this case $\mathcal{D}^*(B)$ has "blocks" $\Gamma_{\mathcal{B}}(\alpha) = \{C \in \mathcal{B} : \Gamma(C) \cap B(\alpha) = \{\alpha\}\}$, for $\alpha \in B$. Recall that $\mathcal{D}^*(B)$ has "block" set $\mathbf{E}(B) = \{\Gamma_{\mathcal{B}}(\alpha) : \alpha \in B\}$.

Lemma 8.3.1 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $\Gamma[B, C] \cong K_2$ for adjacent blocks B, C of $\mathcal{B}$, and let r be the valency of Γ. Then the following* (a)-(c) *hold.*

(a) $\mathbf{E}(B)$ *is a G_B-invariant partition of $\Gamma_{\mathcal{B}}(B)$.*

(b) *If G is faithful on $V(\Gamma)$, then the induced action of G on $\mathcal{B}$ is faithful.*

(c) *If $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$ holds for some pair of distinct vertices α, β of Γ, then $\Gamma \cong n \cdot K_{r+1}$ for some integer n; and in this case $\mathcal{L}(\gamma) = \mathcal{L}(\delta)$ holds for any vertices γ, δ in the same component of Γ.*

Proof (a) Since there is only one edge of Γ between two adjacent blocks of $\mathcal{B}$, it follows from the definition that $\mathbf{E}(B)$ is a partition of $\Gamma_{\mathcal{B}}(B)$. Suppose that $(\Gamma_{\mathcal{B}}(\alpha))^g \cap \Gamma_{\mathcal{B}}(\beta) \neq \emptyset$ for some $\alpha, \beta \in B$ and $g \in G_B$, say $C^g = D$ for some $C \in \Gamma_{\mathcal{B}}(\alpha)$ and $D \in \Gamma_{\mathcal{B}}(\beta)$. Since α is the unique vertex in B adjacent to a vertex in C and since β is the unique vertex in B adjacent to a vertex in D, $C^g = D$ implies $\alpha^g = \beta$ and hence $(\Gamma_{\mathcal{B}}(\alpha))^g = \Gamma_{\mathcal{B}}(\beta)$. Therefore, $\mathbf{E}(B)$ is a G_B-invariant partition of $\Gamma_{\mathcal{B}}(B)$.

(b) Suppose that $g \in G$ fixes setwise each block of $\mathcal{B}$. Then, for each $B \in \mathcal{B}$ and $\alpha \in B$, g fixes in particular each of the blocks in $\Gamma_{\mathcal{B}}(\alpha)$. So it follows from (a) that g fixes each vertex in B. Since this holds for each $B \in \mathcal{B}$, g fixes each vertex of Γ. So, if G is faithful on $V(\Gamma)$, then $g = 1$ and hence G is faithful on $\mathcal{B}$.

(c) Suppose that $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$ for two distinct vertices $\alpha \in B$ and $\beta \in C$. Then $B \neq C$, $C \in \Gamma_{\mathcal{B}}(\alpha)$ and $B \in \Gamma_{\mathcal{B}}(\beta)$, and in particular B, C are adjacent. Moreover, since there is only one edge between B and C, α, β must be adjacent in Γ. So the transitivity of G_α on $\Gamma(\alpha)$ implies that, for each $\gamma \in \Gamma(\alpha)$, there exists $g \in G_\alpha$ such that $\beta^g = \gamma$. Since $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$ we then have $\mathcal{L}(\alpha) = (\mathcal{L}(\alpha))^g = (\mathcal{L}(\beta))^g = \mathcal{L}(\gamma)$. In particular this implies that each block in $\mathcal{L}(\alpha)$ other than $B(\gamma)$ contains a (unique) neighbour of γ, and so any two blocks in $\mathcal{L}(\alpha)$ are adjacent. For distinct vertices $\gamma, \delta \in \Gamma(\alpha)$, say $\delta \in D$, let δ' be the neighbour of γ in the block D. Then by the G-symmetry of Γ there exists $h \in G$ such that $(\alpha, \delta)^h = (\gamma, \delta')$. This implies $(\mathcal{L}(\alpha))^h = \mathcal{L}(\gamma)$ and $(\mathcal{L}(\delta))^h = \mathcal{L}(\delta')$. Since $\mathcal{L}(\alpha) = \mathcal{L}(\delta)$ as shown above, we have $\mathcal{L}(\delta') = \mathcal{L}(\gamma) = \mathcal{L}(\alpha)$. Thus δ' is adjacent to a vertex in B. However, our assumption on $\Gamma[B, D]$ implies that δ is the unique vertex in D adjacent to a vertex in B. So we must have $\delta' = \delta$. Thus we have shown that any two vertices in $\Gamma(\alpha)$ are adjacent. Hence $\{\alpha\} \cup \Gamma(\alpha)$ induces the complete graph K_{r+1}, which must be a connected component of Γ since Γ has valency r. Therefore, Γ is a union of disjoint copies of K_{r+1}. Obviously in this case $\mathcal{L}(\gamma) = \mathcal{L}(\delta)$ holds for any vertices γ, δ in the same component of Γ. $\square$

Part (c) of Lemma 8.3.1 implies that, if $k = 1$ and Γ is not a union of complete graphs, then the sets $\mathcal{L}(\alpha)$ (for $\alpha \in V(\Gamma)$) of blocks of $\mathcal{B}$ are pairwise distinct and thus $\mathcal{D}(\Gamma, \mathcal{B}) = (\mathcal{B}, \{\mathcal{L}(\alpha) : \alpha \in V(\Gamma)\})$. On the other hand, we will see in Example 8.3.1 that the opposite case can occur, that is, it may happen that $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$ for some pair of distinct vertices α, β of Γ. Inspired by part (a) of Lemma 8.3.1, we give the following definition.

Definition 8.3.1 Let $\mathcal{D}$ and G be as in Definition 8.2.1. A G-orbit Θ on the flags of $\mathcal{D}$ is said to be a 1-*feasible* G-orbit if it is feasible and $L \cap N = \{\sigma\}$ holds for distinct $(\sigma, L), (\sigma, N) \in \Theta(\sigma)$, where σ is a point of $\mathcal{D}$.

Now we prove that, up to isomorphism, the class of G-symmetric graphs with $k = 1$ is precisely the class of G-flag graphs $\Gamma(\mathcal{D}, \Theta, \Psi)$ such that Θ is 1-feasible.

Theorem 8.3.1 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $\Gamma[B, C] \cong K_2$ for adjacent blocks B, C of $\mathcal{B}$, and let r be the valency of Γ. Then $\Gamma \cong \Gamma(\mathcal{D}, \Theta, \Psi)$ holds for a certain G-point-transitive* *and* G*-block-transitive* 1*-design* $\mathcal{D}$ *with block size* $r+1$*, a certain* 1*-feasible* G*-orbit* Θ *on the flags of* $\mathcal{D}$*, and a certain self-paired* G*-orbit* Ψ *on* $\mathrm{C}(\mathcal{D}, \Theta)$*.*

Conversely, for any G*-point-transitive and* G*-block-transitive* 1*-design* $\mathcal{D}$ *with block size* $r+1$*, any* 1*-feasible* G*-orbit* Θ *on the flags of* $\mathcal{D}$*, and any self-paired* G*-orbit* Ψ *on* $\mathrm{C}(\mathcal{D}, \Theta)$*, the graph* $\Gamma := \Gamma(\mathcal{D}, \Theta, \Psi)$*, group* G*, partition* $\mathcal{B} := \mathcal{B}(\Theta)$ *and integer* r *satisfy all the conditions above.*

We will show further that, in both parts of this theorem, G is faithful on the vertices of Γ if and only if it is faithful on the points of $\mathcal{D}$.

Proof For the first part, we have seen in Example 8.1.1 that $\mathcal{D}^*(B)$ contains no repeated blocks and that the block size of $\mathcal{D}^*(B)$ is equal to r, the valency of Γ. By Lemma 8.1.1, $\mathcal{D} := \mathcal{D}(\Gamma, \mathcal{B})$ is a G-point-transitive and G-block-transitive 1-design with block size $r+1$. We have shown in the proof of Theorem 8.2.1 that $\Theta := \Theta(\Gamma, \mathcal{B})$ is a feasible G-orbit on the flags of $\mathcal{D}$, that $\Psi := \{((B(\alpha), \mathcal{L}(\alpha)), (B(\beta), \mathcal{L}(\beta))) : (\alpha, \beta) \in \mathrm{Arc}(\Gamma)\}$ is a self-paired G-orbit on $\mathrm{C}(\mathcal{D}, \Theta)$, and that $\Gamma \cong \Gamma(\mathcal{D}, \Theta, \Psi)$. From Lemma 8.3.1(a), we have $\mathcal{L} \cap \mathcal{N} = \{B\}$ for distinct $(B, \mathcal{L}), (B, \mathcal{N}) \in \Theta(B)$. Hence Θ is 1-feasible, and the first part of the theorem is proved. Moreover, by Lemma 8.3.1(b), if G is faithful on $V(\Gamma)$, then it is also faithful on the point set $\mathcal{B}$ of $\mathcal{D}$.

Suppose conversely that $\mathcal{D}$, G, Θ, Ψ and r are as in the second part of the theorem. We have proved in Theorem 8.2.1 that $\Gamma := \Gamma(\mathcal{D}, \Theta, \Psi)$ is a G-symmetric graph, that $\mathcal{B} := \mathcal{B}(\Theta)$ is a nontrivial G-invariant partition of the vertex set Θ of Γ, and that $\mathcal{D}^*(\Theta(\sigma))$ has block size r and contains no repeated blocks, where σ is a point of $\mathcal{D}$. Let $\Theta(\sigma)$, $\Theta(\tau)$ be adjacent blocks of $\mathcal{B}$. Then there exist $(\sigma, L) \in \Theta(\sigma)$ and $(\tau, N) \in \Theta(\tau)$ such that $((\sigma, L), (\tau, N)) \in \Psi$. So we have $\sigma \neq \tau$ and $\sigma, \tau \in L \cap N$. Since Θ is 1-feasible this implies that, for any $(\sigma, L_1) \in \Theta(\sigma) \setminus \{(\sigma, L)\}$ and $(\tau, N_1) \in \Theta(\tau) \setminus \{(\tau, N)\}$, we have $\sigma \notin N_1$ and $\tau \notin L_1$. Thus none of $((\sigma, L), (\tau, N_1))$, $((\sigma, L_1), (\tau, N))$ and $((\sigma, L_1), (\tau, N_1))$ belongs to Ψ. In other words, the edge of Γ joining (σ, L) and (τ, N) is the only edge between $\Theta(\sigma)$ and $\Theta(\tau)$. Hence we have $\Gamma[\Theta(\sigma), \Theta(\tau)] \cong K_2$, and consequently the valency of Γ is equal to the block size r of $\mathcal{D}^*(\Theta(\sigma))$. If an element of G fixes each flag in Θ, then it must fix each point of $\mathcal{D}$. So if G is faithful on the points of $\mathcal{D}$, then it must be faithful on Θ, the vertex set of Γ. This completes the proof of Theorem 8.3.1, and that of the statement immediately following it. □

Analogous to Corollary 8.2.1, we have the following consequence of Theorem 8.3.1.

Corollary 8.3.1 *Let $v \geq 2$ and $r \geq 1$ be integers, and let G be a group. Then the following* (a), (b) *are equivalent.*

(a) Γ *is a G-symmetric graph of valency r which admits a nontrivial G-invariant partition $\mathcal{B}$ of block size v such that $\Gamma[B, C] \cong K_2$ for any two blocks B, C of $\mathcal{B}$ (so $\Gamma_{\mathcal{B}} \cong K_{vr+1}$).*

(b) $\Gamma \cong \Gamma(\mathcal{D}, \Theta, \Psi)$*, for a G-doubly transitive and G-block-transitive* 2-$(vr+1, r+1, \lambda)$ *design $\mathcal{D}$, a* 1*-feasible G-orbit Θ on the flags of $\mathcal{D}$, and a self-paired G-orbit Ψ on* $\mathrm{C}(\mathcal{D}, \Theta)$.

We conclude this chapter by giving the following illustrative examples.

Example 8.3.1 (a) If $\mathcal{D}$ is a G-flag-transitive linear space, then of course the flag set Θ of $\mathcal{D}$ is the only G-orbit on the flags of $\mathcal{D}$. Clearly, Θ satisfies (a) in Definition 8.2.1 and the condition in Definition 8.3.1. So Θ is feasible if and only if it satisfies (b) in Definition 8.2.1, and in this case Θ is 1-feasible. For such a Θ, one can see that any self-paired G-orbit on $\mathrm{C}(\mathcal{D}, \Theta)$ has the form $\Psi = \{((\sigma, L_{\sigma\tau}), (\tau, L_{\sigma\tau})) : (\sigma, \tau) \in \Delta\}$, for some self-paired G-orbit Δ on ordered pairs of distinct points of $\mathcal{D}$, where $L_{\sigma\tau}$ denotes the unique line of $\mathcal{D}$ through σ and τ. For such a Ψ, we set $\Gamma := \Gamma(\mathcal{D}, \Theta, \Psi)$ and $L := L_{\sigma\tau}$ for a fixed $(\sigma, \tau) \in \Delta$. From (b) in Definition 8.2.1, for any $\delta \in L \setminus \{\sigma\}$ there exists $g \in G_{\sigma,L}$ such that $\tau^g = \delta$. So $(\sigma, \delta) = (\sigma, \tau)^g \in \Delta$ and $((\sigma, L), (\delta, L)) = ((\sigma, L), (\tau, L))^g \in \Psi$. It follows that (σ, L) is adjacent in Γ to any $(\delta, L) \in \Theta$ with $\delta \in L \setminus \{\sigma\}$. Therefore, each connected component of Γ is a complete graph induced by a line $L_{\sigma\tau}$, for $(\sigma, \tau) \in \Delta$. Such a graph Γ satisfies the condition in Lemma 8.3.1(c).

(b) In particular, if $\mathcal{D}$ is a G-doubly transitive linear space with $vr + 1$ points and block size $r + 1$, then $\mathcal{D}$ is G-flag-transitive and its flag set Θ is 1-feasible. In this case the only self-paired G-orbit on $\mathrm{C}(\mathcal{D}, \Theta)$ is

$$\Psi := \{((\sigma, L), (\tau, L)) : L \text{ is a line of } \mathcal{D},\ \sigma, \tau \text{ are distinct points on } L\}.$$

Hence $\mathcal{D}$ has a unique G-flag graph $\Gamma(\mathcal{D}, \Theta, \Psi)$ of which each connected component is a complete graph induced by a line of $\mathcal{D}$.

A 1-design $\mathcal{D}$ with block size 2 can be viewed as a regular graph Σ, and vice versa, if we identify the blocks of $\mathcal{D}$ with the edges of Σ. The automorphism groups of the design $\mathcal{D}$ and the graph Σ are the same. Moreover, under this identification each flag (σ, L) of $\mathcal{D}$, say $L = \{\sigma, \tau\}$, can be identified with the arc (σ, τ) of Σ. Hence $\mathcal{D}$ is G-flag-transitive if and only if Σ is G-symmetric.

Example 8.3.2 A G-flag-transitive 1-design $\mathcal{D}$ with block size $r + 1 := 2$ such that each point is incident with $c \geq 2$ blocks can be identified with a G-symmetric graph Σ of valency c. Since $\mathcal{D}$ is G-flag-transitive, the only G-orbit on the flags of $\mathcal{D}$ is the set Θ of all flags of $\mathcal{D}$, that is, the arc set $\mathrm{Arc}(\Sigma)$ of Σ. It is clear that Θ is 1-feasible, and that the only self-paired G-orbit on $\mathrm{C}(\mathcal{D}, \Theta)$ is $\Psi := \{((\sigma, \tau), (\tau, \sigma)) : (\sigma, \tau) \in \mathrm{Arc}(\Sigma)\}$. So we get a unique G-flag graph $\Pi := \Gamma(\mathcal{D}, \Theta, \Psi)$, which has vertex set $\mathrm{Arc}(\Sigma)$ and edges joining (σ, τ) and (τ, σ), for all pairs σ, τ of adjacent vertices of Σ. Clearly, we have $\Pi \cong n \cdot K_2$ and $\Pi_{\mathcal{B}(\Theta)} \cong \Sigma$, where n is the number of edges of Σ. From Theorem 8.3.1, these graphs Π can represent all G-symmetric graphs Γ of valency $r = 1$ such that $V(\Gamma)$ admits a nontrivial G-invariant partition $\mathcal{B}$ with $\Gamma[B, C] \cong K_2$ for adjacent blocks B, C of $\mathcal{B}$. Moreover, any G-symmetric graph Σ with valency at least 2 can appear as the quotient $\Gamma_{\mathcal{B}}$ of such a graph Γ.

The graph Γ in Corollary 8.3.1(a) with the additional property $r = 1$ is precisely the unique G-flag graph (given in Example 8.3.1(b)) of a trivial G-doubly transitive linear space $\mathcal{D}$ with $v + 1$ points. Corollary 8.3.1 and Examples 8.3.1(b), 8.3.2 together imply the characterization of such graphs given in [43, Theorem 4.2].

Chapter 9

The case $k = v - 1 \geq 2$: Construction

If I have presented one corner of the square and they cannot come back to me with the other three, I should not go over the points again.

Confucius (551-479 B.C.), LUN YÜ [THE ANALECTS] 7:8

In this chapter, we continue our study of G-symmetric graphs Γ with $v = k+1 \geq 3$, without necessarily assuming the non-repetition of blocks of $\mathcal{D}(B)$. We will first give a construction of such graphs and then prove that, up to isomorphism, it produces all of them. In particular, if $\mathcal{D}(B)$ contains no repeated blocks, then the construction gives rise to 3-arc graphs introduced in Section 5.2. Note that $v = k + 1 \geq 3$ does not guarantee the non-repetition of the blocks of the dual 1-design of $\mathcal{D}(B)$. Thus in this case we cannot apply the G-flag graph construction to Γ directly. The approach we will use is to consider the graph Γ' defined in Definition 4.1.1, which is G-symmetric and admits the same G-invariant partition $\mathcal{B}$ such that $\Gamma'[B, C] \cong K_2$ for blocks B, C of $\mathcal{B}$ adjacent in $\Gamma'_{\mathcal{B}}$ $(= \Gamma_{\mathcal{B}})$. As in the previous chapter, the construction here requires a G-point-transitive and G-block-transitive 1-design $\mathcal{D}$ with no repeated blocks, and the resultant graph is a certain flag graph of $\mathcal{D}$. The case where in addition $\Gamma_{\mathcal{B}}$ is a complete graph occurs if and only if the design $\mathcal{D}$ involved is a G-doubly transitive 2-design. Since, as a result of the classification of finite simple groups, all the finite doubly transitive groups are known (see Section 2.1), our construction makes it possible to classify all such graphs Γ. As a moderate goal, we will in the last two sections of this chapter classify all such graphs Γ in the case where $\mathcal{D}$ is a classic projective or affine geometry.

9.1 Preliminary discussion

Let Γ be a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $v = k + 1 \geq 3$. As before, for each $\alpha \in V(\Gamma)$, we use $\mathcal{B}(\alpha)$ to denote the set of blocks of $\mathcal{B}$ which are adjacent to $B(\alpha)$ in $\Gamma_{\mathcal{B}}$ but contain no vertex adjacent to α in Γ, that is, $\mathcal{B}(\alpha) = \Gamma_{\mathcal{B}}(B(\alpha)) \setminus \Gamma_{\mathcal{B}}(\alpha)$. In Section 4.3 we have seen that, for $B \in \mathcal{B}$, $\mathbf{B}(B) = \{\mathcal{B}(\alpha) : \alpha \in B\}$ is a G_B-invariant partition of $\Gamma_{\mathcal{B}}(B)$. In Definition 4.1.1 we defined Γ' to be the graph with the same vertices as Γ in which two vertices α, β are adjacent if and only if they are mates, that is, if and only if $B(\alpha) \in \mathcal{B}(\beta)$ and $B(\beta) \in \mathcal{B}(\alpha)$ hold simultaneously. Recall that Γ' is G-symmetric (Theorem 4.1.1) and admits the same G-invariant partition $\mathcal{B}$ such that $\Gamma'[B, C] \cong K_2$ for blocks B, C of $\mathcal{B}$ adjacent in $\Gamma'_{\mathcal{B}}$ $(= \Gamma_{\mathcal{B}})$. In other words, Γ' satisfies the condition of Example 8.1.1, and hence the discussion in Section 8.3 applies to Γ'. Using the notation in Section 8.1, we know that $\mathcal{B}(\alpha) = \Gamma'_{\mathcal{B}}(\alpha)$ and thus $\mathbf{B}(B)$ is equal to $\mathbf{E}(B)$ defined in (8.2) for Γ'. Set

$$\mathcal{L}'(\alpha) := \{B(\alpha)\} \cup \mathcal{B}(\alpha).$$

Then $\mathcal{L}'(\alpha)$ is equal to $\mathcal{L}(\alpha)$ defined in (8.1) for Γ'. From Lemma 8.3.1(c), if there exist two distinct vertices α, β such that $\mathcal{L}'(\alpha) = \mathcal{L}'(\beta)$, then Γ' is a union of disjoint copies of K_{m+1}, where m is the multiplicity of $\mathcal{D}(B)$. In the following we denote by $\mathbf{L}'$ the set of all the distinct $\mathcal{L}'(\alpha)$, for $\alpha \in V(\Gamma)$. Then, as shown in Section 8.1, G induces a natural action on $\mathbf{L}'$ defined by $(\mathcal{L}'(\alpha))^g := \mathcal{L}'(\alpha^g)$ for $\alpha \in V(\Gamma)$ and $g \in G$. Clearly, we have

$$\mathcal{D}(\Gamma', \mathcal{B}) = (\mathcal{B}, \mathbf{L}')$$

and

$$\Theta(\Gamma', \mathcal{B}) = \{(B(\alpha), \mathcal{L}'(\alpha)) : \alpha \in V(\Gamma)\}.$$

Set

$$\mathbf{L}'(B) := \{\mathcal{L}'(\alpha) : \alpha \in B\}.$$

From Lemma 8.1.2(c) the action of G_B on B is permutationally equivalent to the actions of G_B on $\mathbf{B}(B)$, $\mathbf{L}'(B)$ with respect to the bijections defined by $\alpha \mapsto \mathcal{B}(\alpha)$,

$\alpha \mapsto \mathcal{L}'(\alpha)$, for $\alpha \in B$, respectively. By Theorem 4.3.2(b), these actions are doubly transitive. In the following we collect some simple results that we will use in the next section.

Lemma 9.1.1 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ with block size $v = k + 1 \geq 3$. Let m be the multiplicity of $\mathcal{D}(B)$, and let $B \in \mathcal{B}$, $\alpha \in B$ and $C \in \mathcal{B}(\alpha)$. Then the following* (a)-(d) *hold.*

(a) *$\mathcal{D}(\Gamma', \mathcal{B})$ is a 1-design with block size $m + 1$ which admits G as a point- and block-transitive group of automorphisms.*

(b) *$\Theta(\Gamma', \mathcal{B})$ is a G-orbit on the set of flags of $\mathcal{D}(\Gamma', \mathcal{B})$, and the actions of G on $V(\Gamma)$ and $\Theta(\Gamma', \mathcal{B})$ are permutationally equivalent with respect to the bijection defined by $\rho : \alpha \mapsto (B(\alpha), \mathcal{L}'(\alpha))$, for $\alpha \in V(\Gamma)$. Hence we have $G_{B,\mathcal{B}(\alpha)} = G_{B,\mathcal{L}'(\alpha)} = G_\alpha$.*

(c) *$G_{B,\mathcal{L}'(\alpha)}$ is transitive on $\mathcal{B}(\alpha)$, for $\alpha \in B$.*

(d) *$G_{B,C}$ is transitive on $\mathbf{L}'(B) \setminus \{\mathcal{L}'(\alpha)\}$.*

Proof Parts (a)-(c) follow directly from Lemmas 8.1.1, 8.1.2 and Example 8.1.1. Since the actions of G_B on $\mathbf{B}(B)$ and $\mathbf{L}'(B)$ are permutationally equivalent with respect to the bijection defined by $\mathcal{B}(\gamma) \mapsto \mathcal{L}'(\gamma)$, for $\gamma \in B$, part (d) is a restatement of Theorem 4.3.2(b). $\square$

9.2 Construction

We will use the notation and terminology of the previous chapter. Let $\mathcal{D}$ be a G-point-transitive and G-block-transitive 1-design with no repeated blocks and with block size at least 2. Let Θ be a 1-feasible G-orbit on the flags of $\mathcal{D}$. We use $\mathrm{F}(\mathcal{D}, \Theta)$ to denote the set of ordered pairs $((\sigma, L), (\tau, N)) \in \Theta^{(2)}$ which are not in $\mathrm{C}(\mathcal{D}, \Theta)$ but are such that there exist $(\sigma, L') \in \Theta(\sigma)$ and $(\tau, N') \in \Theta(\tau)$ with $((\sigma, L'), (\tau, N')) \in \mathrm{C}(\mathcal{D}, \Theta)$. In other words, $((\sigma, L), (\tau, N)) \in \mathrm{F}(\mathcal{D}, \Theta)$ if and only if $\sigma \notin N$, $\tau \notin L$ but $\sigma \in N'$, $\tau \in L'$ for some $(\sigma, L'), (\tau, N') \in \Theta$. In this case we have $L \neq L'$, $N \neq N'$, and the 1-feasibility of Θ implies that both (σ, L') and (τ, N') are unique. Moreover, for any $(\sigma, L_1), (\tau, N_1) \in \Theta$ with $L_1 \neq L'$ and $N_1 \neq N'$, we have $((\sigma, L_1), (\tau, N_1)) \in \mathrm{F}(\mathcal{D}, \Theta)$. It is easy to see that $\mathrm{F}(\mathcal{D}, \Theta)$ is a G-invariant subset of $\Theta^{(2)}$. For a subset Ψ of $\mathrm{F}(\mathcal{D}, \Theta)$, we set

$$\Psi' := \{((\sigma, L'), (\tau, N')) : ((\sigma, L), (\tau, N)) \in \Psi\}.$$

Then Ψ' is a subset of $\mathrm{C}(\mathcal{D}, \Theta)$. Clearly, if Ψ is a G-orbit on $\mathrm{F}(\mathcal{D}, \Theta)$, then Ψ' is a G-orbit on $\mathrm{C}(\mathcal{D}, \Theta)$; and if Ψ is self-paired, then Ψ' is self-paired as well. (The converses of these assertions are not necessarily true.) To construct G-symmetric graphs with $v = k + 1 \geq 3$, we should impose more conditions on Θ.

Definition 9.2.1 Let $\mathcal{D}$ and G be as above. A G-orbit Θ on the flags of $\mathcal{D}$ is said to be a *strict* 1*-feasible* G-orbit if it is 1-feasible and is such that $|\Theta(\sigma)| \geq 3$ and that, for $(\sigma, L) \in \Theta$ and $\tau \in L \setminus \{\sigma\}$, $G_{\sigma\tau}$ is transitive on $\Theta(\sigma) \setminus \{(\sigma, L)\}$.

The first major result in this chapter is the following theorem.

Theorem 9.2.1 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ with block size $v = k + 1 \geq 3$, and let m be the multiplicity of $\mathcal{D}(B)$. Then $\Gamma \cong \Gamma(\mathcal{D}, \Theta, \Psi)$ holds for a certain G-point-transitive and G-block-transitive* 1*-design with block size $m+1$, a certain strict* 1*-feasible G-orbit Θ on the flags of $\mathcal{D}$, and a certain self-paired G-orbit Ψ on $\mathrm{F}(\mathcal{D}, \Theta)$.*

Conversely, for any G-point-transitive and G-block-transitive 1*-design $\mathcal{D}$ with no repeated blocks and with block size $m+1$, any strict* 1*-feasible G-orbit Θ on the flags of $\mathcal{D}$, and any self-paired G-orbit Ψ on $\mathrm{F}(\mathcal{D}, \Theta)$, the graph $\Gamma := \Gamma(\mathcal{D}, \Theta, \Psi)$, group G, partition $\mathcal{B} := \mathcal{B}(\Theta)$ and integer m satisfy all the conditions above.*

Remark 9.2.1 In both parts of Theorem 9.2.1, G is faithful on the vertices of Γ if and only if it is faithful on the points of $\mathcal{D}$. Moreover, the graph Γ' defined in Definition 4.1.1 for Γ is isomorphic to the G-flag graph $\Gamma(\mathcal{D}, \Theta, \Psi')$.

Proof of Theorem 9.2.1 Suppose that Γ, G and $\mathcal{B}$ are as in the first part of the theorem, and let Γ' be the graph defined in Definition 4.1.1. By Lemma 9.1.1(a)(b), $\mathcal{D} := \mathcal{D}(\Gamma', \mathcal{B})$ is a G-point-transitive and G-block-transitive 1-design with block size $m + 1$, and $\Theta := \Theta(\Gamma', \mathcal{B})$ is a G-orbit on the flags of $\mathcal{D}$. It follows from the definition that $\Theta(B) = \{(B, \mathcal{L}) : \mathcal{L} \in \mathbf{L}'(B)\}$ for $B \in \mathcal{B}$. So $|\Theta(B)| = v \geq 3$ and $\mathcal{L} \cap \mathcal{N} = \{B\}$ holds for distinct flags $(B, \mathcal{L}), (B, \mathcal{N})$ in $\Theta(B)$. For $(B, \mathcal{L}) \in \Theta(B)$, say $\mathcal{L} = \mathcal{L}'(\alpha)$ for some $\alpha \in B$, we have $\mathcal{L} \setminus \{B\} = \mathcal{B}(\alpha)$ and $\Theta(B) \setminus \{(B, \mathcal{L})\} = \{(B, \mathcal{N}) : \mathcal{N} \in \mathbf{L}'(B) \setminus \{\mathcal{L}\}\}$. So Lemma 9.1.1(c)(d) and the argument above imply that Θ is a strict 1-feasible G-orbit on the flags of $\mathcal{D}$.

For an arc (α, β) of Γ, the blocks $B := B(\alpha)$ and $C := B(\beta)$ are adjacent in $\Gamma_{\mathcal{B}}$. So there exist $\alpha' \in B$ and $\beta' \in C$ such that α', β' are mates, that is, $(\alpha', \beta') \in \mathrm{Arc}(\Gamma')$. Thus we have $B \in \mathcal{L}'(\beta')$ and $C \in \mathcal{L}'(\alpha')$. It follows from the definition that $B \notin \mathcal{L}'(\beta)$ and $C \notin \mathcal{L}'(\alpha)$, and therefore we have $((B, \mathcal{L}'(\alpha)), (C, \mathcal{L}'(\beta))) \in \mathrm{F}(\mathcal{D}, \Theta)$. Thus, setting $\Psi := \{((B(\alpha), \mathcal{L}'(\alpha)), (B(\beta), \mathcal{L}'(\beta))) : (\alpha, \beta) \in \mathrm{Arc}(\Gamma)\}$, then $\Psi \subseteq \mathrm{F}(\mathcal{D}, \Theta)$ and Ψ is clearly self-paired. By Lemma 9.1.1(b), the actions of G on $V(\Gamma)$ and Θ are permutationally equivalent with respect to the bijection $\rho : \gamma \mapsto (B(\gamma), \mathcal{L}'(\gamma))$, for $\gamma \in V(\Gamma)$. Since Γ is G-symmetric, this implies that Ψ is a (self-paired) G-orbit on $\mathrm{F}(\mathcal{D}, \Theta)$. It is easily checked that the bijection ρ above defines an isomorphism from Γ to $\Gamma(\mathcal{D}, \Theta, \Psi)$, and hence the first part of Theorem 9.2.1 is proved. In addition, from Theorem 4.3.1(c), if G is faithful on the vertices of Γ, then it is also faithful on the points of $\mathcal{D}$. Clearly, we have $\Psi' = \{((B(\alpha'), \mathcal{L}'(\alpha')), (B(\beta'), \mathcal{L}'(\beta'))) : (\alpha', \beta') \in \mathrm{Arc}(\Gamma')\}$. From the comments before Definition 9.2.1, Ψ' is a self-paired G-orbit on $\mathrm{C}(\mathcal{D}, \Theta)$. Thus, from the proof of Theorem 8.3.1, it follows that $\Gamma' \cong \Gamma(\mathcal{D}, \Theta, \Psi')$.

Suppose conversely that $\mathcal{D}, G, \Theta, \Psi$ and m are as in the second part of the theorem. Let $\Gamma := \Gamma(\mathcal{D}, \Theta, \Psi)$, and let $\mathcal{B} := \mathcal{B}(\Theta)$ be as defined in (8.3). Then it follows from the definition that Γ is a G-symmetric graph with vertex set Θ, and $\mathcal{B}$ is a nontrivial G-invariant partition of Θ with block size $v := |\Theta(\sigma)| \geq 3$, where σ is a point of $\mathcal{D}$. To complete the proof, we need to show that the block size k of the 1-design $\mathcal{D}(\Theta(\sigma))$ induced on the block $\Theta(\sigma)$ of $\mathcal{B}$ satisfies $v = k + 1$, and that the multiplicity of $\mathcal{D}(\Theta(\sigma))$ is equal to m.

Let $\Theta(\sigma), \Theta(\tau)$ be adjacent blocks of $\mathcal{B}$. Then there exist $(\sigma, L) \in \Theta(\sigma)$ and $(\tau, N) \in \Theta(\tau)$ such that $(\sigma, L), (\tau, N)$ are adjacent in Γ, that is, $((\sigma, L), (\tau, N)) \in \Psi$. Since Ψ is a G-orbit on $\mathrm{F}(\mathcal{D}, \Theta)$, we have $\sigma \notin N$, $\tau \notin L$ but there exist $(\sigma, L'), (\tau, N') \in \Theta$ such that $\sigma \in N', \tau \in L'$. Clearly, for any $(\sigma, L_1) \in \Theta(\sigma)$, we have $((\sigma, L_1), (\tau, N')) \notin \mathrm{F}(\mathcal{D}, \Theta)$ and hence (τ, N') is not adjacent in Γ to any vertex in $\Theta(\sigma)$. Similarly, (σ, L') is not adjacent in Γ to any vertex in $\Theta(\tau)$. On the other hand, since Θ is a strict 1-feasible G-orbit and since $\tau \in L' \setminus \{\sigma\}$, it follows from Definition 9.2.1 that $G_{\sigma\tau}$ is transitive on $\Theta(\sigma) \setminus \{(\sigma, L')\}$. Thus, for any $(\sigma, L_1) \in \Theta(\sigma) \setminus \{(\sigma, L')\}$, there exists $g \in G_{\sigma\tau}$ such that $(\sigma, L)^g = (\sigma, L_1)$. Since $\sigma \notin N$ and g fixes σ, we have $\sigma \notin N^g$. But $\sigma \in N'$, so we have $(\tau, N_1) := (\tau, N)^g \in \Theta(\tau) \setminus \{(\tau, N')\}$, and $(\sigma, L_1), (\tau, N_1)$ are adjacent in Γ. Thus each vertex in $\Theta(\sigma) \setminus \{(\sigma, L')\}$ is adjacent in Γ to at least one veretx in $\Theta(\tau) \setminus \{(\tau, N')\}$, that is, $\Gamma(\Theta(\tau)) \cap \Theta(\sigma) = \Theta(\sigma) \setminus \{(\sigma, L')\}$. Hence $v = k + 1$. From the comments before Definition 9.2.1, we know that $\Psi' = ((\sigma, L'), (\tau, N'))^G$ and that Ψ' is a self-paired G-orbit on $\mathrm{C}(\mathcal{D}, \Theta)$. Moreover, the argument above shows that the G-flag graph $\Gamma(\mathcal{D}, \Theta, \Psi')$ is exactly the accompanying graph Γ' of Γ defined in Definition 4.1.1. It follows from Theorem 8.3.1 that Γ' has valency m. In other words, the multiplicity of $\mathcal{D}(\Theta(\sigma))$ is equal to m.

Finally, if an element of G fixes each flag in Θ, then it must fix each point of $\mathcal{D}$. So if G is faithful on the points of $\mathcal{D}$ then it must be faithful on the vertices of Γ. This completes the proof of Theorem 9.2.1, as well as that of Remark 9.2.1. $\Box$

From the proof above, the graph $\Gamma = \Gamma(\mathcal{D}, \Theta, \Psi)$ in Theorem 9.2.1 coexists with the G-flag graph $\Gamma' = \Gamma(\mathcal{D}, \Theta, \Psi')$. For brevity we will call such a graph $\Gamma(\mathcal{D}, \Theta, \Psi)$ a *coexisting G-flag graph*. Now we illustrate our construction of such graphs by examining an important special case. The following example shows that, in this "simplest" case, the construction produces precisely the 3-arc graphs associated with $(G, 2)$-arc transitive graphs.

Example 9.2.1 As mentioned before Example 8.3.2, a G-flag-transitive 1-design $\mathcal{D}$ with block size 2 can be viewed as a G-symmetric graph Σ, and vice versa, if we identify the blocks of $\mathcal{D}$ with the edges of Σ. Under this identification each flag of $\mathcal{D}$ can be identified with an arc of Σ, and hence the valency v of Σ is equal to the number of blocks of $\mathcal{D}$ incident with a given point. We assume $v \geq 3$ in the following. Since $\mathcal{D}$ is G-flag-transitive, the only G-orbit on the flags of $\mathcal{D}$ is the set Θ of all flags of $\mathcal{D}$, that is, the arc set $\mathrm{Arc}(\Sigma)$ of Σ. Clearly, Θ is 1-feasible and $|\Theta(\sigma)| = v \geq 3$. The second condition in Definition 9.2.1 is equivalent to requiring that Σ is $(G, 2)$-arc transitive. Therefore, $\mathcal{D}$ has a strict 1-feasible G-orbit on its flags if and only if Σ is $(G, 2)$-arc transitive, and in this case the only such G-orbit is Θ. The G-invariant partition $\mathcal{B}(\Theta)$ of Θ (defined in (8.3)) can be identified with the G-invariant partition $\mathcal{B}(\Sigma) := \{B(\sigma) : \sigma \in V(\Sigma)\}$ of $\mathrm{Arc}(\Sigma)$ defined in Section 5.2, where $B(\sigma)$ is the set of arcs of Σ initiated at σ. Moreover, an ordered pair $((\sigma, L), (\tau, N)) \in \Theta^{(2)}$, say $L = \{\sigma, \sigma'\}$, $N = \{\tau, \tau'\}$, lies in $\mathrm{F}(\mathcal{D}, \Theta)$ if and only if $(\sigma', \sigma, \tau, \tau')$ is a 3-arc of Σ. So we may identify such a pair $((\sigma, L), (\tau, N))$ with $(\sigma', \sigma, \tau, \tau')$, and thus identify $\mathrm{F}(\mathcal{D}, \Theta)$ with $\mathrm{Arc}_3(\Sigma)$. Hence a self-paired G-orbit

Ψ on $\mathrm{F}(\mathcal{D},\Theta)$ can be identified with a self-paired G-orbit Δ on $\mathrm{Arc}_3(\Sigma)$, and vice versa. Therefore, the flag graph $\Gamma(\mathcal{D},\Theta,\Psi)$ is isomorphic to the 3-arc graph $\Xi(\Sigma,\Delta)$ of Σ with respect to Δ.

Remark 9.2.2 Let Γ be a G-symmetric graph Γ admitting a nontrivial G-invariant partition $\mathcal{B}$ with block size $v = k+1 \geq 3$. Then $\mathcal{D}(B)$ contains no repeated blocks if and only if the 1-design $\mathcal{D} := \mathcal{D}(\Gamma', \mathcal{B})$ has block size 2. In this case we may identify $\mathcal{D}$ with the quotient graph $\Gamma_{\mathcal{B}}$ by identifying each block $\{B, C\}$ of $\mathcal{D}$ with the edge of $\Gamma_{\mathcal{B}}$ joining B and C. So Theorem 9.2.1 and the discussion in Example 9.2.1 imply Theorem 5.2.3 as a consequence.

9.3 Coexisting G-flag graphs of doubly transitive designs

In this section we examine the case where $v = k+1 \geq 3$ and in addition $\Gamma_{\mathcal{B}}$ is a complete graph, that is, $\Gamma_{\mathcal{B}} \cong K_{mv+1}$ (note that $\mathrm{val}(\Gamma_{\mathcal{B}}) = mv$ by Theorem 4.3.1(a)). Similar to Corollaries 8.2.1 and 8.3.1, we have the following consequence of Theorem 9.2.1.

Corollary 9.3.1 *Let $v \geq 3$ and $m \geq 1$ be integers. Then the following* (a), (b) *are equivalent.*

(a) Γ *is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ of block size v such that $v = k+1$ and $\Gamma_{\mathcal{B}} \cong K_{mv+1}$.*

(b) $\Gamma \cong \Gamma(\mathcal{D},\Theta,\Psi)$, *for a G-doubly transitive and G-block-transitive* 2-$(mv+1, m+1, \lambda)$ *design $\mathcal{D}$, a strict* 1*-feasible G-orbit Θ on the flags of $\mathcal{D}$, and a self-paired G-orbit Ψ on* $\mathrm{F}(\mathcal{D},\Theta)$.

Moreover, the integer m is equal to the multiplicity of the 1-design $\mathcal{D}(B)$, and G is faithful on $V(\Gamma)$ if and only if it is faithful on the points of $\mathcal{D}$. Thus, by studying G-doubly transitive, G-block-transitive 2-$(mv+1, m+1, \lambda)$ designs $\mathcal{D}$, it seems feasible to classify all G-symmetric graphs Γ in Corollary 9.3.1. If and only if $m = 1$ such a design $\mathcal{D}$ is a G-doubly transitive 2-$(v+1, 2, 1)$ design, that is, a G-doubly transitive trivial linear space. From Example 9.2.1, in this case the existence of coexisting G-flag graphs of $\mathcal{D}$ requires that $\mathcal{D}$ is G-triply transitive, and such graphs are precisely the 3-arc graphs of the $(G, 2)$-arc transitive graph K_{v+1} and thus are those graphs classified in Theorem 6.6.1. In the following we suppose $m \geq 2$ and $\mathcal{D}$ admits a strict 1-feasible G-orbit Θ on its flags. Let V denote the point set of $\mathcal{D}$. The double transitivity of G on V implies that, for each pair σ, τ of distinct points of $\mathcal{D}$, there exists $(\sigma, L) \in \Theta$ such that $\tau \in L$. Thus, since Θ is 1-feasible, $V \setminus \{\sigma\}$ admits a G_σ-invariant partition of block size m, namely $\mathcal{P} := \{L \setminus \{\sigma\} : (\sigma, L) \in \Theta\}$. In particular, this implies that G is not 2-primitive and hence not 3-transitive on V. Moreover, by the strict 1-feasibility of Θ, for any $P \in \mathcal{P}$ and $\tau \in P$, $G_{\sigma,P}$ is transitive on P and $G_{\sigma\tau}$ is transitive on $\mathcal{P} \setminus \{P\}$. This latter assertion implies that any $G_{\sigma\tau}$-orbit X on $V \setminus P$ intersects with the same number of points in each of the $v-1$ blocks of $\mathcal{P} \setminus \{P\}$, and hence $v - 1$ must be a divisor of $|X|$. (If we use $v_0 = mv + 1$ and $k_0 = m + 1$ to denote the number of points and the block size of $\mathcal{D}$ respectively, then this is equivalent to saying that $(v_0 - k_0)/(k_0 - 1)$ is a divisor of $|X|$.) This necessary condition can be used to exclude some 2-transitive groups G involved, as shown in the following example.

Example 9.3.1 (a) We show that the Higman-Sims group HS cannot serve as the group G above. Suppose otherwise, then since HS is 2- but not 3-transitive with degree 176, we have $mv = 175 = 5^2 \times 7$ and $m \geq 2$ by the discussion above. Hence the only possibilities for (m, v) are $(5, 35), (7, 25), (25, 7), (35, 5)$. For distinct $\sigma, \tau \in V$, we have $\mathrm{HS}_\sigma = \mathrm{PSU}(3, 5) : \mathbb{Z}_2 \geq \mathrm{PSU}(3, 5)$ (split extension, see [24, pp.81]), and so $\mathrm{HS}_{\sigma\tau} = A_6.\mathbb{Z}_2^2 \geq (\mathrm{PSU}(3, 5))_\tau$ (see [24, pp.34]). In the action of $\mathrm{PSU}(3, 5)$ with degree 175, $(\mathrm{PSU}(3, 5))_\tau$ has orbits of lengths $1, 21, 28, 125$, respectively. So the $\mathrm{HS}_{\sigma\tau}$-orbits on $V \setminus \{\sigma, \tau\}$ have lengths at least 21. In view of the necessary condition above, this happens only when $(m, v) = (25, 7)$ or $(35, 5)$. In these two cases there must have an $\mathrm{HS}_{\sigma\tau}$-orbit with length 21, and either there are two remaining $\mathrm{HS}_{\sigma\tau}$-orbits with lengths $28, 125$ respectively, or there is only one remaining $\mathrm{HS}_{\sigma\tau}$-orbit with length $28 + 125$. Note that $v - 1$ is either 6 or 4, but $6 \not| \, 28, 125, 28 + 125$ and $4 \not| \, 125, 28 + 125$. This contradicts our condition above, and hence the group HS can be excluded.

(b) Similarly, we can show that the Conway group Co_3 cannot serve as the group G above. Suppose otherwise, then since Co_3 is 2- but not 3-transitive with degree 276, we have $mv = 275 = 5^2 \times 11$ and $m \geq 2$. Hence $(m, v) = (5, 55), (11, 25), (25, 11)$, or $(55, 5)$. By [24, pp.134], $(\mathrm{Co}_3)_\sigma = \mathrm{McL} : \mathbb{Z}_2 \geq \mathrm{McL}$; and by [24, pp.100] $\mathrm{McL}_\tau = \mathrm{PSU}(4, 3)$ has orbits of lengths $1, 22, 252$ in its action of degree 275. Since Co_3 is not 3-transitive on V, $(\mathrm{Co}_3)_{\sigma\tau}$ must have two orbits on $V \setminus \{\sigma, \tau\}$ and the lengths of them must be $22, 252$, respectively. Using the necessary condition above, we can see that all the possibilities for (m, v) cannot appear. Hence the group Co_3 can be excluded as well.

As a result of the finite simple group classification, all doubly transitive linear spaces are known [51, Theorem 1]. Because of this, it seems possible to classify the flag graphs $\Gamma(\mathcal{D}, \Theta, \Psi)$ appeared in Corollary 9.3.1 for G-doubly transitive linear spaces $\mathcal{D}$, and this will contribute to the classification of all the graphs Γ therein. As an effort towards this project, we will classify in the next two sections such graphs $\Gamma(\mathcal{D}, \Theta, \Psi)$ for two typical G-doubly transitive linear spaces $\mathcal{D}$, namely the projective geometry $\mathrm{PG}(d-1, q)$ $(d \geq 3)$ and affine geometry $\mathrm{AG}(d, q)$ $(d \geq 2)$, where G is a group with $\mathrm{PSL}(d, q) \leq G \leq \mathrm{P\Gamma L}(d, q)$ or $\mathrm{AGL}(d, q) \leq G \leq \mathrm{A\Gamma L}(d, q)$, respectively.

Remark 9.3.1 (a) A G-doubly transitive linear space $\mathcal{D}$ must be G-flag-transitive, and hence the only G-orbit on the flags of $\mathcal{D}$ is the flag set Θ of $\mathcal{D}$. In this case Θ satisfies (b) in Definition 8.2.1 and the condition in Definition 8.3.1 automatically. Hence Θ is strictly 1-feasible if and only if a point is incident with at least three lines and, for two points σ, τ, $G_{\sigma\tau}$ is transitive on the lines incident with σ but not τ. Note that in this case we have $\mathrm{F}(\mathcal{D}, \Theta) = \{((\sigma, L), (\tau, N)) : (\sigma, L), (\tau, N) \in \Theta, \sigma \notin N, \tau \notin L\}$.

(b) Conversely, if the flag set of a G-flag-transitive 2-design $\mathcal{D}$ is 1-feasible, then $\mathcal{D}$ is forced to be a linear space.

9.4 Projective flag graphs

Let $d \geq 3$ be an integer, and let $q = p^e$ with p a prime and $e \geq 1$. The projective geometry $\mathrm{PG}(d-1, q)$ is the geometry obtained by taking n-flats of $\mathrm{AG}(d, q)$ as its $(n-1)$-flats, for $1 \leq n \leq d$. As usual in the literature we will use the same notation to denote the (point, line)-incidence structure of $\mathrm{PG}(d-1, q)$. Then, for any group G with $\mathrm{PSL}(d, q) \leq G \leq \mathrm{P\Gamma L}(d, q)$, $\mathrm{PG}(d-1, q)$ is a G-doubly transitive linear space with $mv+1 := (q^d-1)/(q-1)$ points in which each line contains $m+1 := q+1$ points (see e.g. [84, Theorem 2.5(ii)]). So we have $v = (q^{d-1}-1)/(q-1)$ and $m = q$. The purpose of this section is to classify the coexisting G-flag graphs of $\mathrm{PG}(d-1, q)$, and to characterize them as the only such graphs arising from any G-doubly transitive 2-design.

Recall that we use $V(d, q)$ to denote the d-dimensional linear space of row vectors over $\mathrm{GF}(q)$. Let V denote the point set of $\mathrm{PG}(d-1, q)$. Then $V = \{[\mathbf{x}] : \mathbf{x} \in V(d, q) \setminus \{\mathbf{0}\}\}$, where $[\mathbf{x}]$ denotes the point of $\mathrm{PG}(d-1, q)$ representing non-zero multiples of the vector $\mathbf{x}$. For $1 \leq n \leq d-1$, $n+1$ points of $\mathrm{PG}(d-1, q)$ are said to be *independent* [84, pp. 72] if they do not lie on any $(n-1)$-flat of $\mathrm{PG}(d-1, q)$. In particular, three points of $\mathrm{PG}(d-1, q)$ are *non-collinear* if they are independent, and *collinear* otherwise. We will exploit the following basic result in projective geometry. (See [84, Theorem 2.10(iii)] for a proof in the special case where $G = \mathrm{PGL}(d, q)$. The result in general case can be derived from [25, 1.4.24].)

Lemma 9.4.1 *Suppose* $\mathrm{PSL}(d, q) \leq G \leq \mathrm{P\Gamma L}(d, q)$, *where* $d \geq 3$ *and* q *is a prime power. Then, for any integer* n *with* $1 \leq n \leq d-1$, G *is transitive on the set of ordered* $(n+1)$*-tuples of independent points of* $\mathrm{PG}(d-1, q)$.

Let $\Theta(P; d, q)$ denote the set of flags (that is, (point, line)-flags) of $\mathrm{PG}(d-1, q)$. In the following lemma we will show that $\Theta(P; d, q)$ is strictly 1-feasible. Thus, setting $\mathrm{F}(P; d, q) := \mathrm{F}(\mathrm{PG}(d-1, q), \Theta(P; d, q))$, then from Remark 9.3.1(a) we have

$$\mathrm{F}(P; d, q) = \{((\sigma, L), (\tau, N)) : (\sigma, L), (\tau, N) \in \Theta(P; d, q), \sigma \notin N, \tau \notin L\}.$$

Two distinct lines L, N of $\mathrm{PG}(d-1, q)$ are said to be *intersecting* if there exists a unique point incident with both L and N (that is, L, N lie on the same plane of $\mathrm{PG}(d-1, q)$), and *skew* otherwise. We use $\Psi^{+}(P; d, q)$ (respectively, $\Psi^{\simeq}(P; d, q)$) to denote the set of ordered pairs $((\sigma, L), (\tau, N)) \in \mathrm{F}(P; d, q)$ such that L, N are intersecting (respectively, skew). Then $\Psi^{+}(P; d, q)$ and $\Psi^{\simeq}(P; d, q)$ consist of a partition of $\mathrm{F}(P; d, q)$. Note that $\Psi^{\simeq}(P; d, q) \neq \emptyset$ if and only if $d \geq 4$ (see e.g. [84, pp.71]). So we have $\mathrm{F}(P; 3, q) = \Psi^{+}(P; 3, q)$.

Lemma 9.4.2 *Suppose* $\mathrm{PSL}(d, q) \leq G \leq \mathrm{P\Gamma L}(d, q)$, *where* $d \geq 3$ *and* q *is a prime power. Then the following* (a), (b) *hold.*

(a) *There exists a unique strict* 1*-feasible* G*-orbit on the flags of* $\mathrm{PG}(d-1,q)$*, namely* $\Theta(P;d,q)$.

(b) *If* $d=3$*, then* G *is transitive on* $\mathrm{F}(P;3,q)$*; if* $d\geq 4$*, then* G *has two orbits on* $\mathrm{F}(P;d,q)$*, namely* $\Psi^{+}(P;d,q)$ *and* $\Psi^{\simeq}(P;d,q)$.

Proof (a) Since $\mathrm{PG}(d-1,q)$ is a G-doubly transitive linear space, it is G-flag-transitive, and hence $\Theta(P;d,q)$ is the only candidate for a strict 1-feasible G-orbit on the flags of $\mathrm{PG}(d-1,q)$. In $\mathrm{PG}(d-1,q)$ each point is incident with $(q^{d-1}-1)/(q-1)\geq 3$ lines ([84, Theorem 2.5(iii)]). For distinct points σ,τ, let L be the unique line incident with both σ and τ. Let N_1,N_2 be two lines incident with σ but not τ, and let $\delta_i\in N_i\setminus\{\sigma\}$, $i=1,2$. Then (σ,τ,δ_1), (σ,τ,δ_2) are triples of non-collinear points. So by Lemma 9.4.1 there exists $g\in G$ such that $(\sigma,\tau,\delta_1)^g=(\sigma,\tau,\delta_2)$, and hence $g\in G_{\sigma\tau}$. Since N_i is the unique line incident with σ and δ_i, this implies $N_1^g=N_2$, and hence $\Theta(P;d,q)$ is strictly 1-feasible by Remark 9.3.1(a).

(b) Let $((\sigma_1,L_1),(\tau_1,N_1)),((\sigma_2,L_2),(\tau_2,N_2))\in\Psi^{+}(P;d,q)$. Let δ_i be the common point of L_i and N_i, for $i=1,2$. Then $(\sigma_1,\tau_1,\delta_1)$, $(\sigma_2,\tau_2,\delta_2)$ are triples of non-collinear points. By Lemma 9.4.1 we have $(\sigma_1,\tau_1,\delta_1)^g=(\sigma_2,\tau_2,\delta_2)$ for some $g\in G$. This implies $((\sigma_1,L_1),(\tau_1,N_1))^g=((\sigma_2,L_2),(\tau_2,N_2))$, and hence G is transitive on $\Psi^{+}(P;d,q)$. Similarly, for $((\sigma_1,L_1),(\tau_1,N_1)),((\sigma_2,L_2),(\tau_2,N_2))\in\Psi^{\simeq}(P;d,q)$, we can choose $\sigma_i'\in L_i\setminus\{\sigma_i\}$ and $\tau_i'\in N_i\setminus\{\tau_i\}$, for $i=1,2$. So $(\sigma_1',\sigma_1,\tau_1,\tau_1')$, $(\sigma_2',\sigma_2,\tau_2,\tau_2')$ are quadruples of independent points of $\mathrm{PG}(d-1,q)$. Again by Lemma 9.4.1 we have $(\sigma_1',\sigma_1,\tau_1,\tau_1')^g=(\sigma_2',\sigma_2,\tau_2,\tau_2')$ for some $g\in G$. This implies $((\sigma_1,L_1),(\tau_1,N_1))^g=((\sigma_2,L_2),(\tau_2,N_2))$, and hence G is transitive on $\Psi^{\simeq}(P;d,q)$. Since G preserves relative positions between lines and since $\Psi^{+}(P;d,q)$ and $\Psi^{\simeq}(P;d,q)$ consist of a partition of $\mathrm{F}(P;d,q)$, the assertions in (b) follow immediately. □

Clearly, both $\Psi^{+}(P;d,q)$ and $\Psi^{\simeq}(P;d,q)$ are self-paired. Hence the flag graphs of $\mathrm{PG}(d-1,q)$ with respect to $(\Theta(P;d,q),\Psi^{+}(P;d,q))$, $(\Theta(P;d,q),\Psi^{\simeq}(P;d,q))$ are well-defined. We denote these graphs by $\Gamma^{+}(P;d,q)$, $\Gamma^{\simeq}(P;d,q)$, respectively. (In defining $\Gamma^{\simeq}(P;d,q)$ we require that $d\geq 4$.) From Lemma 9.4.2 they are the only coexisting G-flag graphs of $\mathrm{PG}(d-1,q)$. Moreover, we have the following characterization of such graphs.

Lemma 9.4.3 *Suppose* $\mathrm{PSL}(d,q) \leq G \leq \mathrm{P\Gamma L}(d,q)$*, where* $d \geq 3$ *and* q *is a prime power. Suppose further that* $\mathcal{D}$ *is a 2-design, other than the trivial linear space, which admits* G *as a faithful, doubly transitive group of automorphisms. Then any coexisting* G*-flag graph of* $\mathcal{D}$ *is isomorphic to* $\Gamma^{+}(P;d,q)$ *or* $\Gamma^{\simeq}(P;d,q)$.

Proof The group G has only two faithful permutation representations, namely the natural actions on the points and hyperplanes of $\mathrm{PG}(d-1,q)$. Such representations are interchangable by an outer automorphism of $\mathrm{P\Gamma L}(d,q)$. So in the following it suffices to consider the usual action of G on the point set V of $\mathrm{PG}(d-1,q)$.

Since $\mathcal{D}$ is nontrivial, its block size is at least three. Suppose Θ is a strict 1-feasible G-orbit on the flags of $\mathcal{D}$, and let $(\sigma, L) \in \Theta$. Then, as shown in Section 9.3, we have:

(1) $\{N \setminus \{\sigma\} : (\sigma, N) \in \Theta\}$ is a G_σ-invariant partition of $V \setminus \{\sigma\}$.

We claim further that:

(2) For any $\tau, \delta \in L \setminus \{\sigma\}$, the points σ, τ, δ must be collinear in $\mathrm{PG}(d-1,q)$.

Suppose otherwise, and let ε be a point in a block N of $\mathcal{D}$ with $(\sigma, N) \in \Theta(\sigma)$ and $N \neq L$. Then in $\mathrm{PG}(d-1,q)$ either $\sigma, \tau, \varepsilon$ are non-collinear, or $\sigma, \delta, \varepsilon$ are non-collinear, since otherwise σ, τ, δ would be collinear, which contradicts our assumption. Without loss of generality we may suppose that $\sigma, \tau, \varepsilon$ are non-collinear in $\mathrm{PG}(d-1,q)$. Then by Lemma 9.4.1 there exists $g \in G$ such that $(\sigma, \tau, \delta)^g = (\sigma, \tau, \varepsilon)$. So we have $g \in G_{\sigma\tau}$. Since g fixes τ, by (1) it must fix L setwise. On the other hand, since g maps δ to ε, again from (1), g must map L to N. This is a contradiction and hence (2) is proved. From this it follows that, for each $(\sigma, L) \in \Theta$, the block L of $\mathcal{D}$ consists of some collinear points of $\mathrm{PG}(d-1,q)$. Moreover, we have:

(3) For each $(\sigma, L) \in \Theta$, the block L of $\mathcal{D}$ is a line of $\mathrm{PG}(d-1,q)$.

Suppose otherwise, then from (1), (2) there exists $(\sigma, N_1) \in \Theta$ such that the points of L and N_1 lie on the same line, say L^*, of $\mathrm{PG}(d-1,q)$. Since $d \geq 3$, we can take $(\sigma, N_2) \in \Theta$ such that the points in L and those in N_2 do not lie on the same line of $\mathrm{PG}(d-1,q)$. Take a point $\tau \in L \setminus \{\sigma\}$. Since Θ is strictly 1-feasible, by the second condition in Definition 9.2.1, there exists $g \in G_{\sigma\tau}$ such that $N_1^g = N_2$. Since g fixes σ and τ, it must fix the line L^* of $\mathrm{PG}(d-1,q)$. Hence the points in N_1 are mapped by g to some points on L^*. That is, the points in N_2 must lie on L^*. This is a contradiction and hence (3) is proved.

The claims (1) and (3) together imply that $\Theta(\sigma) = \Theta(P; d, q)(\sigma)$ for each $\sigma \in V$. So we have $\Theta = \Theta(P; d, q)$. In particular each line of $\mathrm{PG}(d-1, q)$ is a block of $\mathcal{D}$. Thus it follows from the definition that $\mathrm{F}(\mathcal{D}, \Theta) = \mathrm{F}(P; d, q)$. From Lemma 9.4.2(b), the result in Lemma 9.4.3 follows. $\square$

Applying Corollary 9.3.1 and Theorem 6.6.1, the discussion above leads to the following classification theorem, which is the main result in this section.

Theorem 9.4.1 *Suppose* $\mathrm{PSL}(d, q) \le G \le \mathrm{P\Gamma L}(d, q)$, *where* $d \ge 2$ *and* $q = p^e$ *with* p *a prime and* $e \ge 1$. *Then, if and only if either* $d \ge 3$ *or* $d = 2$ *and* G *is* 3-*transitive, there exists a* G-*symmetric graph* Γ *with* G *faithful on* $V(\Gamma)$ *which admits a nontrivial* G-*invariant partition* $\mathcal{B}$ *such that* $v = k + 1 \ge 3$ *and* $\Gamma_{\mathcal{B}} \cong K_{mv+1}$, *where* m *is the multiplicity of* $\mathcal{D}(B)$. *Moreover, all the possibilities of such* Γ, G *and the corresponding* m, v *can be classified as follows.*

(a) $\Gamma \cong (q+1) \cdot K_q$, $G = \mathrm{PGL}(2, q).\langle \psi^n \rangle$ *or* $\mathrm{M}(n, q)$ *(for suitable* p, e *and* n*), and* $(m, v) = (1, q)$.

(b) $(\Gamma, G) = (\mathrm{CR}(q; x, n), \mathrm{PGL}(2, q).\langle \psi^t \rangle)$ *and* $(m, v) = (1, q)$, *where* $x \in \mathrm{GF}(q) \setminus \{0, 1\}$, n *is a divisor of* $n(x)$, *and* t *is a divisor of* e *with* $\gcd(n(x), t) = n$.

(c) $(\Gamma, G) = (\mathrm{TCR}(q; x, n), \mathrm{M}(t/2, q))$ *and* $(m, v) = (1, q)$, *where* p *is odd,* $e \ge 2$ *is even,* $x \in \mathrm{GF}(q) \setminus \{0, 1\}$ *with* $n(x)$ *even and* $x - 1$ *a square of* $\mathrm{GF}(q)$, n *is an even divisor of* $n(x)$, *and* t *is a divisor of* e *with* $\gcd(n(x), t) = n$.

(d) $\Gamma = \Gamma^{+}(P; d, q)$ *or* $\Gamma^{\simeq}(P; d, q)$, *where* $d \ge 3$, G *is any doubly transitive subgroup of* $\mathrm{P\Gamma L}(d, q)$, *and* $(m, v) = (q, (q^{d-1} - 1)/(q - 1))$. *(The graph* $\Gamma^{\simeq}(P; d, q)$ *appears only when* $d \ge 4$.*)*

We conclude this section by proving the following properties of the projective flag graphs $\Gamma^{+}(P; d, q)$ and $\Gamma^{\simeq}(P; d, q)$. As before, we denote by $L_{\sigma\tau}$ the unique line of $\mathrm{PG}(d-1, q)$ through two distinct points σ and τ.

Theorem 9.4.2 *Let* $d \ge 3$ *and* q *a prime power, and let* $\Theta := \Theta(P; d, q)$. *Then the following* (a)-(c) *hold.*

(a) *Both* $\Gamma^{+}(P; d, q)$ *and* $\Gamma^{\simeq}(P; d, q)$ *are connected graphs with diameter two and girth three, and with valencies* $(q^{d+1} - q^3)/(q-1)$ *and* $(q^{d-1} - q^2)(q^d - q^2)/(q-1)^2$, *respectively.*

(b) *For distinct blocks* $\Theta(\sigma), \Theta(\tau)$ *of* $\mathcal{B}(\Theta)$, *each vertex of* $\Theta(\sigma)$ *other than* $(\sigma, L_{\sigma\tau})$ *is adjacent to exactly* q *vertices of* $\Theta(\tau)$ *in* $\Gamma^{+}(P; d, q)$, *and adjacent to exactly* $(q^{d-1} - q^2)/(q-1)$ *vertices of* $\Theta(\tau)$ *in* $\Gamma^{\simeq}(P; d, q)$. *In particular, for* $\Gamma := \Gamma^{+}(P; 3, q)$ *we have* $\Gamma[\Theta(\sigma), \Theta(\tau)] \cong K_{q,q}$.

(c) *For* $\mathrm{PSL}(d, q) \leq G \leq \mathrm{P\Gamma L}(d, q)$, *any* G*-symmetric graph with vertex set* Θ *(under the induced action) is isomorphic to either* $\Gamma^{+}(P; d, q)$, *or* $\Gamma^{\simeq}(P; d, q)$, *or* $(q^d - 1)/(q-1) \cdot K_{(q^{d-1}-1)/(q-1)}$ *with connected components the sets of flags incident with a common point, or* $(q^{d-1} - 1)(q^d - 1)/(q-1)(q^2-1) \cdot K_{q+1}$ *with connected components the sets of flags incident with a common line.*

Proof Let $(\sigma, L), (\tau, N) \in \Theta$ be distinct flags of $\mathrm{PG}(d-1, q)$. If $L \neq N$ then, since each line of $\mathrm{PG}(d-1, q)$ contains $q + 1 \geq 3$ points, we can take $\delta \in L \setminus \{\sigma, \tau\}$, $\varepsilon \in N \setminus \{\sigma, \tau\}$ and $\eta \in L_{\delta\varepsilon} \setminus \{\delta, \varepsilon\}$. One can check that the sequence $(\sigma, L), (\eta, L_{\delta\varepsilon}), (\tau, N)$ is a path of $\Gamma^{+}(P; d, q)$ with length two. In particular, if $(\sigma, L), (\tau, N)$ are adjacent in $\Gamma^{+}(P; d, q)$, then the sequence $(\sigma, L), (\eta, L_{\delta\varepsilon}), (\tau, N), (\sigma, L)$ is a triangle. Similarly, if $\sigma \neq \tau$ but $L = N$, then we can take $\delta \in L \setminus \{\sigma, \tau\}$ and a point ε not incident with L. Thus the sequence $(\sigma, L), (\varepsilon, L_{\delta\varepsilon}), (\tau, L)$ is a path of $\Gamma^{+}(P; d, q)$ with length two. Hence $\Gamma^{+}(P; d, q)$ is connected with diameter two and girth three. The definition of $\Gamma^{\simeq}(P; d, q)$ requires that $d \geq 4$. So for any distinct $(\sigma, L), (\tau, N) \in \Theta$, we can choose a line M which is skew with both L and N. For any $\delta \in M$, the sequence $(\sigma, L), (\delta, M), (\tau, N)$ is a path of $\Gamma^{\simeq}(P; d, q)$ with length two. Moreover, if $(\sigma, L), (\tau, N)$ are adjacent in $\Gamma^{\simeq}(P; d, q)$, then the sequence $(\sigma, L), (\delta, M), (\tau, N), (\sigma, L)$ is a triangle. Hence $\Gamma^{\simeq}(P; d, q)$ is connected with diameter two and girth three as well.

For any flag (σ, L) and any point τ not incident with L, there are exactly q lines which are incident with τ and intersect with L at a point other than σ, namely those lines joining τ and one of the points in $L \setminus \{\sigma\}$. Hence there are exactly $v - q - 1$ lines which are incident with τ and skew with L (note that $L_{\sigma\tau}$ is not skew with L), where $v = (q^{d-1} - 1)/(q-1)$ as before. From these the assertions in (b) follow immediately. Note that, for a point τ incident with L, (σ, L) is not adjacent to any vertex of $\Omega(\tau)$ in either $\Gamma^{+}(P; d, q)$ or $\Gamma^{\simeq}(P; d, q)$. Since L contains $q + 1$ points and $\mathrm{PG}(d-1,q)$ has $(q^d-1)/(q-1)$ points in total, from (b) the assertion in (a) concerning the valencies of $\Gamma^+(P;d,q)$ and $\Gamma^{\simeq}(P;d,q)$ follows.

Now let us prove (c). Suppose Γ is a graph with vertex set Θ which is G-symmetric under the induced action of G on Θ. Let $((\sigma,L),(\tau,N))$ be an arc of Γ. If $\sigma=\tau$, then $L\neq N$, and two flags (σ_1,L_1), (τ_1,N_1) are adjacent in Γ if and only if $\sigma_1=\tau_1$ and $L_1\neq N_1$. Since $\mathrm{PG}(d-1,q)$ has $(q^d-1)/(q-1)$ points, and since each point is incident with exactly $(q^{d-1}-1)/(q-1)$ lines, in this case we have $\Gamma\cong(q^d-1)/(q-1)\cdot K_{(q^{d-1}-1)/(q-1)}$. Similarly, if $L=N$, then we have $\Gamma\cong(q^{d-1}-1)(q^d-1)/(q-1)(q^2-1)\cdot K_{q+1}$. In the following we suppose that $\sigma\neq\tau$ and $L\neq N$. Then the G-symmetry of Γ implies that there exists $g\in G$ which interchanges (σ,L) and (τ,N). So we have $\sigma\notin N$ for otherwise we would have $\sigma\in L\cap N$ and thus $\tau=\sigma^g\in(L\cap N)^g=L\cap N$, which implies $\sigma=\tau$ and so contradicts with our assumption. Similarly, we have $\tau\notin L$ and hence $((\sigma,L),(\tau,N))\in\mathrm{F}(P;d,q)$. Thus, since Γ is G-symmetric, its arc set $\mathrm{Arc}(\Gamma)$ is a self-paired G-orbit on $\mathrm{F}(P;d,q)$. Therefore, from Lemma 9.4.2, Γ is isomorphic to either $\Gamma^+(P;d,q)$ or $\Gamma^{\simeq}(P;d,q)$. □

9.5 Affine flag graphs

For an integer $d\geq 2$ and a prime power q, we use the same notation $\mathrm{AG}(d,q)$ to denote the (point, line)-incidence structure of the affine geometry $\mathrm{AG}(d,q)$. Thus, for any group G with $\mathrm{AGL}(d,q)\leq G\leq \mathrm{A\Gamma L}(d,q)$, $\mathrm{AG}(d,q)$ is a G-doubly transitive linear space. The purpose of this section is to classify and characterize the coexisting G-flag graphs of $\mathrm{AG}(d,q)$.

From Lemma 6.5.1 and Remark 9.3.1, it is easily verified that the flag set $\Theta(A;d,q)$ of $\mathrm{AG}(d,q)$ is strictly 1-feasible. Thus, setting

$$\mathrm{F}(A;d,q):=\mathrm{F}(\mathrm{AG}(d,q),\Theta(A;d,q)),$$

we have

$$\mathrm{F}(A;d,q)=\{((\sigma,L),(\tau,N)):(\sigma,L),(\tau,N)\in\Theta(A;d,q),\sigma\notin N,\tau\notin L\}.$$

We call two distinct lines of $\mathrm{AG}(d,q)$ *intersecting* if they share a unique common point, *parallel* if they lie on the same plane but have no point in common, and *skew* in the remaining case. We use $\Psi^+(A; d, q)$ ($\Psi^=(A; d, q)$, $\Psi^{\simeq}(A; d, q)$, respectively) to denote the set of ordered pairs $((\sigma, L), (\tau, N))$ in $\mathrm{F}(A; d, q)$ such that L, N are intersecting (parallel, skew, respectively). Then $\Psi^+(A; d, q)$, $\Psi^=(A; d, q)$ and $\Psi^{\simeq}(A; d, q)$ consist of a partition of $\mathrm{F}(A; d, q)$. (Note that $\Psi^{\simeq}(A; d, q) \neq \emptyset$ if and only if $d \geq 3$, see [84, Theorem 1.15(i)].) Using Lemma 6.5.1 and by a similar argument as in the proof of Lemma 9.4.2, one can prove the following lemma.

Lemma 9.5.1 *Suppose* $\mathrm{AGL}(d, q) \leq G \leq \mathrm{A\Gamma L}(d, q)$, *where* $d \geq 2$ *and* q *is a prime power. Then the following* (a), (b) *hold.*

(a) *There exists a unique strict* 1*-feasible* G*-orbit on the flags of* $\mathrm{AG}(d, q)$, *namely* $\Theta(A; d, q)$.

(b) *If* $d = 2$, *then* G *has two orbits on* $\mathrm{F}(A; d, q)$, *namely* $\Psi^+(A; 2, q)$ *and* $\Psi^=(A; 2, q)$; *if* $d \geq 3$, *then* G *has three orbits on* $\mathrm{F}(A; d, q)$, *namely* $\Psi^+(A; d, q)$, $\Psi^=(A; d, q)$ *and* $\Psi^{\simeq}(A; d, q)$.

Clearly, $\Psi^+(A; d, q)$, $\Psi^=(A; d, q)$ and $\Psi^{\simeq}(A; d, q)$ are all self-paired. Thus the flag graphs of $\mathrm{AG}(d, q)$ with respect to $(\Theta(A; d, q), \Psi)$, for $\Psi = \Psi^+(A; d, q)$, $\Psi^=(A; d, q)$ and $\Psi^{\simeq}(A; d, q)$, are well-defined. We use $\Gamma^+(A; n, q)$, $\Gamma^=(A; n, q)$ and $\Gamma^{\simeq}(A; n, q)$ respectively to denote these graphs. (In defining $\Gamma^{\simeq}(A; d, q)$ we require that $d \geq 3$.) From Lemma 9.5.1, these are the only coexisting G-flag graphs of $\mathrm{AG}(d, q)$, for G as above. Moreover, the following lemma shows that they are in fact the only coexisting G-flag graphs of any G-doubly transitive 2-design. The proof of this result is similar to that of Lemma 9.4.3 and hence is omitted. (In the proof we make use of the following fact: The only faithful permutation representation of G is its natural action on $V(d, q)$.)

Lemma 9.5.2 *Suppose* $\mathrm{AGL}(d, q) \leq G \leq \mathrm{A\Gamma L}(d, q)$, *where* $d \geq 2$ *and* q *is a prime power. Suppose further that* $\mathcal{D}$ *is a* 2*-design which admits* G *as a faithful, doubly transitive group of automorphisms. Then any coexisting* G*-flag graph of* $\mathcal{D}$ *is isomorphic to* $\Gamma^+(A; d, q)$, $\Gamma^=(A; d, q)$, *or* $\Gamma^{\simeq}(A; d, q)$.

Remark 9.5.1 The affine geometry $\mathrm{AG}(d, q)$ has $mv + 1 := q^d$ points, and each line of it contains $m + 1 := q$ points. So we have $v = (q^d - 1)/(q - 1)$ and $m = q - 1$. Thus, $\mathrm{AG}(d, q)$ is the trivial linear space if and only if $q = 2$, which in turn is true if and only if $\mathrm{AGL}(d, q)$ is 3-transitive on $V(d, q)$. Hence, from Example 9.2.1,

$\Gamma^{+}(A;d,2)$, $\Gamma^{=}(A;d,2)$ and $\Gamma^{\simeq}(A;d,2)$ are all 3-arc graphs of the complete graph Σ with vertex set $V(d,2)$. The vertices of these three graphs are ordered pairs $\mathbf{uw}$ of distinct vectors of $V(d,2)$. Since each plane of $\mathrm{AG}(d,2)$ contains exactly four points ([84, Theorem 1.17]), one can see that $\mathbf{uw},\mathbf{yz}$ are adjacent in $\Gamma^{+}(A;d,2)$ if and only if $\mathbf{w}=\mathbf{z}$. So $\Gamma^{+}(A;d,2)$ is isomorphic to $2^d \cdot K_{2^d-1}$ and is the 3-arc graph of Σ with respect to the set of all 3-cycles of Σ. Similarly, $\mathbf{uw},\mathbf{yz}$ are adjacent in $\Gamma^{=}(A;d,2)$ if and only if $\mathbf{u},\mathbf{w},\mathbf{y},\mathbf{z}$ are distinct and $\mathbf{u}-\mathbf{w}=\mathbf{y}-\mathbf{z}$, and they are adjacent in $\Gamma^{\simeq}(A;d,2)$ if and only if $\mathbf{u},\mathbf{w},\mathbf{y},\mathbf{z}$ do not lie on the same plane of $\mathrm{AG}(d,2)$. Thus $\Gamma^{=}(A;d,2)$ and $\Gamma^{\simeq}(A;d,2)$ are, respectively, the 3-arc graphs $\Xi_1(d,2)$ and $\Xi_2(d,2)$ defined in Example 6.5.1.

From Corollary 9.3.1 and the discussion above, we come to the following main result of this section.

Theorem 9.5.1 *Suppose* $\mathrm{AGL}(d,q) \le G \le \mathrm{A\Gamma L}(d,q)$, *where* $d \ge 2$ *and* q *is a prime power. Then there exists a* G*-symmetric graph* Γ *with* G *faithful on* $V(\Gamma)$ *which admits a nontrivial* G*-invariant partition* $\mathcal{B}$ *such that* $v = k+1 \ge 3$ *and* $\Gamma_{\mathcal{B}} \cong K_{mv+1}$. *Moreover, each such graph* Γ *is isomorphic to* $\Gamma^{+}(A;d,q)$, $\Gamma^{=}(A;d,q)$, *or* $\Gamma^{\simeq}(A;d,q)$ *(the third graph appears only when* $d \ge 3$*). In each case we have* $v=(q^d-1)/(q-1)$ *and the multiplicity* m *of* $\mathcal{D}(B)$ *for* $B \in \mathcal{B}$ *is equal to* $q-1$.

By a similar argument as in the proof of Theorem 9.4.2, one can prove the following properties of the affine flag graphs above.

Theorem 9.5.2 *Let* $d \ge 2$ *and* $q \ge 2$ *be a prime power, and set* $\Theta := \Theta(A;d,q)$. *Then the following* (a)-(d) *hold.*

(a) *Both* $\Gamma^{+}(A;d,q)$ *and* $\Gamma^{\simeq}(A;d,q)$ *are connected graphs with diameter two and girth three, and with valencies* $(q-1)(q^d-q)$ *and* $(q^d-q^2)(q^d-q)/(q-1)$, *respectively.*

(b) $\Gamma^{=}(A;d,q)$ *has valency* q^d-q *and contains* $(q^d-1)/(q-1)$ *connected components, each of which is a complete* q^{d-1}*-partite graph with* q *vertices in each part. Moreover,* $\Gamma^{=}(A;d,q)$ *is an almost cover of* K_{q^d}.

(c) *For distinct blocks* $\Theta(\sigma),\Theta(\tau)$ *of* $\mathcal{B}(\Theta)$, *each vertex* (σ,L) *of* $\Theta(\sigma)$ *other than* $(\sigma,L_{\sigma\tau})$ *is adjacent to exactly* $q-1$ *vertices of* $\Theta(\tau)$ *in* $\Gamma^{+}(A;d,q)$, *and adjacent to exactly* $(q^d-q^2)/(q-1)$ *vertices of* $\Theta(\tau)$ *in* $\Gamma^{\simeq}(A;d,q)$. *In particular, for* $\Gamma := \Gamma^{+}(A;2,q)$, $\Gamma[\Theta(\sigma),\Theta(\tau)]$ *is isomorphic to* $K_{q,q}$ *minus a perfect matching.*

(d) *For* $\mathrm{AGL}(d, q) \leq G \leq \mathrm{A\Gamma L}(d, q)$, *any* G*-symmetric graph with vertex set* Θ *(under the induced action) is isomorphic to* $\Gamma^{+}(A; d, q)$, *or* $\Gamma^{\simeq}(A; d, q)$, *or* $\Gamma^{=}(A; d, q)$, *or* $q^d \cdot K_{(q^d-1)/(q-1)}$ *with connected components the sets of flags incident with a common point, or* $q^{d-1}(q^d - 1)/(q - 1) \cdot K_q$ *with connected components the sets of flags incident with a common line.*

Chapter 10

Local actions

To say that you know when you do know and say that you do not know when you do not know – that is the way to acquire knowledge.
Confucius (551-479 B.C.), LUN YÜ [THE ANALECTS] 2:17

10.1 Introduction

In Section 3.2 we defined $G_{(B)}$ and $G_{[B]}$ to be the kernels of the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$, respectively. In this chapter, we will study actions induced by these two kernels. In particular, we will investigate the action of $G_{[B]}$ on B and the actions of $G_{(B)}$ on $\Gamma_{\mathcal{B}}(B)$, $\Gamma(\alpha)$ and $\Gamma_{\mathcal{B}}(\alpha)$ (where $\alpha \in B$), and the influence of these "local actions" on the structure of Γ. It is expected that the investigation in this chapter would provide a basis for future study of imprimitive symmetric graphs. For our purpose it seems natural to distinguish whether one of $G_{(B)}$, $G_{[B]}$ is a subgroup of the other. With respect to this, we have the following (not necessarily exclusive) possibilities: (i) $G_{[B]} \leq G_{(B)}$; (ii) $G_{[B]} \not\leq G_{(B)}$; (iii) $G_{(B)} \leq G_{[B]}$; (iv) $G_{(B)} \not\leq G_{[B]}$; (v) $G_{[B]} \not\leq G_{(B)}$ and $G_{(B)} \not\leq G_{[B]}$. Setting $M = G_{(B)}G_{[B]}$, then $M \trianglelefteq G_B$ and we have Figure 6 in the lattice of subgroups of G_B.

We will put our discussion in a general setting and consider the following subgroups of G_B. Let $d := \mathrm{diam}(\Gamma_{\mathcal{B}})$, which can be finite (if $\Gamma_{\mathcal{B}}$ is connected) or ∞ (otherwise). For each integer i with $0 \leq i < d{+}1$, let $\Gamma_{\mathcal{B}}(i, B)$ denote the set of blocks of $\mathcal{B}$ with distance in $\Gamma_{\mathcal{B}}$ no more than i from B. Then $\Gamma_{\mathcal{B}}(i, B)$ is G_B-invariant and hence G_B induces a natural action on $\Gamma_{\mathcal{B}}(i, B)$. We will use $G_{[i,B]}$ to denote the kernel of this action, so in particular we have $G_B = G_{[0,B]}$ and $G_{[B]} = G_{[1,B]}$. Figure 7 illustrates the relationships among these groups $G_{[i,B]}$ in the lattice of subgroups of G_B when d is finite.

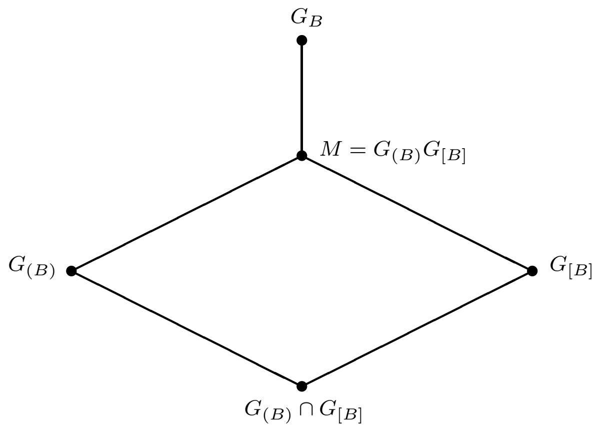


FIGURE 6 $G_{(B)}$ and $G_{[B]}$

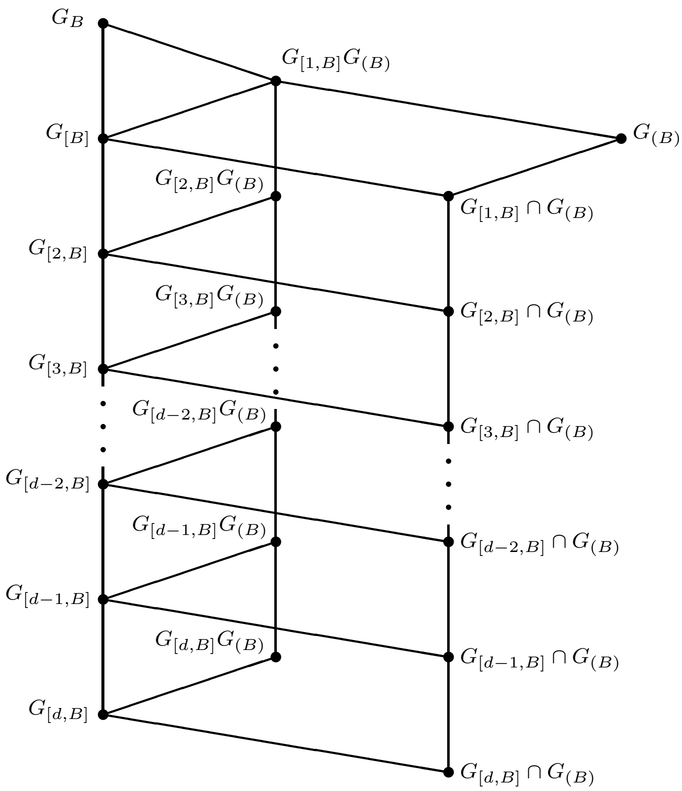


FIGURE 7 Relationships among $G_{[i,B]}$'s

The main results of this chapter are as follows. In Section 10.2, we will show (Theorem 10.2.1) that each $G_{[i,B]}$ induces a G-invariant partition $\mathcal{B}_i$ of $V(\Gamma)$ such that the sequence $\mathcal{B} = \mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_i, \ldots$ is a tower possessing some nice "level structure" properties, where as in [66] a sequence of G-invariant partitions is said to be a *tower* if each partition is a refinement of the previous partition. We will show (Theorem 10.2.2) further that, if $G_{[i,B]} \leq G_{(B)}$ for some $i \geq 1$ then G is faithful on $\mathcal{B}$; whilst if $G_{[i,B]} \not\leq G_{(B)}$ for some $i \geq 1$ then either $\mathcal{B}_i$ is a genuine refinement of $\mathcal{B}$ or Γ is a multicover of $\Gamma_{\mathcal{B}}$. In Section 10.3 we will study a special case where, for any $C, D \in \Gamma_{\mathcal{B}}(B)$, either $\Gamma(C) \cap B = \Gamma(D) \cap B$, or $\Gamma(C) \cap \Gamma(D) \cap B = \emptyset$. Based on the discussions in these two sections, we then study in Section 10.4 the case where Γ is G-locally quasiprimitive. In this case we will show (Corollary 10.4.1) amongst other things that, if $\mathcal{B}$ is a minimal G-invariant partition, then either $k = 1$, or $G_{[B]} = G_{(B)}$, or Γ is a multicover of $\Gamma_{\mathcal{B}}$. Recall that, for $\alpha \in V(\Gamma)$, we use $G_{[\alpha]}$ to denote the subgroup of G_α fixing setwise each block $C \in \Gamma_{\mathcal{B}}(\alpha)$. So $G_{[\alpha]}$ induces a natural action on $\Gamma(\alpha) \cap C$. In Section 10.4 we will also study G-locally quasiprimitive graphs Γ such that $G_{[\alpha]}$ is transitive on $\Gamma(\alpha) \cap C$, and prove that in this case either Γ is a bipartite graph or $\Gamma[B, C]$ is a matching.

As in most part of this thesis, we will identify in this chapter the blocks of $\mathcal{D}(B)$ with the subsets $\Gamma(C) \cap B$ of B (with multiplicity m), for $C \in \Gamma_{\mathcal{B}}(B)$. We conclude this introductory section by making the following observations.

Lemma 10.1.1 *Suppose Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$. Let $B \in \mathcal{B}$ and $\alpha \in B$, and set $d := \mathrm{diam}(\Gamma_{\mathcal{B}})$. Then the following* (a)-(f) *hold.*

(a) $G_{(B)} \lhd G_B$.

(b) $G_{(B)} \unlhd G_\alpha$.

(c) $G_{[\alpha]} \unlhd G_\alpha$.

(d) $G_{[i,B]} \unlhd G_B$ *for each integer i with $0 \leq i < d + 1$. In particular, we have $G_{[B]} \unlhd G_B$.*

(e) $G_{[i,B]} \unlhd G_{[i-1,B]}$ *for each integer i with $1 \leq i < d + 1$.*

(f) *If i is at least the diameter of the connected components of $\Gamma_{\mathcal{B}}$, then $G_{[i,B]}$ is equal to the kernel of the induced action of G on the component of $\Gamma_{\mathcal{B}}$ containing B. In particular, if $\Gamma_{\mathcal{B}}$ is connected, then $G_{[d,B]}$ is equal to the kernel of the action of G on $\mathcal{B}$.*

Proof Since $G_{(B)}$ is the kernel of the action of G_B on B, we have $G_{(B)} \trianglelefteq G_B$. Since G_B is transitive on B whilst $G_{(B)}$ is not, we have $G_{(B)} \neq G_B$ and (a) is proved. From this and $G_{(B)} \leq G_\alpha \leq G_B$, we get (b) immediately. Similarly, (c) follows from the fact that $G_{[\alpha]}$ is the kernel of the action of G_α on $\Gamma_{\mathcal{B}}(\alpha)$. Since $G_{[i,B]}$ is the kernel of the action of G_B on $\Gamma_{\mathcal{B}}(i, B)$, we have $G_{[i,B]} \trianglelefteq G_B$ for each i with $0 \leq i < d+1$. In particular, we have $G_{[B]} = G_{[1,B]} \trianglelefteq G_B$ and thus (d) is proved. For $1 \leq i < d+1$, since $G_{[i,B]} \leq G_{[i-1,B]} \leq G_B$ and since $G_{[i,B]} \trianglelefteq G_B$ by (d), we get (e) immediately. The G-symmetry of $\Gamma_{\mathcal{B}}$ implies that its connected components are isomorphic and hence have the same diameter. If i is no less than this diameter, then $\Gamma_{\mathcal{B}}(i, B)$ is equal to the set of blocks in the component of $\Gamma_{\mathcal{B}}$ containing B and G induces an action on $\Gamma_{\mathcal{B}}(i, B)$. Hence the validity of the statements in (f) follows. □

10.2 G-invariant partitions induced by $G_{[i,B]}$

We first prove the following general result, which shows that each normal subgroup of G_B induces a refinement of the given G-invariant partition $\mathcal{B}$.

Lemma 10.2.1 *Suppose Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$, and let $B \in \mathcal{B}$. Then each normal subgroup N of G_B induces a G-invariant partition $\mathcal{B}_N^*$ of $V(\Gamma)$. Moreover, $\mathcal{B}_N^*$ is a refinement of $\mathcal{B}$ and the following* (a)-(c) *hold.*

(a) *$\mathcal{B}_N^*$ is the trivial partition $\{\{\alpha\} : \alpha \in V(\Gamma)\}$ if and only if $N \leq G_{(B)}$.*

(b) *$\mathcal{B}_N^*$ coincides with $\mathcal{B}$ if and only if N is transitive on B.*

(c) *If N is a normal subgroup of G, then $\mathcal{B}_N^*$ coincides with the G-normal partition $\mathcal{B}_N$ of $V(\Gamma)$ induced by N (defined after Lemma 2.2.2).*

Proof Since $N \trianglelefteq G_B$ and G_B is transitive on B, Lemma 2.2.2 implies that $B^* := \alpha^N$ (for some $\alpha \in B$) is a block of imprimitivity for G_B in B. Since $\mathcal{B}$ is a G-invariant partition of $V(\Gamma)$, this implies that B^* is a block of imprimitivity for G in $V(\Gamma)$. Hence B^* induces a G-invariant partition of $V(\Gamma)$, namely,

$$\mathcal{B}_N^* := \{(B^*)^g : g \in G\}. \tag{10.1}$$

The validity of (a)-(c) follows from the definition of $\mathcal{B}_N^*$ immediately. □

Remark 10.2.1 For distinct blocks $B, C \in \mathcal{B}$, there exists $g \in G$ such that $B^g = C$. So $(G_B)^g = G_C$ by Lemma 2.1.1(a), and hence $N \trianglelefteq G_B$ if and only if $N^g \trianglelefteq G_C$. It is easy to see that $\mathcal{B}^*_{N^g} = \mathcal{B}^*_N$. So, in studying the G-invariant partition $\mathcal{B}^*_N$, we can start from any chosen block $B \in \mathcal{B}$.

In Lemma 10.1.1(d) we have seen that $G_{[i,B]}$ is a normal subgroup of G_B, for each integer i with $0 \le i < \mathrm{diam}(\Gamma_{\mathcal{B}}) + 1$. So it follows from Lemma 10.2.1 that $G_{[i,B]}$ induces a G-invariant partition

$$\mathcal{B}_i := \{B_i^g : g \in G\} \tag{10.2}$$

of $V(\Gamma)$ which is a refinement of $\mathcal{B}$, where $B_i := \alpha^{G_{[i,B]}}$ (for some $\alpha \in B$) is a typical block of $\mathcal{B}_i$. Let v_i, r_i, b_i, k_i, s_i denote the parameters with respect to $\mathcal{B}_i$, as defined in Section 3.2. Since $\mathcal{B}_0$ is precisely the original partition $\mathcal{B}$, we have $(v_0, r_0, b_0, k_0, s_0) = (v, r, b, k, s)$. The following theorem gives some "level structure" properties concerning these partitions.

Theorem 10.2.1 *Suppose Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$. Let $B \in \mathcal{B}$ and set $d := \mathrm{diam}(\Gamma_{\mathcal{B}})$. Then for each integer i with $0 \le i < d+1$, $G_{[i,B]}$ induces a G-invariant partition $\mathcal{B}_i$, defined in (10.2), which is a refinement of $\mathcal{B}$. Moreover, for $1 \le i < d+1$, $\mathcal{B}_i$ is a refinement of $\mathcal{B}_{i-1}$ and the following* (a)-(d) *hold.*

(a) *v_i is a common divisor of v_{i-1} and k_{i-1}, s_i is a divisor of s_{i-1}, and r_{i-1} is a divisor of r_i (with $s_{i-1}/s_i = r_i/r_{i-1}$).*

(b) *Each block of the 1-design $\mathcal{D}(B_{i-1})$ (for $B_{i-1} \in \mathcal{B}_{i-1}$) is a disjoint union of some blocks of $\mathcal{B}_i$. More precisely, for adjacent blocks B_{i-1}, C_{i-1} of $\Gamma_{\mathcal{B}_{i-1}}$, $G_{[i,B]}$ leaves $\Gamma(C_{i-1}) \cap B_{i-1}$ invariant and the $(G_{[i,B]})$-orbits on $\Gamma(C_{i-1}) \cap B_{i-1}$ form a $(G_{B_{i-1},C_{i-1}})$-invariant partition of $\Gamma(C_{i-1}) \cap B_{i-1}$.*

(c) *$\Gamma_{\mathcal{B}_{i-1}}(\alpha) = \Gamma_{\mathcal{B}_{i-1}}(\beta)$ for any vertices α, β in the same block of $\mathcal{B}_i$.*

(d) *For each integer j with $0 \le j < i$, the set $\mathcal{B}_i$ admits a G-invariant partition $\mathbf{B}_{ij}$ such that $\Gamma_{\mathcal{B}_j} \cong (\Gamma_{\mathcal{B}_i})_{\mathbf{B}_{ij}}$ and that the parameters $\mathtt{v}_{ij}, \mathtt{r}_{ij}, \mathtt{b}_{ij}, \mathtt{k}_{ij}, \mathtt{s}_{ij}$ with respect to $\mathbf{B}_{ij}$ satisfy $\mathtt{v}_{ij} = v_j/v_i, \mathtt{k}_{ij} = k_j/v_i, \mathtt{b}_{ij} = b_j, \mathtt{r}_{ij} = r_j, \mathtt{s}_{ij} = b_i/r_j$.*

Proof Let $\alpha \in B$ and $B_i := \alpha^{G_{[i,B]}}$, and let $\mathcal{B}_i$ be as defined in (10.2) for each i. Then, since $G_{[i,B]} \trianglelefteq G_B$ by Lemma 10.1.1(d), Lemma 10.2.1 implies that $\mathcal{B}_i$ is a G-invariant partition of $V(\Gamma)$ and is a refinement of $\mathcal{B}$. For $1 \leq i < d+1$, since $G_{[i,B]} \trianglelefteq G_{[i-1,B]}$ (Lemma 10.1.1(e)), it follows that $\mathcal{B}_i$ is a refinement of $\mathcal{B}_{i-1}$. Consequently, v_i is a divisor of v_{i-1}.

Now suppose C_{i-1} is a block of $\mathcal{B}_{i-1}$ adjacent to B_{i-1} in $\Gamma_{\mathcal{B}_{i-1}}$, and let C be the block of $\mathcal{B}$ containing C_{i-1}. Then there exist $\beta \in \Gamma(C_{i-1}) \cap B_{i-1}$ and $\gamma \in \Gamma(B_{i-1}) \cap C_{i-1}$ such that β, γ are adjacent in Γ. By the definition of $\mathcal{B}_{i-1}$, we have $B_{i-1} = \beta^{G_{[i-1,B]}}$ and $C_{i-1} = \gamma^{G_{[i-1,C]}}$, and by Lemma 3.2.3(c) we have $\Gamma(C_{i-1}) \cap B_{i-1} = \beta^{G_{B_{i-1},C_{i-1}}}$ and $\Gamma(B_{i-1}) \cap C_{i-1} = \gamma^{G_{B_{i-1},C_{i-1}}}$. Note that B, C are adjacent blocks of $\mathcal{B}$. So we have $\Gamma_{\mathcal{B}}(i-1, C) \subseteq \Gamma_{\mathcal{B}}(i, B)$ and hence $G_{[i,B]} \leq G_{[i-1,C]}$. This implies that $G_{[i,B]}$ fixes C_{i-1} setwise. Since $G_{[i,B]} \trianglelefteq G_{[i-1,B]}$, $G_{[i,B]}$ also fixes B_{i-1} setwise. Thus, we have $G_{[i,B]} \leq G_{B_{i-1},C_{i-1}}$. This implies $G_{[i,B]} \trianglelefteq G_{B_{i-1},C_{i-1}}$ since $G_{B_{i-1},C_{i-1}} \leq G_B$ and $G_{[i,B]} \trianglelefteq G_B$ (Lemma 10.1.1(d)). So $G_{[i,B]}$ leaves $\Gamma(C_{i-1}) \cap B_{i-1}$ invariant and, again by Lemma 2.2.2, the $(G_{[i,B]})$-orbits on $\Gamma(C_{i-1}) \cap B_{i-1}$ constitute a $(G_{B_{i-1},C_{i-1}})$-invariant partition of $\Gamma(C_{i-1}) \cap B_{i-1}$. Thus, each block $\Gamma(C_{i-1}) \cap B_{i-1}$ of the 1-design $\mathcal{D}(B_{i-1})$ is a disjoint union of some blocks of $\mathcal{B}_i$. This implies in particular that v_i is a divisor of k_{i-1}, and so v_i is a common divisor of v_{i-1} and k_{i-1}. One can see that each block C_{i-1} of $\Gamma_{\mathcal{B}_{i-1}}(\beta)$ contains the same number of blocks of $\Gamma_{\mathcal{B}_i}(\beta)$. Hence r_{i-1} is a divisor of r_i. Since $r_{i-1}s_{i-1} = r_i s_i = \mathrm{val}(\Gamma)$, this implies that s_i is a divisor of s_{i-1}.

If δ, ε are in the same block of $\mathcal{B}_i$, without loss of generality we may suppose that $\delta, \varepsilon \in B_i$. Then since B_i is a $(G_{[i,B]})$-orbit there exists $x \in G_{[i,B]}$ such that $\delta^x = \varepsilon$, and hence $(\Gamma_{\mathcal{B}_{i-1}}(\delta))^x = \Gamma_{\mathcal{B}_{i-1}}(\varepsilon)$. On the other hand, the elements of $G_{[i,B]}$ fix setwise each block C_{i-1} in $\Gamma_{\mathcal{B}_{i-1}}(B_{i-1})$ since $G_{[i,B]} \trianglelefteq G_{B_{i-1},C_{i-1}}$, as shown above. In particular, x fixes setwise each block in $\Gamma_{\mathcal{B}_{i-1}}(\delta)$ since $\Gamma_{\mathcal{B}_{i-1}}(\delta) \subseteq \Gamma_{\mathcal{B}_{i-1}}(B_{i-1})$. Thus, we have $\Gamma_{\mathcal{B}_{i-1}}(\delta) = (\Gamma_{\mathcal{B}_{i-1}}(\delta))^x = \Gamma_{\mathcal{B}_{i-1}}(\varepsilon)$.

Let j be an integer with $0 \leq j < i$. Since for each ℓ with $j+1 \leq \ell \leq i$ the partition $\mathcal{B}_\ell$ is a refinement of the partition $\mathcal{B}_{\ell-1}$, as shown above, we know that $\mathcal{B}_i$ is a refinement of $\mathcal{B}_j$ and hence each block C_j of $\mathcal{B}_j$ is a union of some blocks of $\mathcal{B}_i$. Denote $\mathbb{C}_{ij} = \{B_i^z : B_i^z \subseteq C_j, z \in G\}$, the set of blocks of $\mathcal{B}_i$ contained in C_j. Then $\mathbf{B}_{ij} := \{\mathbb{C}_{ij} : C_j \in \mathcal{B}_j\}$ is a partition of $\mathcal{B}_i$. We claim further that $\mathbf{B}_{ij}$ is a G-invariant partition of $\mathcal{B}_i$ under the induced action of G on $\mathcal{B}_i$. In fact, if $\mathbb{C}_{ij}^g \cap \mathbb{C}_{ij} \neq \emptyset$ for some $g \in G$, say $(B_i^x)^g = B_i^y$ for some $B_i^x, B_i^y \in \mathbb{C}_{ij}$, then $B_i^x, B_i^y \subseteq C_j$ and hence $(B_i^x)^g = B_i^y \subseteq C_j$. Since C_j is a block of imprimitivity for G in $V(\Gamma)$, this implies that g fixes C_j setwise. Therefore, we have $\mathbb{C}_{ij}^g = \{(B_i^z)^g : B_i^z \subseteq C_j, z \in G\} = \mathbb{C}_{ij}$ and hence $\mathbf{B}_{ij}$ is G-invariant indeed. Clearly, the mapping $\psi : C_j \mapsto \mathbb{C}_{ij}$ is a bijection from $\mathcal{B}_j$ to $\mathbf{B}_{ij}$. By the definition of a quotient graph, one can see that $\Gamma_{\mathcal{B}_j} \cong (\Gamma_{\mathcal{B}_i})_{\mathbf{B}_{ij}}$ with respect to ψ. Clearly, we have $\mathtt{v}_{ij} = v_j/v_i, \mathtt{k}_{ij} = k_j/v_i, \mathtt{b}_{ij} = b_j$ and $\mathtt{r}_{ij}\mathtt{s}_{ij} = \mathrm{val}(\Gamma_{\mathcal{B}_i}) = b_i$. From $\mathtt{v}_{ij}\mathtt{r}_{ij} = \mathtt{b}_{ij}\mathtt{k}_{ij}$, we get $(v_j/v_i)\mathtt{r}_{ij} = b_j(k_j/v_i)$, which in turn implies $\mathtt{r}_{ij} = r_j$ since $v_j r_j = b_j k_j$. Finally, we have $\mathtt{s}_{ij} = b_i/\mathtt{r}_{ij} = b_i/r_j$ and the proof is complete. $\square$

Remark 10.2.2 If $G_{[i,B]} \trianglelefteq G$ for $B \in \mathcal{B}$, then from Lemma 10.2.1(c), $\mathcal{B}_i$ is the G-normal partition of $V(\Gamma)$ induced by $G_{[i,B]}$. In this case Γ is a multicover of $\Gamma_{\mathcal{B}_i}$ (see [66, Section 1] or [71, Theorem 4.1]). This happens in particular for $\Gamma_{\mathcal{B}_1}$ when $\Gamma_{\mathcal{B}}$ is a complete graph since in this case we have $d = 1$ and $G_{[B]}$ is the kernel of the action of G on $\mathcal{B}$.

Theorem 10.2.2 *Suppose the triple* $(\Gamma, G, \mathcal{B})$ *is as in Theorem 10.2.1. Let* $B \in \mathcal{B}$ *and let an integer* i *satisfy* $1 \le i < d+1$, *where* $d := \mathrm{diam}(\Gamma_{\mathcal{B}})$. *The one of the following* (a), (b) *holds.*

(a) $G_{[i,B]} \le G_{(B)}$, *in this case G is faithful on $\mathcal{B}$ if in addition G is faithful on* $V(\Gamma)$.

(b) $G_{[i,B]} \not\le G_{(B)}$, *and either*

  (i) $G_{[i,B]}$ *induces the G-invariant partition $\mathcal{B}_i$ of $V(\Gamma)$, defined in (10.2), which is a genuine refinement of $\mathcal{B}$ such that v_i is a common divisor of v and k, s_i is a divisor of s, and r is a divisor of r_i; or*

  (ii) Γ *is a multicover of* $\Gamma_{\mathcal{B}}$ *and* $G_{[i,B]}$ *is transitive on* B.

Proof Suppose that $G_{[i,B]} \le G_{(B)}$. Then, since G is transitive on $\mathcal{B}$ and since $G_{[i,B^g]} = (G_{[i,B]})^g$ and $G_{(B^g)} = (G_{(B)})^g$ for any $g \in G$, we have $G_{[i,C]} \le G_{(C)}$ for all blocks $C \in \mathcal{B}$. Thus, if g is in the kernel of the action of G on $\mathcal{B}$, then $g \in G_{[i,C]}$ in particular and hence $g \in G_{(C)}$. In other words, g fixes each vertex in C. Since this holds for all C, it follows that g fixes each vertex of Γ. So, if G is faithful on $V(\Gamma)$, then $g = 1$ and hence G is faithful on $\mathcal{B}$ as well.

Now suppose that $G_{[i,B]} \not\leq G_{(B)}$. Then, by Lemma 10.2.1(a), the partition $\mathcal{B}_i$ of $V(\Gamma)$ induced by $G_{[i,B]}$ is a nontrivial G-invariant partition of $V(\Gamma)$. So we know from Lemma 10.2.1(b) and Theorem 10.2.1 that, either $\mathcal{B}_i$ is a genuine refinement of $\mathcal{B}$, or $G_{[i,B]}$ is transitive on B. In the former case, it follows from Theorem 10.2.1(a) that v_i is a common divisor of v and k, s_i is a divisor of s and r is a divisor of r_i, and hence (i) in (b) occurs. Since $G_{[i,B]}$ fixes setwise the block B and each block $C \in \Gamma_{\mathcal{B}}(B)$, it also fixes setwise $\Gamma(C) \cap B$. So in the latter case where $G_{[i,B]}$ is transitive on B, we must have $\Gamma(C) \cap B = B$, that is, Γ is a multicover of $\Gamma_{\mathcal{B}}$ and hence (ii) in (b) occurs. □

Note that, if case (i) in Theorem 10.2.2(b) occurs, then at least one of the $\mathbf{B}_{ij}$ given in Theorem 10.2.1(d), say $\mathbf{B}_{i0}$, is a nontrivial partition of $\mathcal{B}_i$. If case (ii) in Theorem 10.2.2(b) occurs, then from Lemma 10.2.1(b), the partition $\mathcal{B}_i$ induced by $G_{[i,B]}$ coincides with $\mathcal{B}$. Applying Theorem 10.2.2 to $G_{[B]}$, we have the following consequence.

Corollary 10.2.1 *Suppose the triple* $(\Gamma, G, \mathcal{B})$ *is as in Theorem 10.2.1. Then one of the following (a), (b) holds.*

(a) $G_{[B]} \leq G_{(B)}$, *in this case G is faithful on $\mathcal{B}$ if in addition G is faithful on* $V(\Gamma)$.

(b) $G_{[B]} \not\leq G_{(B)}$, *and either*

(i) $G_{[B]}$ *induces a G-invariant partition of $V(\Gamma)$, namely $\mathcal{B}_1$ defined in (10.2) for $i = 1$, which is a genuine refinement of $\mathcal{B}$ such that v_1 is a common divisor of v and k, s_1 is a divisor of s, and r is a divisor of r_1; or*

(ii) Γ *is a multicover of* $\Gamma_{\mathcal{B}}$ *and* $G_{[B]}$ *is transitive on* B.

If the vertices in B are "distinguishable" in some sense, for example if $\Gamma_{\mathcal{B}}(\alpha) \neq \Gamma_{\mathcal{B}}(\beta)$ for distinct $\alpha, \beta \in B$, then case (a) in Corollary 10.2.1 occurs. (This happens for G-symmetric graphs with $k = v - 1 \geq 1$. See Theorems 4.2.1 and 4.3.1(c).) If $\mathcal{B}$ is chosen to be a minimal nontrivial G-invariant partition of $V(\Gamma)$, then case (b)(i) in Corollary 10.2.1 does not appear. We conclude this section by giving the following example which shows that case (b)(ii) in Corollary 10.2.1 occurs if G is not quasiprimitive on $V(\Gamma)$ and if $\mathcal{B}$ is a nontrivial G-normal partition of $V(\Gamma)$.

Example 10.2.1 Suppose Γ is a G-symmetric graph, with G faithful but not quasiprimitive on $V(\Gamma)$. Then there exists a nontrivial normal subgroup N of G which is intransitive on $V(\Gamma)$, so the G-normal partition $\mathcal{B}_N$ of $V(\Gamma)$ induced by N (see Lemma 2.2.2) is nontrivial. Let Γ_N be the quotient graph of Γ with respect to $\mathcal{B}_N$. Since N is contained in the kernel of the action of G on $\mathcal{B}_N$, G is not faithful on $\mathcal{B}_N$. So from Corollary 10.2.1 we must have $G_{[B]} \not\leq G_{(B)}$ for $B \in \mathcal{B}_N$. Since $N \trianglelefteq G_{[B]}$, we have $B = \alpha^N \subseteq \alpha^{G_{[B]}} \subseteq B$ for $\alpha \in B$, which implies $\alpha^{G_{[B]}} = B$. Hence $G_{[B]}$ is transitive on B, and consequently we come to the result (see e.g. [71, Theorem 4.1]) that Γ is a multicover of Γ_N. Thus, case (b)(ii) in Corollary 10.2.1 occurs.

10.3 Two blocks of $\mathcal{D}(B)$ incident with either the same or disjoint subsets of B

In Corollary 10.2.1 we have seen that, if $G_{[B]} \not\leq G_{(B)}$, then either Γ is a multicover of $\Gamma_{\mathcal{B}}$, or we get a genuine refinement of $\mathcal{B}$. Note that G_B is transitive on $\Gamma_{\mathcal{B}}(B)$ and $G_{(B)} \lhd G_B$ by Lemma 10.1.1(a). So in the opposite case where $G_{(B)} \not\leq G_{[B]}$, Lemma 2.2.2 implies that the $G_{(B)}$-orbits on $\Gamma_{\mathcal{B}}(B)$ form a nontrivial G_B-invariant partition of $\Gamma_{\mathcal{B}}(B)$. Since $G_{(B)}$ fixes B pointwise, any two blocks in the same $G_{(B)}$-orbit on $\Gamma_{\mathcal{B}}(B)$ induce repeated blocks of $\mathcal{D}(B)$. In some cases, blocks in distinct $G_{(B)}$-orbits on $\Gamma_{\mathcal{B}}(B)$ may induce disjoint blocks of $\mathcal{D}(B)$. For example, in Remark 10.3.1 below we will see that this happens in particular when Γ is G-locally quasiprimitive and $G_{(B)} \not\leq G_{[B]}$. This motivates us to study the case where, for any $C, D \in \Gamma_{\mathcal{B}}(B)$, either $\Gamma(C) \cap B = \Gamma(D) \cap B$, or $\Gamma(C) \cap \Gamma(D) \cap B = \emptyset$. In this case, the multiplicity m of $\mathcal{D}(B)$ is equal to r. This seemingly trivial case is by no means trivial because it contains the following two very difficult but important subcases:

(i) $k = 1$;

(ii) $k = v$.

We have studied the first subcase (i) in Section 8.3, where we gave a construction of such graphs from certain kinds of G-point- and G-block-transitive 1-designs. In the second subcase (ii), Γ is a multicover of $\Gamma_{\mathcal{B}}$. Our study in this section shows that (see Remark 10.3.2(a) below), in some sense, the study of G-symmetric graphs with blocks $\Gamma(C) \cap B$ of $\mathcal{D}(B)$ (for $C \in \Gamma_{\mathcal{B}}(B)$) satisfying the condition above can be reduced to the study of these two subcases. The results obtained here will be used in the next section.

Lemma 10.3.1 *Suppose* Γ *is a* G*-symmetric graph admitting a nontrivial* G*-invariant partition* $\mathcal{B}$*. Let* $B \in \mathcal{B}$*,* $\alpha \in B$*, and let* (a), (b), (c) *be the following statements. Then* (a) *implies* (b)*, and* (b) *in turn implies* (c)*.*

(a) $G_{(B)} \not\leq G_{[B]}$*, and either* G_α *or* $(G_B)_{\Gamma_{\mathcal{B}}(\alpha)}$ *is quasiprimitive on* $\Gamma_{\mathcal{B}}(\alpha)$*;*

(b) $G_{(B)}$ *is transitive on* $\Gamma_{\mathcal{B}}(\alpha)$*;*

(c) *for* $C, D \in \Gamma_{\mathcal{B}}(B)$*, either* $\Gamma(C) \cap B = \Gamma(D) \cap B$ *or* $\Gamma(C) \cap \Gamma(D) \cap B = \emptyset$*.*

Proof (a) $\Rightarrow$ (b) Suppose $G_{(B)} \not\leq G_{[B]}$. Then there exist $x \in G_{(B)}$ and $C, D \in \Gamma_{\mathcal{B}}(B)$ with $C \neq D$ such that $C^x = D$. Let $\alpha \in \Gamma(C) \cap B$, so that $C \in \Gamma_{\mathcal{B}}(\alpha)$. Since x fixes each vertex in B and hence fixes α in particular, we have $(\Gamma(\alpha) \cap C)^x = \Gamma(\alpha) \cap D$. Since $\Gamma(\alpha) \cap C \neq \emptyset$, we have $\Gamma(\alpha) \cap D \neq \emptyset$ and hence $D \in \Gamma_{\mathcal{B}}(\alpha)$. Thus the action of $G_{(B)}$ on $\Gamma_{\mathcal{B}}(\alpha)$ is nontrivial. On the other hand, since $G_{(B)} \lhd G_B$ (Lemma 10.1.1(a)) and $G_{(B)} \leq (G_B)_{\Gamma_{\mathcal{B}}(\alpha)} \leq G_B$, we have $G_{(B)} \trianglelefteq (G_B)_{\Gamma_{\mathcal{B}}(\alpha)}$. So if $(G_B)_{\Gamma_{\mathcal{B}}(\alpha)}$ is quasiprimitive on $\Gamma_{\mathcal{B}}(\alpha)$, then $G_{(B)}$ must be transitive on $\Gamma_{\mathcal{B}}(\alpha)$. Similarly, since $G_{(B)} \trianglelefteq G_\alpha$ (Lemma 10.1.1(b)) and $G_{(B)}$ acts on $\Gamma_{\mathcal{B}}(\alpha)$ in a nontrivial way, the quasiprimitivity of G_α on $\Gamma_{\mathcal{B}}(\alpha)$ implies the transitivity of $G_{(B)}$ on $\Gamma_{\mathcal{B}}(\alpha)$.

(b) $\Rightarrow$ (c) The assumption (b) and Lemma 3.2.6(b)(ii) together imply that $G_{(B)}$ is transitive on $\Gamma_{\mathcal{B}}(\alpha)$ for each $\alpha \in B$. That is, for any $C, D \in \Gamma_{\mathcal{B}}(\alpha)$, there exists $x \in G_{(B)}$ such that $C^x = D$. This implies $(\Gamma(C) \cap B)^x = \Gamma(D) \cap B$. However, since x fixes each vertex in B, we have $(\Gamma(C) \cap B)^x = \Gamma(C) \cap B$. So $\Gamma(C) \cap B = \Gamma(D) \cap B$. In other words, if two blocks $\Gamma(C) \cap B, \Gamma(D) \cap B$ of $\mathcal{D}(B)$ have a common vertex α, then $\Gamma(C) \cap B = \Gamma(D) \cap B$. Hence (c) is true. $\square$

Remark 10.3.1 Clearly, the quasiprimitivity of G_α on $\Gamma(\alpha)$ implies the quasiprimitivity of G_α on $\Gamma_{\mathcal{B}}(\alpha)$. So, if Γ is a G-locally quasiprimitive graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $G_{(B)} \not\leq G_{[B]}$, then by Lemma 10.3.1, either $\Gamma(C) \cap B = \Gamma(D) \cap B$ or $\Gamma(C) \cap \Gamma(D) \cap B = \emptyset$, for any $C, D \in \Gamma_{\mathcal{B}}(B)$.

The main result in this section is the following theorem.

Theorem 10.3.1 *Suppose* Γ *is a* G*-symmetric graph admitting a nontrivial* G*-invariant partition* $\mathcal{B}$*. Suppose further that, for any* $C, D \in \Gamma_{\mathcal{B}}(B)$*, either* $\Gamma(C) \cap B =$

$\Gamma(D) \cap B$ or $\Gamma(C) \cap \Gamma(D) \cap B = \emptyset$. Then $V(\Gamma)$ admits a second G-invariant partition $\mathcal{B}^* := \{(B^*)^g : g \in G\}$, where B^* is a block of $\mathcal{D}(B)$. Moreover, the following (a)-(c) *hold.*

(a) *$\mathcal{B}^*$ is a refinement of $\mathcal{B}$, and it is a genuine refinement of $\mathcal{B}$ if and only if $2 \leq k \leq v - 1$.*

(b) *Γ is a multicover of $\Gamma_{\mathcal{B}^*}$, k is a divisor of v, and the parameters v^*, r^*, b^*, k^*, s^* with respect to $\mathcal{B}^*$ satisfy $v^* = k^* = k, b^* = r^* = r, s^* = s$.*

(c) *There exists a G-invariant partition $\mathbf{B}$ of $\mathcal{B}^*$ such that $(\Gamma_{\mathcal{B}^*})_{\mathbf{B}} \cong \Gamma_{\mathcal{B}}$ and the parameters $\mathtt{v}$, $\mathtt{r}$, $\mathtt{b}$, $\mathtt{k}$, $\mathtt{s}$ with respect to $\mathbf{B}$ satisfy $\mathtt{v} = v/v^*$, $\mathtt{k} = \mathtt{s} = 1$, $\mathtt{b} = b$ and $\mathtt{r} = r$.*

Proof Our assumption on $\mathcal{D}(B)$ implies that the set of subsets of B of the form $\Gamma(C) \cap B$, for $C \in \Gamma_{\mathcal{B}}(B)$, is a partition of B, which we denote by $\mathcal{P}(B)$. Thus the blocks of $\mathcal{P}(B)$ have size k and k divides v. Let $B^* := \Gamma(C) \cap B$ be a typical block of $\mathcal{P}(B)$, where $C \in \Gamma_{\mathcal{B}}(B)$. Since G_B is transitive on $\Gamma_{\mathcal{B}}(B)$ and since $(B^*)^g = \Gamma(C^g) \cap B$ for $g \in G_B$, we have $\mathcal{P}(B) = \{(B^*)^g : g \in G_B\}$ and hence $\mathcal{P}(B)$ is a G_B-invariant partition of B. We claim further that $\mathcal{B}^* := \{(B^*)^g : g \in G\}$ defines a G-invariant partition of $V(\Gamma)$. In fact, if $(B^*)^g \cap B^* \neq \emptyset$ for some $g \in G$, then $B^g \cap B \neq \emptyset$ since $B^* \subseteq B$ and $(B^*)^g \subseteq B^g$. But B is a block of imprimitivity for G in $V(\Gamma)$, so we have $B^g = B$ and hence $g \in G_B$. Thus $(B^*)^g \subseteq B$ and $(B^*)^g$ is a block of $\mathcal{P}(B)$ having nonempty intersection with B^*. Since $\mathcal{P}(B)$ is a G_B-invariant partition of B, as shown above, this implies $(B^*)^g = B^*$. Therefore, B^* is a block of imprimitivity for G in $V(\Gamma)$ and so $\mathcal{B}^*$ is a G-invariant partition of $V(\Gamma)$. It is easily checked that $\mathcal{B}^* = \bigcup_{B \in \mathcal{B}} \mathcal{P}(B)$. Clearly, $\mathcal{B}^*$ is a refinement of $\mathcal{B}$, and it is a genuine refinement of $\mathcal{B}$ if and only if $2 \leq k \leq v - 1$. Since $\Gamma_{\mathcal{B}}$ is G-symmetric, there exists $h \in G$ which interchanges B and C. So $\Gamma(B) \cap C = (\Gamma(C) \cap B)^h = (B^*)^h \in \mathcal{B}^*$, and hence each vertex in B^* is adjacent to at least one vertex in $(B^*)^h$. Therefore, Γ is a multicover of $\Gamma_{\mathcal{B}^*}$, and hence $v^* = k^* = k, b^* = r^* = r, s^* = s$. Finally, it is straightforward to show that $\mathbf{B} := \{\mathcal{P}(B) : B \in \mathcal{B}\}$ is a G-invariant partition of $\mathcal{B}^*$ and that $(\Gamma_{\mathcal{B}^*})_{\mathbf{B}} \cong \Gamma_{\mathcal{B}}$. Also, it is clear that the parameters $\mathtt{v}$, $\mathtt{r}$, $\mathtt{b}$, $\mathtt{k}$, $\mathtt{s}$ with respect to $\mathbf{B}$ are as specified in (c). $\square$

Remark 10.3.2 (a) The partition $\mathcal{B}^*$ in Theorem 10.3.1 is equal to the trivial partition $\{\{\alpha\} : \alpha \in V(\Gamma)\}$ if and only if $k = 1$, and is equal to $\mathcal{B}$ if and only if

$k = v$. In the general case where $2 \le k \le v-1$, $\mathcal{B}^*$ is a genuine refinement of $\mathcal{B}$, and as $k^* = v^*$, the partition $(\mathcal{B}^*)^*$ resulting from applying Theorem 10.3.1 to $\mathcal{B}^*$, is equal to $\mathcal{B}^*$. Moreover, the quotient graph $\Gamma_{\mathcal{B}^*}$ admits a G-invariant partition, namely $\mathbf{B}$, for which $k = 1$ and thus the construction given in Section 8.3 applies to $\Gamma_{\mathcal{B}^*}$.

(b) Setting $i = 1$ in Theorem 10.2.1(b), we know immediately that the partition $\mathcal{B}_1$ (defined in (10.2) for $i = 1$) is a refinement of $\mathcal{B}^*$. Moreover, $\mathcal{B}_1$ admits a G-invariant partition $\mathbf{B}_1 := \{\mathcal{P}(B^*) : B^* \in \mathcal{B}^*\}$, where $\mathcal{P}(B^*) := \{\alpha^{G_{[B]}} \subseteq B^* : \alpha \in B^*\}$, such that $(\Gamma_{\mathcal{B}_1})_{\mathbf{B}_1} \cong \Gamma_{\mathcal{B}^*}$ and $\Gamma_{\mathcal{B}_1}$ is a multicover of $\Gamma_{\mathcal{B}^*}$, and that the parameters $\mathtt{v}_1, \mathtt{r}_1, \mathtt{b}_1, \mathtt{k}_1, \mathtt{s}_1$ with respect to $\mathbf{B}_1$ satisfy $\mathtt{v}_1 = \mathtt{k}_1 = k/v_1$, $\mathtt{r}_1 = \mathtt{b}_1 = r$, $\mathtt{s}_1 = b_1/r$.

10.4 Locally quasiprimitive graphs

We now apply the results obtained in the last two sections to G-locally quasiprimitive graphs. Such graphs were studied initially in [66, 67], and more recent results were obtained in [54]. In this case we have the following theorem, which can be viewed as a generalization of [43, Lemma 3.4].

Theorem 10.4.1 *Suppose Γ is a G-locally quasiprimitive graph which admits a nontrivial G-invariant partition $\mathcal{B}$. Suppose further that $G_{[B]} \ne G_{(B)}$.*

(a) *If $G_{(B)} \not\le G_{[B]}$, then $G_{(B)}$ is transitive on $\Gamma(\alpha)$ for each $\alpha \in B$. Moreover, either*

  (i) *$k = 1$ and $G_{[B]} < G_{(B)}$; or*

  (ii) *$k \ge 2$, k divides v, and $V(\Gamma)$ admits a second nontrivial G-invariant partition $\mathcal{B}^*$ such that $\mathcal{B}^*$ is a refinement of $\mathcal{B}$, Γ is a multicover of $\Gamma_{\mathcal{B}^*}$ and the parameters v^*, r^*, b^*, k^*, s^* with respect to $\mathcal{B}^*$ satisfy $v^* = k^* = k, b^* = r^* = r, s^* = s$.*

(b) *If $G_{[B]} \not\le G_{(B)}$, then $G_{[B]}$ induces a nontrivial G-invariant partition $\mathcal{B}_1$ of $V(\Gamma)$ (defined in (10.2) for $i = 1$) such that $\mathcal{B}_1$ is a refinement of $\mathcal{B}$, v_1 is a common divisor of v and k, s_1 is a divisor of s, and r is a divisor of r_1.*

Proof (a) Suppose $G_{(B)} \not\leq G_{[B]}$. Then there exist $x \in G_{(B)}$ and distinct blocks C, D of $\Gamma_{\mathcal{B}}(B)$ such that $C^x = D$. Let $\alpha \in \Gamma(C) \cap B$, so $\Gamma(\alpha) \cap C \neq \emptyset$. Since x fixes each vertex in B, it fixes α in particular and hence maps a vertex in $\Gamma(\alpha) \cap C$ to a vertex in $\Gamma(\alpha) \cap D$. Since $G_{(B)} \trianglelefteq G_\alpha$ (Lemma 10.1.1(b)), this implies that $G_{(B)}^{\Gamma(\alpha)}$ is a nontrivial normal subgroup of $G_\alpha^{\Gamma(\alpha)}$. Therefore, by the G-local quasiprimitivity of Γ, we conclude that $G_{(B)}$ is transitive on $\Gamma(\alpha)$. From Lemma 3.2.6(b)(i), this assertion is true for all vertices α in B.

If $k = 1$, then $\Gamma_{\mathcal{B}}(\alpha) \cap \Gamma_{\mathcal{B}}(\beta) = \emptyset$ for distinct $\alpha, \beta \in B$. Hence, if $g \in G_B$ fixes each block $C \in \Gamma_{\mathcal{B}}(B)$ setwise, then it also fixes each vertex in B. So we have $G_{[B]} < G_{(B)}$ in this case.

If $k \geq 2$, then by Remark 10.3.1, for any $C, D \in \Gamma_{\mathcal{B}}(B)$, either $\Gamma(C) \cap B = \Gamma(D) \cap B$ or $\Gamma(C) \cap \Gamma(D) \cap B = \emptyset$. Hence Theorem 10.3.1 applies, and the partition $\mathcal{B}^*$ defined therein is a nontrivial G-invariant partition of $V(\Gamma)$ and is a refinement of $\mathcal{B}$. The truth of the remaining statements in (ii) follows from Theorem 10.3.1(b).

(b) Now we suppose $G_{[B]} \not\leq G_{(B)}$. Then $B_1 := \alpha^{G_{[B]}}$ has length at least two, where $\alpha \in B$. Hence it follows from Theorem 10.2.1 that the partition $\mathcal{B}_1$ (defined in (10.2) for $i = 1$) is a nontrivial G-invariant partition of $V(\Gamma)$ and is a refinement of $\mathcal{B}$, and that the parameters v_1, s_1, r_1 with respect to $\mathcal{B}_1$ have the required properties. □

For minimal nontrivial G-invariant partitions, we have the following consequence of Theorem 10.4.1.

Corollary 10.4.1 *Suppose Γ is a G-locally quasiprimitive graph, with G faithful on $V(\Gamma)$. Suppose further that $\mathcal{B}$ is a minimal nontrivial G-invariant partition of $V(\Gamma)$. Then one of the following* (a)-(c) *holds.*

(a) $G_{[B]} = G_{(B)}$*, in this case G is faithful on $\mathcal{B}$;*

(b) $G_{[B]} < G_{(B)}$ *and $k = 1$;*

(c) Γ *is a multicover of* $\Gamma_{\mathcal{B}}$.

Moreover, if $\Gamma_{\mathcal{B}}$ is a complete graph, then the occurrence of (a) *implies $G_{[B]} = G_{(B)} = 1$; if $G_{[B]} \not\leq G_{(B)}$, then the occurrence of* (c) *implies that $G_{[B]}$ is transitive on B.*

Proof In the case where $G_{(B)} = G_{[B]}$, G is faithful on $\mathcal{B}$ by Corollary 10.2.1(a). Suppose $G_{(B)} \neq G_{[B]}$. Then either $G_{(B)} \not\leq G_{[B]}$ or $G_{[B]} \not\leq G_{(B)}$. In the former case, Theorem 10.4.1(a) applies. If (i) in Theorem 10.4.1(a) appears, then we have $k = 1$ and $G_{[B]} < G_{(B)}$, and hence (b) above occurs. If (ii) in Theorem 10.4.1(a) appears, then by the minimality of $\mathcal{B}$, the partition $\mathcal{B}^*$ therein must coincide with $\mathcal{B}$; hence Γ is a multicover of $\Gamma_{\mathcal{B}}$ and (c) holds. In the latter case where $G_{[B]} \not\leq G_{(B)}$, by Corollary 10.2.1 and the minimality of $\mathcal{B}$, we know that Γ is a multicover of $\Gamma_{\mathcal{B}}$ (hence (c) above occurs), and moreover $G_{[B]}$ is transitive on B.

Now suppose that $\Gamma_{\mathcal{B}}$ is a complete graph, and that case (a) occurs. Then $G_{[B]}$ is the kernel of the action of G on $\mathcal{B}$ and hence $G_{[B]} = G_{(B)} \lhd G$. This implies that $G_{(B)} = g^{-1}G_{(B)}g = G_{(B^g)}$ for any $g \in G$. Since B^g runs over all blocks of $\mathcal{B}$ when g runs over G, this means that $G_{(B)}$ fixes each vertex of Γ, and hence by the faithfulness of G on $V(\Gamma)$ we get $G_{[B]} = G_{(B)} = 1$. □

Recall that $G_{[\alpha]}$ is the subgroup of G_α fixing setwise each block in $\Gamma_{\mathcal{B}}(\alpha)$. So $G_{[\alpha]}$ induces an action on $\Gamma(\alpha) \cap C$, for each $C \in \Gamma_{\mathcal{B}}(\alpha)$. As exemplified in the following lemma, it may happen that $G_{[\alpha]}$ is transitive on $\Gamma(\alpha) \cap C$, or, equivalently, $\Gamma(\alpha) \cap C$ is a $(G_{[\alpha]})$-orbit on $\Gamma(\alpha)$.

Lemma 10.4.1 *Suppose Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$, and let $\alpha \in V(\Gamma)$. If G_α is regular on $\Gamma_{\mathcal{B}}(\alpha)$, then $G_{[\alpha]}$ is transitive on $\Gamma(\alpha) \cap C$, for each $C \in \Gamma_{\mathcal{B}}(\alpha)$.*

Proof For any $C \in \Gamma_{\mathcal{B}}(\alpha)$ and $\beta, \gamma \in \Gamma(\alpha) \cap C$, by the G-symmetry of Γ there exists $x \in G_\alpha$ such that $\beta^x = \gamma$, and hence x fixes C setwise. Since by our assumption G_α acts regularly on $\Gamma_{\mathcal{B}}(\alpha)$, this implies that $D^x = D$ for all $D \in \Gamma_{\mathcal{B}}(\alpha)$, and hence $x \in G_{[\alpha]}$. Thus, any vertex β in $\Gamma(\alpha) \cap C$ can be mapped to any other vertex γ in $\Gamma(\alpha) \cap C$ by an element of $G_{[\alpha]}$. In other words, $G_{[\alpha]}$ is transitive on $\Gamma(\alpha) \cap C$. □

We conclude this section by studying G-locally quasiprimitive graphs Γ such that $G_{[\alpha]}$ is transitive on $\Gamma(\alpha) \cap C$, for $C \in \Gamma_{\mathcal{B}}(\alpha)$. In this case we have the following theorem which is a counterpart of Corollary 3.2.1.

Theorem 10.4.2 *Suppose Γ is a G-locally quasiprimitive graph admitting a nontrivial G-invariant partition $\mathcal{B}$. Suppose further that $G_{[\alpha]}$ is transitive on $\Gamma(\alpha) \cap B$, for some $\alpha \in V(\Gamma)$ and $B \in \Gamma_{\mathcal{B}}(\alpha)$. Then either*

(a) *$\Gamma[B, C] \cong k \cdot K_2$ is a matching of k edges, for adjacent blocks B, C of $\mathcal{B}$; or*

(b) Γ *is a bipartite graph with each part of the bipartition of a connected component contained in some block of* $\mathcal{B}$*, and for any* $C, D \in \Gamma_{\mathcal{B}}(B)$*, either* $\Gamma(C) \cap B = \Gamma(D) \cap B$ *or* $\Gamma(C) \cap \Gamma(D) \cap B = \emptyset$ *(hence* $v = bk$*).*

Proof From Lemma 3.2.6(c)(ii), our assumption on $G_{[\alpha]}$ implies that $G_{[\alpha]}$ is transitive on $\Gamma(\alpha) \cap C$ for each $C \in \Gamma_{\mathcal{B}}(\alpha)$. So, if $G_{[\alpha]}^{\Gamma(\alpha)} = 1$, then we have $|\Gamma(\alpha) \cap C| = 1$. That is, $\Gamma[B, C]$ is a matching for adjacent blocks B, C of $\mathcal{B}$, and hence (a) holds.

In the following we suppose that $G_{[\alpha]}^{\Gamma(\alpha)} \neq 1$. Then, since $G_{[\alpha]}^{\Gamma(\alpha)} \trianglelefteq G_{\alpha}^{\Gamma(\alpha)}$ by Lemma 10.1.1(c) and since Γ is G-locally quasiprimitive by our assumption, $G_{[\alpha]}$ must be transitive on $\Gamma(\alpha)$. However, $G_{[\alpha]}$ fixes $\Gamma(\alpha) \cap C$ setwise for each $C \in \Gamma_{\mathcal{B}}(\alpha)$. So we must have $r = |\Gamma_{\mathcal{B}}(\alpha)| = 1$ and hence $\Gamma(\alpha) \subseteq C$ for some C. Let B be the block of $\mathcal{B}$ containing α. Then, since G is transitive on arcs of Γ, for any $\beta \in \Gamma(\alpha)$ there exists an element of G which interchanges α and β and hence interchanges B and C. Hence $\Gamma(\alpha) \subseteq C$ implies $\Gamma(\beta) \subseteq B$. Similarly, $\Gamma(\beta) \subseteq B$ implies $\Gamma(\gamma) \subseteq C$ for any $\gamma \in \Gamma(\beta)$. Continuing this process, one can see that $\Gamma[B, C]$ consists of connected components of Γ, and hence each such component is a bipartite graph with the two parts of the bipartition contained in B, C, respectively. This, together with the fact $r = 1$, implies that Γ is a bipartite graph with $v = bk$, and that either $\Gamma(C) \cap B = \Gamma(D) \cap B$ or $\Gamma(C) \cap \Gamma(D) \cap B = \emptyset$ for any $C, D \in \Gamma_{\mathcal{B}}(B)$. □

From Lemma 10.4.1, the results in Theorem 10.4.2 hold in particular when Γ is a G-locally quasiprimitive graph such that G_{α} is regular on $\Gamma_{\mathcal{B}}(\alpha)$ for $\alpha \in V(\Gamma)$. In this case, if Γ is not a bipartite graph, then $\Gamma[B, C]$ is a matching and hence, by Lemma 3.2.4(b), G_{α} is regular on $\Gamma(\alpha)$. Hence G is regular on the arcs of Γ if in addition Γ is connected. Examples of such graphs include G-Frobenius graphs [32, Definition 1.2] arising from self-paired G-orbitals of a Frobenius group G.

Chapter 11

Local actions: Heritage of the labelling method

He who by reanimating the Old can gain knowledge of the New is qualified to teach others.

Confucius (551-479 B.C.), Lun Yü [The Analects] 2:11

Continuing our study on "local actions", we will investigate in this chapter the particular case where the induced actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent. That is, we will study G-symmetric graphs Γ admitting a nontrivial G-invariant partition $\mathcal{B}$ such that the following [P] holds for some, and hence all (see Lemma 3.2.6(a)), blocks B of $\mathcal{B}$.

[P] The induced actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent with respect to some bijection $\rho : B \to \Gamma_{\mathcal{B}}(B)$.

From a geometric point of view, this requires that the automorphism group of $\mathcal{D}(B)$ induced by G_B (Lemma 3.2.5) acts in essentially the same way on the points and the blocks of the 1-design $\mathcal{D}(B) = (B, \Gamma_{\mathcal{B}}(B), \mathrm{I})$. Clearly, any G-symmetric graph such that $k = v - 1 \geq 2$ and $\mathcal{D}(B)$ contains no repeated blocks possesses this property (see Example 11.1.1 below), and this observation is one of the motivations for the study in this chapter. Recall that in this case $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive (Theorem 5.1.2); we will characterize such graphs as the only graphs Γ satisfying [P] such that $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive (Theorems 11.3.1(c)). Under the assumption [P], we will develop a labelling technique similar to that used in Section 5.1, and we will show that $\mathcal{D}(B)$ plays a more active role in influencing Γ, $\Gamma_{\mathcal{B}}$ and $\Gamma[B, C]$. We will study the case where in addition the bijection ρ in [P] preserves the incidence relation of $\mathcal{D}(B)$ in the sense that, for $\alpha \in B$ and $C \in \Gamma_{\mathcal{B}}(B)$, $\alpha \mathrm{I} C$ if and only if $\rho^{-1}(C)\mathrm{I}\rho(\alpha)$. Finally, based on the labelling technique, we will prove that the class of G-symmetric graphs satisfying [P] is precisely the class of 3-arc graphs $\Xi(\Sigma, \Delta)$ of G-symmetric graphs Σ with respect to self-paired G-orbits Δ on $\mathrm{Arc}_3(\Sigma)$. Therefore, this chapter may be viewed as an extension of Chapter 5. *To avoid triviality, we assume* $\mathrm{val}(\Gamma) > 1$ *in this chapter.*

11.1 Examples

Example 11.1.1 Let Γ be a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - 1 \geq 2$ and $\mathcal{D}(B)$ contains no repeated blocks. Then, for each $\alpha \in B$, the set $\mathcal{B}(\alpha)$ defined in (4.1) contains a unique block C_α and, by Theorem 4.3.2(a), [P] is satisfied for the bijection $\rho : \alpha \mapsto C_\alpha$ from B to $\Gamma_{\mathcal{B}}(B)$.

Graphs in Example 11.1.1 have been studied in Chapters 5-7. The following example shows that, besides these graphs, there exist other G-symmetric graphs for which [P] is satisfied. Note that in this example we have $k < v - 1$ and $\mathcal{D}(B)$ contains repeated blocks.

Example 11.1.2 Let $\mathrm{PG}(2, 2)$ be the Fano plane whose points $1, 2, \ldots, 7$ are as shown in Figure 8. Let X be the set of ordered pairs of distinct points of $\mathrm{PG}(2, 2)$. Then $G := \mathrm{PGL}(3, 2)$ is transitive on X ([10, Theorem 2.5.4]). Define Γ to be the graph with vertex set X in which $\alpha\beta, \gamma\delta \in X$ are adjacent if and only if (i) $\alpha, \beta, \gamma, \delta$ are distinct, and (ii) β, δ and the unique point collinear with α, γ are distinct and are collinear in $\mathrm{PG}(2, 2)$. For example, $17, 26$ are adjacent in Γ since the unique point collinear with $1, 2$ is 3 and since $7, 6, 3$ are collinear in $\mathrm{PG}(2, 2)$. Similarly, one can see that $\Gamma(17) = \{26, 62, 35, 53\}$. Note that the pointwise stabilizer G_{17} of $1, 7$ in G contains an element which exchanges 2 and 6, and exchanges 3 and 5; also G_{17} contains an element which exchanges 2 and 3, and exchanges 6 and 5. So G_{17} is transitive on $\Gamma(17)$, and hence Γ is G-symmetric. One can see that $\Gamma \cong 7 \cdot K_{2,2,2}$ and $\mathcal{B} := \{B(\sigma) : \sigma \text{ is a point of } \mathrm{PG}(2, 2)\}$ is a G-invariant partition of X, where $B(\sigma) := \{\sigma\tau : \tau \text{ is a point of } \mathrm{PG}(2, 2) \text{ with } \tau \neq \sigma\}$. We have $\Gamma_{\mathcal{B}} \cong K_7$,

$\Gamma[B(\sigma), B(\tau)] \cong 4 \cdot K_2$ for $\sigma \neq \tau$, $\mathcal{D}(B(1))$ is a 1-$(6,4,4)$ design, and the traces of the blocks of $\mathcal{D}(B(1))$ are $\{12,13,14,17\}$, $\{14,15,16,17\}$, $\{12,13,16,15\}$ with each repeated twice. Thus the block size of $\mathcal{D}(B(1))$ is less than $|B(1)|-1$. Clearly, the induced actions of $G_{B(\sigma)}$ on $B(\sigma)$ and $\Gamma_{\mathcal{B}}(B(\sigma))$ are permutationally equivalent with respect to the bijection $\rho : \sigma\tau \mapsto B(\tau)$. Note that $\sigma\tau$ is adjacent to a vertex in a block $B(\delta)$ if and only if $\sigma\delta$ is adjacent to a vertex in the block $B(\tau)$.

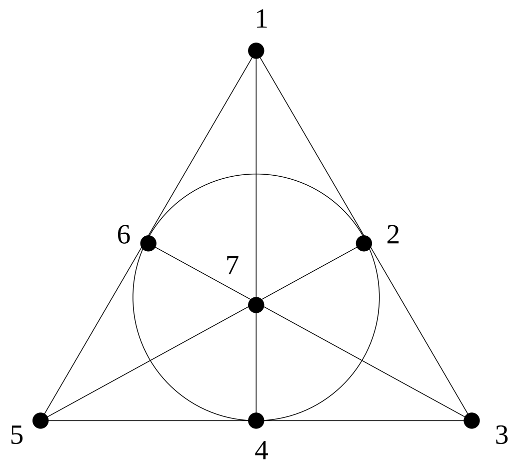


FIGURE 8 Fano plane

11.2 The labelling technique

As a fundamental fact, we now show that [P] holds if and only if the vertices of Γ can be labelled in a natural way by the arcs of $\Gamma_{\mathcal{B}}$. For convenience, we call a mapping $\mu : V(\Gamma) \to \mathrm{Arc}(\Gamma_{\mathcal{B}})$ *compatible* with $\mathcal{B}$ if, for any $\alpha \in V(\Gamma)$, the arc $\mu(\alpha)$ of $\Gamma_{\mathcal{B}}$ is initiated at the block $B(\alpha)$ containing α.

Lemma 11.2.1 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$. Then* [P] *holds if and only if the actions of G on $V(\Gamma)$ and* $\mathrm{Arc}(\Gamma_{\mathcal{B}})$ *are permutationally equivalent with respect to some bijection $\mu : V(\Gamma) \to$* $\mathrm{Arc}(\Gamma_{\mathcal{B}})$ *compatible with $\mathcal{B}$. Moreover, in this case we have $b = v \geq 2$, $G_{[B]} = G_{(B)}$, and G is faithful on $\mathcal{B}$ if G is faithful on $V(\Gamma)$.*

Proof Suppose first that [P] holds for some $B \in \mathcal{B}$ and a bijection $\rho : B \to \Gamma_{\mathcal{B}}(B)$, and let α be a fixed vertex of B. Then, since Γ is G-vertex-transitive, each vertex of Γ has the form α^x for some $x \in G$. We will show that $\mu : \alpha^x \mapsto (B^x, (\rho(\alpha))^x)$, $x \in G$, defines a bijection from $V(\Gamma)$ to $\mathrm{Arc}(\Gamma_{\mathcal{B}})$ which is compatible with $\mathcal{B}$. In fact, if $\alpha^x = \alpha^y$ for some $x, y \in G$, then $xy^{-1} \in G_\alpha$ $(\leq G_B)$, and hence $B^{xy^{-1}} = B$ and $(\rho(\alpha))^{xy^{-1}} = \rho(\alpha^{xy^{-1}}) = \rho(\alpha)$. Therefore, we have $\mu(\alpha^x) = \mu(\alpha^y)$ and thus μ is well-defined. Secondly, if $\mu(\alpha^x) = \mu(\alpha^y)$ for two vertices α^x, α^y, then $xy^{-1} \in G_B$ since $B^x = B^y$. This, together with $(\rho(\alpha))^x = (\rho(\alpha))^y$, implies that $\rho(\alpha) = (\rho(\alpha))^{xy^{-1}} = \rho(\alpha^{xy^{-1}})$. Note that $xy^{-1} \in G_B$ implies $\alpha^{xy^{-1}} \in B$, and that ρ is a bijection from B to $\Gamma_{\mathcal{B}}(B)$. So we have $\alpha^{xy^{-1}} = \alpha$, implying $\alpha^x = \alpha^y$ and hence μ is injective. Since G is transitive on arcs of $\Gamma_{\mathcal{B}}$, μ is in fact a bijection from $V(\Gamma)$ to $\mathrm{Arc}(\Gamma_{\mathcal{B}})$. Since B and $\rho(\alpha)$ are adjacent blocks and $B^x = (B(\alpha))^x = B(\alpha^x)$, B^x and $(\rho(\alpha))^x$ are adjacent blocks and hence μ is compatible with $\mathcal{B}$. It follows from the definition that the actions of G on $V(\Gamma)$ and $\mathrm{Arc}(\Gamma_{\mathcal{B}})$ are permutationally equivalent with respect to μ. Moreover, the definition of μ does not depend on the choice of $\alpha \in B$. In fact, for another vertex $\beta \in B$ and any vertex of Γ, say $\gamma = \alpha^x = \beta^y$ for some $x, y \in G$, we have $B^x = B(\alpha^x) = B(\beta^y) = B^y$ and hence $xy^{-1} \in G_B$. So $(\rho(\alpha))^{xy^{-1}} = \rho(\alpha^{xy^{-1}}) = \rho(\beta)$, implying $(B, \rho(\alpha))^x = (B, \rho(\beta))^y$ and indeed the definition of μ is independent of the choice of $\alpha \in B$.

Now suppose conversely that the actions of G on $V(\Gamma)$ and $\mathrm{Arc}(\Gamma_{\mathcal{B}})$ are permutationally equivalent with respect to a bijection $\mu : V(\Gamma) \to \mathrm{Arc}(\Gamma_{\mathcal{B}})$ which is compatible with $\mathcal{B}$. Then $(B, \rho(\alpha)) = \mu(\alpha)$, for $\alpha \in B$, defines a bijection $\rho : B \to \Gamma_{\mathcal{B}}(B)$. It is easily checked that the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent with respect to ρ.

Finally, if [P] holds, then $b = |\Gamma_{\mathcal{B}}(B)| = |B| = v \geq 2$ and $G_{[B]} = G_{(B)}$ for each $B \in \mathcal{B}$. So if G is faithful on $V(\Gamma)$ then it is faithful on $\mathcal{B}$ as well. $\square$

Lemma 11.2.1 implies that, under the assumption [P], each vertex α of Γ can be uniquely labelled by an ordered pair "BC" of adjacent blocks of $\Gamma_{\mathcal{B}}$, where $(B, C) = \mu(\alpha)$. In the following we will identify α with the *label* "BC", so we have $G_{\text{“}BC\text{”}} = G_{B,C}$. Since $(\mu(\alpha))^x = \mu(\alpha^x)$, it follows that

$$\text{“}BC\text{”}^x = \text{“}B^x C^x\text{”} \tag{11.1}$$

for $x \in G$ and "BC" $\in V(\Gamma)$. One can see that the block B is precisely the set of those vertices of Γ whose labels have the first coordinate B, that is, $B = \{\text{“}BC\text{”} : (B, C) \in \mathrm{Arc}(\Gamma_{\mathcal{B}})\}$. Note that each vertex α = "BC" of Γ has a unique *mate*

$\alpha' :=$ "CB", and that $z : \alpha \mapsto \alpha'$ defines an involution on $V(\Gamma)$. Also, z centralises G since "BC"^{zx} = "CB"^{x} = "C^xB^x" = "B^xC^x"^{z} = "BC"^{xz} for any $x \in G$. Since G preserves $\mathcal{B}$ invariant whilst it is easy to see that $B^z = \{\alpha' : \alpha \in B\} \notin \mathcal{B}$, we have $z \notin G$. Clearly, $\{\{\alpha, \alpha'\} : \alpha \in V(\Gamma)\}$ is a $(G \times \langle z \rangle)$-invariant partition of $V(\Gamma)$, and the graph Γ' with vertex set $V(\Gamma)$ and arc set $\{(\alpha, \alpha') : \alpha \in V(\Gamma)\}$ is G-symmetric. We record these basic results in the following theorem, which will be used repeatedly in our later discussion. The validity of these results for the graphs in Example 11.1.1 has been established in Section 5.1.

Theorem 11.2.1 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent, for some $B \in \mathcal{B}$. Let $\mu : V(\Gamma) \to \mathrm{Arc}(\Gamma_{\mathcal{B}})$ be the bijection guaranteed by Lemma 11.2.1. Then the following* (a)-(d) *hold.*

(a) *Each vertex α of Γ can be labelled uniquely by an ordered pair "BC" of adjacent blocks of $\Gamma_{\mathcal{B}}$, where $(B, C) = \mu(\alpha)$. Moreover, we have $G_{\text{"}BC\text{"}} = G_{B,C}$ and "BC"^{x} = "B^xC^x" for "BC" $\in V(\Gamma)$ and $x \in G$.*

(b) *Each vertex α = "BC" has a unique mate $\alpha' :=$ "CB", the mapping $z : \alpha \mapsto \alpha'$ defines an involution such that $z \notin G$ and z centralises G, $\mathcal{P} := \{\{\alpha, \alpha'\} : \alpha \in V(\Gamma)\}$ is a $(G \times \langle z \rangle)$-invariant partition of $V(\Gamma)$, and the graph Γ' with vertex set $V(\Gamma)$ and arc set $\{(\alpha, \alpha') : \alpha \in V(\Gamma)\}$ is G-symmetric.*

(c) *$\mathcal{B}^* := \{B^* : B \in \mathcal{B}\}$ (where $B^* := B^z$) is a G-invariant partition of $V(\Gamma)$, $G_B = G_{B^*}$, and the actions of G_B on B and B^* are transitive and permutationally equivalent with respect to the restriction of z on B.*

(d) *There is no edge of Γ joining vertices of B and B^*. In particular, for each arc ("BC", "DE") of Γ, (C, B, D, E) is a 3-arc of $\Gamma_{\mathcal{B}}$.*

Proof The truth of (a) and (b) has been shown above, and from this we get (c) by a routine argument. To prove (d), we assume B, C are two adjacent blocks of $\Gamma_{\mathcal{B}}$. If "CB" is adjacent to "BC", then, since $\mathrm{val}(\Gamma) > 1$, "CB" is adjacent to a vertex "B_1C_1" distinct from "BC". By the G-symmetry of Γ, there exists $x \in G$ such that ("CB", "BC")^x = ("CB", "B_1C_1"), which implies $C = C^x = C_1, B = B^x = B_1$. This is a contradiction and hence each vertex "CB" of $V(\Gamma)$ is not adjacent to its mate "BC". Similarly, if "CB" is adjacent to a vertex "BD" $\in B \setminus \{$"BC"$\}$, then we can take a vertex "B_1D_1" which is distinct from "BD" and is adjacent to "CB". By the G-symmetry of Γ, we have ("CB", "BD")^x = ("CB", "B_1D_1") for some $x \in G$, and hence $B = B^x = B_1$. On the other hand, there exists $y \in G$ such that ("CB", "BD")^y = ("B_1D_1", "CB"). This implies $C = B^y = D_1$, and hence "B_1D_1" = "BC". Again, this is a contradiction and hence there is no edge of Γ between B and B^*. In particular, if ("BC", "DE") is an arc of Γ, then $C \neq D, B \neq E$ and hence (C, B, D, E) is a 3-arc of $\Gamma_{\mathcal{B}}$. □

The following theorem is a counterpart of Theorem 5.1.2(a)(b).

Theorem 11.2.2 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent, for some $B \in \mathcal{B}$. Then one, and only one, of the following* (a), (b) *occurs.*

(a) *Any two adjacent vertices have labels involving four distinct blocks. In this case, each block of $\mathcal{B}^*$ is an independent set of Γ.*

(b) *Any two adjacent vertices of Γ share the same second coordinate. In this case, Γ is disconnected with each block of $\mathcal{B}^*$ consisting of connected components of Γ, and moreover we have* $\mathrm{girth}(\Gamma_{\mathcal{B}}) = 3$, $\Gamma[B, C] \cong k \cdot K_2$ *and* $\mathrm{val}(\Gamma) = |D^{G_{B,C}}|$, *where $B, C, D \in \mathcal{B}$ such that "CB", "DB" are adjacent in Γ. In particular, $\Gamma[B^*] \cong K_v$ if and only if $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive, and in this case we have $\Gamma \cong n(v+1) \cdot K_v$, $\Gamma[B, C] \cong (v-1) \cdot K_2$ and $\Gamma_{\mathcal{B}} \cong n \cdot K_{v+1}$ for an integer n, and the group induced on the vertex set of a connected component of $\Gamma_{\mathcal{B}}$ is 3-transitive.*

Proof If there exist two adjacent vertices of Γ, say "CB", "DB", which share the same second coordinate. Then, since Γ is G-symmetric, by Theorem 11.2.1(a) any arc of Γ has the form ("C^xB^x", "D^xB^x"), for some $x \in G$, and hence any two adjacent vertices of Γ share the same second coordinate. Thus, either (a) or (b) occurs. It is easy to see that (a) occurs if and only if each block of $\mathcal{B}^*$ is an independent set of Γ.

In the following we suppose that (b) occurs, and let "CB", "DB" be adjacent vertices. Then any two adjacent vertices of Γ lie in the same block of $\mathcal{B}^*$. Hence the subgraph $\Gamma[E^*]$ induced by each $E^* \in \mathcal{B}^*$ consists of connected components of Γ. Clearly, we have $\mathrm{girth}(\Gamma_{\mathcal{B}}) = 3$ since (B, C, D, B) is a triangle of $\Gamma_{\mathcal{B}}$. By our assumption, "CB" is the unique vertex in C adjacent to "DB". So we have

$\Gamma[C, D] \cong k \cdot K_2$. Moreover, a vertex "$D_1B$" $\in B^*$ is adjacent to "CB" in $\Gamma \Leftrightarrow$ there exists $g \in G$ such that ("CB", "DB")^g = ("CB", "D_1B") $\Leftrightarrow$ there exists $g \in G_{B,C}$ such that $D^g = D_1$. Thus, we have $\mathrm{val}(\Gamma) = |D^{G_{B,C}}|$. In particular, $\Gamma[B^*] \cong K_v \Leftrightarrow G_{B,C}$ is transitive on $\Gamma_{\mathcal{B}}(B) \setminus \{C\} \Leftrightarrow G_B$ is 2-transitive on $\Gamma_{\mathcal{B}}(B) \Leftrightarrow \Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. In this case, the argument above shows that (i) $\Gamma \cong |\mathcal{B}^*| \cdot K_v$, (ii) $\{B\} \cup \Gamma_{\mathcal{B}}(B)$ induces the complete graph K_{v+1} which is a connected component of $\Gamma_{\mathcal{B}}$ (note that $b = v$ by Lemma 11.2.1), and (iii) G induces a 3-transitive group on $\{B\} \cup \Gamma_{\mathcal{B}}(B)$. Therefore, we have $\Gamma_{\mathcal{B}} \cong n \cdot K_{v+1}$ and $\Gamma \cong n(v+1) \cdot K_v$ for an integer n. Counting the number of edges of Γ in two ways, we get $(n(v+1)v/2)k = n(v+1)(v(v-1)/2)$, which implies $k = v - 1$ and hence $\Gamma[C, D] \cong (v-1) \cdot K_2$. □

Note that case (a) in Theorem 11.2.2 occurs when $\mathrm{girth}(\Gamma_{\mathcal{B}}) \geq 4$. If $\mathrm{girth}(\Gamma_{\mathcal{B}}) \geq 5$, then we get the following generalizations of Theorem 5.1.3 and Corollary 5.1.2 – the proofs are very much similar and hence ommitted.

Theorem 11.2.3 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent, for some $B \in \mathcal{B}$. Suppose further that* $\mathrm{girth}(\Gamma_{\mathcal{B}}) \geq 5$. *Then the following* (a)-(c) *hold.*

(a) $\Gamma[\{\alpha, \alpha'\}, \{\beta, \beta'\}] \cong K_2$ *for adjacent blocks $\{\alpha, \alpha'\}$ and $\{\beta, \beta'\}$ of $\mathcal{P}$.*

(b) $\Gamma[B^*, C^*]$ *is a matching for adjacent blocks B^*, C^* of $\mathcal{B}^*$; in particular we have* $\Gamma[B^*, C^*] \cong K_2$ *if* $\mathrm{girth}(\Gamma_{\mathcal{B}}) \geq 7$.

(c) *The involution $z : \alpha \mapsto \alpha'$ ($\alpha \in V(\Gamma)$) defines a graph monomorphism from Γ to the complement $\overline{\Gamma}$. Moreover, z induces graph monomorphisms from $\Gamma_{\mathcal{B}}$ to $\overline{\Gamma_{\mathcal{B}^*}}$, and from $\Gamma_{\mathcal{B}^*}$ to $\overline{\Gamma_{\mathcal{B}}}$, defined by $B \mapsto B^*$, and $B^* \mapsto B$, respectively.*

Corollary 11.2.1 *With the same assumptions as in Theorem 11.2.3, we have* $\mathrm{val}(\Gamma) \leq (|V(\Gamma)| - 2)/4$ *and* $\mathrm{val}(\Gamma_{\mathcal{B}^*}) \leq (|V(\Gamma)|/v) - v - 1$. *If in addition* $\mathrm{girth}(\Gamma_{\mathcal{B}}) \geq 7$, *then* $\mathrm{val}(\Gamma) \leq (|V(\Gamma)|/v^2) - (1/v) - 1$.

Remark 11.2.1 Let k^* denote the block size of the 1-design $\mathcal{D}(B^*)$. If $k^* = 1$, then $\mathrm{val}(\Gamma_{\mathcal{B}^*}) = v \cdot \mathrm{val}(\Gamma) > v = |B^*|$ and hence the actions of G_{B^*} on B^* and $\Gamma_{\mathcal{B}^*}(B^*)$ cannot be permutationally equivalent. From Theorem 11.2.3(b), this is the case in particular when $\mathrm{girth}(\Gamma_{\mathcal{B}}) \geq 7$. Thus the G-invariant partition $\mathcal{B}^*$ may not satisfy [P]. Moreover, in the case where $k^* = 1$, the construction given in Section 8.3 applies, and so Γ is isomorphic to a certain G-flag graph of the 1-design $\mathcal{D}(\Gamma, \mathcal{B}^*)$.

11.3 The 1-design $\mathcal{D}(B)$

Part (d) of Theorem 11.2.1 is equivalent to saying that, if (“BC”, D) is a flag of $\mathcal{D}(B)$, then $C \neq D$ and hence (C, B, D) is a 2-arc of $\Gamma_{\mathcal{B}}$. Denote by $\mathrm{arc}_2(\Gamma_{\mathcal{B}})$ the set of all such 2-arcs of $\Gamma_{\mathcal{B}}$, that is,

$$\mathrm{arc}_2(\Gamma_{\mathcal{B}}) := \{(C, B, D) : \text{“}BC\text{”}\,\mathrm{I}D\}.$$

As before, denote by $\mathcal{D}^*(B)$ the dual 1-design of $\mathcal{D}(B)$. Then we have the following theorem which conveys more information about the 1-design $\mathcal{D}(B)$.

Theorem 11.3.1 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent, for some $B \in \mathcal{B}$. Then the following* (a)-(d) *hold.*

(a) *Both $\mathcal{D}(B)$ and $\mathcal{D}^*(B)$ are 1-(v, k, k) designs.*

(b) $\mathrm{arc}_2(\Gamma_{\mathcal{B}})$ *is a G-orbit on* $\mathrm{Arc}_2(\Gamma_{\mathcal{B}})$, *$k = |C^{G_{B,D}}|$, and $k + m \leq v$, where $(C, B, D) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$ and m is the multiplicity of $\mathcal{D}(B)$.*

(c) *The following conditions* (i)-(iv) *are equivalent:*

(i) $\Gamma_{\mathcal{B}}$ *is $(G, 2)$-arc transitive;*

(ii) $\mathrm{arc}_2(\Gamma_{\mathcal{B}}) = \mathrm{Arc}_2(\Gamma_{\mathcal{B}})$*;*

(iii) $k = v - 1$*;*

(iv) $k = v - 1$ *and $\mathcal{D}(B)$ contains no repeated blocks.*

(d) $\Gamma[B, C] \cong K_{k,k}$ *if and only if $G_{B,C,D}$ is transitive on $\Gamma(B) \cap D$ for* $(C, B, D) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$. *In particular, $\Gamma[B, C] \cong K_{v-1,v-1}$ if and only if $\Gamma_{\mathcal{B}}$ is $(G, 3)$-arc transitive.*

Proof (a) That $\mathcal{D}(B)$ is a 1-design implies $vr = bk$. Since $b = v$ (Lemma 11.2.1), we have $r = k$ and hence both $\mathcal{D}(B)$ and $\mathcal{D}^*(B)$ are 1-(v, k, k) designs.

(b) Let $(C, B, D), (C_1, B_1, D_1) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$. Then “$BC$” is adjacent to a vertex $\beta \in D$ and “B_1C_1” is adjacent to a vertex $\beta_1 \in D_1$. So “B^xC^x” is adjacent to $\beta^x \in D^x$ for any $x \in G$. Thus $(C^x, B^x, D^x) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$ and hence $\mathrm{arc}_2(\Gamma_{\mathcal{B}})$ is G-invariant. On the other hand, since Γ is G-symmetric, there exists $y \in G$ such that (“BC”, $\beta)^y$ = (“B_1C_1”, β_1), which implies $(C, B, D)^x = (C_1, B_1, D_1)$ and hence G

is transitive on $\mathrm{arc}_2(\Gamma_{\mathcal{B}})$. Therefore, $\mathrm{arc}_2(\Gamma_{\mathcal{B}})$ is a G-orbit on $\mathrm{Arc}_2(\Gamma_{\mathcal{B}})$. From this we have: "$BE$" $\in B$ is adjacent to a vertex in $D \Leftrightarrow (E, B, D) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}}) \Leftrightarrow$ there exists $x \in G$ such that $(C, B, D)^x = (E, B, D) \Leftrightarrow$ there exists $x \in G_{B,D}$ such that $C^x = E$. So we have $k = |C^{G_{B,D}}|$. Now suppose $D_1, \ldots, D_m \in \Gamma_{\mathcal{B}}(B)$ are repeated blocks of $\mathcal{D}(B)$, so we have $\Gamma(D_1) \cap B = \cdots = \Gamma(D_m) \cap B$. Then by Theorem 11.2.1(d), none of the m distinct vertices "BD_1", ..., "BD_m" of B is in $\Gamma(D_1) \cap B$, and hence $k + m \leq v$ follows.

(c) Clearly, (i) and (ii) are equivalent since $\mathrm{arc}_2(\Gamma_{\mathcal{B}})$ is a G-orbit on $\mathrm{Arc}_2(\Gamma_{\mathcal{B}})$. Note that $k = v - 1$ implies $k = v - 1 \geq 2$ for otherwise we would have $\mathrm{val}(\Gamma) = 1$, contradicting our assumption on the valency of Γ. From the argument in the proof of (b), we have: $k = v - 1 \Leftrightarrow k = v - 1 \geq 2 \Leftrightarrow G_{B,D}$ is transitive on $\Gamma_{\mathcal{B}}(B) \setminus \{D\}$ $\Leftrightarrow G_B$ is 2-transitive on $\Gamma_{\mathcal{B}}(B) \Leftrightarrow \Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. So (i) and (iii) are equivalent. Clearly, (iv) implies (iii). Conversely, since $k + m \leq v$ as we have shown above, $k = v - 1$ implies $m = 1$ and hence $\mathcal{D}(B)$ has no repeated blocks. The equivalence of (i)-(iv) is then established.

(d) Let $(C, B, D) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$. Then by the G-symmetry of Γ, $G_{\text{“}BC\text{”},D} = G_{B,C,D}$ is transitive on $\Gamma(\text{“}BC\text{”}) \cap D \neq \emptyset$. Clearly, we have: $\Gamma[B, D] \cong K_{k,k} \Leftrightarrow \Gamma(\text{“}BC\text{”}) \cap D = \Gamma(B) \cap D \Leftrightarrow G_{\text{“}BC\text{”},D}$ is transitive on $\Gamma(B) \cap D \Leftrightarrow G_{B,C,D}$ is transitive on $\Gamma(B) \cap D$. In particular, from (c) above and Theorem 11.2.1(d) we have: $\Gamma[B, D] \cong K_{v-1,v-1} \Leftrightarrow k = v - 1$ and $G_{B,C,D}$ is transitive on $D \setminus \{\text{“}DB\text{”}\} \Leftrightarrow \Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive and $G_{B,C,D}$ is transitive on $\Gamma_{\mathcal{B}}(D) \setminus \{B\} \Leftrightarrow \Gamma_{\mathcal{B}}$ is $(G, 3)$-arc transitive. □

Remark 11.3.1 (a) Applying Theorem 11.3.1(c) to the graphs Γ in Example 11.1.1, we recover the result (Theorem 5.1.2) that, if $k = v - 1 \geq 2$ and $\mathcal{D}(B)$ contains no repeated blocks, then $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. Furthermore, Theorem 11.3.1(c) shows that, under the assumption [P], this is the only case where $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. Part (d) of Theorem 11.3.1 implies that in such a case $\Gamma_{\mathcal{B}}$ is $(G, 3)$-arc transitive if and only if $\Gamma[B, C] \cong K_{v-1,v-1}$ (Theorem 5.3.1), and that this is the only case where $\Gamma_{\mathcal{B}}$ is $(G, 3)$-arc transitive.

(b) In Theorem 11.2.2(b) we have proved that, if adjacent vertices of Γ share the same second coordinate, then $\Gamma[B^*] \cong K_v$ if and only if $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. By Theorem 11.3.1(c), this in turn is true if and only if $k = v - 1 \geq 2$ and $\mathcal{D}(B)$ contains no repeated blocks. So the assertions in the last sentence of Theorem

11.2.2(b) concerning Γ, $\Gamma[B,C]$, $\Gamma_{\mathcal{B}}$ and the group induced on a component of $\Gamma_{\mathcal{B}}$ can be derived from Theorem 5.1.2(b).

11.4 The case where ρ is incidence-preserving

In this section, we study the case where the bijection ρ in [P] is *incidence-preserving*, that is, it satisfies

$$\alpha \mathrm{I} D \Leftrightarrow \rho^{-1}(D)\mathrm{I}\rho(\alpha) \tag{11.2}$$

for $\alpha \in B$ and $D \in \Gamma_{\mathcal{B}}(B)$. Using labels for the vertices of Γ, this condition can be restated as

$$\text{“}BC\text{”}\mathrm{I}D \Leftrightarrow \text{“}BD\text{”}\mathrm{I}C \tag{11.3}$$

for distinct $C, D \in \Gamma_{\mathcal{B}}(B)$, which in turn is equivalent to saying that

$$(C,B,D) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}}) \Leftrightarrow (D,B,C) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}}). \tag{11.4}$$

Thus, in view of Theorem 11.3.1(b), one of the above holds if and only if $\mathrm{arc}_2(\Gamma_{\mathcal{B}})$ is a self-paired G-orbit on $\mathrm{Arc}_2(\Gamma_{\mathcal{B}})$. By Theorem 11.3.1(c) this is the case in particular when Γ is as in Example 11.1.1. However, there are other cases for which (11.3) is satisfied. This happens for the graph Γ in Example 11.1.2, where (11.3) is satisfied (see the last sentence in that example) but $\Gamma_{\mathcal{B}}$ is not $(G,2)$-arc transitive by Theorem 11.3.1(c) and the fact that $4 = k < v - 1 = 5$.

The additional requirement above implies immediately that $\mathcal{D}(B)$ is a self-dual 1-design, as stated below.

Proposition 11.4.1 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that, for some $B \in \mathcal{B}$, the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent with respect to an incidence-preserving bijection ρ. Then $\mathcal{D}(B)$ is a self-dual* 1-(v,k,k) *design and ρ induces a polarity of $\mathcal{D}(B)$.*

Proof Let ψ be the bijection from $B \cup \Gamma_{\mathcal{B}}(B)$ to $\Gamma_{\mathcal{B}}(B) \cup B$ defined by $\psi(\alpha) = \rho(\alpha)$, $\psi(C) = \rho^{-1}(C)$ for $\alpha \in B$ and $C \in \Gamma_{\mathcal{B}}(B)$. Then $\psi(B) = \Gamma_{\mathcal{B}}(B)$, $\psi(\Gamma_{\mathcal{B}}(B)) = B$, and (11.2) implies that $\alpha \mathrm{I} C \Leftrightarrow \psi(C)\mathrm{I}\psi(\alpha) \Leftrightarrow \psi(\alpha)\mathrm{I}^*\psi(C)$. Thus, ψ is an isomorphism from $\mathcal{D}(B)$ to $\mathcal{D}^*(B)$ and hence $\mathcal{D}(B)$ is self-dual. Clearly, we have $\psi^2 = 1$ and hence ψ is a polarity of $\mathcal{D}(B)$. $\square$

For brevity we call a chordless 6-cycle in a given graph a *hexagon*. Recall that in Section 11.2 we defined Γ' to be the graph with vertex set $V(\Gamma)$ and edge set $\{\{\alpha, \alpha'\} : \alpha \in V(\Gamma)\}$. In the case where (b) in Theorem 11.2.2 occurs, we have the following result which is interesting from a combinatorial point of view.

Theorem 11.4.1 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that, for some $B \in \mathcal{B}$, the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent with respect to an incidence-preserving bijection ρ. Suppose further that adjacent vertices of Γ have the same second coordinate. Then there exists a G-invariant set $\mathcal{H}$ of hexagons of the graph $\Gamma \cup \Gamma'$ such that*

(a) *the edges of each hexagon of $\mathcal{H}$ lie in Γ and Γ' alternatively;*

(b) *each edge of Γ belongs to a unique hexagon of $\mathcal{H}$, and each edge of Γ' belongs to exactly k hexagons of $\mathcal{H}$; and*

(c) *any two hexagons of $\mathcal{H}$ have at most one common edge.*

Proof Let $\{$"BC", "DC"$\}$ be an edge of Γ. Then "BC"$\mathrm{I}D$ and "DC"$\mathrm{I}B$. From (11.3) and our assumption on labels of adjacent vertices, it follows that "BD" is adjacent to "CD" and "DB" is adjacent to "CB". It is easy to see that $h\{$"BC", "DC"$\}$:= ("BC", "DC", "CD", "BD", "DB", "CB", "BC") is a hexagon, and that its edges belong to Γ and Γ' alternatively. (See Figure 9, where the dashed lines represent edges of Γ'.) Set

$$\mathcal{H} := \{h\{\text{“}BC\text{”}, \text{“}DC\text{”}\} : (\text{“}BC\text{”}, \text{“}DC\text{”}) \in \mathrm{Arc}(\Gamma)\}.$$

Since both Γ and Γ' are G-symmetric, $\mathcal{H}$ is G-invariant. One can see that

$$h\{\text{“}BC\text{”}, \text{“}DC\text{”}\} = h\{\text{“}CD\text{”}, \text{“}BD\text{”}\} = h\{\text{“}DB\text{”}, \text{“}CB\text{”}\},$$

and that this is the unique hexagon in $\mathcal{H}$ containing the edge $\{$"BC", "DC"$\}$ of Γ. By Theorem 11.2.2(b), we have $\Gamma[B, D] \cong k \cdot K_2$. When $\{$"BC", "DC"$\}$ runs over all the edges of $\Gamma[B, D]$, we get k hexagons $h\{$"BC", "DC"$\}$ and these are the only members of $\mathcal{H}$ containing the edge $\{$"BD", "DB"$\}$ of Γ'. So both (a) and (b) are true. The validity of (c) follows immediately from the definition of the hexagons of $\mathcal{H}$. $\square$

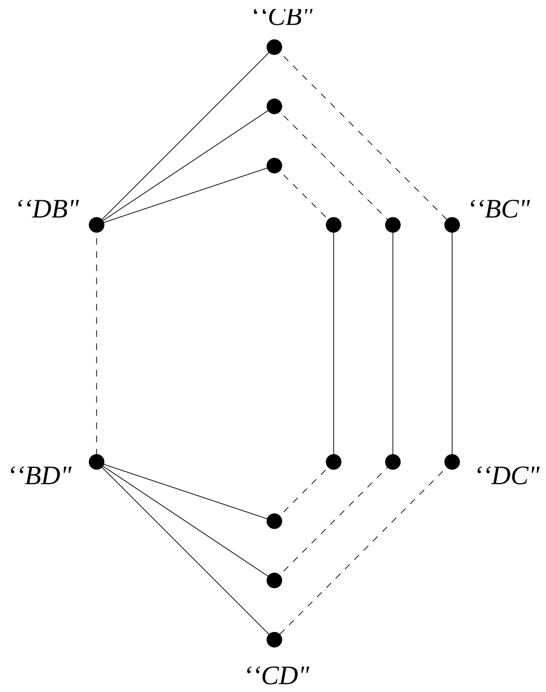


FIGURE 9 Hexagons in $\Gamma \cup \Gamma'$

The case where two adjacent vertices of Γ have labels involving four distinct blocks seems to be much more complicated, even under our additional assumption that ρ is incidence-preserving. So in the following we concentrate on the extreme case where $\Gamma[B,C] \cong k \cdot K_2$ is a matching. In this case, we show that there exists a G-orbit on n-cycles of $\Gamma_{\mathcal{B}}$, for some even integer $n \geq 4$, which determines completely the adjacency of Γ.

Theorem 11.4.2 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that, for some $B \in \mathcal{B}$, the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent with respect to an incidence-preserving bijection ρ. Suppose further that adjacent vertices of Γ have labels involving four distinct blocks and that $\Gamma[B,C] \cong k \cdot K_2$ for adjacent blocks B, C of $\mathcal{B}$. Then there exist an even integer $n \geq 4$ and a G-orbit $\mathcal{E}$ on n-cycles of $\Gamma_{\mathcal{B}}$ such that the following* (a)-(c) *hold:*

(a) *each 2-arc of $\Gamma_{\mathcal{B}}$ is contained in at most one n-cycle of $\mathcal{E}$;*

(b) *a 2-arc of $\Gamma_{\mathcal{B}}$ is contained in an n-cycle of $\mathcal{E}$ if and only if it lies in* $\mathrm{arc}_2(\Gamma_{\mathcal{B}})$*; and*

(c) *two vertices "BC", "DE" of Γ are adjacent if and only if (C,B,D,E) is a 3-arc of $\Gamma_{\mathcal{B}}$ contained in an n-cycle of $\mathcal{E}$.*

Proof Let $(B_0, B_1, B_2) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$, that is, "$B_1B_0$"I$B_2$. Then, since $\Gamma[B_1, B_2] \cong k \cdot K_2$ by our assumption, there exists a unique block $B_3 \in \Gamma_{\mathcal{B}}(B_2)$ such that "B_2B_3" is the unique vertex in B_2 adjacent to "B_1B_0". This implies $(B_3, B_2, B_1) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$ and hence $(B_1, B_2, B_3) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$ by (11.4). Thus "B_2B_1"IB_3 and hence there exists a unique block $B_4 \in \Gamma_{\mathcal{B}}(B_3)$ such that "B_3B_4" is the unique vertex in B_3 adjacent to "B_2B_1". This in turn implies that $(B_4, B_3, B_2) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$ and hence $(B_2, B_3, B_4) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$. Inductively, suppose that $B_0, B_1, B_2, \ldots, B_i$ have been determined for some $i \geq 3$ such that $(B_{j-1}, B_j, B_{j+1}), (B_{j+1}, B_j, B_{j-1}) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$ for $j = 1, 2, \ldots, i-1$, and that "B_jB_{j-1}" is adjacent to "$B_{j+1}B_{j+2}$" for $j = 1, 2, \ldots, i-2$. Then in particular "$B_{i-1}B_{i-2}$"IB_i and hence there exists a unique block $B_{i+1} \in \Gamma_{\mathcal{B}}(B_i)$ such that "B_iB_{i+1}" is the unique vertex in B_i adjacent to "$B_{i-1}B_{i-2}$". Thus we have $(B_{i+1}, B_i, B_{i-1}) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$ and hence $(B_{i-1}, B_i, B_{i+1}) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$. Continuing this process, we see that each 2-arc (B_0, B_1, B_2) in $\mathrm{arc}_2(\Gamma_{\mathcal{B}})$ determines a unique sequence $B_0, B_1, B_2, \ldots, B_i, B_{i+1}, \ldots$ of blocks of $\mathcal{B}$ such that $(B_{i-1}, B_i, B_{i+1}), (B_{i+1}, B_i, B_{i-1}) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$ and "B_iB_{i-1}" is adjacent to "$B_{i+1}B_{i+2}$" for each $i \geq 1$. Our assumption on labels of adjacent vertices of Γ implies that any four consecutive blocks in this sequence are pairwise distinct. Since we have only a finite number of blocks in $\mathcal{B}$, this sequence must contain repeated terms. Let B_n be the first block in the sequence which coincides with one of the preceding blocks. Then $n \geq 4$ and we claim that B_n must coincide with B_0. Suppose to the contrary that $B_n = B_\ell$ for some integer ℓ with $\ell \geq 1$. Then, since $\mathrm{arc}_2(\Gamma_{\mathcal{B}})$ is a G-orbit on $\mathrm{Arc}_2(\Gamma_{\mathcal{B}})$ (Theorem 11.3.1(b)), there exists $x \in G$ such that $(B_\ell^x, B_{\ell+1}^x, B_{\ell+2}^x) = (B_0, B_1, B_2)$. By the construction above, one can see that the sequence determined by $(B_\ell^x, B_{\ell+1}^x, B_{\ell+2}^x)$ is $B_\ell^x, B_{\ell+1}^x, B_{\ell+2}^x, \ldots, B_{\ell+i}^x, \ldots$. So by the uniqueness of the sequence determined by (B_0, B_1, B_2) we must have $B_{\ell+i}^x = B_i$ for each $i \geq 0$. In particular, we have $B_n^x = B_{\ell+(n-\ell)}^x = B_{n-\ell}$. On the other hand, $B_n = B_\ell$ implies that $B_n^x = B_\ell^x = B_0$. Thus we have $B_{n-\ell} = B_0$, which contradicts the minimality of n. So B_n must coincide with B_0 and we get an n-cycle $C(B_0, B_1, B_2) := (B_0, B_1, B_2, \ldots, B_{n-1}, B_0)$ of $\Gamma_{\mathcal{B}}$. Note that $(B_2, B_1, B_0) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$ implies that there exists a unique block $C \in \Gamma_{\mathcal{B}}(B_0)$ such that "B_0C" is the unique vertex in B_0 adjacent to "B_1B_2". So we have $(C, B_0, B_1) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$ and, by the construction above, the sequence determined by (C, B_0, B_1) is $C, B_0, B_1, B_2, \ldots, B_i, \ldots$. Since the first repeated block in this sequence is C, as shown above, we must have $C = B_{n-1}$ and hence $(B_{n-1}, B_0, B_1), (B_1, B_0, B_{n-1}) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$ and "B_0B_{n-1}" is adjacent to "B_1B_2". In a similar way, one can show that $(B_{n-2}, B_{n-1}, B_0), (B_0, B_{n-1}, B_{n-2}) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$ and "$B_{n-1}B_{n-2}$" is adjacent to "B_0B_1". Therefore, reading the subscripts modulo n (here and in the remainder of the proof), we have $(B_{i-1}, B_i, B_{i+1}), (B_{i+1}, B_i, B_{i-1}) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$ and "B_iB_{i-1}" is adjacent to "$B_{i+1}B_{i+2}$" for each $i \geq 1$. Hence n must be an even integer and, by definition, all these 2-arcs contained in $C(B_0, B_1, B_2)$ determine the same n-cycle, namely $C(B_0, B_1, B_2)$. By Theorem 11.3.1(b) any 2-arc in $\mathrm{arc}_2(\Gamma_{\mathcal{B}})$ has the form (B_0^x, B_1^x, B_2^x) for some $x \in G$, and by definition we have $C(B_0^x, B_1^x, B_2^x) = (B_0^x, B_1^x, B_2^x, \ldots, B_{n-1}^x, B_0^x) = (C(B_0, B_1, B_2))^x$. This implies that $\mathcal{E} := \{C(E, D, B) : (E, D, B) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})\}$ is a G-orbit on n-cycles of $\Gamma_{\mathcal{B}}$. Note that, for a given 2-arc (E, D, B) of $\mathrm{arc}_2(\Gamma_{\mathcal{B}})$, $C(E, D, B)$ is the unique n-cycle in $\mathcal{E}$ containing (E, D, B). So (a) and (b) are true. If "DE", "BC" are adjacent in Γ, then $(E, D, B) \in \mathrm{arc}_2(\Gamma_{\mathcal{B}})$ and by the argument above (E, D, B, C) is a 3-arc contained in $C(E, D, B)$. Conversely, from the definition of the n-cycles in $\mathcal{E}$, for each 3-arc (E, D, B, C) contained in an n-cycle of $\mathcal{E}$, "DE", "BC" are adjacent in Γ and hence (c) follows. □

Remark 11.4.1 Suppose Γ, G, $\mathcal{B}$ and ρ are as in Example 11.1.1. Suppose further that Γ almost covers $\Gamma_{\mathcal{B}}$. Then ρ is incidence-preserving, as mentioned at the beginning of this section, and $\mathrm{arc}_2(\Gamma_{\mathcal{B}}) = \mathrm{Arc}_2(\Gamma_{\mathcal{B}})$ by Theorem 11.3.1(c). So in this case Theorem 11.4.2 implies that $\Gamma_{\mathcal{B}}$ is a near n-gonal graph with respect to $\mathcal{E}$. Thus, Theorem 11.4.2 can be taken as a generalization of the first assertion of Theorem 7.0.2(b), and the proof is similar in spirit to that of the "only if" part of Theorem 7.3.1.

11.5 Reconstruction of Γ, and 3-arc graphs again

By using the labelling technique established in Section 11.2, we now prove that any G-symmetric graph Γ satisfying [P] can be reconstructed from the quotient graph $\Gamma_{\mathcal{B}}$ and the action of G on $\mathcal{B}$, namely Γ is isomorphic to a 3-arc graph of $\Gamma_{\mathcal{B}}$ with respect to a certain self-paired G-orbit on 3-arcs of $\Gamma_{\mathcal{B}}$. Conversely, we prove that, for any G-symmetric graph Σ and any self-paired G-orbit Δ on $\mathrm{Arc}_3(\Sigma)$, the 3-arc graph $\Xi(\Sigma, \Delta)$ satisfies the condition [P]. The proof of the following theorem is essentially the same as the proofs of Theorems 5.2.1 and 5.2.2.

Theorem 11.5.1 *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent, so the vertices of Γ are labelled by ordered pairs of adjacent blocks of $\Gamma_{\mathcal{B}}$. Then $\Gamma \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ holds for Δ the (self-paired) G-orbit on $\mathrm{Arc}_3(\Gamma_{\mathcal{B}})$ containing the 3-arc (C, B, D, E), where $(\text{“}BC\text{”}, \text{“}DE\text{”})$ is an arc of Γ.*

Conversely, for any G-symmetric graph Σ and any self-paired G-orbit Δ on $\mathrm{Arc}_3(\Sigma)$, the graph $\Gamma := \Xi(\Sigma, \Delta)$, group G and partition $\mathcal{B} := \mathcal{B}(\Sigma)$ satisfy all the conditions above. Moreover, we have $\Gamma_{\mathcal{B}} \cong \Sigma$.

Proof Let Γ, G and $\mathcal{B}$ be as in the first part of the theorem. Let $(\text{“}BC\text{”}, \text{“}DE\text{”})$ be a fixed arc of Γ. Then by Theorem 11.2.1(d), (C, B, D, E) is a 3-arc of $\Gamma_{\mathcal{B}}$. Let Δ be the G-orbit on $\mathrm{Arc}_3(\Gamma_{\mathcal{B}})$ containing (C, B, D, E). Since Γ is G-symmetric, there exists $x \in G$ such that $(\text{“}BC\text{”}, \text{“}DE\text{”})^x = (\text{“}DE\text{”}, \text{“}BC\text{”})$. So $(E, D, B, C) = (C, B, D, E)^x \in \Delta$ by (11.1), and hence Δ is self-paired. Again by the G-symmetry of Γ and (11.1), we have: $(C_1, B_1, D_1, E_1) \in \Delta \Leftrightarrow$ there exists $x \in G$ such that $(C_1, B_1, D_1, E_1) = (C, B, D, E)^x \Leftrightarrow$ there exists $x \in G$ such that $(\text{“}B_1C_1\text{”}, \text{“}D_1E_1\text{”}) = (\text{“}BC\text{”}, \text{“}DE\text{”})^x \Leftrightarrow (\text{“}B_1C_1\text{”}, \text{“}D_1E_1\text{”}) \in \mathrm{Arc}(\Gamma)$. Therefore, the mapping $\text{“}B_1C_1\text{”} \mapsto (B_1, C_1)$, for $\text{“}B_1C_1\text{”} \in V(\Gamma)$, establishes a graph isomorphism from Γ to $\Xi(\Gamma_{\mathcal{B}}, \Delta)$.

Now suppose Σ is a G-symmetric graph and Δ is a self-paired G-orbit on $\mathrm{Arc}_3(\Sigma)$, and let $(\tau, \sigma, \sigma', \tau') \in \Delta$. Then from Lemma 5.2.1, $\Gamma := \Xi(\Sigma, \Delta)$ is a G-symmetric graph with $\mathcal{B} := \mathcal{B}(\Sigma)$ a nontrivial G-invariant partition of $V(\Gamma)$, where $\mathcal{B}(\Sigma) := \{B(\alpha) : \alpha \in V(\Sigma)\}$ is as defined in Section 5.2. If $B(\alpha)$ and $B(\alpha')$ are adjacent blocks of $\mathcal{B}$, then there exist $(\alpha, \beta) \in B(\alpha)$ and $(\alpha', \beta') \in B(\alpha')$ such that $(\alpha, \beta), (\alpha', \beta')$ are adjacent in Γ, and hence $(\beta, \alpha, \alpha', \beta') \in \Delta$. In particular, we have $(\alpha, \alpha') \in \mathrm{Arc}(\Sigma)$. Conversely, suppose $(\alpha, \alpha') \in \mathrm{Arc}(\Sigma)$. Then since Σ is G-symmetric there exists $g \in G$ such that $(\sigma, \sigma')^g = (\alpha, \alpha')$. Setting $\tau^g = \beta$ and $(\tau')^g = \beta'$, then $(\beta, \alpha, \alpha', \beta') = (\tau, \sigma, \sigma', \tau')^g \in \Delta$. So $(\alpha, \beta) \in B(\alpha)$ is adjacent to $(\alpha', \beta') \in B(\alpha')$ in Γ and hence $B(\alpha)$ and $B(\alpha')$ are adjacent blocks of $\mathcal{B}$. Thus, $\alpha \mapsto B(\alpha)$ defines an isomorphism from Σ to $\Gamma_{\mathcal{B}}$. From Lemma 5.2.1(d), the actions of $G_{B(\sigma)}$ on $B(\sigma)$ and $\Sigma(\sigma)$ are permutationally equivalent with respect to the bijection $(\sigma, \sigma') \mapsto \sigma'$. So the actions of $G_{B(\sigma)}$ on $B(\sigma)$ and $\Gamma_{\mathcal{B}}(B(\sigma))$ are permutationally equivalent with respect to the bijection $\rho : (\sigma, \sigma') \mapsto B(\sigma')$, for $(\sigma, \sigma') \in B(\sigma)$. $\square$

Remark 11.5.1 (a) Theorem 11.5.1 is a counterpart of the second part of Theorem 5.2.3, where $\Gamma_{\mathcal{B}}$ and Σ are assumed to be $(G, 2)$-arc transitive.

(b) From Theorem 11.4.2(c) and the proof above one can see that, under the assumptions of Theorem 11.4.2, the self-paired G-orbit Δ on $\mathrm{Arc}_3(\Gamma_{\mathcal{B}})$ such that $\Xi(\Gamma_{\mathcal{B}}, \Delta) \cong \Gamma$ is precisely the set of all 3-arcs of $\Gamma_{\mathcal{B}}$ contained in some n-cycle of $\mathcal{E}$.

Conversely, if, for a G-symmetric graph Σ, there exist an even integer $n \geq 4$ and a G-orbit $\mathcal{E}$ on n-cycles of Σ such that each 2-arc of Σ is contained in at most one n-cycle of $\mathcal{E}$, and that the set of 2-arcs of Σ contained in some n-cycle of $\mathcal{E}$ is a G-orbit on $\mathrm{Arc}_2(\Sigma)$, then one can check that the following (i)-(iii) hold:

(i) the set Δ of 3-arcs of Σ contained in some n-cycle of $\mathcal{E}$ is a self-paired G-orbit on $\mathrm{Arc}_3(\Sigma)$, and thus $\Gamma := \Xi(\Sigma, \Delta)$ is well-defined;

(ii) for $\mathcal{B} := \mathcal{B}(\Sigma)$, the bijection ρ from $B(\sigma)$ to $\Gamma_{\mathcal{B}}(B(\sigma))$ defined at the end of the proof of Theorem 11.5.1 is incidence-preserving; and

(iii) $\Gamma[B(\sigma), B(\sigma')] \cong k \cdot K_2$ for adjacent blocks $B(\sigma)$, $B(\sigma')$ of $\mathcal{B}$.

These results together give the inverse of Theorem 11.4.2 and generalize the second assertion in Theorem 7.0.2(b).

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Appendix A

The Chinese originals of the quotes

The quotes used in the thesis are translated from some Chinese classics [1]. Listed below are the Chinese originals and the chapters of the thesis where these quotes are used.

[1]We referenced the translations of these classics in the following books: 1. A Source Book in Chinese Philosophy, translated and compiled by Wing-Tsit Chan, Princeton University Press, Princeton, NJ, 1963; 2. The Analects of Confucius, translated and annotated by Arthur Waley, George Allen & Unwin Ltd, London, 1938; 3. Confucius: The Great Digest and the Unwobbling Pivot, translated and commented by Ezra Pound, Peter Owen, London, 1952.