Topological Phases and Quantum Computation

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Quantum computation, like any other storage or manipulation of information, is prone to errors. Quantum error correction aims to detect and correct such errors, and some topological systems correspond to quantum error correction methods.

The toric code

Consider a square lattice with spins placed on the edges as in Figure 1. We identify the top and bottom edges, and the left and right edges of the lattice, hence embedding it on a torus. The Hamiltonian is given by

$$H = -J_e \sum_s A_s - J_m \sum_p B_p$$

where s runs over all vertices (stars) of the lattice and p runs over all plaquettes (faces). The star operator A_s acts on the four spins surrounding a vertex s, and the plaquette operator B_p acts on the four spins surrounding a plaquette p,

$$A_s = \prod_{j \in \text{star}(s)} \sigma_j^x, \qquad B_p = \prod_{j \in \partial p} \sigma_j^z.$$

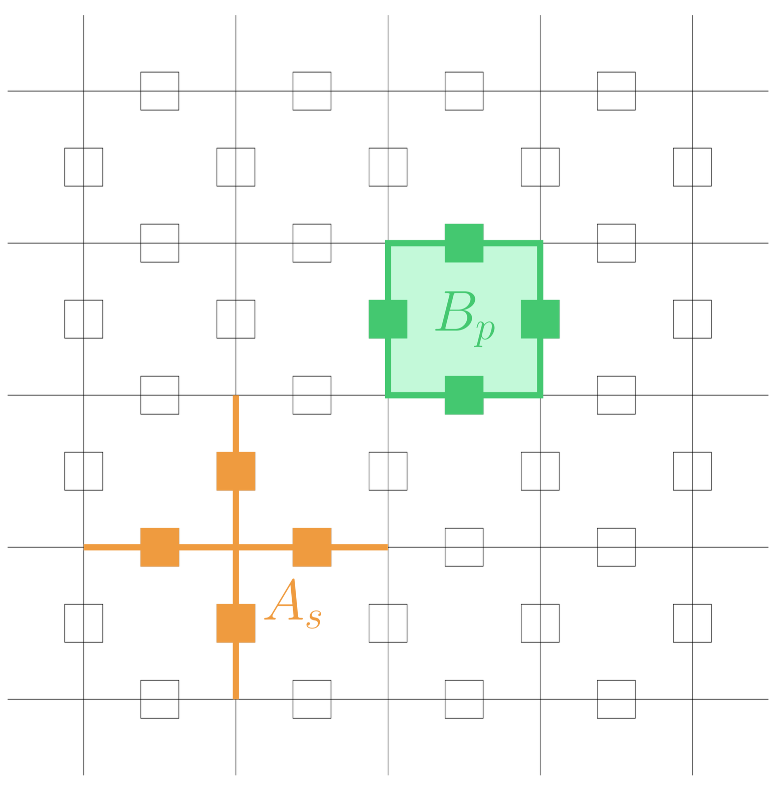

Figure 1: A section of the toric code

All the operators commute with each other, so we can diagonalise the Hamiltonian term by term.

Ground states and excitations

The ground states $|\psi\rangle$ of H have the minimum energy value. Each A_s and B_p term has a maximum eigenvalue of 1, so we have

$$A_s |\psi\rangle = |\psi\rangle\,, \qquad B_p |\psi\rangle = |\psi\rangle\,.$$

We can define electric and magnetic path operators as

$$W_l^{(e)} = \prod_{j \in l} \sigma_j^z, \qquad W_{l^*}^{(m)} = \prod_{j \in l^*} \sigma_j^x.$$

respectively, where the path l lies in the lattice, but the path l^* lies in the dual lattice as shown in Figure 2.

Applying these path operators to a ground state over a closed loop gives us a ground state. Contractible loops give the same ground state. However, we have two non-contractible cycles of a torus, so with our two path operators we can produce four different ground states.

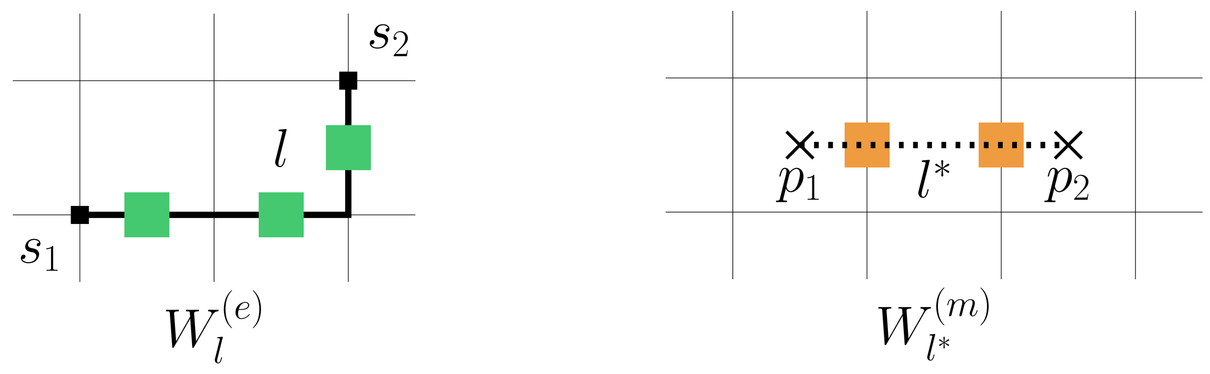

Figure 2: Electric and magnetic path operators

GROUND STATE DEGENERACY: The toric code exhibits a ground state degeneracy of 4^g when embedded on a surface of genus g.

Applying the path operators over a path that is not closed introduces excitations: electric charges or magnetic vortices. The path operators commute with every A_s and B_p except at the end points, where they anticommute. This introduces an extra $2J_e$ of energy to create a pair of electric charges, and an extra $2J_m$ for a pair of magnetic vortices.

Braiding

Electric charges (e) and magnetic vortices (m) are bosons since path operators of the same type commute with one another, so their mutual statistics can be found from Figure 3a. Consider taking a charge e around a vortex m as in Figure 3b.

(a) Bosonic self-statistics *(b) A charge e taken around a vortex m*

Figure 3

This maps a state $|\xi\rangle$ to $\left(\prod_{j \in l} \sigma_j^z\right) |\xi\rangle$ which equals $-|\xi\rangle$ because of the plaquette containing the vortex. From this we can derive that composite $e-m$ particles are fermions:

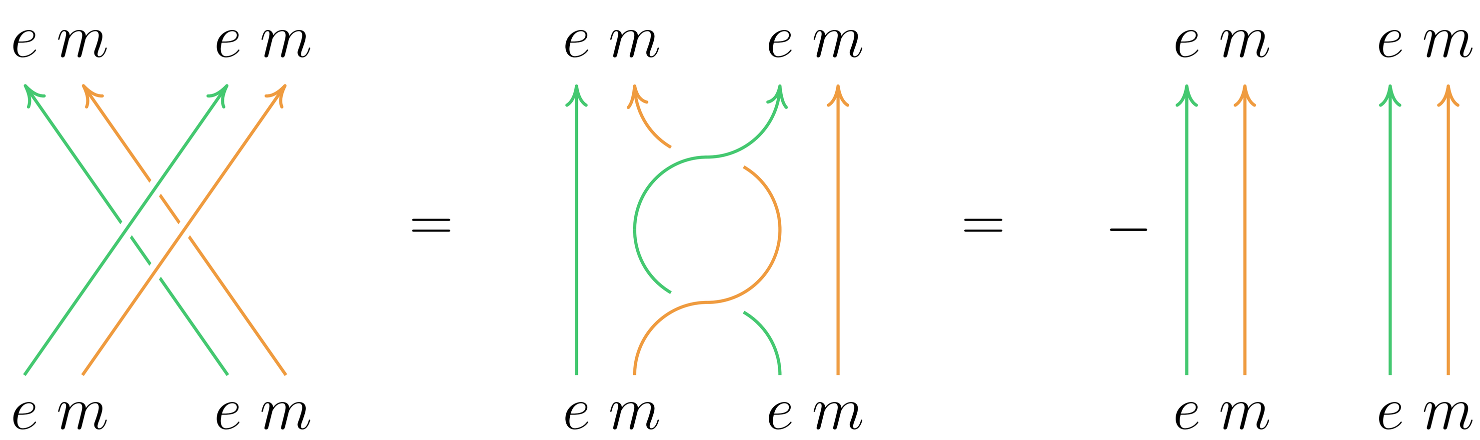

Figure 4: Mutual statistics of a composite $e-m$ particle

References:

Kitaev, A. and Laumann, C. (2009). Topological phases and quantum computation. *arXiv*, **0904.2771v1**.