Topological phases of SU(N) spin chains and their realization in ultra-cold atom gases

Thomas Quella

University of Cologne

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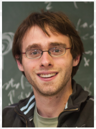

Based on publications with K. Duivenvoorden
[Phys.Rev.B86 (2012) 235142 and Phys.Rev.B87 (2013) 125145]
and work in progress...

Related work: arXiv/1304.7234

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Center of Excellence "Quantum Matter and Materials" (QM2)
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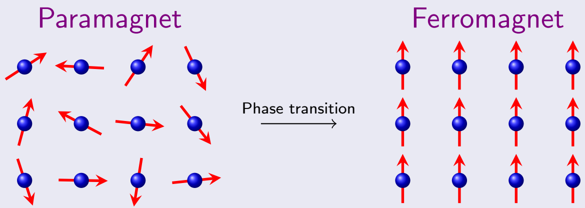
Landau's paradigm

Landau's paradigm

Different phases are characterized by different patterns of symmetry breaking and the associated order parameters

Illustration: Magnets



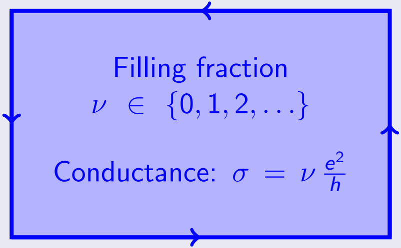
	Paramagnet	Ferromagnet
Order parameter	$\langle\vec{S}\rangle = 0$	$\langle\vec{S}\rangle \neq 0$
Rotation symmetry	preserved	broken

Beyond Landau's paradigm

Topological order

There are phase transitions which do not involve symmetry breaking but rather a change in a non-local order

Illustration: The integer quantum Hall effect



Characteristics:

- Quantized edge current
- ν is a topological invariant
- Robustness against disorder

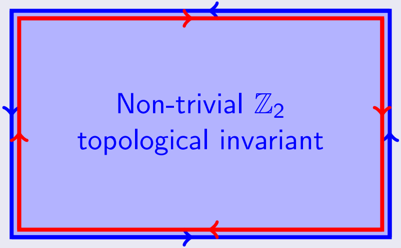
Beyond Landau's paradigm

Topological order

There are phase transitions which do not involve symmetry breaking but rather a change in a non-local order

Illustration: Topological insulators (here: 2D)

[Kane,Mele]



Characteristics:

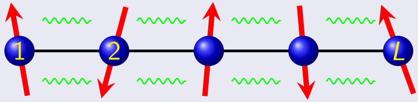
- Spin polarized edge currents
  Spin up/Spin down
- Time-reversal symmetry
- Robustness against disorder

Spin models

The AKLT model

[Affleck,Kennedy,Lieb,Tasaki '87–'88]

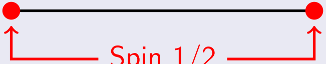
Anti-ferromagnetic $S{=}1$ spin model in 1D



$$H = J\sum_{\langle ij\rangle}\Big[\vec{S}_i\vec{S}_j + \tfrac{1}{3}(\vec{S}_i\vec{S}_j)^2\Big]$$

Peculiar property: Massless boundary modes

Open BC



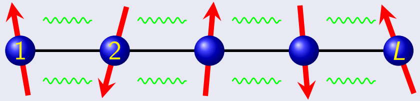
Spin 1/2
(fractionalization)

Periodic BC

Virtual boundary modes
(through bipartite entanglement)

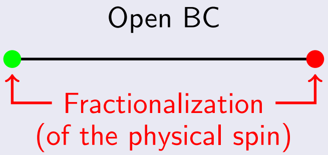
Results on anti-ferromagnetic gapped SU(N) spin chains

Anti-ferromagnetic SU(N) spin model in 1D



Spin operators: $\vec{S}_i \in su(N)$

Classification of gapped symmetry protected topological phases [Duivenvoorden,TQ]

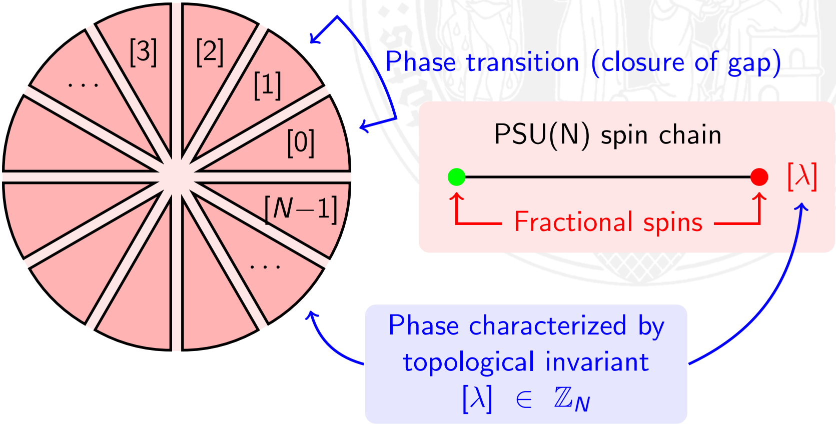


Main result:*
The symmetry can fractionalize in up to N topologically distinct ways

* Note: We have derived similar results for all simple Lie groups G

Sketch of the physical situation

Space of gapped PSU(N) spin chains [* PSU(N) = "variant of SU(N)"]



Symmetry fractionalization

What is symmetry fractionalization?

Symmetry of emerging boundary spins

different from

Symmetry of physical spins

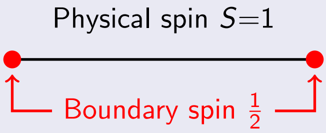
To be covered in this talk

- Symmetry fractionalization for continuous groups
- Detection of topological order
- Cold atom realization
- Outlook

Symmetry Fractionalization

Symmetry fractionalization in the SU(2) AKLT model

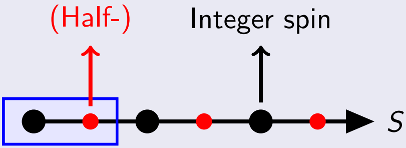
Example: AKLT model



SU(2) ✓

SU(2) ✓

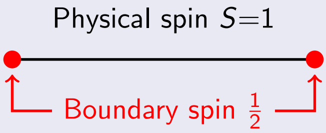
Representations of SU(2)



- ● Reps of SO(3)
- ● Not reps of SO(3)

Symmetry fractionalization in the SU(2) AKLT model

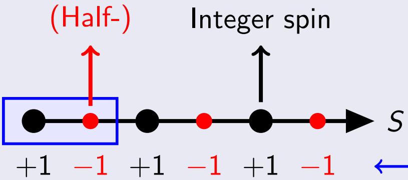
Example: AKLT model



SU(2) ✓ SO(3) ✓

SU(2) ✓ SO(3) X

Representations of SU(2) and SO(3)



- Reps of $SO(3) = SU(2)/\mathbb{Z}_2$
- Projective reps

Action of $-\mathbb{I} \in \mathbb{Z}_2 \subset SU(2)$

Topological phases protected by SO(3) symmetry

The key player

Center of SU(2) $\quad\longleftrightarrow\quad$ $\mathbb{Z}_2 = \{\pm\mathbb{I}\}$

Classification: Two distinct topological phases

[Chen,Gu,Wen] [Schuch,Perez-Garcia,Cirac]

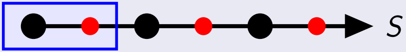
Boundary modes:

Trivial phase [0] — $\mathbb{Z}_2$ acts trivially

Haldane phase [1] — $\mathbb{Z}_2$ acts non-trivially

$\mathbb{Z}_2$-invariant

$$[S] \equiv 2S \mod 2$$



Alternative perspective

[Pollmann,Berg,Turner,Oshikawa]

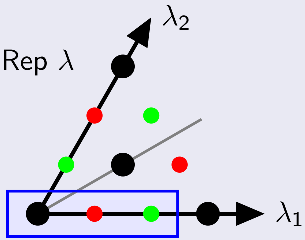
Haldane phases can also be thought of as being protected by time-reversal symmetry, inversion symmetry or $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathrm{SO}(3)$

Symmetry fractionalization in SU(N) spin models

Example: SU(3)

Center: $\mathbb{Z}_3 = \{\mathbb{I}, \omega\mathbb{I}, \omega^2\mathbb{I}\} \subset SU(3)$ generated by $\omega = e^{\frac{2\pi i}{3}}$

Representations of SU(3)

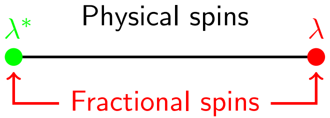


Action of $\omega\mathbb{I}$: $\omega^{[\lambda]} = \exp(\frac{2\pi i}{3}[\lambda])$

$[\lambda]$ 0 1 2 $\equiv \lambda_1 + 2\lambda_2 \mod 3$

determines rep of $\mathbb{Z}_3$

$\Rightarrow$ Can have symmetry fractionalization for physical spins in $PSU(3) = SU(3)/\mathbb{Z}_3$



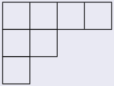
- Linear rep of PSU(N) $= SU(N)/\mathbb{Z}_N$
- Projective rep λ
- Dual projective rep λ^*

Main result

PSU(N) admits N distinct topological phases: $[\lambda] \in \mathbb{Z}_N$ [Duivenvoorden,TQ]

Characterization of representations

$$\lambda = (\lambda_1, \ldots, \lambda_{N-1}) \quad \Rightarrow \quad [\lambda] = \sum_{k=1}^{N-1} k\lambda_k \mod N$$

$\lambda =$  $\Rightarrow$ $[\lambda] = \text{Boxes}(\lambda) \mod N$

Detecting Topological Order

Detecting topological order in spin chains

The story so far...

- We found N different topological phases for PSU(N) spin chains
- The MPS approach provides representatives for each class

Open problem...

Given the unique groundstate $|\psi\rangle$ of an arbitrary gapped PSU(N)-invariant system, can one reconstruct its topological phase?

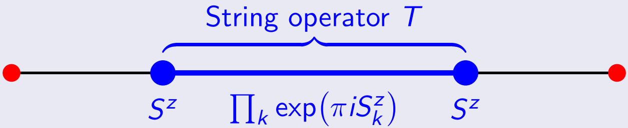
Still to come...

- A non-local string order parameter for SU(2) [Den Nijs,Rommelse]
- The generalization to SU(N) [Duivenvoorden,TQ]

A string order parameter for SU(2)

Non-local string order

[Den Nijs,Rommelse]



Result in the thermodynamic limit

$$\langle T \rangle \;\to\; R \quad \text{with} \quad \begin{cases} R = 0 & \text{, trivial phase} \\ R \neq 0 & \text{, Haldane phase (generically)} \end{cases}$$

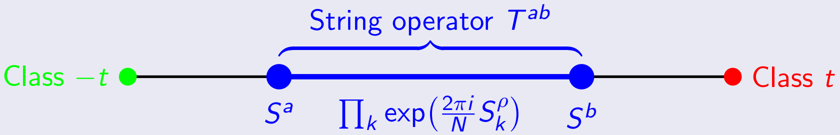
Remark

$\langle T \rangle$ measures diluted anti-ferromagnetic order

A string order parameter for SU(N)

Non-local string order (matrix valued!)

[Duivenvoorden,TQ]



Result in the thermodynamic limit (in a specific basis)

[Duivenvoorden,TQ]

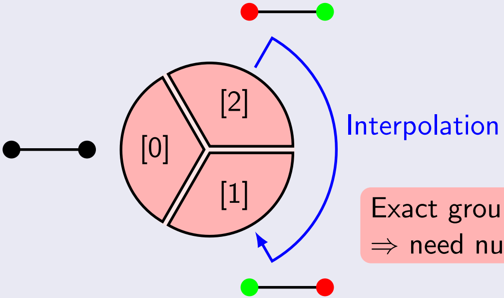
$$\langle T^{ab} \rangle \;\rightarrow\; R\,\omega^{t(a-b)} \qquad (\text{with } R = 0 \text{ for } t = 0)$$

Idea of proof

MPS representation, transfer matrix methods and Weyl group gymnastics

An interpolating Hamiltonian for PSU(3)

The space of gapped PSU(3) spin chains



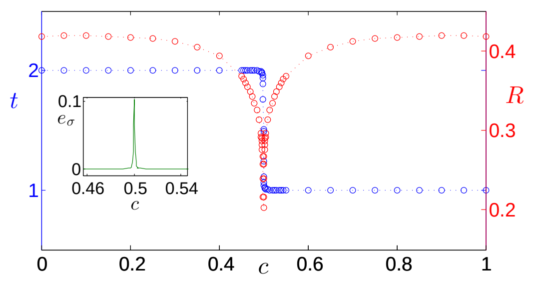
Exact groundstate unknown
$\Rightarrow$ need numerical DMRG

Interpolating Hamiltonian

$$H_{\text{two-site}}(c) \;=\; \vec{S}_1\vec{S}_2 + \underbrace{(1-2c)}_{\in[-1,1]}\underbrace{d_{ABC}\left(S_1^A S_1^B S_2^C - S_1^A S_2^B S_2^C\right)}_{\text{cubic Casimir of } su(3)}$$

Numerical results for PSU(3)

DMRG analysis



Periodic BC
Length: 20 sites
Bond dimension: 400

Deviations:
Finite size effects

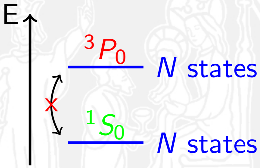
Excellent agreement with expected formula

$$T^{ab} \to -R \begin{pmatrix} 1 & \omega^{-t} \\ \omega^{t} & 1 \end{pmatrix} \qquad \text{with} \qquad t \in \{0, 1, 2\}$$

Cold atom realization of SU(N) spin chains

From cold atoms to SU(N) Heisenberg spin chains

Atom	^{171}Yb	^{173}Yb	^{43}Ca	^{87}Sr
Nuclear spin I	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$
$N = 2I + 1$	2	6	8	10



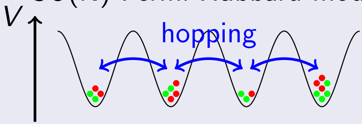
Features of alkaline-earth atoms

- Large nuclear spin
- Decoupling of electronic and nuclear spin

Dynamics in an optical lattice

[Gorshkov,Hermele,Gurarie,Xu,Julienne,Ye,Zoller,Demler,Lukin,Rey]

SU(N) Fermi-Hubbard model



$$\xrightarrow[\text{half-filling}]{\text{strong coupling}}$$

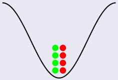
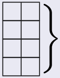
SU(N) Heisenberg model

$$H = J \sum \vec{S}_i \vec{S}_{i+1}$$

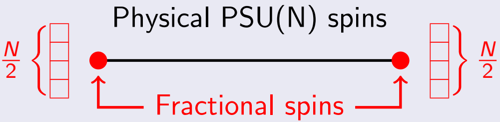
Realization of non-trivial topological phases

Physical spin (for even N)

[Nonne,Moliner,Capponi,Lecheminant,Totsuka]

Half-filling:  $\Rightarrow$ Physical spin:  $\frac{N}{2}$ (class [0])

Symmetry fractionalization (in a closely related AKLT-type model)



Topological class $\left[\frac{N}{2}\right]$

Conjecture (based on the analysis of PSU(4))

Both models are in the same non-trivial topological phase

[Nonne,Moliner,Capponi,Lecheminant,Totsuka] [TQ: Analysis of SU(6) and beyond (work in progress)]

Discrete versus continuous symmetries

Stabilization of Haldane phases

Different symmetries – same effect

[Pollmann,Berg,Turner,Oshikawa]

Symmetries that protect the Haldane phase:

- Full rotation symmetry SO(3) [Chen,Gu,Wen] [Schuch,Perez-Garcia,Cirac]
- π rotations around x, y, z axes $\longrightarrow$ $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathrm{SO}(3)$
- Time-reversal symmetry
- Inversion symmetry

Generalization to arbitrary simple Lie groups

[Duivenvoorden,TQ]

- For PSU(N) use subgroup $\mathbb{Z}_N \times \mathbb{Z}_N$ see also [Else,Bartlett,Doherty]
- For SO(N) use subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ see also [Haegeman,Perez-Garcia,Cirac,Schuch]
- For PSO(2N+2) use subgroup $\mathbb{Z}_4 \times \mathbb{Z}_4$
- ...

Summary and Outlook

Summary and Outlook

Summary

In 1D systems, topological phases are in one-to-one correspondence with classes of projective representations of the on-site symmetry group

[Chen,Gu,Wen] [Schuch,Perez-Garcia,Cirac] [Duivenvoorden,TQ]

Results for PSU(N) spin chains

[Duivenvoorden,TQ] [Duivenvoorden,TQ]

- They exhibit N distinct topological phases
- These can be distinguished by a non-local string order parameter
- A realization in cold atom systems should be achievable [Nonne at al] [TQ]

Outlook

- Explanation by "hidden symmetry breaking" [Else,Bartlett,Doherty] [Duivenvoorden,TQ]
- Generalizations to supersymmetric spin chains [Michalski,TQ]

Extra slides

Evaluation for continuous groups

Realization of continuous groups

Every compact simple Lie group with Lie algebra $\mathfrak{g}$ can be realized as $\mathsf{G}_\Gamma = \mathsf{G}/\Gamma$ where G is simply-connected and $\Gamma \subset \mathcal{Z}(\mathsf{G})$ is a central subgroup

Projective representations

$$H^2\big(\mathsf{G}/\Gamma, \mathsf{U}(1)\big) \;=\; \mathrm{Hom}\big(\pi_1\big(\mathsf{G}/\Gamma\big), \mathsf{U}(1)\big) \;=\; \mathrm{Hom}\big(\Gamma, \mathsf{U}(1)\big)$$

Example: Projective group $\mathsf{PG} = \mathsf{G}/\mathcal{Z}(\mathsf{G})$

$$H^2\big(\mathsf{PG}, \mathsf{U}(1)\big) \;\cong\; \mathcal{Z}(\mathsf{G})$$

Summary for classical Lie groups

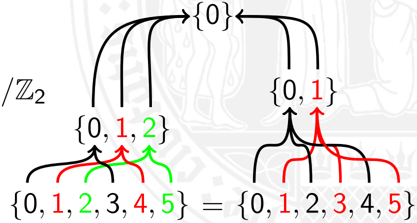
Table of (possible) topological phases

$\mathfrak{g}$	$su(N)$	$sp(2N)$	$so(2N{+}1)$	$so(4N{+}2)$	$so(4N)$
G	SU(N)	Sp(2N)	Spin(2N+1)	Spin(4N+2)	Spin(4N)
$\mathcal{Z}(G)$	$\mathbb{Z}_N$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

Hierarchies of topological phases

$$
\begin{tikzcd}
 & SU(6) \arrow[dl, "\mathbb{Z}_3"'] \arrow[dr, "\mathbb{Z}_2"] & \\
SU(6)/\mathbb{Z}_3 \arrow[dr, "\mathbb{Z}_2"'] & & SU(6)/\mathbb{Z}_2 \arrow[dl, "\mathbb{Z}_3"] \\
 & PSU(6) &
\end{tikzcd}
$$

Symmetry group



Topological classes