Distance Labellings of Graphs

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outline of this talk

- Channel assignment and distance labelling

  channel assignment, distance labelling,
  connection with chromatic number

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- Distance two labelling

  Δ^2-conjecture, planar graphs, outerplanar graphs

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  distance three labelling for trees, recent progress, hypercubes

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  a general approach, Hamming graphs, hypercubes, recent progress

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- Summary

motivations

- frequency assignment

motivations

- frequency assignment

- colouring power graphs

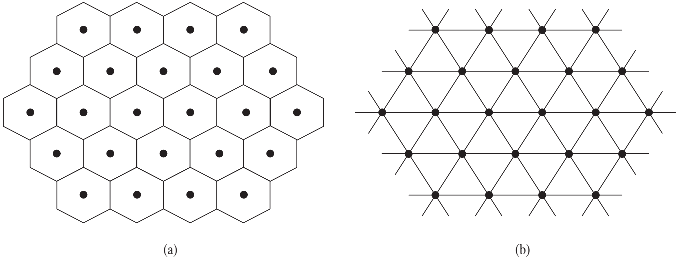
channel assignment

The area covered by a cellular communication system is divided into regions, called cells.

Each cell is served by a base station within the cell, where a transmitter serves customers in the cell.

Interference graph: vertices represent transmitters; two vertices are adjacent if and only if they 'interfer' with each other.

Considering only interference caused by geographical proximity, two vertices are adjacent in the interference graph if and only if the corresponding cells have a common boundary.



(a) A cellular system; (b) the corresponding interference graph.

The available bandwidth is divided into slots, called channels and represented by integers $0, 1, 2, \ldots$.

The same channel can be used by different transmitters which are distant enough geographically.

The channel assignment problem asks for assigning a channel or a set of channels to each transmitter such that

- interference is kept at an acceptable level, and
- the span is minimized, where the span is the difference between the largest and smallest channels used.

There are various models for the channel assignment problem.

distance labelling

Definition

Let G be a finite or infinite graph.
Let $h_1, h_2, \ldots, h_d \geq 0$ be integers. (Often we assume $h_1 \geq h_2 \geq \cdots \geq h_d$.)

An $L(h_1, h_2, \ldots, h_d)$-labelling of G is a mapping

$$\phi : V(G) \to \{0, 1, 2, \ldots\}$$

such that, for $i = 1, 2, \ldots, d$ and $u, v \in V(G)$,

$$d(u, v) = i \Rightarrow |\phi(u) - \phi(v)| \geq h_i.$$

Definition

The span of G w.r.t. ϕ is defined as

$$sp(G, \phi) := \max_{v \in V(G)} \phi(v) - \min_{v \in V(G)} \phi(v).$$

(We can always assume $\min_{v \in V(G)} \phi(v) = 0$.)

The $\lambda_{h_1, h_2, \ldots, h_d}$-number of G is defined as

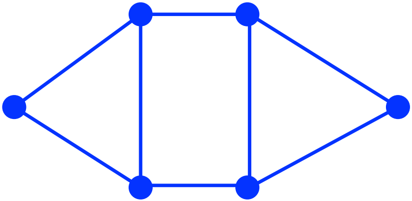
$$\lambda_{h_1, h_2, \ldots, h_d}(G) := \min_{\phi} sp(G, \phi) = \min_{\phi} \max_{v \in V(G)} \phi(v).$$

Call

$$\lambda(G) := \lambda_{2,1}(G)$$

the λ-number of G.

$L(2,1)$-labelling: an example



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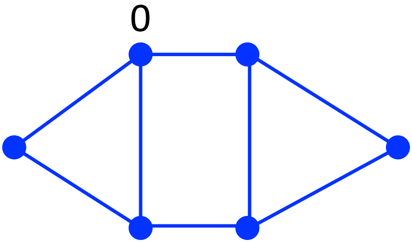

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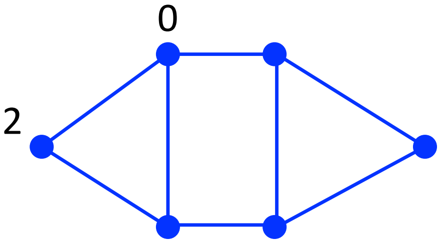

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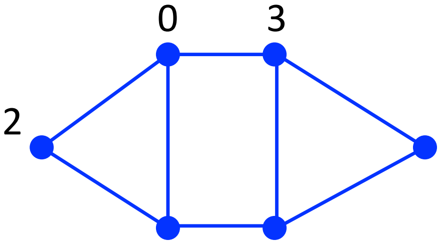

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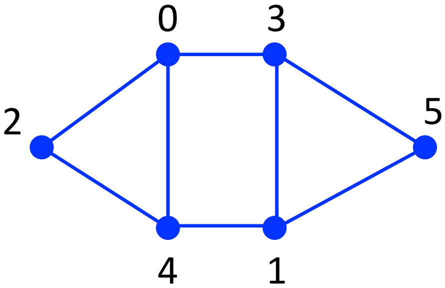

complexity and connection with chromatic number

The decision problem corresponding to λ is NP-complete for general graphs, and polynomial solvable for trees.

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$\vdots$

- $\lambda_{1,\dots,1}(G) = \chi(G^d) - 1$

Δ^2-conjecture

Conjecture

(Griggs & Yeh 1992)
For any graph G with $\Delta \geq 2$, $\lambda(G) \leq \Delta^2$.

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For any graph G with $\Delta \geq 2$, $\lambda(G) \leq \Delta^2$.

This has been confirmed for

- chordal graphs (Sakai)
- outerplanar graphs (multiple authors)
- generalized Petersen graphs (Georges & Mauro)
- Hamiltonian graphs with $\Delta \leq 3$ (Kang)
- two families of Hamming graphs (Chang, Lu & Z, Z)

$\vdots$

Theorem

For any graph G,

- $\lambda(G) \leq \Delta^2 + 2\Delta$ *(Griggs & Yeh 1992)*

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Theorem

(Havet, Reed & Sereni 2012) *For any $h \geq 1$, there exists a constant $\Delta(h)$ such that any graph with $\Delta \geq \Delta(h)$ has an $L(h, 1)$-labelling with span $\leq \Delta^2$.*

In particular, the Δ^2-conjecture is true for sufficiently large Δ.

planar graphs

Theorem

Let G be a planar graph.

- $\lambda_{p,q}(G) \leq (4q-2)\Delta + 10p + 38q - 24$
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  (Molloy & Salavatipour 2005)
- $\lambda_{p,q}(G) \leq q\Delta + 2p - 2$ *if girth* ≥ 7 *and* $\Delta \geq 190 + 2\lceil p/q \rceil$
  (Dvořák, Kŕal, Nejedlý & Škrekovski 2007+)

Theorem

(Bella, Kŕal, Mohar & Quittnerová 2007)
The Δ^2-conjecture is true for planar graphs with $\Delta \neq 3$.

outerplanar graphs

Theorem

(Bodlaender, Kloks, Tan & Leeuwen 2000) For any outerplanar graph G,

$$\lambda(G) \leq \Delta + 8$$

Conjecture

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Theorem

This is

- *true if $\Delta \geq 15$ (Liu & Zhu 2005)*
- *true if $\Delta \geq 8$ (Calamoneri & Petreschi 2004)*
- *false if $\Delta = 3$ (Calamoneri & Petreschi 2004)*

Theorem

(Calamoneri & Petreschi 2004) For outerplanar graphs G with $\Delta = 3$,

$$\lambda(G) \leq \Delta + 5.$$

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Question

(Calamoneri & Petreschi 2004) Is the bound

$$\lambda(G) \leq \Delta + 5$$

tight for outerplanar graphs with $\Delta = 3$?

Theorem

(Li & Z 2011+) For every outerplanar graph G with $\Delta = 3$,

$$\lambda(G) \leq \Delta + 3 = 6.$$

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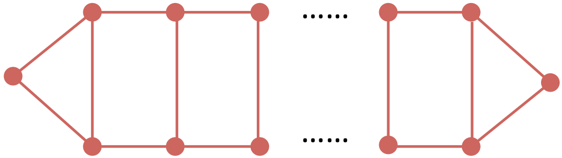
The bound is attainable by infinitely many outerplanar graphs.



A family of outerplanar graphs with $\lambda = 6$

idea of the proof

- Suppose otherwise. Let G be a smallest counterexample.

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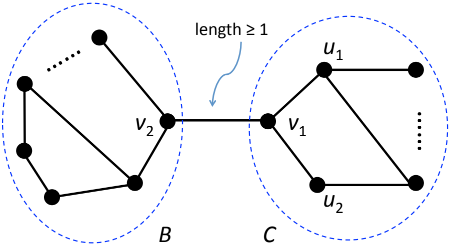

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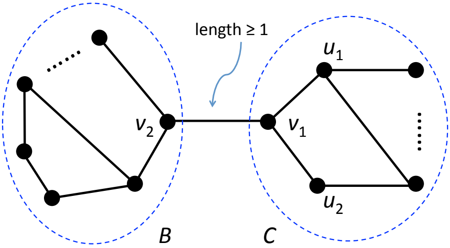
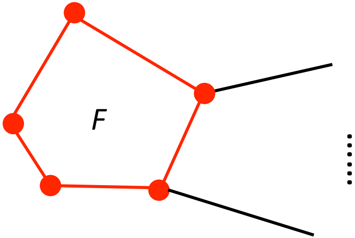


- We can 'extend' a 6-$L(2,1)$-labelling of a 'short' path to a 6-$L(2,1)$-labelling of G, a contradiction.

idea of the proof

- G contains a face F of length ≥ 4.



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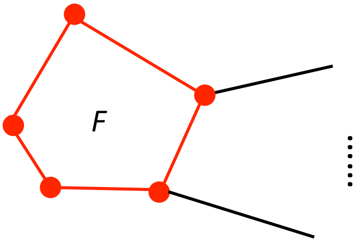

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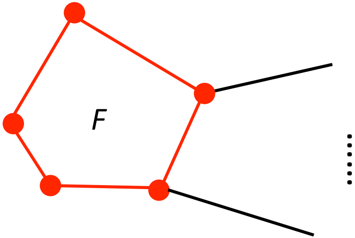

- The boundary of F has a 'good' 6-$L(2,1)$-labelling.
- Extend this to a 6-$L(2,1)$-labelling of G, a contradiction.

related results

Conjecture

(Wegner 1977) For any planar graph G,

$$\chi(G^2) \leq \begin{cases} 7, & \text{if } \Delta = 3 \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7 \\ \lfloor 3\Delta/2 \rfloor + 1, & \text{if } \Delta \geq 8 \end{cases}$$

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Open for general case with best bound $5\Delta/3 + 77$ (Molloy et al)

Theorem

(Thomassen 2001) Wegner's conjecture is true for $\Delta = 3$. That is, for any planar graph G with $\Delta = 3$, we have

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We proved

$$\lambda_{2,1}(G) \leq 6.$$

Since $\lambda_{1,1} \leq \lambda_{2,1}$, our result can be viewed as a generalisation of Thomassen's result for outerplanar graphs.

distance three labelling for trees

In the following we will focus on distance three labellings.

Definition

Define

$$\Delta_2(G) := \max_{uv \in E(G)} (d(u) + d(v)).$$

If G is infinite, then $\Delta_2(G) = \infty$ iff $\{d(u)\}_{u \in V(G)}$ is unbounded, and in this case $\lambda_{h,1,1}(G) = \infty$.

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We only consider finite trees and infinite trees with Δ finite.

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We only consider finite trees and infinite trees with Δ finite.

We always use T to denote a finite tree with diameter at least 3 or an infinite tree with a finite maximum degree.

bounds

Theorem

(King, Ras & Z 2010) For any $h \geq 1$, we have

$$\max\left\{\max_{uv \in E(T)} \min\{d(u), d(v)\} + h - 1, \Delta_2(T) - 1\right\}$$

$$\leq \lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 1$$

Moreover, the lower bound is attainable for any $h \geq 1$ and the upper bound is attainable for any $h \geq 3$.

improving the upper bound

Define

$$\delta^*(T) = \min_{u \in V(T),\, d(u) \geq 2} d(u).$$

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Theorem

(King, Ras & Z 2010) *If* $h \leq \delta^*(T)$, *then*

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Theorem

(King, Ras & Z 2010) If $h \leq \delta^*(T)$, then

$$\lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 2.$$

The condition $h \leq \delta^*(T)$ is sufficient but not necessary to ensure the upper bound.

E.g. the upper bound is valid if T has only one ‘heavy’ edge.

Theorem

(King, Ras & Z 2010) Let T be a finite caterpillar of diameter at least three or an infinite caterpillar of finite maximum degree. If $h \geq 2$, then

$$\lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 2$$

and the bound is sharp.

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and the bound is sharp.

Moreover, if there exists no vertex on the spine with degree $\Delta_2(T) - 2$, then

$$\lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 3;$$

if there exist consecutive vertices u, v, w on the spine such that $d(u) = d(w) = \Delta_2 - 2$ and $d(v) = 2$, then

$$\lambda_{h,1,1}(T) = \Delta_2(T) + h - 2.$$

Since $\delta^*(T) \geq 2$, we have

Corollary

(King, Ras & Z 2010)

$$\Delta_2(T) - 1 \leq \lambda_{2,1,1}(T) \leq \Delta_2(T).$$

This is the counterpart of

$$\Delta(T) + 1 \leq \lambda_{2,1}(T) \leq \Delta(T) + 2$$

(Griggs & Yeh 1992).

Corollary

$$\chi(T^3) = \lambda_{1,1,1}(G) + 1 = \Delta_2(T).$$

If T is finite, this can also be deduced from the following facts:
(1) T^3 is chordal with clique number $\Delta_2(T)$;
(2) for chordal graphs, chromatic number = clique number
(G chordal and n odd $\Rightarrow G^n$ chordal. Since a finite tree T is chordal, T^3 is chordal.)

proving the upper bound

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- $L_i(v) := \{w \in V(T_v) : d(v,w) = i\}, \ \ i = 0, 1, \ldots,$
- index the vertices of T_u such that the unique path between u and a vertex $a_1 a_2 \cdots a_{i-1} a_i \in L_i(u)$ is

$$u, a_1, a_1 a_2, a_1 a_2 a_3, \ldots, a_1 a_2 \cdots a_{i-1} a_i$$

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- index the vertices of T_v such that the unique path between v and a vertex $b_1 b_2 \cdots b_{i-1} a_i \in L_i(v)$ is

$$u, b_1, b_1 b_2, b_1 b_2 b_3, \ldots, b_1 b_2 \cdots b_{i-1} b_i$$

initialization

Define

$$\phi(u) = 0$$

$$\phi(v) = \Delta_2 + h - 1$$

$$\phi(a_1) = \Delta_2 + h - 1 - a_1, \quad a_1 = 1, 2, \ldots, d(u) - 1$$

$$\phi(b_1) = b_1, \quad b_1 = 1, 2, \ldots, d(v) - 1$$

Since $\Delta_2 + h - 1 - (d(u) - 1) = d(v) + h$, we have

$$\phi(N(u) \setminus \{v\}) = [d(v) + h, \Delta_2 + h - 2]$$

$$\phi(N(v) \setminus \{u\}) = [1, d(v) - 1].$$

labelling T_u

Prove the following for $i \geq 1$ by induction:

(a) if i is odd, then for all $a_1 \cdots a_{i-1} \in L_{i-1}(u)$ we can label independently the vertices of $N(a_1 \cdots a_{i-1}) \setminus \{a_1 \cdots a_{i-2}\}$ by the $d(a_1 \cdots a_{i-1}) - 1$ largest available integers in

$$[\Delta_2 + h - 1 - d(a_1 \cdots a_{i-1}), \Delta_2 + h - 1]$$

such that the $L(h, 1, 1)$-conditions are satisfied among vertices of T_u up to level $L_i(u)$;

(b) if i is even, then for all $a_1 \cdots a_{i-1} \in L_{i-1}(u)$ we can label independently the vertices of $N(a_1 \cdots a_{i-1}) \setminus \{a_1 \cdots a_{i-2}\}$ by the $d(a_1 \cdots a_{i-1}) - 1$ smallest available integers in

$$[0, d(a_1 \cdots a_{i-1})]$$

such that the $L(h, 1, 1)$-conditions are satisfied among vertices of T_u up to level $L_i(u)$.

labelling T_v

Prove the following for $i \geq 1$ by induction:

(c) if i is odd, then for all $b_1 \cdots b_{i-1} \in L_{i-1}(v)$ we can label independently the vertices of $N(b_1 \cdots b_{i-1}) \setminus \{b_1 \cdots b_{i-2}\}$ by the $d(b_1 \cdots b_{i-1}) - 1$ smallest available integers in

$$[0, d(a_1 \cdots a_{i-1})]$$

such that the $L(h,1,1)$-conditions are satisfied among vertices of T_v up to level $L_i(v)$;

(d) if i is even, then for all $b_1 \cdots b_{i-1} \in L_{i-1}(v)$ we can label independently the vertices of $N(b_1 \cdots b_{i-1}) \setminus \{b_1 \cdots b_{i-2}\}$ by the $d(b_1 \cdots b_{i-1}) - 1$ largest available integers in

$$[\Delta_2 + h - 1 - d(b_1 \cdots b_{i-1}), \Delta_2 + h - 1]$$

such that the $L(h,1,1)$-conditions are satisfied among vertices of T_v up to level $L_i(v)$.

questions and conjecture

Question

(King, Ras & Z 2010)

(a) *Given $h \geq 3$, characterise those finite trees T with diameter at least three such that $\lambda_{h,1,1}(T) = \Delta_2(T) + h - 1$.*

(b) *Characterise finite trees T with diameter at least three such that $\lambda_{2,1,1}(T) = \Delta_2(T)$.*

$N(n)$: # pairwise non-isomorphic trees with n vertices and diameter at least three
$N_1(n)$: # such trees with $\lambda_{2,1,1} = \Delta_2 - 1$

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Conjecture

(King, Ras & Z 2010) $\lim_{n\to\infty} \frac{N_1(n)}{N(n)} = 1$.

$N(n)$: # pairwise non-isomorphic trees with n vertices and diameter at least three
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Conjecture

(King, Ras & Z 2010) $\lim_{n\to\infty} \frac{N_1(n)}{N(n)} = 1.$

Question

(King, Ras & Z 2010) For a fixed integer $h \geq 2$, is the problem of determining $\lambda_{h,1,1}$ for finite trees solvable in polynomial time?

$L(2,1,1)$-labeling with fixed span

$L(i,j,k)$-LABELING:

Instance: a graph G and an integer λ
Question: does G have an $L(i,j,k)$-labelling with span λ?

$L(2,1,1)$-labeling with fixed span

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Question: does G have an $L(i,j,k)$-labelling with span λ?

Theorem

(Fiala, Golovach, Kratochvíl, Lidický & Paulusmab 2011)
$L(2,1,1)$-LABELING *is NP-complete for every fixed* $\lambda \geq 5$ *and is solvable in linear time for all* $\lambda \leq 4$.

$L(2,1,1)$-labeling for graphs of bounded treewidth

Theorem

(B. Courcelle 1990) Every problem definable in Monadic Second-Order Logic (MSOL) can be solved in linear time on graphs of bounded treewidth.

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Theorem

(FGKLP 2011) $L(2,1,1)$-LABELING *for every fixed* λ *is solvable in linear time for graphs of bounded treewidth.*

In particular, $L(2,1,1)$-LABELING for trees can be solved in linear time if λ is fixed (i.e. not part of the input).

The same results hold for the $L(h_1,\ldots,h_d)$-Labelling Problem.

$L(2,1,1)$-labeling when λ is part of the input

Theorem

(FGKLP 2011) $L(2,1,1)$-LABELING *is NP-complete for the class of trees.*

$L(2,1,1)$-labeling when λ is part of the input

Theorem

(FGKLP 2011) $L(2,1,1)$-LABELING *is NP-complete for the class of trees.*

This together with

$$\Delta_2(T) - 1 \leq \lambda_{2,1,1}(T) \leq \Delta_2(T)$$

implies:

Corollary

Unless $NP = P$, there is no good (i.e. polynomial time verifiable) characterisation of finite graphs T with $\lambda_{2,1,1}(T) = \Delta_2(T)$.

elegant labelling

The proof of our upper bound $\Delta_2(T) + h - 1$ is by construction of an $L(h, 1, 1)$-labelling with some extra property.

This motivated FGKLP to introduce the following concept.

Definition

(FGKLP 2011) An $L(h_1, h_2, \ldots, h_d)$-labelling ϕ of G with span λ is called elegant if for every vertex u there exists an interval I_u mod $(\lambda + 1)$ such that $\phi(N(u)) \subseteq I_u$, and for every edge $uv \in E(G)$, $I_u \cap I_v = \emptyset$.

$L(h_1, h_2, h_3)$-labelling of hypercubes

The d-dimensional cube Q_d has 01-words of length d as its vertices such that two words are adjacent iff they differ at exactly one position.

Denote

$$p = p(d) := \lceil \log_2(d+1) \rceil$$

$$q = q(d) := \max\{d + 1 + \lceil \log_2(d+1) \rceil - 2^{\lceil \log_2(d+1) \rceil}, 0\}.$$

Then $q \leq p$ and

$$2^{p-1} \leq d \leq 2^p - 1.$$

Note that d is a power of 2 iff $d = 2^{p-1}$.

Theorem

(Z 2008) For any $d \geq 3$ *and* $h_1 \geq h_2 \geq h_3 \geq 1$,

$$\begin{aligned} h_2(d-1) + h_1 &\leq \lambda_{h_1,h_2,h_3}(Q_d) \\ &\leq \begin{cases} 2^p(h_3+n) + 2^q(h_1-n) - h_1, & d \neq 2^{p-1} \\ (2^p-2)n + h_1, & d = 2^{p-1} \end{cases} \end{aligned}$$

where $n := \max\{h_2, \lceil h_1/2 \rceil\}$. *Moreover, we give 'balanced'* $L(h_1, h_2, h_3)$*-labellings of* Q_d *using* $2^{\lceil \log_2 d \rceil + 1}$ *labels whose spans are equal to the upper bound above. In addition, if* $h_1 \leq 2$, *then*

$$\lambda_{h_1,h_2,h_3}(Q_d) \geq 2(d-1) + h_1.$$

Proof:

LB: Relatively easy; UB: A little bit group theory

Corollary

(Z 2008) Let $d \geq 3$. If $d \neq 2^{p-1}$, then

$$2d \leq \lambda_{2,1,1}(Q_d) \leq 2^{p+1} + 2^q - 2;$$

if $d = 2^{p-1}$, then

$$\lambda_{2,1,1}(Q_d) = 2d$$

and Q_d admits a balanced $L(2,1,1)$-labelling with span $2d$ and exactly one hole.

labelling Cayley graphs

Let $\Gamma(G, X)$ denote the Cayley graph on a group G with respect to a connection set X.

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Definition

A subgroup $H \leq G$ is said to avoid X if

$$H \cap X = \emptyset, \quad H \cap X^2 = \{1\},$$

where $X^2 = \{xx' : x, x' \in X\}$.

The trivial subgroup $\{1\}$ avoids every connection set of G.

Assume $j \geq k \geq 1$ in the following.

Theorem

(Z 2006) Let G be a finite abelian group. For any connection set X of G and any subgroup H of G which avoids X, we have

$$\begin{aligned}\lambda_{j,k}(\Gamma(G,X)) \leq \; & |G:H| \max\{k, \lceil j/2 \rceil\} + \\ & |G:\langle G - HX\rangle| \min\{j-k, \lfloor j/2 \rfloor\} - j.\end{aligned}$$

In particular,

$$\lambda(\Gamma(G,X)) \leq |G:H| + |G:\langle G - HX\rangle| - 2.$$

Corollary

Under the same condition as above, if in addition $G - HX$ generates G, then

$$\lambda_{j,k}(\Gamma(G,X)) \leq (|G:H|-1)\max\{k, \lceil j/2 \rceil\};$$

in particular,

$$\lambda(\Gamma(G,X)) \leq |G:H|-1$$

and $\Gamma(G,X)$ admits a 'no-hole' $L(2,1)$-labelling which uses $|G:H|$ labels.

Hamming graphs and hypercubes

The results above were used to produce upper bounds or exact values of $\lambda_{j,k}$ for hypercubes and Hamming graphs.

Their applications to other families of Cayley graphs have not been explored.

Hamming graph: $H_{n_1,n_2,\ldots,n_d} = K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_d}$

$V(H_{n_1,n_2,\ldots,n_d}) = Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_d}$; two d-tuples in $Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_d}$ are adjacent if and only if they differ in exactly one coordinate.

Hypercube: $Q_d = H_{2,2,\ldots,2}$ (d factors)

$d \leq 5$: $\lambda(Q_d)$ is known

$d \geq 5$: $d + 3 \leq \lambda(Q_d) \leq 2d$ (Griggs & Yeh + Jonas)

Denote

$$n = 1 + \lfloor \log_2 d \rfloor, \quad t = \min\{2^n - d - 1, n\}$$

Theorem

(Z, 2006) Let Γ be a connected d-regular graph whose automorphism group contains a vertex-transitive abelian subgroup. Then, for any $j \geq k \geq 1$,

$$\lambda_{j,k}(\Gamma) \leq 2^n \max\{k, \lceil j/2 \rceil\} + 2^{n-t} \min\{j - k, \lfloor j/2 \rfloor\} - j.$$

In particular, if $2k \geq j$, *then*

$$\lambda_{j,k}(\Gamma) \leq 2^n k + 2^{n-t}(j - k) - j.$$

Thus,

$$\lambda(\Gamma) \leq 2^n + 2^{n-t} - 2.$$

Corollary

For any $d \geq 1$ and $j \geq k \geq 1$,

$$\lambda_{j,k}(Q_d) \leq 2^n \max\{k, \lceil j/2 \rceil\} + 2^{n-t} \min\{j-k, \lfloor j/2 \rfloor\} - j.$$

In particular, if $2k \geq j$, then

$$\lambda_{j,k}(Q_d) \leq 2^n k + 2^{n-t}(j-k) - j.$$

Taking $j = 2, k = 1$, we get

$$\lambda(Q_d) \leq 2^n + 2^{n-t} - 2$$

(Whittlesey, Georges & Mauro, 1995)

Convention: $n_1 \geq n_2 \geq \cdots \geq n_d \geq 2$, $d \geq 2$

Theorem

(Z 2006) Suppose $n_1 > d \geq 2$, n_2 divides n_1, and each prime factor of n_1 is no less than d. Then, for any $n_3, \ldots, n_d \leq n_2$ and j, k with $2k \geq j \geq k \geq 1$,

$$\lambda_{j,k}(H_{n_1,n_2,\ldots,n_d}) = (n_1 n_2 - 1)k;$$

in particular,

$$\lambda(H_{n_1,n_2,\ldots,n_d}) = n_1 n_2 - 1.$$

Corollary

Let $n = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$.
If $2 \le d \le p_i$ *for each* i *and* $\sum_{i=1}^{t}(p_i - d + r_i) \ge 2$, *then for any* j, k *with* $2k \ge j \ge k \ge 1$,

$$\lambda_{j,k}(H_{n,n,\ldots,n}) = (n^2 - 1)k.$$

This implies a result of Georges, Mauro & Stein (2000) as a special case.

a sandwich theorem

Theorem

(Chang, Lu and Z 2009) *If* $n_1 \geq N(n_2, \ldots, n_d)$ *is sufficiently large (where* $N(n_2, \ldots, n_d)$ *is a specific function), then for any graph* G *such that*

$$H_{n_1,n_2} \subseteq G \subseteq H_{n_1,n_2,\ldots,n_d},$$

we have

$$\lambda(G) = \lambda_{1,1}(G) \; (= \chi(G) - 1) = n_1 n_2 - 1.$$

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we have

$$\lambda(G) = \lambda_{1,1}(G)\ (= \chi(G) - 1) = n_1 n_2 - 1.$$

In fact, we proved that for G the values of 8 invariants are equal to $n_1 n_2 - 1$, and we give a labelling of G which is optimal for all these invariants simultaneously.

recent progress

Consider a group $\Gamma = \langle S|R \rangle$.

The presentation $\langle S|R \rangle$ is called *N*-balanced if for every $s \in S$ and $(w = 1) \in R$, $exp_s(w) \equiv 0 \mod N$, where $exp_s(w)$ is the sum of the exponents (positive or negative) on occurrences of s in w.

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Theorem

(Bahls 2011) If $ss' \neq 1$ for all $s, s' \in S$ and the presentation $\Gamma = \langle S|R\rangle$ is $(2(n+h)-1)$-balanced, then

$$\lambda(\Gamma(G, S)) \leq 2(n + h - 1)$$

and equality holds if $h \leq 2n$.

summary

- There are many interesting topics in the area of distance labelling.
- There are nice connections with chromatic number and theory of colourings.
- Combinatorial, probabilistic and algebraic approaches have been used to solve problems pertaining to distance labelling.
- A number of papers have been published in this area, but for sure more will be produced in future.

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THANK YOU!