The Cantor Set and The Cantor Function

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We recall the classical construction of a continuous, non-decreasing function whose derivative is zero almost everywhere: the Cantor function. From a probabilistic viewpoint, this leads to an example of a random variable whose distribution function is continuous but with zero derivative almost everywhere (with respect to the Lebesgue measure). In particular, this random variable is continuous but cannot have a density function. Such a random variable/distribution is said to be *singular*. Before describing the Cantor function, it is helpful to first recall the construction of a closely related object: the Cantor set.

The Cantor Set

The *Cantor set* is a closed subset of $[0,1]$ which is constructed in the following procedure. We start with the entire interval $[0,1]$. At step 1, we divide the interval into three sub-intervals of equal length $1/3$, and remove the middle open interval $(1/3, 2/3)$. Let C_1 denote what is left, i.e.

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1].$$

At step 2, we apply the same removal procedure to each sub-interval obtained from the previous step. Namely, we divide $[0, 1/3]$ into three sub-intervals of equal length $1/3^2$ and remove the middle open interval, and do the same thing for the interval $[2/3, 1]$. Let C_2 denote what is left. Apparently

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1].$$

We continue this procedure inductively. After this entire removal procedure performed sequentially, what is left is the definition of the Cantor set C. Mathematically, if we let C_n denote the closed subset obtained at the end of the n-th step of removal, then $C = \cap_{n=1}^{\infty} C_n$.

The first observation about the Cantor set is that it has zero Lebesgue measure. To see this, we compute the total measure of what is removed. If we let I_n be the union of the open sub-intervals removed from step n, then I_n consists of a total number of 2^{n-1} intervals, each having length $\frac{1}{3^n}$. Therefore, the Lebesgue measure of I_n is

$$|I_n| = 2^{n-1} \times \frac{1}{3^n},$$

and the total measure of the subset being removed (i.e. $I \triangleq \cup_{n=1}^{\infty} I_n$) is

$$|I| = \sum_{n=1}^{\infty} |I_n| = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1.$$

In other words, the Cantor set $C = [0,1]\backslash I$ has zero Lebesgue measure.

Using the ternary expansion of real numbers, the heuristics behind the above construction becomes more straight forward. Recall that, a real number $x \in [0,1]$ admits an expansion

$$x = 0.x_1x_2x_3\cdots$$

where $x_n = 0, 1, 2$. This expansion is called the *ternary expansion* of x (or the expansion of x in base 3). In terms of this expansion, $x_1 = 0$ (respectively, $x_1 = 1$ or $x_1 = 2$) means x falls in the first sub-interval (respective, the second or the third) in the first step of the ternary sub-division of $[0,1]$. Similarly, x_2 records which sub-interval that x belongs to in the next step of ternary sub-division and so forth. Using this interpretation, it is clear from the construction of the Cantor set that $x \in C$ *if and only if the ternary expansion of x does not contain the digit* 1. Indeed, if there is a "1" in the expansion of x, that means x belongs to the middle interval in some step of ternary sub-division, and that interval is removed by the construction of C.

The Cantor function

Now we describe the construction of the *Cantor function* $G(x)$. The strategy of constructing G is to first specify its values on $[0,1]\backslash C = \cup_{n=1}^{\infty} I_n = I$ (the open intervals being removed), and then extend G uniquely to the entire interval $[0,1]$. The function G will be constant on each sub-interval in I. To specify its values, we first consider I_1. I_1 is simply given by the interval $(1/3, 2/3)$. We define G on I_1 to take the constant value $\frac{1}{2}$. Next, we consider I_2, which is given by two intervals

$$I_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}).$$

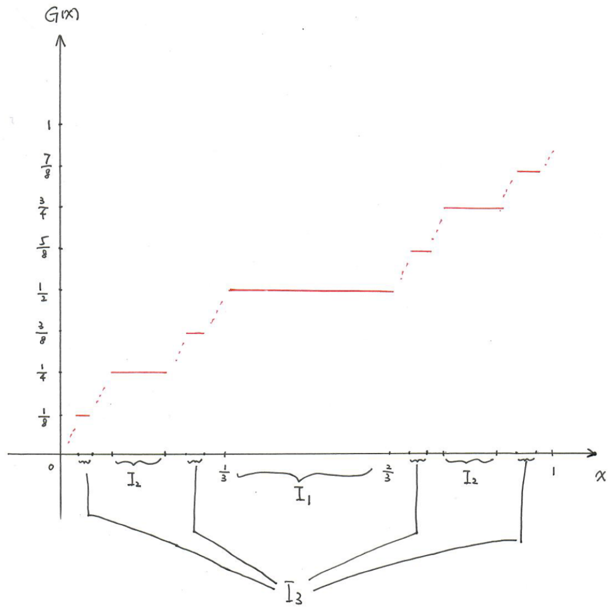

Figure 1: The Cantor Function

We define G to take the values $\frac{1}{4}, \frac{3}{4}$ on each of these two intervals respectively. Inductively, at the n-th step, I_n consists of 2^{n-1} open sub-intervals, and we define G to take the values

$$\frac{1}{2^n}, \frac{3}{2^n}, \frac{5}{2^n}, \cdots, \frac{2^n-1}{2^n}$$

on each of these intervals in the natural order. Inductively, this specifies the definition of G on I. Figure 1 illustrates the construction of G on I_n for $n = 1, 2, 3$. Note that G maps I onto the set of dyadic partition points of $(0, 1)$, i.e.

$$G(I) = \{\frac{m}{2^n} : n \geqslant 1,\ 1 \leqslant m \leqslant 2^n - 1\}.$$

It is helpful to think through how this mapping procedure works geometrically.

There is a neat algebraic expression for the above construction of G. For given $x \in I$, there must be a "1" in its ternary expansion $x = 0.x_1x_2x_3\cdots$. Let $N_x \triangleq \inf\{n \geqslant 1 : x_n = 1\}$. Then one can check that

$$G(x) = \frac{1}{2^{N_x}} + \sum_{n=1}^{N_x-1} \frac{x_n/2}{2^n}. \tag{1}$$

There are two equivalent ways of extending the definition G to the entire interval $[0,1]$. The *first way* is to show that G is uniformly continuous on I. It is then a simple consequence of the denseness of I in $[0,1]$ (as a subset of full Lebesgue measure) that G admits a unique continuous extension to $[0,1]$. The *second way*, which is more explicit, is to observe that the right hand side of (1) remains meaningful even when $x \notin I$. Indeed, if the digits x_n's contains 0 and 2 only, we have $N_x = \infty$ and in this case we can simply define

$$G(x) \triangleq \sum_{n=1}^{\infty} \frac{x_n/2}{2^n}. \tag{2}$$

In other words, we can take the expression (1) as the definition of G for all $x \in [0,1]$. It can be checked that G is a well defined, continuous and non-decreasing function on $[0,1]$. Now using the fact that I has Lebesgue measure one, and G is constant on each sub-interval of I, we see that $G' = 0$ on I, and thus G has a zero derivative for almost everywhere.

There is an insightful observation from this Cantor function G. Recall that every point in $x \in C$ is represented by a ternary expansion consisting of 0 and 2 only. In view of the corresponding value of the Cantor function given by (2), in this case $G(x)$ is precisely a binary expansion of a real number in $[0,1]$. This shows that, G maps the Cantor set C onto $[0,1]$. As a consequence, the cantor set has cardinality the same as $[0,1]$, although it has zero Lebesgue measure!