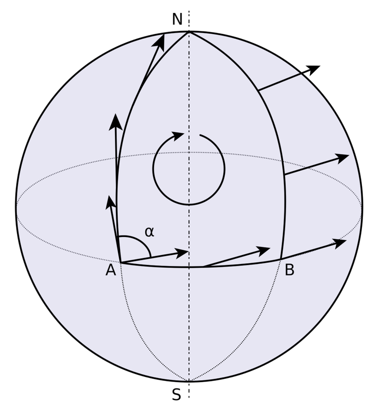
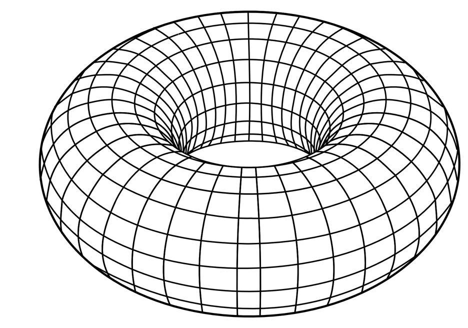


Topological Invariants in Quantum Systems



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Introduction

The physics of a quantum system generally depends on parameters which govern the microscopic interactions. The variation of these parameters may cause the system to exhibit qualitatively distinct behaviours—**phases of matter**. The mathematical language of **topology** can be used to distinguish some phases with a discrete **invariant**.

Such topological phases are appealing due to their **inherent immunity** to quantum errors caused by small fluctuations. By exploiting this immunity, **quantum computing** may be more achievable.

The Berry Phase

A quantum state $|\psi\rangle$ is an equivalence class of rays in a complex Hilbert space $\mathcal{H}$. Insisting on normalised states still leaves us with

$$\eta|\psi\rangle \sim |\psi\rangle, \quad \eta \in U(1).$$

Let the Hamiltonian H of the system depend on parameters $\mathbf{R} = (R_1, R_2, \dots)$. Consider the evolution over a **closed path** C parameterised by $\mathbf{R}(t)$. If the evolution is assumed **adiabatic**, then there exists a complete ordered orthonormal set of instantaneous eigenstates $|n(\mathbf{R}(t))\rangle$. Over time t, $|n(\mathbf{R}(0))\rangle$ evolves into

$$|\psi(t)\rangle = \exp\left[\frac{1}{i\hbar}\int_0^t E_n(\mathbf{R}(\tau))\,\mathrm{d}\tau\right]\exp\left(i\gamma_n(t)\right)|n(\mathbf{R}(t))\rangle. \tag{1}$$

The left-hand exponential is the internal chronometer of the system. Due to the Schrödinger equation, the other exponent satisfies

$$\dot{\gamma}_n = i\,\langle n(\mathbf{R}(t))|\nabla_{\mathbf{R}}|n(\mathbf{R}(t))\rangle \cdot \dot{\mathbf{R}}.$$

The total contribution from this term over the loop C is the **Berry phase**

$$\gamma_n = i\oint_C \langle n(\mathbf{R})|\nabla_{\mathbf{R}}|n(\mathbf{R})\rangle \cdot \mathrm{d}\mathbf{R}. \tag{2}$$

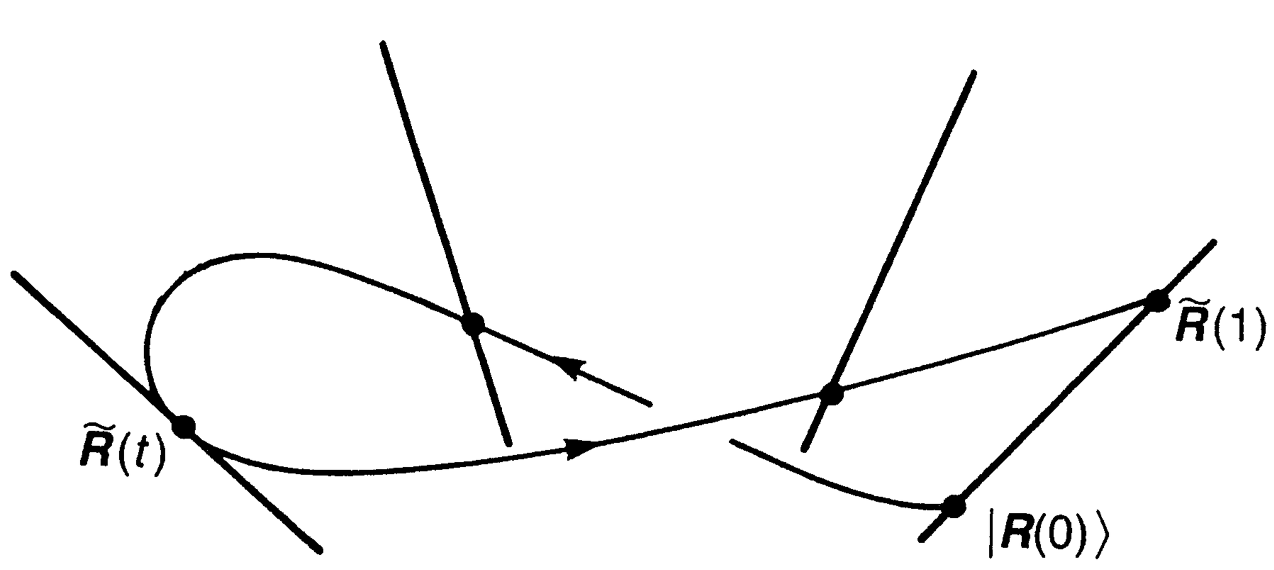

Figure 1: An illustration of parallel transport.[4]

By defining the **Berry connection** 1-form

$$\mathcal{A}_n = i\,\langle n(\mathbf{R})|\mathrm{d}|n(\mathbf{R})\rangle, \tag{3}$$

we see that the Berry phase takes the form of a **holonomy**. Under a different choice of basis states, $|n\rangle \mapsto \exp(i\mu(\mathbf{R}))\,|n\rangle$, we observe $\mathcal{A}_n \mapsto \mathcal{A}_n + i\mathrm{d}\mu$. This dependence means that $\mathcal{A}_n$ is not physically observable, since it depends on our "gauge" we choose to measure with. However, its corresponding **curvature form** $\mathcal{F}_n = \mathrm{d}\mathcal{A}_n$ is **gauge-invariant**. Stokes' theorem allows us to write the Berry phase as

$$\gamma_n = \int_\Sigma \mathcal{F}_n, \tag{4}$$

where $C = \partial\Sigma$. Thus the Berry phase is gauge-invariant, and intrinsic to the topology of parameter space.

Would You Like To Know More?

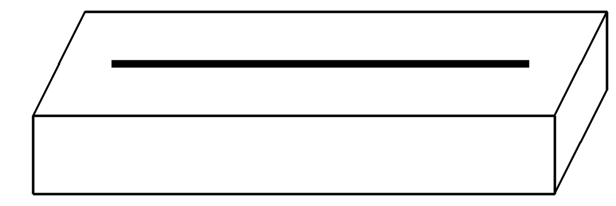

Figure 2: The Kitaev chain setup.

The Kitaev Chain

Consider a 1D lattice of L fermion sites over a superconducting substrate. The Hamiltonian for this system is given by

$$H = \sum_{j=1}^{L-1}\left(\Delta c_j c_{j+1} - t c_{j+1}^\dagger c_j + h.c.\right) - \sum_{j=1}^{L}\mu\left(c_j^\dagger c_j - \frac{1}{2}\right). \tag{5}$$

We split each fermionic mode into two **Majorana modes**:

$$\vartheta_{2j-1} = c_j + c_j^\dagger; \quad \vartheta_{2j} = -i\left(c_j - c_j^\dagger\right). \tag{6}$$

They satisfy the relations

$$\vartheta_\ell^\dagger = \vartheta_\ell, \quad \{\vartheta_\ell, \vartheta_k\} = 2\delta_{\ell k},$$

and thus generate a Clifford algebra. The Hamiltonian in this basis is

$$H = \frac{i}{2}\sum_j (t+\Delta)\,\vartheta_{2j}\vartheta_{2j+1} + (\Delta - t)\,\vartheta_{2j-1}\vartheta_{2j+2} - \mu\vartheta_{2j-1}\vartheta_{2j}. \tag{7}$$

Consider two special cases: a) $\mu < 0, \Delta = t = 0$; b) $\mu = 0, \Delta = t > 0$. The respective Hamiltonians are:

$$\text{a)} \quad H = -\frac{i\mu}{2}\sum_j \vartheta_{2j-1}\vartheta_{2j}; \quad \text{b)} \quad H = it\sum_j \vartheta_{2j}\vartheta_{2j+1}. \tag{8}$$

We observe two types of pairings of the ϑ, depicted below:

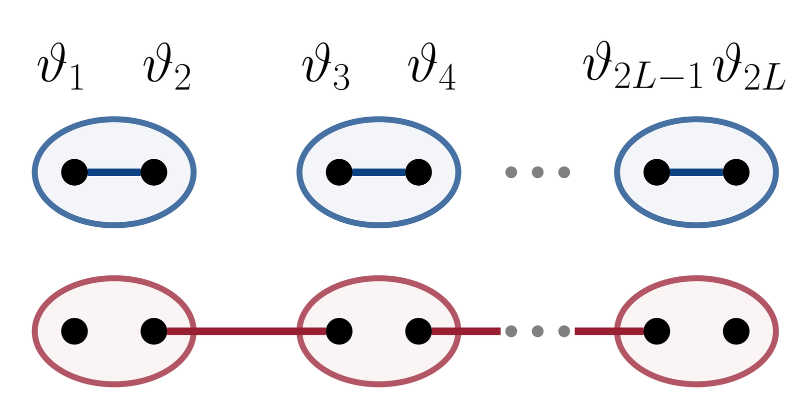

Figure 3: Large ellipses are physical lattice sites. Horizontal lines are fermionic modes.

Case b) reveals two Majorana **edge zero modes**, $\vartheta_1, \vartheta_{2L}$. They combine to a non-local zero energy fermionic mode. Even **beyond the special case**, we can still construct a fermionic zero mode Ψ which:

- commutes with the Hamiltonian: $[H, \Psi] = 0$.
- anticommutes with the fermionic parity operator: $\{(-1)^F, \Psi\} = 0$.
- remains "normalised" even as $L \to \infty$: $\Psi^\dagger\Psi = 1$.

In fact, the system features two phases: the **trivial phase** in the region $2|t| < |\mu|$; and the **topological phase** for $|\mu| < 2|t|$. The topological phase exhibits edge zero modes, which are absent in the trivial phase.

The existence of these edge zero modes is a $\mathbb{Z}_2$ **topological invariant** of the system.

Acknowledgements

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