Geometry and Topology in Quantum Mechanics

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Introduction

Quantum mechanically, the state $|\psi\rangle$ of a physical system can be represented by an element of a Hilbert space $\mathcal{H}$, modulo the equivalence relation

$$|\psi\rangle \sim e^{i\theta}|\psi\rangle$$

In this project we explored some interesting consequences of this equivalence relation.

The Berry Phase

Suppose that we have a Hamiltonian $H(\boldsymbol{R})$ which depends on an adiabatically varying family of parameters $\boldsymbol{R}(t)$, and that this Hamiltonian has an instantaneous basis of nondegenerate eigenfunctions $|n(\boldsymbol{R}(t))\rangle$, with energies $E_n(t)$. A state $|\psi\rangle$ prepared in the state $|n(\boldsymbol{R}(0))\rangle$ will evolve to

$$|\psi(t)\rangle = \exp\left\{-i\int_0^t E_n(t')dt'\right\}\exp(i\gamma_n t)\,|n(\boldsymbol{R}(t))\rangle \tag{1}$$

where the factor $\exp(i\gamma_n t)$ is called[1] the Berry phase, or geometric phase. Substituting this expression for $|\psi\rangle$ into the Schrodinger equation we obtain an equation for γ_n:

$$\frac{d\gamma_n}{dt} = i\,\langle n(\boldsymbol{R})|\nabla_{\boldsymbol{R}}\, n(\boldsymbol{R}(t))\rangle \cdot \frac{d\boldsymbol{R}}{dt} \tag{2}$$

If the changing parameters $\boldsymbol{R}(t)$ trace out a closed loop C in parameter space, then the total geometric phase change will be given by

$$\gamma_n(C) = i\oint_C \langle n(\boldsymbol{R})|\nabla_{\boldsymbol{R}}\, n(\boldsymbol{R}(t))\rangle \cdot d\boldsymbol{R} \tag{3}$$

This is an interesting result because although the system has returned to exactly the same place in parameter space the wave function has not returned to its original value, but rather picked up an additional phase factor $\exp(i\gamma_n t)$ (as well as the standard dynamical phase factor). Mathematically, this is an example of holonomy. A picture demonstrating the $SO(2)$ holonomy of the sphere is shown; a vector parallel transported around a sphere is rotated when it is returned to its original position. Although its origin is very different (coming from the Schrodinger equation instead of the geometry of the sphere), Berry's phase is also an example of $SO(2)$ holnomy. We now explore several consequences of Berry's result.

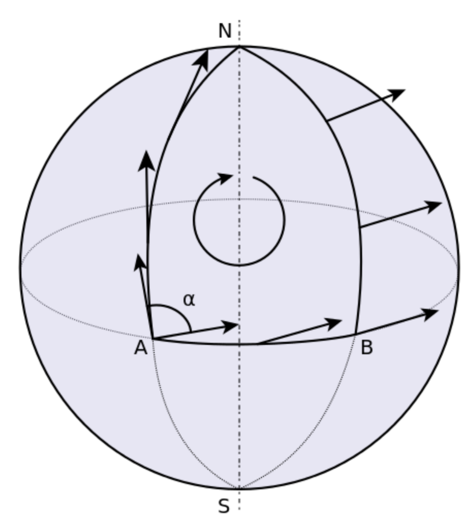

Figure 1: Holonomy on a sphere[2]

The Aharonov-Bohm effect

The Aharonov-Bohm effect is an interesting quantum mechanical effect with no classical explanation: it shows that particles can experience magnetic effects in a region where the magnetic field is zero. This effect can be seen to be a consequence of particles picking up geometric phases. The physical situation is depicted in the below diagram. On the left, a particle is transported in a box centred at $\boldsymbol{R}(t)$ around a current carrying wire:

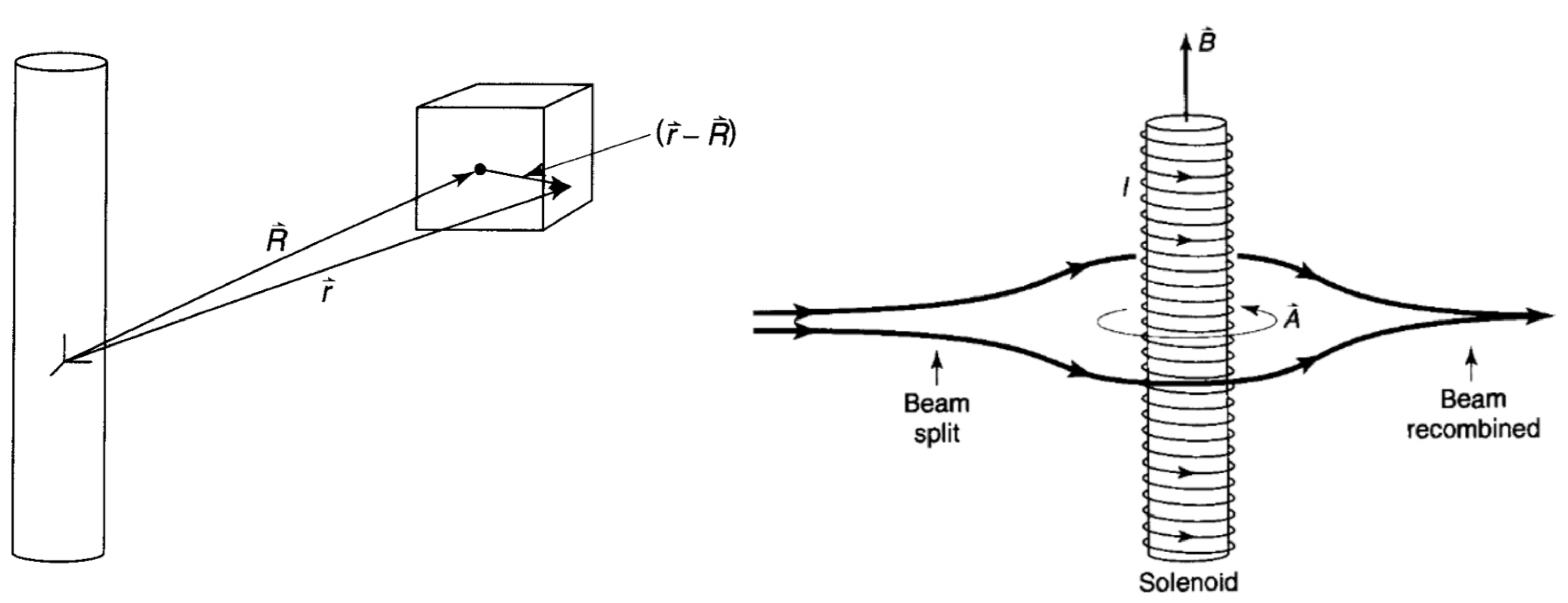

Figure 2: A schematic representation of the Aharonov-Bohm effect[3]

For a particle transported all the way around the solenoid n times we obtain

$$\gamma = q\oint_C \boldsymbol{A}(\boldsymbol{R})\cdot d\boldsymbol{R} = nq\Phi \tag{4}$$

where Φ is the magnetic flux, q is the charge of the particle and $\boldsymbol{A}$ is the vector potential. This is where the diagram on the right of figure 2 comes into play: particles which travel around opposite sides of the solenoid will pick up different phase factors, leading to an interference effect, despite the fact that the particles never entered the solenoid, which is the only region where the magnetic field is nonzero. The appearance of the winding number n shows that this effect reflects the topology of the path of the electron.

2π twist in the Kitaev Model

The Kitaev model[4] consists of a chain of sites, each of which can either be empty or filled by an electron, sitting on a superconducting surface. A general Hamiltonian describing such a system is given by

$$H = \sum_j \left(-w(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) - \mu(a_j^\dagger a_j - \tfrac{1}{2}) + \Delta a_j a_{j+1} + \Delta^* a_{j+1}^\dagger a_j^\dagger\right). \tag{5}$$

where w is a "hopping amplitude" describing the tendency of electrons to move from site to site, μ a chemical potential term describing whether it is energetically favourable for sites to be occupied, and Δ is the induced superconducting gap, which encapsulates the freedom of pairs of electrons to transfer back and forth between the wire and the superconducting surface. There are two special cases which are of particular interest, $a)$ where $w = \Delta = 0$, and $b)$ where $\mu = 0,\ w = |\Delta| > 0$. The two cases are shown below using the Majorana fermion representation. This involves introducing the operators

$$c_{2j-1} = a_j + a_j^\dagger, \qquad c_{2j} = -i\left(a_j - a_j^\dagger\right) \tag{6}$$

So there are two Majorana operators corresponding to each electron site. A pictorial representation of the two cases is:

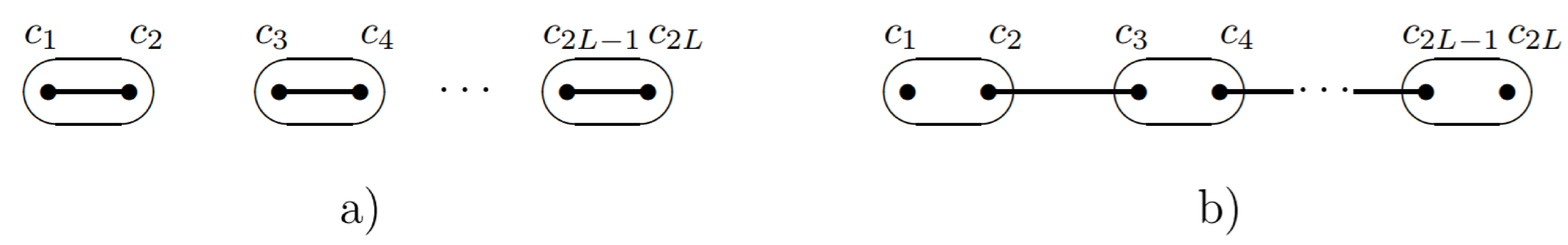

Figure 3: The two topologically distinct phases of the model[4]

We say that two surfaces are topologically distinct if one cannot be continuously deformed into the other (without ripping or gluing). Analogously, the two phases of the model are topologically distinct as their Hamiltonians cannot be deformed into one other (without closing the bulk energy gap). The bulk energy is the energy required to excite the system from its ground state.

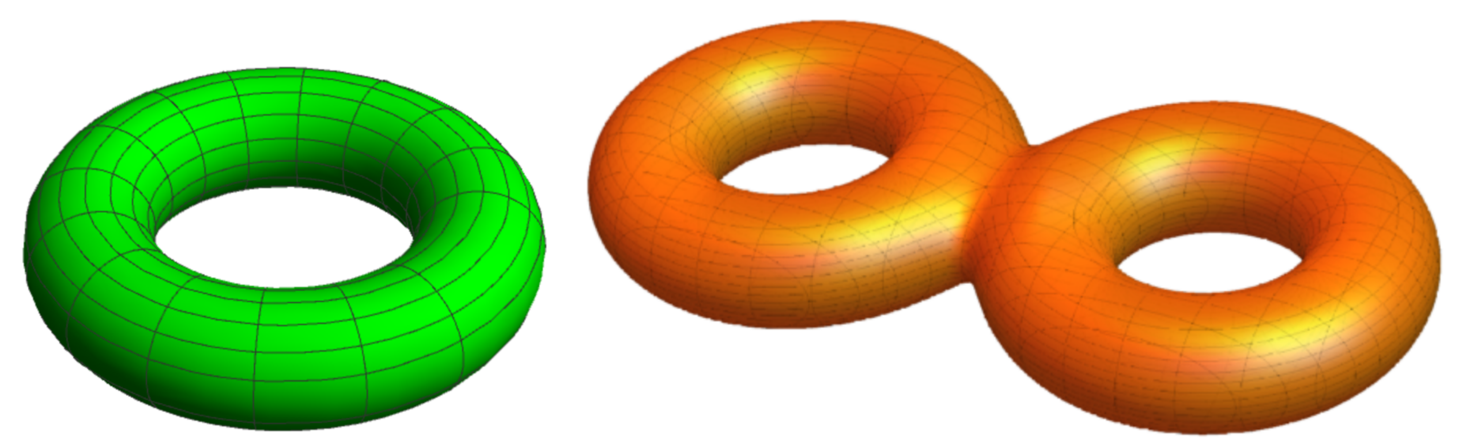

Figure 4: Topologically distinct surfaces

Topologically distinct surfaces can be characterised by topological invariants. For example, the two tori shown are distinguished by their Euler characteristics χ. We can look for an analogous quantity which distinguishes the two phases of the model. Suppose that we introduce periodic boundary conditions and a twist into the model:

$$a_j \to a_j \exp(i\theta j/L), \quad a_{L+1} = a_1 \exp(i\theta) \tag{7}$$

We can calculate the Berry phase as θ varies from 0 to 2π. In case $a)$ we calculate $\gamma_a = 0$, while in $b)$ (at least for small values of L) we obtain $\gamma_b = \pi/2$. We hypothesise that the Berry phase is a topological invariant which distinguishes the two phases of the model.

Thanks to Thomas Quella for his time and many helpful explanations throughout the project.

References

[1] Michael Berry, *Quantal phase factors accompanying adiabatic changes*, 1983

[2] Wikipedia, `https://en.wikipedia.org/wiki/Holonomy`

[3] D.J. Griffiths, *Introduction to Quantum Mechanics, 2nd ed.* Pearson education, 2004.

[4] Alexei Kitaev, *Unpaired Majorana fermions in quantum wires*, `arXiv:cond-mat/0010440`, 2000.