A non-abelian generalization of the massless Thirring / free boson correspondence

Thomas Quella

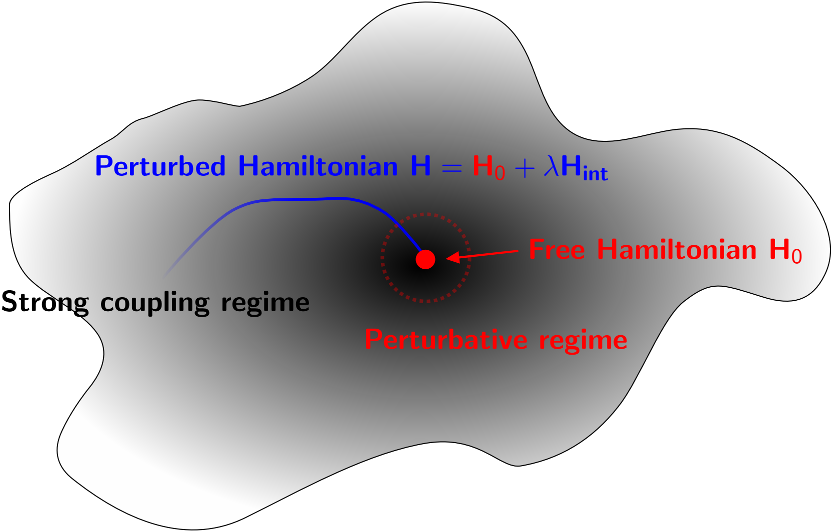
University of Amsterdam

Condensed Matter Theory Lunch Seminar 31.3.2010

Based on arXiv:0809.1046 (with V. Mitev and V. Schomerus)

[This research received funding from an Intra-European Marie-Curie Fellowship]

Strong coupling phenomena



Quantum Chromodynamics

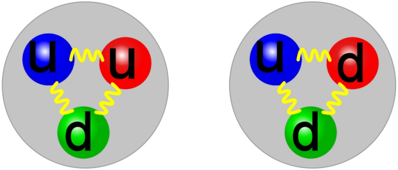
Quark confinement

Quarks cannot be isolated...

Bound states:

- Baryons (3 quarks)
- Mesons (quark-antiquark)

Proton



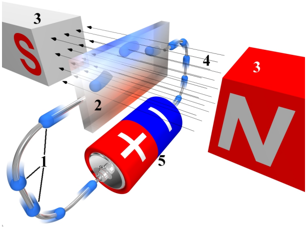
Neutron

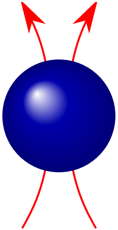
Weak coupling degrees of freedom are not appropriate!

Disordered fermions

Fractional quantum Hall effect

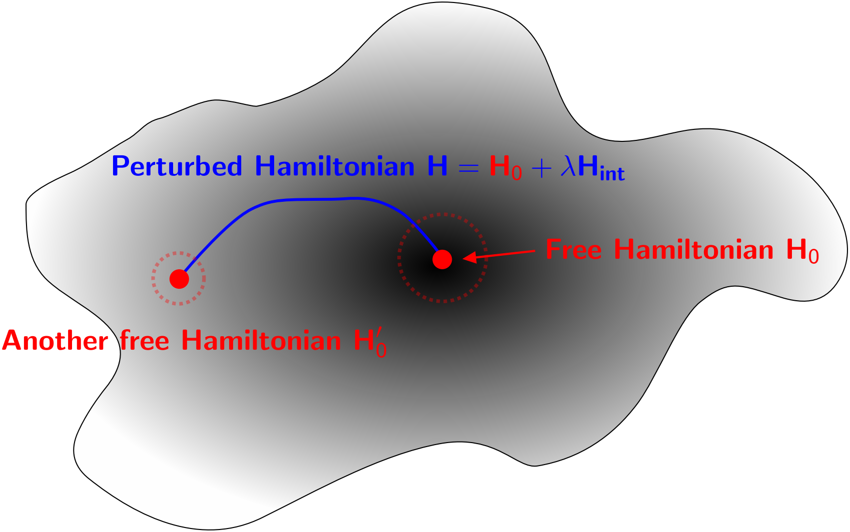
→ Composite fermions (fermions with flux quanta) [Jain '89]





Weak coupling degrees of freedom are not appropriate

The ideal situation: A duality

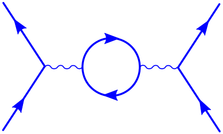


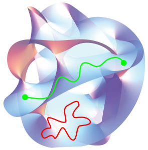
Maldacena's AdS/CFT Conjecture:

Approach to QCD-like theories via gravity / string theory

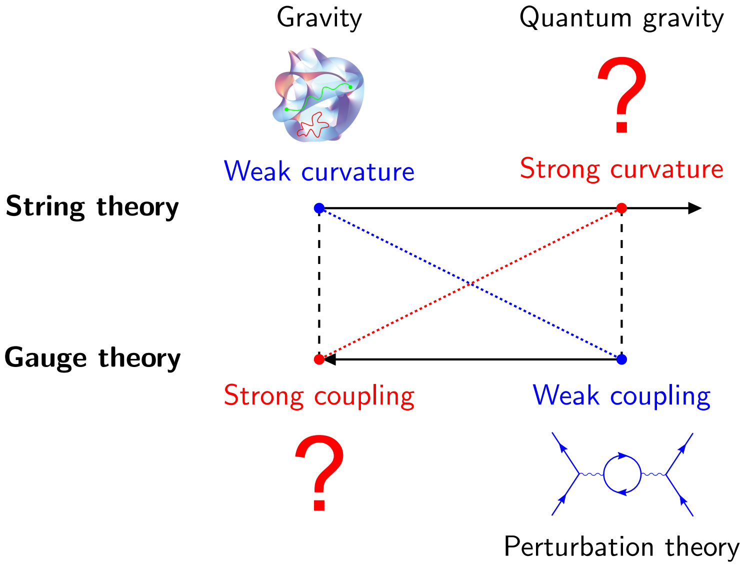
Parts of the AdS/CFT dictionary

Gauge theory	String theory / Gravity
D dimensions	$D+1$ dimensions
Scale invariance	Asympotical AdS-geometry
Non-trivial coupling	Quantum corrections to geometry
Supersymmetry	





Strong versus weak coupling



Applications of the AdS/CFT correspondence

High energy physics and quantum gravity

- QCD: MHV amplitudes
- Black hole physics: Microstate counting & entropy
- Quark gluon plasmas: Transport properties

Condensed matter physics

- Fluid dynamics [Policastro,Son,Starinets '01] [Kovtun,Son,Starinets '05] [...]
- Superconductors [Hartnoll,Herzog,Horowitz '08] [...]
- Cold atoms [Son '08] [Adams,Balasubramanian,McGreevy '08] [...]
- Quantum Hall physics [Davis,Kraus,Shah '08] [Fujita,Li,Ryu,Takayanagi '09] [...]
- Topological insulators [Karch '09] [Ryu,Takayanagi '10] [...]
- ... [Sachdev et al] [...]

Outline of the talk

Weak coupling descriptions of strong coupling phenomena!

Outline of this talk

1. The Thirring model
    - Duality with a circle theory
2. Supercoset σ-models
    - Applications
    - Conformal invariance
3. A particular example
    - Supersphere σ-models
    - Duality with a Gross-Neveu model
4. Application to statistical physics
    - Loop ensembles

The Thirring Model

The Thirring model

Action functional

$$\mathcal{S}[\psi] \;=\; \int \Big\{ \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{g}{2} \underbrace{(\bar{\psi}\gamma^{\mu}\psi)(\bar{\psi}\gamma_{\mu}\psi)}_{\text{in 2D: } \sim(\bar{\psi}\psi)^2} \Big\} d^2x$$

[Thirring '58]

Properties

- Fundamental field: Dirac fermion ψ
- Two couplings
  - Mass m
  - Quartic interactions g
- $U(1)$ symmetry

A close relative: The Gross-Neveu model

Action functional

$$\mathcal{S}[\psi^a] \;=\; \int \Big\{ \bar{\psi}_a (i\partial\!\!\!/ - m)\psi^a - \frac{g^2}{2N}\big(\bar{\psi}_a\psi^a\big)^2 \Big\} d^2x$$

[Gross,Neveu '74]

Properties

- Fundamental fields: N Dirac fermions ψ^a
- Two couplings
  - Mass m
  - Quartic interactions g
- $U(N)$ symmetry (for Majorana fermions: $O(N)$)

Action functional

$$\mathcal{S}[\psi] \;=\; \int\Big\{i\bar{\psi}\partial\!\!\!/\psi - \frac{g}{2}\underbrace{\big(\bar{\psi}\gamma^\mu\psi\big)\big(\bar{\psi}\gamma_\mu\psi\big)}_{\text{in 2D: }\sim(\bar{\psi}\psi)^2}\Big\}d^2x$$

[Johnson '61] [Hagen '67] [Klaiber '69]

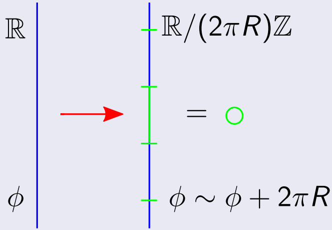
Properties

- Scale invariance for each value of g
- Bosonization: $\bar{\psi}\gamma^\mu\psi \sim \epsilon^{\mu\nu}\,\partial_\nu\phi$
- The action can be replaced by a single free boson

$$\mathcal{S}[\phi] \;=\; \frac{1}{2}\int\Big\{\partial_\mu\phi\,\partial^\mu\phi\Big\}d^2x$$

How to incorporate the coupling g?

Compactification on circle of radius R

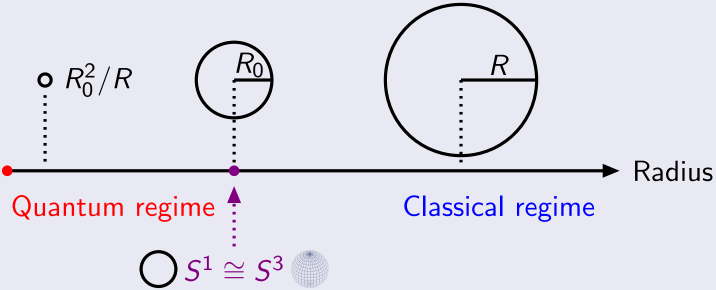


This corresponds to the quotient $\mathbb{R} \to \mathbb{R}/(2\pi R)\mathbb{Z}$

The free boson QFT has a geometric interpretation!

The compactified free boson

Free boson theories

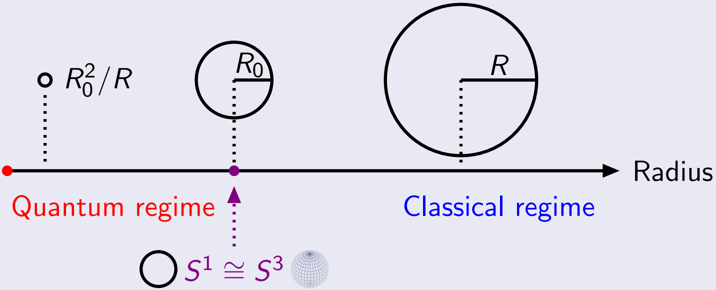


Two lessons

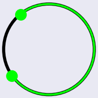
- There is an equivalence: $R \leftrightarrow R_0^2/R$ ("T-duality")
- In the quantum regime geometry starts to loose its meaning

The compactified free boson

Free boson theories



An open string partition function



$$Z(q,z|R) = \mathrm{tr}\left[z^{\mathrm{P}}\, q^{\mathrm{Energy}(R)}\right] = \frac{1}{\eta(q)} \sum_{w\in\mathbb{Z}} z^w\, q^{\frac{w^2}{2R^2}}$$

Geometrical QFTs: σ-models and their Applications

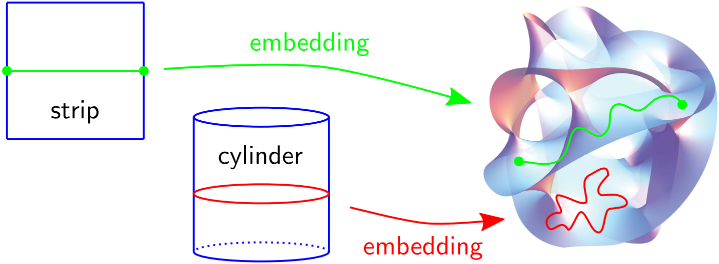
σ-models in a nutshell

World-sheet

2D surface
(w/wo boundaries or handles)

Target space

(Pseudo-)Riemannian manifold
(extra structure: gauge fields, ...)



$\boxed{\sigma\text{-models} = \text{(quantum) field theories}}$

Adding supersymmetry...

Appearances of superspace σ-models

- String theory
  - Quantization of strings in flux backgrounds [Berkovits]
  - String theory / gauge theory correspondence [Maldacena]
  - Moduli stabilization in string phenomenology [KKLT] [...]
- Disordered systems
  - Quantum Hall systems
  - Self avoiding random walks, polymer physics, ...
  - Efetov's supersymmetry trick

Conformal invariance

- String theory: Diffeomorphism + Weyl invariance
- Statistical physics: Critical points / 2^{nd} order phase transitions

Appearance of supercosets

String backgrounds as supercosets...

Minkowski	$\mathrm{AdS}_5 \times S^5$	$\mathrm{AdS}_4 \times \mathbb{CP}^3$	$\mathrm{AdS}_2 \times S^2$
$\frac{\text{super-Poincaré}}{\text{Lorentz}}$	$\frac{\mathrm{PSU}(2,2\|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)}$	$\frac{\mathrm{OSP}(6\|2,2)}{\mathrm{U}(3)\times\mathrm{SO}(1,3)}$	$\frac{\mathrm{PSU}(1,1\|2)}{\mathrm{U}(1)\times\mathrm{U}(1)}$

[Metsaev,Tseytlin] [Berkovits,Bershadsky,Hauer,Zhukov,Zwiebach] [Arutyunov,Frolov]

Supercosets in statistical physics...

IQHE	Dense polymers	Dense polymers
(non-conformal)	$S^{2S+1\|2S}$	$\mathbb{CP}^{S-1\|S}$
$\frac{\mathrm{U}(1,1\|2)}{\mathrm{U}(1\|1)\times\mathrm{U}(1\|1)}$	$\frac{\mathrm{OSP}(2S+2\|2S)}{\mathrm{OSP}(2S+1\|2S)}$	$\frac{\mathrm{U}(S\|S)}{\mathrm{U}(1)\times\mathrm{U}(S-1\|S)}$

[Weidenmüller] [Zirnbauer]

[Read,Saleur] [Candu,Jacobsen,Read,Saleur]

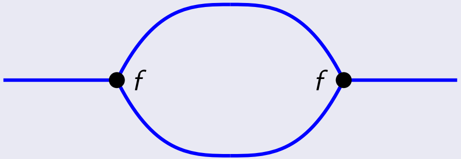
A unifying construction

Definition of the cosets

$$G/H : \quad gh \sim g$$

Some additional requirements for conformal invariance

- $H \subset G$ is invariant subgroup under an automorphism
- Ricci flatness ("super Calabi-Yau") $\Leftrightarrow$ vanishing Killing form

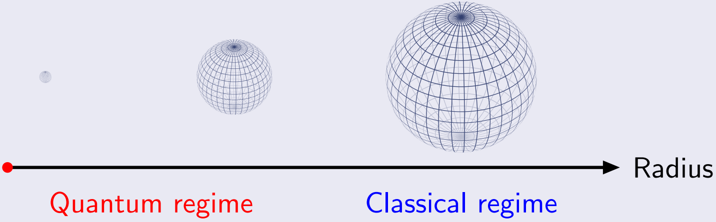


$= 0$

Examples: Cosets of $\mathrm{PSU}(N|N)$, $\mathrm{OSP}(2S+2|2S)$, $\mathrm{D}(2,1;\alpha)$.

Properties of conformal supercoset models

The moduli space of generic supercoset theories

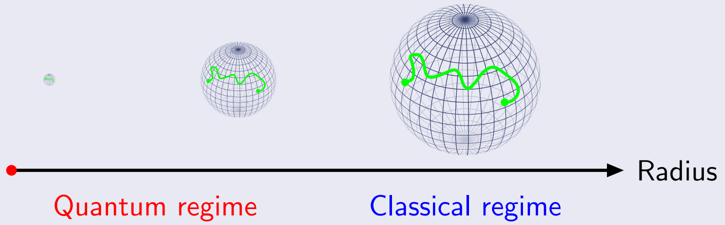


Properties at a glance

- Supersymmetry G: $g \mapsto kg$ (realized geometrically)
- Conformal invariance [Kagan,Young] [Babichenko]
- Integrability [Pohlmeyer] [Lüscher] ... [Bena,Polchinski,Roiban] [Young]

Properties of conformal supercoset models

The moduli space of generic supercoset theories



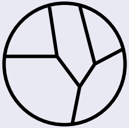
The general open string partition function

$$Z(q,z|R) \;=\; \mathrm{tr}\left[z^{\mathrm{Cartan}}\, q^{\mathrm{Energy}(R)}\right] \;=\; \sum_{\Lambda} \underbrace{\psi_\Lambda(q,R)}_{\text{Dynamics}}\;\underbrace{\chi_\Lambda(z)}_{\text{Symmetry}}$$

Sketch of conformal invariance

The β-function vanishes identically...

$$\beta = \sum_{\substack{\text{certain}\\ G\text{-invariants}}}$$



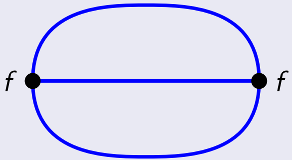
$$= 0$$

Ingredients:

Invariant metric: $\kappa^{\mu\nu}$

Structure constants: $f^{\mu\nu\lambda}$

The β-function vanishes identically...



$= 0$

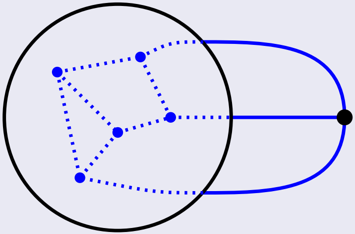
Ingredients:

Invariant metric: $\kappa^{\mu\nu}$

Structure constants: $\cancel{f^{\mu\nu\lambda}}$

Sketch of conformal invariance

The β-function vanishes identically...

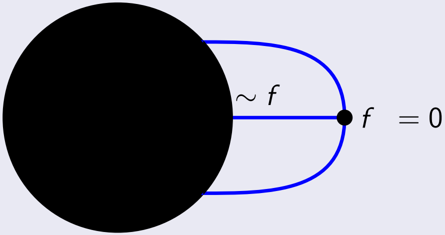


There is a unique invariant rank 3 tensor!

[Bershadsky,Zhukov,Vaintrob'99] [Babichenko'06]

Sketch of conformal invariance

The β-function vanishes identically...



There is a unique invariant rank 3 tensor!

[Bershadsky,Zhukov,Vaintrob'99] [Babichenko'06]

Supersphere σ-Models

The supersphere $S^{3|2}$

Realization of $S^{3|2}$ as a submanifold of flat superspace $\mathbb{R}^{4|2}$

$$\vec{X} = \begin{pmatrix} \vec{x} \\ \eta_1 \\ \eta_2 \end{pmatrix} \qquad \text{with} \qquad \vec{X}^2 = \vec{x}^2 + 2\eta_1\eta_2 = R^2$$

Symmetry

$$\mathrm{O}(4) \times \mathrm{SP}(2) \xrightarrow[\text{symmetrization}]{\text{super-}} \mathrm{OSP}(4|2)$$

Realization as a supercoset

$$S^{3|2} = \frac{\mathrm{OSP}(4|2)}{\mathrm{OSP}(3|2)}$$

The supersphere σ-model

Action functional

$$\mathcal{S}_\sigma \;=\; \int \partial_\mu \vec{X} \cdot \partial^\mu \vec{X} \qquad \text{with} \qquad \vec{X}^2 = R^2$$

The space of states for freely moving open strings

$$\prod X^{a_i} \prod \partial_t X^{b_j} \prod \partial_t^2 X^{c_k} \cdots \qquad \text{and} \qquad \vec{X}^2 = R^2$$

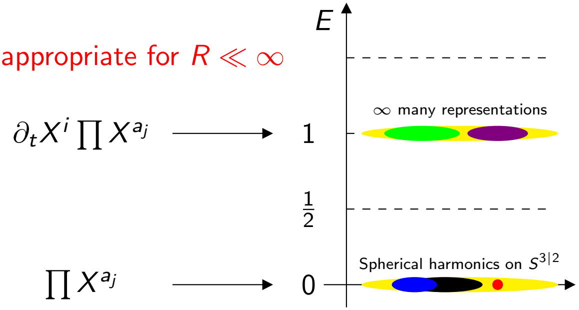
$\Rightarrow$ Products of coordinate fields and their derivatives

Large volume partition function

- “Single particle energies” add up $\rightarrow$ # derivatives
- Partition function is pure combinatorics [Candu,Saleur] [Mitev,TQ,Schomerus]

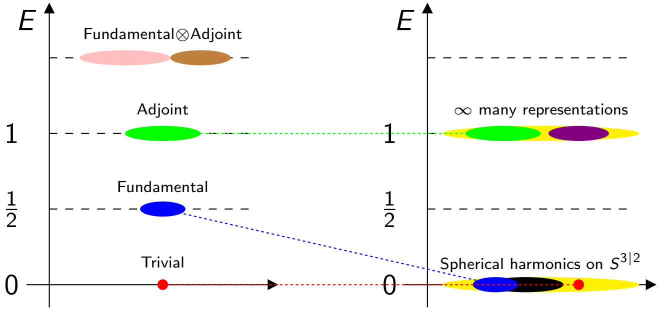
Sketch of the large volume partition function

Basis not appropriate for $R \ll \infty$



Weak curvature ($R = \infty$)

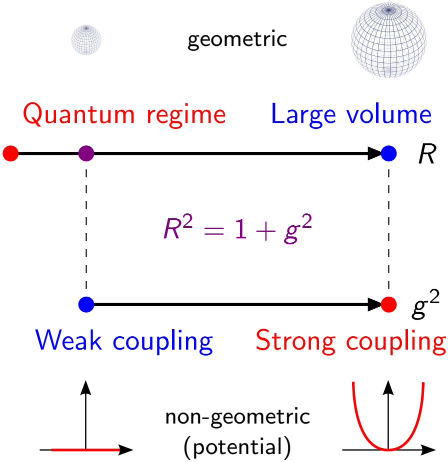
Sketch of the large volume partition function



Quantum regime ($R = 1$) | **Weak curvature ($R = \infty$)**

A Duality for Supersphere σ-Models

A world-sheet duality for superspheres?



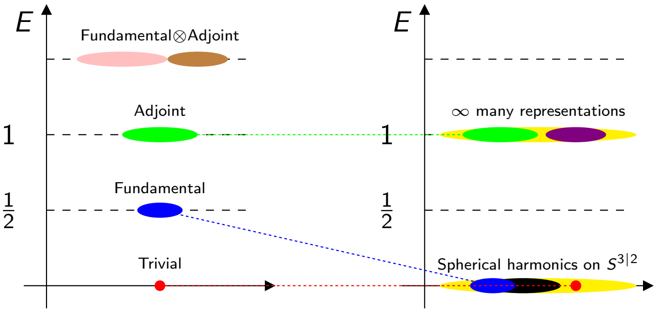
Supersphere σ-model

$$Z_\sigma(q, z|R) \;\Big\|\; Z_{\mathrm{GN}}(q, z|g^2)$$

OSP$(4|2)$ Gross-Neveu model

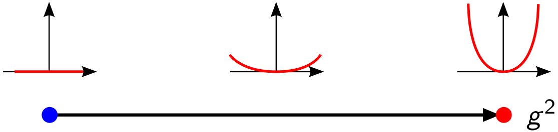
[Candu,Saleur]^2 [Mitev,TQ,Schomerus]

In the two extreme limits the spectrum has the form...



Quantum regime ($R = 1$) **Weak curvature ($R = \infty$)**

Evidence for the duality



Weak coupling

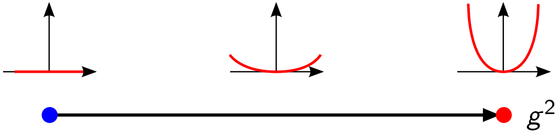
Strong coupling

Lattice formulation

Free fields / WZW model
Kac-Moody symmetry

Free theory
Combinatorics

[Candu,Saleur]^2 [Mitev,TQ,Schomerus]



Weak coupling

Strong coupling

Lattice formulation

Free fields / WZW model
Kac-Moody symmetry

Free theory
Combinatorics

$$\boxed{\text{Goal:} \quad Z_{\mathrm{GN}}(q,z|g^2) \;=\; \sum_\Lambda \psi_\Lambda^\sigma(q,g^2)\,\chi_\Lambda(z)}$$

[Candu,Saleur] [Mitev,TQ,Schomerus]

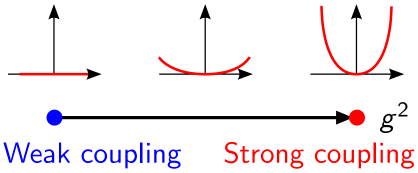
OSP(4|2) Gross-Neveu Model

Field content

- Fundamental OSP(4|2)-multiplet $(\psi_1, \psi_2, \psi_3, \psi_4, \beta, \gamma)$
- All these fields have scaling dimension $1/2$

Formulation as a Gross-Neveu model

$$\mathcal{S}_{\text{GN}} = \mathcal{S}_{\text{free}} + g^2\, \mathcal{S}_{\text{int}} \qquad \begin{cases} \mathcal{S}_{\text{free}} = \int \left[\psi\bar{\partial}\psi + \beta\bar{\partial}\gamma + h.c.\right] \\ \mathcal{S}_{\text{int}} = \int \left[\psi\bar{\psi} + \beta\bar{\gamma} - \gamma\bar{\beta}\right]^2 \end{cases}$$



Goal: $Z_{\text{GN}}(q, z|g^2)$

An open string spectrum

Formulation as a deformed OSP(4|2) WZW model

$$\mathcal{S}_{\mathrm{GN}} = \mathcal{S}_{\mathrm{WZW}} + g^2\,\mathcal{S}_{\mathrm{def}} \qquad \text{with} \qquad \mathcal{S}_{\mathrm{def}} = \int \mathrm{str}(\mathrm{J}\bar{\mathrm{J}})$$

Solution at $g = 0$

- At $g = 0$ there is an OSP(4|2) Kac-Moody algebra symmetry

$$\mathrm{J}^\mu(z)\,\mathrm{J}^\nu(w) = \frac{k\,\delta^{\mu\nu}}{(z-w)^2} + \frac{i f^{\mu\nu}{}_\lambda \mathrm{J}^\lambda(w)}{z-w}$$

- Partition functions can be constructed using combinatorics

An open string partition function for $g = 0$

$$Z_{\mathrm{GN}}(g^2 = 0) = \sum_\Lambda \underbrace{\psi_\Lambda^{\mathrm{WZW}}(q)}_{\text{energy levels}} \quad \underbrace{\chi_\Lambda(z)}_{\text{OSP(4|2) content}}$$

An open string spectrum

Formulation as a deformed OSP(4|2) WZW model

$$\mathcal{S}_{\text{GN}} = \mathcal{S}_{\text{WZW}} + g^2\,\mathcal{S}_{\text{def}} \qquad \text{with} \qquad \mathcal{S}_{\text{def}} = \int \text{str}(\text{J}\bar{\text{J}})$$

Solution at $g = 0$

- At $g = 0$ there is an OSP(4|2) Kac-Moody algebra symmetry

$$\text{J}^\mu(z)\,\text{J}^\nu(w) = \frac{k\,\delta^{\mu\nu}}{(z-w)^2} + \frac{if^{\mu\nu}{}_\lambda \text{J}^\lambda(w)}{z-w}$$

- Partition functions can be constructed using combinatorics

An open string partition function for all g

$$Z_{\text{GN}}(g^2) = \sum_\Lambda \underbrace{q^{-\frac{1}{2}\frac{g^2}{1+g^2}C_\Lambda}}_{\text{anomalous dimension}} \underbrace{\psi_\Lambda^{\text{WZW}}(q)}_{\text{energy levels}} \underbrace{\chi_\Lambda(z)}_{\text{OSP}(4|2)\text{ content}}$$

An annulus partition function

Specific boundary conditions in the OSP(4|2) WZW model...

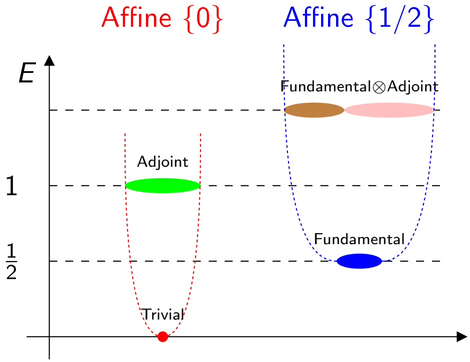
For a certain class of open strings one obtains

$$Z_{\mathrm{GN}}(g^2=0) \;=\; \underbrace{\chi_{\{0\}}(q,z)}_{\text{vacuum}} + \underbrace{\chi_{\{1/2\}}(q,z)}_{\text{fundamental}}$$

The problem (yet again...)

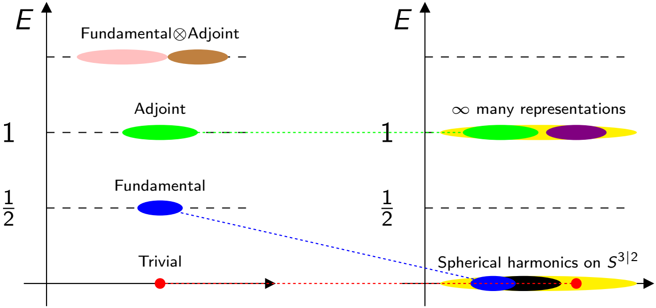
$$\boxed{\text{Organize this into representations of OSP(4|2)!}}$$

What did we achieve now?



$\boxed{\textbf{WZW model } (g = 0)}$

$$Z_{\mathrm{GN}}(g^2) = \sum_{\Lambda} \underbrace{q^{-\frac{1}{2}\frac{g^2}{1+g^2}C_\Lambda}}_{\text{anomalous dimension}} \underbrace{\psi_\Lambda^{\mathrm{WZW}}(q)}_{\text{energy levels}} \underbrace{\chi_\Lambda(z)}_{\mathrm{OSP}(4|2)\text{ content}}$$



WZW model ($g = 0$)

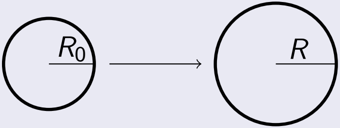
Strong deformation ($g = \infty$)

Supersphere σ-model at $R \to \infty$

Quasi-abelian Deformations

Radius deformation of the free boson revisited

Consider a deformation...



$$R = R_0\sqrt{1+\gamma}$$

Freely moving open strings on a circle of radius R...

$$Z(q,z|R) \;=\; \frac{1}{\eta(q)}\sum_{w\in\mathbb{Z}} z^w\, q^{\frac{w^2}{2R^2}} \;=\; \frac{1}{\eta(q)}\sum_{w\in\mathbb{Z}} q^{\frac{w^2}{2R_0^2(1+\gamma)}}\,\chi_w(z)$$

Anomalous dimensions

$$\delta_\gamma E_w \;=\; \frac{w^2}{2R_0^2}\left[\frac{1}{1+\gamma}-1\right] \;=\; -\frac{\gamma}{1+\gamma}\,\frac{w^2}{2R_0^2} \;=\; -\frac{\gamma}{1+\gamma}\,C_2(w)$$

Quasi-abelianness of supergroup WZW theories

The effective deformation for conformal dimensions

- The combinatorics of the perturbation series is determined by the current algebra

$$\mathrm{J}^\mu(z)\,\mathrm{J}^\nu(w) = \frac{k\,\delta^{\mu\nu}}{(z-w)^2} + \frac{i f^{\mu\nu}{}_\lambda \mathrm{J}^\lambda(w)}{z-w} \sim \frac{k\,\delta^{\mu\nu}}{(z-w)^2}$$

- Vanishing Killing form $\Rightarrow$ the perturbation is quasi-abelian (for the purposes of calculating anomalous dimensions)

[Bershadsky,Zhukov,Vaintrob] [TQ,Schomerus,Creutzig]

- In the OSP(4|2) WZW model a representation Λ shifts by

$$\delta E_\Lambda(g^2) \;=\; -\frac{1}{2}\frac{g^2 C_\Lambda}{1+g^2}$$

Superspheres and Loop Ensembles

A statistical model for dense polymers

A little excursion

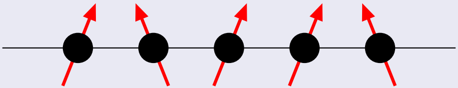
There is a statistical lattice model whose continuum limit is in the same universality class as the supersphere σ-models [Candu,Saleur]

$$S^{2S+1|2S} = \frac{\mathrm{OSP}(2S+2|2S)}{\mathrm{OSP}(2S+1|2S)}$$

Requirements

- $\mathrm{OSP}(R|2S)$ invariance $\to$ super spin chain

$$V \otimes \cdots \otimes V \ =$$



- Scale invariance for $R = 2S + 2$.

A statistical model for dense polymers

Ingredients for a supersymmetric lattice model

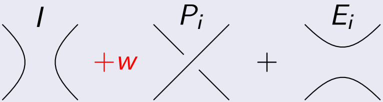
Start with fundamental $\mathrm{OSP}(R|2S)$ representation V

$$\dim(V) \;=\; R + 2S \qquad\qquad \mathrm{sdim}(V) = \; R - 2S \;=\; N$$

Local interactions

Two site transfer matrix

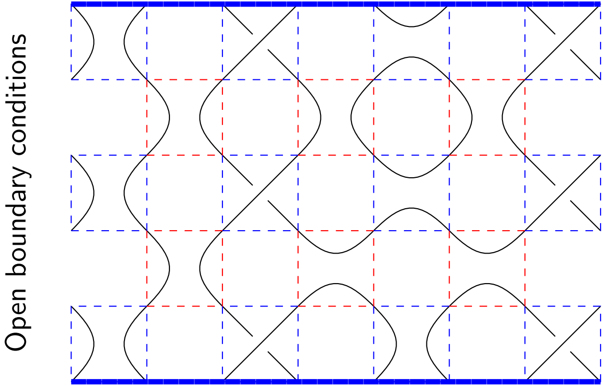
$$t_i \;=\; I + wP_i + E_i \;=$$



[Candu,Saleur]

Note: This spin model is **not** integrable!!!

A typical configuration: Dense intersecting loops

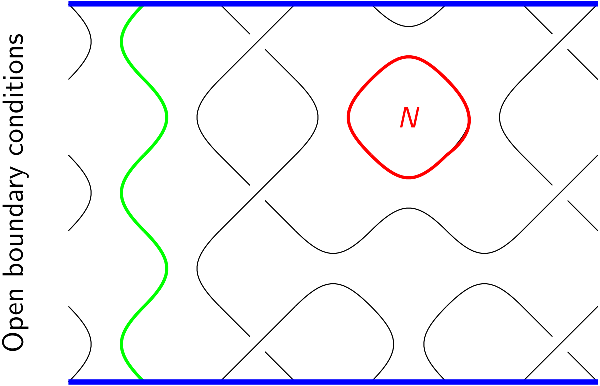


Transfer matrix

$$T = \prod_{2\ \text{rows}} t_i(w)$$

Partition function: $Z(w) = \mathrm{str}\left[T^{\beta}\right]$

A typical configuration: Dense intersecting loops



Partition function: $\quad Z_D(w) = \mathrm{str}\left[D\,T^{\beta}\right]$

Conclusions

Conclusions and Outlook

Conclusions

- Using supersymmetry we determined the full spectrum of anomalous scaling dimensions for certain annulus partition functions in Gross-Neveu models as a function of the moduli
- Our results provided strong evidence for a duality between supersphere σ-models and Gross-Neveu models
- The latter generalizes the duality between the massless Thirring model and the compactified free boson

Outlook

- Conformal invariance $\leftrightarrow$ Integrability
- Cylinder partition function and correlation functions?
- Application to other geometries and phenomena...

Projective Superspaces

New features

- Family contains supertwistor space $\mathbb{CP}^{3|4}$ → [Witten]
- Non-trivial topology

$\Rightarrow$ Monopoles and θ-term

- Symplectic fermions as a subsector [Candu,Creutzig,Mitev,Schomerus]

θ-term $\Rightarrow$ twists

- σ-model brane spectrum can be argued to be

$$Z_{R,\theta}(q,z) = \underbrace{q^{-\frac{1}{2}\lambda(R,\theta)\left[1-\lambda(R,\theta)\right]}}_{\text{twist}} \sum_{\Lambda} \underbrace{q^{f(R,\theta)C_\Lambda}}_{\text{Casimir}} \underbrace{\psi_\Lambda^\infty(q)\,\chi_\Lambda(z)}_{\text{result for } R\to\infty}$$

[Candu,Mitev,TQ,Saleur,Schomerus]

- Currently no duality is known...