The Chern-Gauss-Bonnet Theorem

An Introduction to Global Geometry and Topology

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Abstract

In these notes, we study the *Chern-Gauss-Bonnet theorem* which is among the most profound results in geometry and topology. We will discuss four different but closely related approaches to this theorem. The first approach is the original intrinsic proof of S.S. Chern. The second approach is based on the Chern-Weil theory which was developed in a series of seminal works by S.S. Chern in the 1940s. The deep geometric insights and fundamental techniques from these two approaches shaped the later development of global differential geometry in the last century. The third approach has a strong analytic flavour that is based on the analysis of the heat equation. This approach is in spirit close to the modern analytic approach to the Atiyah-Singer index theorem. The last approach we will discuss is based on Quillen's superconnection technique, which can be viewed as a superconnection analogue of the Chern-Weil approach.

Along our path towards the aforementioned approaches, we will encounter several landmark results in geometry and topology, such as intersection theory, Lefschetz fixed point formula, Poincaré-Hopf index theorem, Poincaré duality, Thom isomorphism, de Rham theorem, Chern-Weil theorem, Hodge theory etc. To make these notes relatively self-contained, we will discuss the major geometric and topological results that are essential to our study, which may not always be covered in a standard course. These notes can thus be viewed as an introduction to geometry and topology (or a second course) with an essential aim towards the Chern-Gauss-Bonnet theorem.

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1 Introduction

The classical Gauss-Bonnet theorem provides a deep connection between geometry (curvature) and topology (the Euler characteristic) for surfaces. It asserts that

> *the integral of the Gaussian curvature over a compact, oriented, two-dimensional Riemannian manifold without boundary is equal to* 2π *times the Euler characteristic of the manifold.*

The Gaussian curvature is a local geometric object while the Euler characteristic is a global topological quantity. The Gauss-Bonnet theorem provides a surprising link between the integral of a local geometric quantity and global topology. It suggests some kind of geometric rigidity that is constraint by topology: curvature behaves like a conserved quantity whose "total" is dictated by topology. The theorem is therefore a profound result from this perspective.

A first attempt of generalising the Gauss-Bonnet theorem to higher dimensions appeared in the seminal paper of H. Hopf [Hop25] in 1925, who proved that the integral of the total curvature of a hypersurface of Euclidean space is a topological invariant. This remarkable insight was among the very first developments that brought the global viewpoint into differential geometry and opened the era of global differential geometry. A more precise form of the Gauss-Bonnet theorem in higher dimensions was obtained independently by C.B. Allendoerfer [All40] and W. Fenchel [Fen40] in 1940. In these two papers, the authors proved that on a closed, oriented Riemannian manifold which is isometrically embedded in an Euclidean space, the integral of the Lipschitz-Killing curvature is a constant multiple of the Euler characteristic. In the later work of C.B. Allendoerfer and A. Weil [AW43], the isometric embedding assumption was removed and a precise version of the Gauss-Bonnet theorem for an arbitrary closed, oriented Riemannian manifold was established. The main idea in [AW43] was to triangulate the manifold into cells that could be isometrically embedded into Euclidean space so that the earlier results of [All40, Fen40] could be applied (Nash's embedding theorem was not yet available during that time). Though the intrinsic formulation of the Gauss-Bonnet theorem was already known in [AW43], the proofs in the aforementioned works were clearly of extrinsic nature and were therefore not entirely satisfactory from a geometric viewpoint.

The first intrinsic proof of the Gauss-Bonnet theorem in higher dimensions appeared in the landmark paper of S.S. Chern [Che44] in 1944. Stated in a concise and elegant form, the theorem asserts that

> *the integral of the (geometric) Euler form over a closed, oriented, even-dimensional Riemannian manifold is equal to the Euler characteristic of the manifold.*

This result is nowadays referred to as the *Chern-Gauss-Bonnet theorem* (CGB in short). In the subsequent seminal papers [Che45, Che46], Chern further developed his remarkable ideas and techniques in [Che44] into a new height, leading to a fundamental theory of characteristic classes which is nowadays known as the Chern-Weil theory. These works had far-reaching impact on the modern development of geometry and topology, in particular in areas such as index theory, gauge theory and global analysis etc. On the other hand, the heat equation approach of [MS67, Pat71] is closely tied to the modern analytic approach to the Atiyah-Singer index theory, where CGB appears as a fundamental example.

1.1 Overview of fundamental ideas

In the present notes, we discuss four different but closely related approaches to the Chern-Gauss-Bonnet theorem, each having its own merit and significance. In what follows, we summarise the philosophy behind these approaches in a conceptual way. Since we have not yet introduced any mathematical concept, it is impossible to give a very precise explanation about the methodology at this stage. More detailed discussion will be given in the relevant chapters below.

Chern's original intrinsic approach

The first approach [Che44] is the original intrinsic proof of S.S. Chern. The rather deep insight of Chern was that the *Euler form lifts to an exact form* $d\Pi$ *on the sphere bundle*. Let V be a smooth vector field on the base manifold which has only isolated zeros. Through the aforementioned lifting, the total integral of the Euler form turns out to be localised near the zeros of V as a consequence of Stokes' theorem. In addition, due to the specific construction of Π, the localised integral picks up the index of the vector field V at each zero. It follows that the total integral of the Euler form is equal to the sum of the indices of V at its zeros. The latter is precisely the Euler characteristic as a consequence of the Poincaré-Hopf index theorem. This proves the CGB theorem.

The Chern-Weil perspective

The second approach [Che46] is based on the Chern-Weil theory, which takes a deep step beyond Chern's original proof. It provides a systematic way of constructing suitable cohomology classes that detect the topological twisting of a vector bundle (characteristic classes). In the context of CGB, it provides a natural way of constructing the geometric Euler form and reveals in a fundamental way why this Euler form lifts to an exact form on the sphere bundle (which was a key insight in Chern's original proof). This exactness property turns out to be rather explicit and quantitative (the Chern-Weil theorem), which is described by means of the so-called transgression formula. Through explicit analysis based on this formula, one can show that the cohomology class of the geometric Euler form coincides with the topological Euler class, whose total integral yields the Euler characteristic due to a standard topological result. This proves CGB.

The heat equation approach

The above two approaches have a quite geometric and topological nature. The third approach [MS67, Pat71] is quite analytic on the other hand. It is based on the analysis of the heat equation. The starting point is a beautiful and profound geometric result known as the Hodge theorem. The Hodge theorem asserts that the space of harmonic forms is isomorphic to the de Rham cohomology in each degree, hence relating the Euler characteristic to the dimensions of the spaces of harmonic forms. By using the Hodge theorem, the Euler characteristic can be expressed as the supertrace of the heat semigroup on forms and the latter is a supertrace integral of the heat kernel. In its small-time limit, only one particular term in the heat kernel expansion would survive, which is precisely the Euler form through explicit analysis. As a consequence, the Euler characteristic is the total integral of the Euler form. This is exactly CGB.

The superconnecion approach

The fourth approach [ZF22] is based on Quillen's superconnection technique (cf. [Qui85, MQ86]). This approach is in spirit very close to the Chern-Weil approach discussed earlier. The key insight is that the Euler form can be realised as the so-called Chern character form associated with the induced superconnection on the (complexified) exterior algebra bundle. By applying a version of the Chern-Weil theorem for superconnections, one concludes that the cohomology class of the Chern character (hence of the Euler form) is independent of the choice of superconnections. Now pick a smooth vector field V with isolated zeros. By deforming the superconnection in a clever way, it turns out that the integral of the Euler form is localised at the zeros of V, yielding their indices through explicit computation. The proof of CGB is again finished by applying the Poincaré-Hopf theorem.

1.2 Organisation and guideline

I have no intention of writing a comprehensive book on geometry and topology, given that there are already too many excellent texts on this topic. There is only one essential motivation and objective in these notes: *to understand the Chern-Gauss-Bonnet theorem in depth.* We will take this opportunity to introduce the landmark results in geometry and topology that will be used along our path towards proving the theorem. This will make the present notes relatively self-contained. Though it is not strictly necessary, it would be helpful if the reader is familiar with some basic algebraic topology, differential and Riemannian geometry. Some basic concepts and tools from these topics that will be used in the main text are summarised in the appendix.

Organisation

(i) *The Gauss-Bonnet theorem.* In Section 1.3, we discuss the Gauss-Bonnet theorem for surfaces following the connection-based approach of [ST67]. This section is not strictly necessary for the purpose of CGB but will provide some important geometric intuition towards the higher dimensional situation.

(ii) *Chern's intrinsic proof.* In Chapter 5, we present Chern's original intrinsic proof of CGB. The argument relies on a basic topological result: the Poincaré-Hopf index theorem. Our approach to this index theorem contains two major steps. Chapter 2 is devoted to the development of the first step, which requires some intersection theory from differential topology. Chapter 3 is devoted to the development of the second step, which requires the Lefschetz fixed point formula from algebraic topology. This is certainly not the shortest path of proving the Poincaré-Hopf index theorem itself. We decide to take this approach because it provides a nice opportunity of motivating and encountering various fundamental results in topology. In Chapter 4, we discuss the differential counterpart of the main algebraic topology results in Chapter 3.

(iii) *The Chern-Weil perspective.* In Chapter 7, we develop basics of the Chern-Weil theory and present the second approach to CGB from the Chern-Weil perspective. This provides deeper insight into Chern's original proof in Chapter 5. The Chern-Weil approach relies on the basic theory of connections on principal bundles, which will be studied in Chapter 6.

(iv) *The heat equation approach.* In Chapter 10, we develop the heat equation proof of CGB. The argument relies on the Hodge theory and explicit analysis of heat kernels on vector bundles, which will be studied in Chapter 8 and Chapter 9 respectively.

(v) *The superconnection approach.* Chapter 11 is devoted the fourth proof of CGB following Quillen's technique of superconnections on super vector bundles. This can be viewed as a superconnection version of the Chern-Weil approach developed in Chapter 7.

Reader's guideline

To understand Chern's original proof, there is no need to read Chapter 2 and Chapter 3 provided that the reader takes the Poincaré-Hopf index theorem as granted.

Chapter 6 is somehow necessary to properly understand the Chern-Weil proof of CGB, unless the reader is familiar with the basic language of connections on principal bundles. Chapter 4 and Chapter 8 can be skipped, provided that the reader assumes the de Rham theorem and the Hodge theorem. The construction of the heat kernel in Chapter 9 is not strictly necessary. However, its small-time expansion (Proposition 9.4 and Corollary 9.4), which is developed along the path of proving its existence, is somehow necessary to understand the heat equation proof of CGB (Patodi's local index formula; see Theorem 10.2). Chapter 11 is quite independent and does not rely on earlier chapters in any essential way.

Notation and conventions

To make the presentation smoother, throughout the rest of these notes we will adopt the following conventions / notation unless otherwise stated.

1. By an (n-)manifold, we always mean an (n-dimensional) *differentiable* manifold. A *closed* manifold is a compact manifold without boundary. Unless otherwise stated, functions / vector fields / tensor fields / differential forms / sections are always assumed to be smooth. The evaluation of a tensor field ξ (e.g. a vector field or a differential form) at a location x is denoted as either $\xi(x)$ or ξ_x.

2. We always adopt Einstein's convention of summation. Namely, any pair of repeated indices in an expression are summed automatically over their domain of definition. For instance,
$$R^i_{jkl}a_ib^j \triangleq \sum_{i,j} R^i_{jkl}a_ib^j.$$
This also includes the case of multiindices, for instance,
$$a_I\omega^I \triangleq \sum_{I=(i_1,\cdots,i_r)} a_I\omega^I.$$

3. Depending on the context, the notation $\langle\cdot,\cdot\rangle$ can refer to the Riemannian metric or the pairing between cotangent and tangent vectors or more generally, the action of a linear functional on a vector.

4. (P)ONB means (positively oriented) orthonormal basis. (P)ONF means (positively oriented) orthonormal frame field.

5. For the multiindex $I = (i_1, \cdots, i_n)$, we set
$$\mathrm{sgn}(I) \triangleq \mathrm{sgn}(i_1,\cdots,i_n) \triangleq \begin{cases} 1, & \text{if } I \text{ is an even permutation of } \{1,\cdots,n\}; \\ -1, & \text{if } I \text{ is an odd permutation of } \{1,\cdots,n\}; \\ 0, & \text{if } I \text{ is not a permutation of } \{1,\cdots,n\}. \end{cases}$$

We also use $\operatorname{sgn}(\sigma)$ to denote the sign of a permutation σ.

6. Unless otherwise stated, the notation $\widehat{\cdot}$ means removing the current object. For instance,

$$\theta^1 \wedge \cdots \wedge \widehat{\theta^i} \wedge \cdots \theta^n \triangleq \theta^1 \wedge \cdots \wedge \theta^{i-1} \wedge \theta^{i+1} \wedge \cdots \wedge \theta^n,$$

$$\omega(X_1, \cdots, \widehat{X_i}, \cdots X_n) \triangleq \omega(X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n).$$

7. The notation $A \Subset B$ means that the closure of A is contained in the interior of B (i.e. $\bar{A} \subseteq \mathring{B}$).

1.3 The Gauss-Bonnet theorem for surfaces

We begin our journey by establishing the classical Gauss-Bonnet formula for surfaces. The main theorem is recaptured as follows.

Theorem 1.1 (The Gauss-Bonnet Theorem)**.** *Let M be a closed, oriented, two-dimensional Riemannian manifold. Then*

$$\frac{1}{2\pi}\int_M K \mathrm{vol}_M = \chi(M). \tag{1.1}$$

Here K is the Gaussian curvature of the Levi-Civita connection, vol_M is the Riemannian volume form and $\chi(M)$ is the Euler characteristic of M.

The intrinsic definition of the Gaussian curvature will be given in Section 1.3.3. The definition of the Euler characteristic will be given in (1.2). In the sequel, we will adopt the approach of [ST67] to prove Theorem 1.1. This approach is very close to the spirit of connections on principal bundles, which will be elaborated in Chapter 6. The heart of the argument, which has a geometric nature, lies at a proper interpretation of the Gaussian curvature (see Theorem 1.7).

1.3.1 A combinatorial formula for the Euler characteristic

The Euler characteristic is a topological invariant which counts the signed total number of "holes" in all dimensions. Its precise definition is given as follows.

Definition 1.1. Let M be a closed n-manifold. The *Euler characteristic* of M is the integer defined by

$$\chi(M) \triangleq \sum_{k=0}^{n} (-1)^k \dim H^k(M;\mathbb{R}), \tag{1.2}$$

where $H^k(M;\mathbb{R})$ is the k-th singular cohomology group of M with real coefficients.

Remark 1.1. One can replace the singular cohomology by the de Rham cohomology. Indeed, according the de Rham theorem the two types of cohomology groups are isomorphic (see Theorem 4.5). The finite dimensionality of $H^k(M;\mathbb{R})$ is addressed in Proposition 4.1. One can also replace $\mathbb{R}$ by any other field with characteristic zero and replace cohomology by homology.

In this subsection, we recall a simple combinatorial formula for the Euler characteristic. This formula will be useful for our later proof of the Gauss-Bonnet theorem. We first introduce the concept of a simplicial complex.

Definition 1.2. Let V be a real vector space. A (closed) *k-simplex* in V is a closed subset of V which admits the form

$$[v_0, v_1, \cdots, v_k] \triangleq \Big\{\sum_{i=0}^{k} a_i v_i : a_i \geqslant 0, \sum_i a_i = 1\Big\},$$

where $v_0, v_1, \cdots, v_k$ are $k+1$ elements in V such that the vectors

$$v_1 - v_0, \cdots, v_k - v_0$$

are linearly independent. An *open simplex* in V is the interior of a closed simplex. By definition, it has the form

$$(v_0, v_1, \cdots, v_k) \triangleq \Big\{\sum_{i=0}^{k} a_i v_i : a_i > 0, \sum_i a_i = 1\Big\}.$$

The *dimension* of a k-simplex (either open or close) is k. The *closed faces* (respectively, *open faces*) of a closed simplex $[v_0, v_1, \cdots, v_k]$ are the closed simplices $[v_{j_0}, v_{j_1}, \cdots, v_{j_l}]$ (respectively, open simplices $(v_{j_0}, v_{j_1}, \cdots, v_{j_l})$), where $\{j_0, j_1, \cdots, j_l\}$ is a nonempty subset of $\{0, 1, \cdots, k\}$.

Notation. We often use the symbol (s) or s to denote an open simplex and use $[s]$ to denote a closed one.

Example 1.1. A 0-simplex is a point (vertex). A 1-simplex is an open (finite) line segment. A 2-simplex is an open solid triangle on a plane.

Definition 1.3. A *simplicial complex* K is a finite set of open simplices in some Euclidean space $\mathbb{R}^N$ such that the following properties hold true.

(i) If an open simplex $(s) \in K$, then all open faces of $[s] \in K$;

(ii) If $(s_1), (s_2) \in K$ and $(s_1) \cap (s_2) \neq \emptyset$, then $(s_1) = (s_2)$.

The *dimension* of K is the maximum dimension of the open simplices in K. We use $|K|$ to denote the (compact) point set union of the open simplices of K equipped with the relative topology as a subset of $\mathbb{R}^N$.

Next, we recall the notation of triangulation in the context of manifolds.

Definition 1.4. A *smooth triangulation* of a manifold M is a pair (K, h) where K is a simplicial complex and $h : |K| \to M$ is a homeomorphism such that for every $(s) \in K$, the map $h|_{[s]} : [s] \to M$ admits an extension h_s to a neighbourhood U of $[s]$ in the vector subspace of $[s]$ such that $h_s : U \to M$ is a smooth embedded submanifold.

It is a well-known result that *every closed manifold admits a smooth triangulation.* We will not prove this fact and will refer the interested reader to [Mun66] for a discussion. The following result provides a combinatorial way of computing the Euler characteristic in terms of a triangulation. Its proof, which requires the use of simplicial homology, will be given in Section 4.7.1 (see Proposition 4.5, Definition 4.11 and Remark 4.10).

Theorem 1.2. *Let (K, h) be a smooth triangulation of M. Then the Euler characteristic of M can be computed as*

$$\chi(M) = \sum_{k=0}^{\dim K} (-1)^k \alpha_k, \tag{1.3}$$

where α_k denotes the number of open k-simplices in K. In particular, for a closed, 2D manifold M with a smooth triangulation as a 2D simplicial complex, one has

$$\chi(M) = \#\ \textit{of vertices} - \#\ \textit{of edges} + \#\ \textit{of 2D faces}.$$

Remark 1.2. Since $\chi(M)$ is a topological invariant, the formula (1.3) does not depend on the actual triangulation of M.

1.3.2 Metric connection and parallel transport

Let M be an oriented, 2D Riemannian manifold. In this subsection, we recall the basics of metric connection. This will provide some essential tools for our later study of the Gaussian curvature. To keep the presentation smooth, we will only provide proofs that are not contained in the more general results developed in Chapter 6.

The idea of a connection, or a covariant derivative, is to provide an intrinsic way of differentiating tensor fields (most basically, vector fields) along arbitrary directions on a manifold. For a smooth function $f : M \to \mathbb{R}$, one can canonically define its directional derivative at a location $x \in M$ along a direction $v \in T_xM$ without introducing any geometric structure. Indeed, pick a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $x_0 = x$ and $\dot{x}_0 = v$. The directional derivative $\nabla_v f$ is simply given by

$$\nabla_v f \triangleq \lim_{t \to 0} \frac{1}{t}(f(\gamma_t) - f(x)).$$

Since f is $\mathbb{R}$-valued, the difference $f(\gamma_t) - f(x)$ makes sense in the usual way and the above limit is well-defined.

Now suppose that Y is a vector field on M. If one tries to follow the above route to define the "directional derivative" of Y at x along v, i.e. by naively setting

$$\text{“}\nabla_v Y \triangleq \lim_{t\to 0} \frac{1}{t}(Y_{\gamma_t} - Y_x)\text{”}, \tag{1.4}$$

one immediately recognises that the difference $Y_{\gamma_t} - Y_x$ makes no sense (Y_{γ_t} and Y_x live on different tangent spaces). A natural attempt to fix this idea is to "parallel transport" the vector Y_{γ_t} along the curve γ back to the starting point $x_0 = x$ so that one obtains a new vector $\tilde{Y}_t \in T_x M$. The limit

$$\lim_{t\to 0} \frac{1}{t}(\tilde{Y}_t - Y_x)$$

would make sense (at least formally), because $\tilde{Y}_t$ and Y_x now live on the same vector space $T_x M$.

From the above perspective, a connection is essentially a notion of *parallel transport* (of tangent vectors) along smooth curves in M. In the Riemannian context, a *metric* connection is a notion of parallel transport which preserves lengths and angles. This is a very natural condition to impose.

Suppose that $\gamma : [0,1] \to M$ is a smooth curve in M and $v \in T_{\gamma_0} M$ is a vector with unit length. We wish to introduce a parallel transport of v along γ, i.e. a vector $v_t \in T_{\gamma_t} M$ with unit length for every t. One can think of (γ_0, v) as an element of the *sphere bundle* SM (see Definition 1.5 below). The aforementioned parallel transport becomes the search for a *lift* of the base curve γ to a curve $u_t = (\gamma_t, v_t)$ $(0 \leqslant t \leqslant 1)$ in $S(M)$ starting at (γ_0, v). As we will see, such a lift is uniquely determined by requiring that the tangent vector $\dot{u}_t$ is *horizontal* for all t. In other words, a metric connection should be viewed as an assignment of a *horizontal subspace* $\mathcal{H}_u \subseteq T_u SM$ to each point $u \in SM$ which satisfies some structural and consistency relations. This is the infinitesimal viewpoint of connection which we will make precise shortly.

The sphere bundle

Recall that M is an oriented, 2D Riemannian manifold.

Definition 1.5. The *sphere bundle* over M is defined by

$$SM \triangleq \{(x,v) : x \in M, v \in T_x M, |v|_{T_x M} = 1\}.$$

We denote $\pi : SM \to M$ as the canonical projection.

The space SM is a three-dimensional manifold. The easiest way to see this is to realise that $SM = f^{-1}(1)$, where $f : TM \to \mathbb{R}$ is the smooth function on the tangent bundle TM defined by $f(x, v) \triangleq \langle v, v\rangle_{T_xM}$. It is easy to check that $(df)_{(x,v)} \neq 0$ for all $(x, v) \in f^{-1}(1)$. It follows from the implicit function theorem that $SM = f^{-1}(1)$ is a closed submanifold of TM with dimension $4 - 1 = 3$. Note that for each $x \in M$, the fiber $\pi^{-1}(x)$ is the unit circle on T_xM. The space SM is locally trivial, in the sense that for any $x \in M$, there exists an open neighbourhood U of x such that $\pi^{-1}(U)$ is diffeomorphic to $U \times S^1$. Here $S^1 = \{e^{it} : t \in \mathbb{R}\}$ denote the circle group where the product is given by complex multiplication. Note that S^1 is an abelian Lie group whose Lie algebra is just $\mathbb{R}$ with exponential map given by $t \mapsto e^{it}$.

An essential structure of SM is that it admits an action by S^1 from the right. Given $u = (x, v) \in SM$ and $g = e^{it} \in S^1$, we define

$$R_g(u) \triangleq ug \triangleq (x, ve^{it}),$$

where ve^{it} is the unit vector in the plane T_xM obtained by rotating v counter-clockwise by an angle of t with respect to the given orientation on T_xM. This action depends only on t mod 2π and is thus well-defined. It is plain to check that $(ug_1)g_2 = u(g_1g_2)$. In addition, any two elements (x, u) and (x, v) is related by $v = ug$ for a unique $g \in S^1$.

Remark 1.3. In the spirit of Sections 6.9 and 6.10, the "correct" bundle for metric connections to live is the *special orthonormal frame bundle*

$$\mathrm{SO}(M) \triangleq \{(x; \varepsilon_1, \varepsilon_2) : x \in M, \{\varepsilon_1, \varepsilon_2\} \text{ is a positive ONB of } T_xM\}.$$

This bundle admits a right-action by the rotation group

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

which is defined by a change of basis:

$$(x; \varepsilon_1, \varepsilon_2)g \mapsto (x; (\varepsilon_1, \varepsilon_2) \cdot g), \quad (x; \varepsilon_1, \varepsilon_2) \in \mathrm{SO}(M), g \in \mathrm{SO}(2).$$

The Lie algebra of SO(2) is

$$\mathfrak{so}(2) = \left\{ \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

In the current 2D setting, it is obvious that $SM \cong \mathrm{SO}(M)$, because a positive ONB $\{\varepsilon_1, \varepsilon_2\}$ of T_xM is uniquely determined by the unit vector e_1 along with the given orientation. Also note that $\mathrm{SO}(2) \cong S^1$ and $\mathfrak{so}(2) \cong \mathbb{R}$ in the obvious way.

The fundamental vector field and the canonical form

There are two canonical objects on SM which are both important to our discussion.

The first one is the fundamental vector field. A tangent vector $V \in T_uS(M)$ is said to be *vertical* if it is tangential to the circle $\pi^{-1}(x)$. The space of vertical vectors at u is denoted as $\mathcal{V}_u$. Obviously, $\dim \mathcal{V}_u = 1$. At each point $u = (x, v)$, there is a distinguished vertical vector defined by

$$V_u \triangleq \frac{d}{dt}\Big|_{t=0}(x, ve^{it}). \tag{1.5}$$

This is the unique unit tangent vector to the circle $\pi^{-1}(x)$ at v whose direction is consistent with the orientation. By varying u, one obtains a vertical vector field

$$V : SM \ni u \mapsto V_u \in \mathcal{V}_u.$$

Due to one-dimensionality, every vertical vector field W can be written as $W = fV$ with some real-valued function f on SM.

Definition 1.6. The vector field V constructed above is called the *fundamental vector field* on SM.

The other one is the canonical form. This is an $\mathbb{R}^2$-valued 1-form $\theta = (\theta^1, \theta^2)$ which is defined in the following way. Let $X \in T_uSM$ where $u = (x, v) \in SM$. Then we set $(\theta^1(X), \theta^2(X))$ to be the coordinates of $(d\pi)_uX \in T_xM$ with respect to the PONB $\{v, iv\}$, i.e.

$$(d\pi)_uX = \theta^1(X) \cdot v + \theta^2(X) \cdot iv.$$

Definition 1.7. The $\mathbb{R}^2$-valued 1-form θ is called the *canonical form* on SM.

Proposition 1.1. *(i)* $\mathcal{V}_u = \ker\theta_u = \{X \in T_uSM : \theta^1(X)_u = \theta^2(X)_u = 0\}$.

(ii) One has

$$R_g^*\begin{pmatrix}\theta^1\\ \theta^2\end{pmatrix} = \begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix}\theta^1\\ \theta^2\end{pmatrix}$$

for all $g = e^{it} \in S^1$.

(iii) The form $\theta^1 \wedge \theta^2$ *descends to the volume form* vol_M *on* M, *namely,* $\theta^1 \wedge \theta^2 = \pi^*\mathrm{vol}_M$.

Proof. In view of Remark 1.3, this is a special case of Proposition 5.2. □

Connections on SM

We now give the precise definition of a connection on SM (cf. Definition 6.8).

Definition 1.8. A *connection* Γ on SM, or a *metric connection* on M, is an assignment of a linear subspace (*horizontal subspace*) $\mathcal{H}_u$ of $T_u(SM)$ at every $u \in SM$ which satisfies the following properties.

(i) $T_u(SM) = \mathcal{V}_u \oplus \mathcal{H}_u$ for all $u \in SM$.

(ii) $\mathcal{H}_{ug} = (dR_g)_u \mathcal{H}_u$ for all $u \in SM$ and $g \in S^1$.

(iii) The assignment $u \mapsto \mathcal{H}_u$ is smooth.

Let Γ be a connection on SM. One can construct a 1-form $\omega \in \Omega^1(SM)$ from Γ in the following way. Let $X \in T_u SM$ and write $X = Y + Z$ where $Y \in \mathcal{V}_u$ and $Z \in \mathcal{H}_u$. There is a unique $\lambda \in \mathbb{R}$ such that $Y = \lambda V_u$, where V is the fundamental vector field defined by (1.5). Set $\omega(X)_u \triangleq \lambda$. This defines a linear functional on $T_u SM$, which yields a 1-form on SM by varying u. One has the following basic theorem.

Theorem 1.3. *The* 1*-form* ω *satisfies the following two properties:*

(i) $\omega(V)_u = 1$ *for all* $u \in SM$*;*

(ii) $R_g^* \omega = \omega$ *for all* $g \in S^1$.

Conversely, given any 1*-form* ω *on* SM *satisfying (i) and (ii), there is an associated connection* Γ *defined by* $\mathcal{H}_u \triangleq \ker \omega_u = \{X \in T_u SM : \omega(X)_u = 0\}$.

Proof. This is a special case of Theorem 6.1 and Proposition 6.5. □

Remark 1.4. On a general principal bundle, Property (ii) should be stated as $R_g^* \omega = \mathrm{Ad}(g^{-1})\omega$ (cf. Theorem 6.1). One does not see the adjoint action here because S^1 is abelian. This is a very special feature of two dimensions.

As we discussed at the beginning, a connection Γ on SM induces a notion of parallel transport of tangent vectors along smooth curves in M. The key result to make this precise is Theorem 1.4 below .

Definition 1.9. A piecewise smooth curve $u_t \in SM$ is said to be *horizontal* if $\dot{u}_t \in \mathcal{H}_{u_t}$ at every differentiable point t.

Theorem 1.4. *Let* $\gamma : [a, b] \to M$ *be a piecewise smooth curve. For each* $u = (\gamma_a, v) \in \pi^{-1}(\gamma_a)$*, there exists a unique horizontal curve* $u : [a, b] \to SM$ *such that* $u_a = u$ *and* $\pi(u_t) = \gamma_t$ *for all* t.

Proof. This is a special case of Theorem 6.2 and Corollary 6.1. □

By varying $u \in \pi^{-1}(\gamma_a)$ in Theorem 1.4, one obtains a map

$$\sigma_\gamma : \pi^{-1}(\gamma_a) \mapsto \pi^{-1}(\gamma_b),\ u \mapsto u_b.$$

This extends to a map

$$\tau_\gamma : T_{\gamma_a}M \to T_{\gamma_b}M,\ v \mapsto \|v\|\sigma_\gamma(v/\|v\|).$$

It will be proved in Proposition 6.19 that τ_γ is a linear isometry (the essential reason is that the connection Γ is assumed to be defined on SM in the first place). In addition, it does not depend on the parametrisation of γ.

Definition 1.10. The map τ_γ is called the *parallel transport* along the curve γ.

The notion of parallel transport induces an intrinsic way of differentiating smooth vector fields on M along arbitrary directions (the covariant derivative). Let $x \in M$ and $v \in T_xM$. Let Y be a smooth vector field on M. We now modify the naive limit (1.4) into the following precise definition:

$$\nabla_v Y \triangleq \lim_{t\to 0} \frac{1}{t}\big(\tau_{0,t}Y_{\gamma_t} - Y_x\big),$$

where $\gamma : (-\varepsilon, \varepsilon) \to M$ is any smooth curve satisfying $\gamma_0 = x$, $\dot{\gamma}_0 = v$ and $\tau_{0,t}$ is the parallel transport from $T_{\gamma_t}M$ to T_xM along γ. It can be shown that the above limit exists and is independent of the choice of γ representing (x, v). For two vector fields $X, Y \in \Gamma(TM)$, we define

$$(\nabla_X Y)(x) \triangleq \nabla_{X_x} Y, \quad x \in M.$$

This gives rise to a *connection* (or a *covariant derivative*)

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM),\ (X, Y) \mapsto \nabla_X Y$$

on the tangent bundle in the sense of Definition 6.12 (see Proposition 6.8). This is a metric connection in the sense that it is compatible with the Riemannian metric:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

for all $X, Y, Z \in \Gamma(TM)$. This is again due to the fact that ∇ comes from a connection on the sphere bundle SM (see Proposition 6.19).

1.3.3 Torsion and curvature forms

In this subsection, we define the torsion and curvature forms of a metric connection and derive their basic structure equations. We will also introduce the key concept of Gaussian curvature which appears in the Gauss-Bonnet formula (1.1). Let Γ be a connection on SM whose connection form is ω.

Definition 1.11. The *torsion form* of Γ is the $\mathbb{R}^2$-valued 2-form Θ on SM defined by

$$\Theta(X,Y) \triangleq d\theta(hX,hY), \quad X,Y \in \Gamma(TSM),$$

The *curvature form* of Γ is the real-valued 2-form Ω on SM defined by

$$\Omega(X,Y) \triangleq d\omega(hX,hY), \quad X,Y \in \Gamma(TSM).$$

Here $h\cdot$ denotes the projection onto the horizontal component.

Remark 1.5. Θ and Ω are the exterior covariant derivatives of θ and ω respectively (cf. Definitions 6.14, 6.17 and 6.22).

By definition, the forms Θ and Ω are horizontal (this means $\Theta(X,Y) = 0$ if at least one of X,Y is vertical and the same for Ω). They satisfy the following *structure equations of Cartan.*

Theorem 1.5. *(i) (The first structure equation)*

$$d\theta^1 = \omega \wedge \theta^2 + \Theta^1, \; d\theta^2 = -\omega \wedge \theta^1 + \Theta^2.$$

(ii) (The second structure equation) $d\omega = \Omega$.

Proof. Through the identification

$$\omega \leftrightarrow \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \; \Omega \leftrightarrow \begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix}$$

and also observing that $\omega \wedge \omega = 0$, the result is a special case of Theorem 6.6. $\square$

According to Theorem 6.5, the family $\{\theta^1, \theta^2, \omega\}$ of 1-forms defines a basis of T_u^*SM at every $u \in SM$. In particular, the family

$$\omega \wedge \theta^1, \omega \wedge \theta^2, \theta^1 \wedge \theta^2 \tag{1.6}$$

provides a basis of $\Lambda^2 T_u^*SM$ at every $u \in SM$. Therefore, any 2-form on SM can be written as a unique linear combination among the elements in (1.6). Since both Θ and Ω are horizontal by definition , they cannot contain an ω-component. In other words, they must have the form

$$\Theta^i = h_i \theta^1 \wedge \theta^2, \; \Omega = -\tilde{K}\theta^1 \wedge \theta^2$$

for some functions $h_i, \tilde{K} \in C^\infty(SM)$ $(i = 1,2)$. The above minus sign is a convention to ensure that $\tilde{K}$ descends to the sectional curvature of ∇ on M (see Lemma 1.1 and Remark 1.6). The structure equations can thus be viewed as the expansion of $d\theta$ and $d\omega$ with respect to the basis (1.6).

Lemma 1.1. *The function* $\tilde{K}$ *descends to* M*. More precisely, there exists* $K \in C^\infty(M)$ *such that* $\tilde{K} = K \circ \pi$*. In addition, one has* $\Omega = -\pi^*(K\mathrm{vol}_M)$*.*

Proof. Since ω is R_g-invariant for all $g \in S^1$, so is $d\omega = \Omega$ (the second structure equation). Since Ω is also horizontal, it follows that Ω descends to a 2-form on M, i.e. $\Omega = \pi^*\tau$ for some $\tau \in \Omega^2(M)$. On the other hand, recall from Proposition 1.1 (iii) that $\theta^1 \wedge \theta^2$ descends to SM (in fact, $\theta^1 \wedge \theta^2 = \pi^*\mathrm{vol}_M$). Therefore, the function $\tilde{K}$ must also descend to a function on M and the relation $\Omega = -\pi^*(K\mathrm{vol}_M)$ follows. □

Definition 1.12. The function K is called the *Gaussian curvature* of the connection Γ.

Remark 1.6. The Gaussian curvature $K(x)$ is precisely the sectional curvature of ∇ with respect to the plane T_xM; one can show that

$$K(x) = \langle R(\varepsilon_1, \varepsilon_2)\varepsilon_2, \varepsilon_1\rangle_{T_xM}$$

where $R(\cdot,\cdot)\cdot$ is the curvature tensor of ∇ defined by (6.46) and $\{\varepsilon_1, \varepsilon_2\}$ is any ONB of T_xM.

Among others, there is a special metric connection which has a vanishing torsion. This simplifies the first structure equation and is particularly useful in Riemannian geometry.

Theorem 1.6 (Fundamental Theorem of Riemannian Geometry)**.** *There exists a unique metric connection whose torsion form is identically zero.*

Proof. This is a special case of Theorem 6.9. □

Definition 1.13. The metric connection given by the above theorem is called the *Levi-Civita connection* on M.

Under the Levi-Civita connection, the structure equations are expressed as

$$d\theta^1 = \omega \wedge \theta^2,\ d\theta^2 = -\omega \wedge \theta^1,\ d\omega = \Omega.$$

Throughout the rest, we will always work with the Levi-Civita connection.

1.3.4 Geometric interpretation of Gaussian curvature

In this subsection, we give a geometric interpretation of the Gaussian curvature. Essentially, the Gaussian curvature measures the amount of rotation obtained by parallel transporting a vector along a closed curve which bounds a topological disk. Here we formulate a 2-simplex version which is directly applicable to our later proof of the Gauss-Bonnet theorem (cf. Remark 1.7 below for the more general case). Let Γ be the Levi-Civita connection on SM whose connection form is ω.

Let $[s] = [v_0, v_1, v_2]$ be a closed 2-simplex. Let $h : [s] \to M$ be a smooth embedding which admits a smooth extension to a neighbourhood of $[s]$. Suppose that $[s]$ is oriented in the way that h is orientation-preserving on the interior of $[s]$. We label the three vertices v_0, v_1, v_2 so that the induced boundary orientation on $\partial[s]$ is $v_0 \to v_1 \to v_2$. Define the curve $\gamma : \partial[s] \to M$ by restricting h to the boundary $\partial[s] \triangleq v_0 \to v_1 \to v_2 \to v_0$.

Theorem 1.7. *Let $w \in T_{h(v_0)}M$ be a unit vector. Then one has*

$$\tau_\gamma(w) = w \exp\Big(i \int_{[s]} h^*(K \mathrm{vol}_M)\Big),$$

where $\tau_\gamma : T_{h(v_0)}M \to T_{h(v_0)}M$ is the parallel transport along γ. In other words, the parallel transport τ_γ is the counterclockwise rotation by an angle of $\int_{[s]} h^*(K\mathrm{vol}_M)$ with respect to the orientation of $T_{h(v_0)}M$.

We first state a lemma which will be useful for the proof of Theorem 1.7.

Lemma 1.2. *Let $\alpha : [a, b] \to M$ be a smooth curve. Suppose that $\tilde{\alpha}, \tilde{\beta} : [a, b] \to SM$ are smooth curves which satisfy $\pi(\tilde{\alpha}) = \pi(\tilde{\beta}) = \alpha$ and $\tilde{\alpha}_a = \tilde{\beta}_a$. Then there exists a unique smooth function $\theta : [a, b] \to \mathbb{R}$ such that $\theta_a = 0$ and $\tilde{\beta}_t = \tilde{\alpha}_t e^{i\theta_t}$ for all $t \in [a, b]$. Suppose further that $\tilde{\alpha}$ is horizontal. Then one has $\omega(\dot{\tilde{\beta}}_t) = \dot{\theta}_t$ for all $t \in [a, b]$.*

Proof. Write $\tilde{\beta}_t = \tilde{\alpha}_t g_t$ for $g_t \in S^1$. Then $g : [a, b] \to S^1$ is a smooth function with $g_a = 1$. Since $\mathbb{R}$ is simply-connected, the function g admits a unique smooth lift $\theta : [a, b] \to \mathbb{R}$ satisfying $\theta_a = 0$ and $g_t = e^{i\theta_t}$. The first claim thus follows.

For the second claim, let us first differentiate the relation $\tilde{\beta}_t = \tilde{\alpha}_t g_t$ to obtain that

$$\dot{\tilde{\beta}}_t = \dot{\tilde{\alpha}}_t g_t + \tilde{\beta}_t g_t^{-1} dg_t. \tag{1.7}$$

Now suppose that $\tilde{\alpha}$ is horizontal. Then the first term of (1.7) is also horizontal and is thus annihilated by ω. In addition, by using $g_t = e^{i\theta_t}$ it is easily seen that the second term of (1.7) is just the vertical vector $\dot{\theta}_t V_{\tilde{\beta}_t}$. Therefore, one has

$$\omega\big(\dot{\tilde{\beta}}_t\big) = \omega\big(\dot{\tilde{\alpha}}_t g_t\big) + \omega\big(\tilde{\beta}_t g_t^{-1} dg_t\big) = 0 + \dot{\theta}_t = \dot{\theta}_t,$$

which proves the second claim. □

Proof of Theorem 1.7. Fix $w \in T_{h(v_0)}M$. Let $\tilde{\gamma} : \partial[s] \to SM$ be the horizontal lift of γ starting at $(h(v_0), w)$. Denote the endpoint of $\tilde{\gamma}$ by $(h(v_0), w')$. Our goal is to show that

$$w' = w \exp\Big(i \int_{[s]} h^*(K \mathrm{vol}_M)\Big). \tag{1.8}$$

The main idea is to construct a lift $\tilde{h} : [s] \to SM$ of the map h (i.e. $\pi \circ \tilde{h} = h$) in a way that

$$\tilde{h}|_{[v_0,v_1]\cup[v_1,v_2]} = \tilde{\gamma}|_{[v_0,v_1]\cup[v_1,v_2]}, \tag{1.9}$$

where $[v_i, v_{i+1}]$ denotes the edge of $[s]$ connecting v_i to v_{i+1}.

Assume such a lift $\tilde{h}$ exists. Then one can write

$$\begin{aligned}\int_{[s]} h^*(K\mathrm{vol}_M) &= \int_{[s]} \tilde{h}^* \circ \pi^*(K\mathrm{vol}_M) = -\int_{[s]} \tilde{h}^* d\omega \\ &= -\int_{[s]} d\tilde{h}^*\omega = -\int_{\partial[s]} \tilde{h}^*\omega.\end{aligned}$$

The second equality follows from the second structure equation and Lemma 1.1. The last equality follows from Stokes' theorem. According to (1.9) and the construction of $\tilde{\gamma}$, one knows that $\tilde{h}|_{[v_0,v_1]\cup[v_1,v_2]}$ is horizontal. In particular, $\omega(\dot{\tilde{h}}_t) = 0$ for all $t \in [v_0, v_1] \cup [v_1, v_2]$. It follows that

$$\int_{[s]} h^*(K\mathrm{vol}_M) = -\int_{[v_2,v_0]} \omega\big(\dot{\tilde{\beta}}_t\big), \tag{1.10}$$

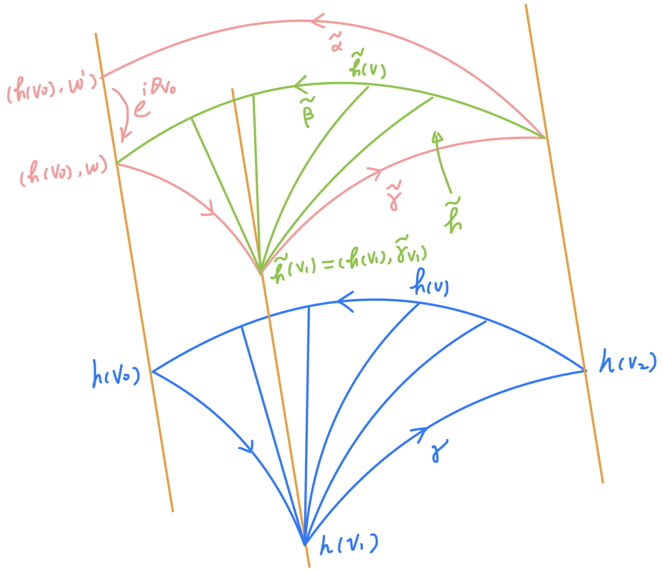
where $\tilde{\beta} \triangleq \tilde{h}|_{[v_2,v_0]}$. Let us set $\tilde{\alpha} \triangleq \tilde{\gamma}|_{[v_2,v_0]}$. Since both $\tilde{\alpha}, \tilde{\beta}$ are lifts of $\alpha \triangleq \gamma|_{[v_2,v_0]}$ with the same initial point and $\tilde{\alpha}$ is horizontal, one knows from Lemma 1.2 that $\tilde{\beta}_t = \tilde{\alpha}_t e^{i\theta_t}$ $(t \in [v_2, v_0])$ where θ_t satisfies $\theta_{v_2} = 0$ and $\omega(\tilde{\beta}_t) = \dot{\theta}_t$ for all $t \in [v_2, v_0]$. By substituting this into (1.10), one obtains that

$$\int_{[s]} h^*(K\mathrm{vol}_M) = -\int_{[v_2,v_0]} \dot{\theta}_t = -\theta_{v_0}.$$

Therefore,

$$\tilde{\alpha}_{v_0} = \tilde{\beta}_{v_0} e^{-i\theta_{v_0}} = \tilde{\beta}_{v_0} \exp\Big(i \int_{[s]} h^*(K\mathrm{vol}_M)\Big).$$

But $\tilde{\beta}_{v_0} = (h(v_0), w)$ and $\tilde{\alpha}_{v_0} = (h(v_0), w')$ by construction. This gives the desired relation (1.8).



Now it remains to construct the map $\tilde{h}$ properly. For each $v \in [v_0, v_2]$, we define $\tilde{h}|_{[v_1,v]}$ to be the horizontal lift of $h|_{[v_1,v]}$ starting at $(h(v_1), \tilde{\gamma}_{v_1})$, where $[v_1, v]$ denotes the segment joining v_1 to v. By varying v on the edge $[v_0, v_2]$, one obtains a map $\tilde{h} : [s] \to SM$ which lifts h. On the edge $[v_1, v_0]$, since the starting point of $\tilde{h}$ is the endpoint of $\tilde{\gamma}|_{[v_0,v_1]}$, it follows from the uniqueness of horizontal lift that $\tilde{h}|_{[v_1,v_0]}$ equals the reversal of $\tilde{\gamma}|_{[v_0,v_1]}$. Namely, $\tilde{h}|_{[v_0,v_1]} = \tilde{\gamma}|_{[v_0,v_1]}$. For the same reason, one also has $\tilde{h}|_{[v_1,v_2]} = \tilde{\gamma}|_{[v_1,v_2]}$. This shows that $\tilde{h}$ satisfies the required property (1.9). □

Remark 1.7. One can also prove (in fact, a more general version of) Theorem 1.7 from the covariant derivative perspective. Let ∇ be the Levi-Civita connection on TM. Let $\gamma : [0, 1] \to M$ be a piecewise smooth, simple closed curve in M with $\gamma_0 = \gamma_1 = x$. Suppose that γ bounds a domain D whose closure is contained in a topological disk. We assume that the orientation of γ is consistent with the orientation on D induced from M. This ensures that Stokes' theorem takes the following form:

$$\int_\gamma \eta = \int_D d\eta$$

for any $\eta \in \Omega^1(\bar{D})$. Let $w \in T_xM$ and let w' be the parallel transport of w along γ. We claim that

$$w' = w \exp\Big(i \int_D K \mathrm{vol}_M\Big). \tag{1.11}$$

To prove this, let $\{e_1, e_2\}$ be a local PONF on a neighbourhood U of $\bar{D}$. Such a frame field exists because $\bar{D}$ is contained in a topological disk; for instance, one can pick a PONB of the tangent space at the center of the disk and parallel transport it along radii. Since ∇ is compatible with the metric, its local connection form on U with respect to $\{e_1, e_2\}$ takes values in $\mathfrak{so}(2)$ (see Appendix B.9). As a result, there exists a 1-form $\eta \in \Omega^1(U)$ such that

$$\nabla e_1 = -\eta \otimes e_2, \ \nabla e_2 = \eta \otimes e_1. \tag{1.12}$$

The relation $\Omega = -\pi^*(K\mathrm{vol}_M)$ descends to the relation

$$d\eta = -K\mathrm{vol}_M \tag{1.13}$$

on U. Let $\{W_t\}_{0\leqslant t\leqslant 1}$ be the parallel transport of w along γ. One can write

$$W_t = (\cos\varphi_t)e_1(x_t) + (\sin\varphi_t)e_2(x_t),$$

where $(\varphi_t)_{0\leqslant t\leqslant 1}$ is a uniquely defined continuous function once the initial value φ_0 is fixed (the angle from $e_1(x)$ to w). By using the relation (1.12), it is easily seen that the parallel property

$$\nabla_{\dot{x}_t} W_t = 0 \iff \varphi_t' = -\eta(\dot{x}_t).$$

It follows from (1.13) and Stokes' theorem that

$$\varphi_1 - \varphi_0 = -\int_\gamma \eta = -\int_D d\eta = \int_D K\mathrm{vol}_M.$$

This gives the relation (1.11), noting that by definition $\varphi_1 - \varphi_0$ is the angle of rotation obtained from the parallel transport along γ. An immediate consequence of (1.11) is the following interpretation of the Gaussian curvature:

$$K(x) = \lim_{D\to 0} \frac{\text{“Rotation angle by parallel transport along } \partial D\text{”}}{\mathrm{vol}_M(D)},$$

where the limit $D \to 0$ is taken by contracting γ to the trivial curve fixing the base point x.

1.3.5 The Gauss-Bonnet theorem for 2-simplices

In this subsection, we establish the Gauss-Bonnet theorem for 2-simplices. The formula involves the geodesic curvature of a curve which we shall first define. As before, we work with the Levi-Civita connection on M.

Definition 1.14. Let $\gamma : I \to M$ be a smooth curve which is parametrised at unit speed (i.e. $\|\dot{\gamma}_t\| = 1$ for all t), where I a subinterval of $\mathbb{R}$. We say that γ is a *geodesic* if the lifted curve $\tilde{\gamma}_t \triangleq (\gamma_t, \dot{\gamma}_t) \in SM$ is horizontal.

Remark 1.8. A curve γ is a geodesic if and only if its unit tangent vector field is parallel along γ, i.e. $\nabla_{\dot{\gamma}_t}\dot{\gamma}_t = 0$ for all t.

Definition 1.15. Let $\gamma : I \to M$ be a smooth curve which is parametrised at unit speed. The *geodesic curvature* of γ at time t is defined by $k_g(t) \triangleq \omega(\dot{\tilde{\gamma}}_t)$, where $\tilde{\gamma}_t \triangleq (\gamma_t, \dot{\gamma}_t)$ as before. Its *total geodesic curvature* is

$$\tau_g(\gamma) \triangleq \int_I k_g(t)dt.$$

Intuitively, the total geodesic curvature measures the extent to which γ fails to be a geodesic. Indeed, if γ is a geodesic, one has $k_g(t) \equiv 0$. It is also clear that the definition of $\tau_g(\gamma)$ extends to the situation where γ is piecewise smooth.

Let us make the same assumptions on the 2-simplex $[s] = [v_0, v_1, v_2]$ and the map h as in Theorem 1.7. The Gauss-Bonnet theorem for 2-simplices further expresses the rotation angle $\int_{[s]} h^*(K\mathrm{vol}_M)$ of the parallel transport in terms of the total geodesic curvature of the boundary curve and the interior angles of $h[s]$.

Theorem 1.8. *One has*

$$\int_{[s]} h^*(K\mathrm{vol}_M) = -\tau_g(\gamma) + \sum(\textit{interior angles of } h[s]) - \pi,$$

where γ is the boundary curve of $h[s]$ with unit speed parametrisation.

Proof. We first make a useful observation. Suppose $\alpha : [a, b] \to M$ is a smooth curve at unit speed. Let $\tilde{\alpha}_t \triangleq (\alpha_t, \dot{\alpha}_t)$ be its canonical lift and let $\bar{\alpha}_t$ be its horizontal lift starting at $(\alpha_0, \dot{\alpha}_0)$. According to Lemma 1.2, one can write $\tilde{\alpha}_t = \bar{\alpha}_t e^{i\theta_t}$, where $\theta_a = 0$ and $\omega(\dot{\tilde{\alpha}}_t) = \dot{\theta}_t$ for all t. It follows that

$$\tau_g(\alpha) = \int_a^b \omega(\dot{\tilde{\alpha}}_t)dt = \theta_b - \theta_a = \theta_b.$$

In other words, the angle from $\tau_\alpha(\dot{\alpha}_a)$ to $\dot{\alpha}_b$ is precisely the total geodesic curvature of α. Here $\tau_\alpha(\dot{\alpha}_a)$ is the parallel transport of $\dot{\alpha}_a$ to $T_{\alpha_b}M$ along α.

Under the above observation, the proof of the theorem becomes very easy. Let $\gamma^{(j)} : [a_j, a_{j+1}] \to M$ be the curve $h|_{[v_j, v_{j+1}]}$ ($j = 0, 1, 2$ with $v_3 \triangleq v_0$) which is reparametrised at unit speed. Let w' be the tangent vector obtained by parallel transporting the initial vector $\dot{\gamma}_0$ along the entire boundary curve γ. Recall from Theorem 1.7 that

$$w' = \dot{\gamma}_0 \exp\Big(i \int_{[s]} h^*(K\mathrm{vol}_M)\Big).$$

To reset w' back to the initial $\dot{\gamma}_0$, one needs to apply a rotation of angle

$$\tau_g(\gamma) + \sum(\text{exterior angles of } \gamma). \tag{1.14}$$

In fact, let w_1 be the vector obtained by parallel transporting $\dot{\gamma}_0$ along $\gamma^{(0)}$. According to the previous fact, the angle from w_1 to $\dot{\gamma}^{(1)}_{a_1+}$ is equal to

$$\tau_g(\gamma^{(0)}) + \text{“exterior angle at } \gamma_{a_1}\text{”}.$$

By rotating w_1 by this angle at γ_{a_1}, it is ensured that the starting vector for the next stage of parallel transport (i.e. along $\gamma^{(1)}$) is $\dot{\gamma}^{(1)}_{a_1+}$. Arguing in the same manner, when one travels along the entire curve γ and attempts to reset the final vector w' back to $\dot{\gamma}_0$, the angle (1.14) is precisely the additional rotation required. To summarise, one has obtained that

$$\int_{[s]} h^*(K\mathrm{vol}_M) + \tau_g(\gamma) + \sum(\text{exterior angles of } \gamma) = 2k\pi \tag{1.15}$$

for some $k \in \mathbb{Z}$.

We claim that $k = 1$. This is most easily seen by using a continuity argument. Suppose for now g_0 is an Euclidean metric on $h[s]$, which is induced from $[s] \subseteq \mathbb{R}^2$ through the map h. Then both K and $\tau_g(\gamma)$ are zero in this case. The relation (1.15) reduces to the obvious Euclidean trigonometric identity

$$\sum(\text{exterior angles of a triangle}) = 2\pi.$$

Now let us consider the family of metrics $g_\rho \triangleq (1-\rho)g_0 + \rho g$,where g is the original metric on M. It is not hard to show that the quantities

$$K_\rho, \tau_{g_\rho}(\gamma), (\text{exterior angles of } \gamma)_\rho,$$

all being defined with respect to g_ρ, depend continuously on the parameter ρ. Therefore, the integer k_ρ appearing in (1.15) (now also depending on ρ) must also be a continuous function of ρ. As a consequence, $k_\rho \equiv 1$. This proves the relation (1.15).

The result of the theorem now follows from the simple fact that the sum of an interior angle and the corresponding exterior angle is equal to π. □

1.3.6 The Gauss-Bonnet theorem for surfaces

Proof of Theorem 1.1. Fix a triangulation (K, h) of M. Let n_i be the number of open i-simplices in K $(i = 0, 1, 2)$. According to Theorem 1.2, one has

$$\chi(M) = n_0 - n_1 + n_2. \tag{1.16}$$

The above relation can be further simplified with the following observation: every 2-simplex has three edges and every 1-simplex is a face of precisely two distinct 2-simplices. This leads to the relation that $3n_2 = 2n_1$, thus yielding

$$\chi(M) = n_0 - \frac{3n_2}{2} + n_2 = n_0 - \frac{n_2}{2} \tag{1.17}$$

from the formula (1.16).

On the other hand, one has from Theorem 1.8 that

$$\int_M K\mathrm{vol}_M = \sum_{\text{2-simplices }[s]} \Big(-\tau_g(\partial[s]) + \sum(\text{interior angles of } h[s]) - \pi\Big).$$

For the total geodesic curvature term, one has

$$\sum_{\text{2-simplices}} \tau_g(\partial[s]) = 0.$$

This is because every 1-simplex, as a face of a 2-simplex, appears exactly twice but with opposite orientations. In addition, the sum of interior angles at each 0-simplex is always 2π. Therefore, one concludes along with (1.17) that

$$\begin{aligned}\int_M K\mathrm{vol}_M &= \sum_{\text{2-simplices }[s]} \Big(\sum(\text{interior angles of } h[s]) - \pi\Big)\\ &= 2\pi n_0 - n_2\pi = 2\pi\chi(M).\end{aligned}$$

This proves the Gauss-Bonnet formula for surfaces.

□

Example 1.2. For a closed, oriented surface M_g with genus $g \geqslant 0$ (a 2-sphere with g handles attached), under any Riemannian metric one has

$$\frac{1}{2\pi}\int_{M_g} K\mathrm{vol} = 2 - 2g,$$

because the RHS is known to be the Euler characteristic of M_g.

Remark 1.9. The Gauss-Bonnet formula (1.1) holds for any metric connection. The essential reason is that the 2-forms $K_1\mathrm{vol}_M$ and $K_2\mathrm{vol}_M$ arising from two metric connections differ by an exact form. Indeed, let ω_1, ω_2 be two metric connections on SM. Then $\omega_1 - \omega_2$ is horizontal and R_g-invariant for all $g \in S^1$. As a result, $\omega_1 - \omega_2 = \pi^*\alpha$ for some $\alpha \in \Omega^1(M)$. For the corresponding curvature forms, one has

$$\pi^*(K_2\mathrm{vol}_M - K_1\mathrm{vol}_M) = \Omega_1 - \Omega_2 = d\omega_1 - d\omega_2 = d\pi^*\alpha = \pi^* d\alpha,$$

which implies that $K_2\mathrm{vol}_M - K_1\mathrm{vol}_M = d\alpha$. It follows from Stokes' theorem that

$$\int_M \big(K_2\mathrm{vol}_M - K_1\mathrm{vol}_M\big) = \int_M d\alpha = 0.$$

In other words, the cohomology class of $K\mathrm{vol}_M$ (and thus the integral $\int_M K\mathrm{vol}_M$) does not depend on the choice of the metric connection. This fact will be studied in a more fundamental way (and in a much more general context) from the perspective of Chern-Weil theory in Chapter 7.

We conclude this introductory chapter with a nice topological application of the Gauss-Bonnet theorem. It suggests that the Euler characteristic is the obstruction to an everywhere nonvanishing (smooth) vector field on a closed, oriented surface.

Corollary 1.1. *Let M be a closed, oriented surface. Suppose that M admits an everywhere nonvanishing smooth vector field. Then $\chi(M)=0$.*

Proof. Suppose that such a vector field V exists. Fix any Riemannian metric on M. One can then define a smooth map $\bar{V}:M\rightarrow SM$ by setting $\bar{V}(x)\triangleq(x,V_x/\|V_x\|)$. By applying $\bar{V}^*$ to the relation $d\omega=-\pi^*(K\mathrm{vol}_M)$, one finds that

$$d(\bar{V}^*\omega)=-(\pi\circ\bar{V})^*(K\mathrm{vol}_M)=-K\mathrm{vol}_M.$$

In particular, $K\mathrm{vol}_M$ is an exact form, hence having zero integral over M. It follows from the Gauss-Bonnet formula that $\chi(M)=0$. □

Example 1.3. The 2-sphere has Euler characteristic two. As a result, it does not admit any smooth nonvanishing vector field. This is the famous *Hairy Ball Theorem* (which is a nice mathematical demonstration that life can never be perfect). The same is true for any closed, oriented surface with genus $g\geqslant 2$.

2 Intersection theory

Chern's original proof of the CGB theorem relies on a classical result in topology: the *Poincaré-Hopf index theorem* for vector fields. This is stated as follows.

Theorem 2.1 (The Poincaré-Hopf Theorem)**.** *Let M be a closed, oriented manifold and let V be a smooth vector field on M with only isolated zeros. Then the sum of indices over all zeros of V is equal to the Euler characteristic of M:*

$$\sum_{x\in\mathcal{Z}(V)} \mathrm{ind}_x(V) = \chi(M), \tag{2.1}$$

where $\mathcal{Z}(V) \triangleq \{x \in M : V(x) = 0\}$ denotes the zero set of V.

The number $\mathrm{ind}_x(V)$, which is known as the *index* of V at x, will be defined in Section 2.7. The sum in (2.1) is well-defined since $\mathcal{Z}(V)$ is a finite set as a consequence of compactness. Our approach to proving (2.1) is based on *intersection theory* and the *Lefschetz fixed point formula*. Let $\Delta \triangleq \{(x, x) : x \in M\}$ denote the diagonal of the product space $M \times M$. Let $I(\Delta, \Delta)$ be the *self-intersection number* of Δ inside $M \times M$, which will be defined in Definition 2.5. The proof of (2.1) contains two major steps.

Step 1. $\sum \mathrm{ind}_x(V) = I(\Delta, \Delta)$.

Step 2. $I(\Delta, \Delta) = \chi(M)$.

The proof of Step 1 has a geometric nature, which uses ideas and tools from intersection theory. This will be developed in the current chapter. On the other hand, the proof of Step 2 is quite topological (and it has nothing to do with the vector field V). As we will see, it is a direct consequence of the more general *Lefschetz fixed point formula*, whose proof relies on several fundamental tools from algebraic topology such as Poincaré duality, Thom isomorphism and intersection product formula. We will develop this step in the next chapter.

Our approach to the Poincaré-Hopf theorem is not the shortest one. We choose this longer path so that the reader will have the opportunity to encounter several landmark results in classical algebraic and differential topology, most of which being of fundamental importance on their own.

2.1 A heuristic interpretation of the Poincaré-Hopf theorem

We use a special situation to illustrate why the formula (2.1) is expected to hold. Let M be a closed surface and assume that it is triangulated as a simplicial complex of dimension two (see Definition 1.4). Consider a vector field V which is constructed in the following way (cf. Remark 2.6). Let x_s denote the center of each simplex s

in the given triangulation (a face, an edge or a vertex). The vector field V is set to vanish at all the x_s's. If s is an edge, the vector field V flows from x_s into the two vertices of s along the edge in the obvious way. If s is a face, the flow lines of V within s all start from x_s and end at one of the vertices of s, except for the three special ones which end at the center of the edges.

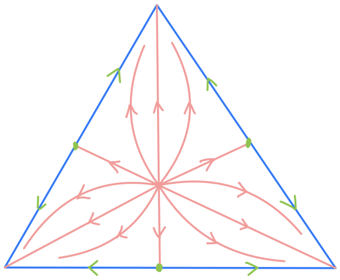


Figure 2.1: A vector field with isolated zeros.

Apparently,

$$x_s \text{ is a } \begin{cases} \text{source}, & \text{if } s \text{ is a face;} \\ \text{saddle point}, & \text{if } s \text{ is an edge;} \\ \text{sink}, & \text{if } s \text{ is a vertex.} \end{cases}$$

It will be apparent from Definition 2.11 of the index $\operatorname{ind}_{x_s}(V)$ that

$$\operatorname{ind}_{x_s}(V) = \begin{cases} 1, & \text{if } s \text{ is a sink or a source;} \\ -1, & \text{if } s \text{ is a saddle point.} \end{cases}$$

As a consequence, one has

$$\begin{aligned} \sum_s \operatorname{ind}_{x_s}(V) &= \#\text{ of sinks} + \#\text{ of sources} - \#\text{ of saddle points} \\ &= \#\text{ of vertices} + \#\text{ of faces} - \#\text{ of edges} \\ &= \chi(M). \end{aligned}$$

This is the Poincaré-Hopf formula (2.1).

Remark 2.1. An immediate application of the Poincaré-Hopf formula is again the Hairy Ball Theorem, which asserts that the two-sphere S^2 does not admit nonvanishing smooth vector fields (cf. Corollary 1.1 and Example 1.3 for a geometric proof based on the Gauss-Bonnet formula). Indeed, if such a vector field V existed, the LHS of (2.1) would just be zero since there were no zeros of V. However, one knows that $\chi(S^2) = 2$. This leads to a contradiction to the formula (2.1).

2.2 Transversal intersection

Let X, Y, Z be oriented manifolds without boundary. We assume that $f : X \to Y$ is a given smooth map and Z is a closed, embedded submanifold of Y.

Definition 2.1. We say that f is *transversal to* Z, denoted as $f \pitchfork Z$, if

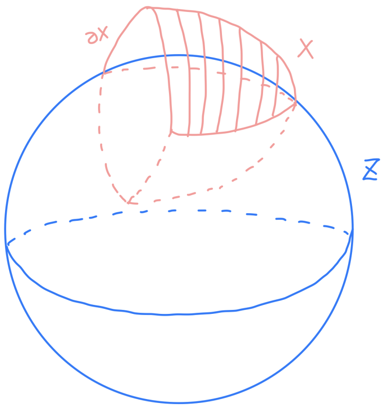
$$\operatorname{Im}(df)_x + T_{f(x)}Z = T_{f(x)}Y \tag{2.2}$$

for any $x \in f^{-1}Z$.

Remark 2.2. Sometimes it is needed to allow X to have boundary. In this case, in addition to the condition (2.2) for interior points one also requires that

$$(df)_x(T_x\partial X) + T_{f(x)}Z = T_{f(x)}Y$$

for boundary points $x \in \partial X \cap f^{-1}Z$. Unless otherwise stated, we always assume that X has no boundary.

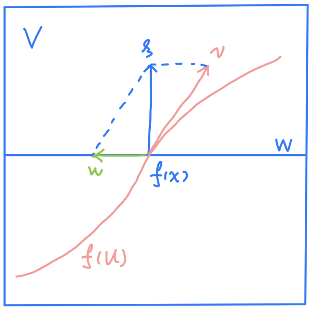


Proposition 2.1. *Suppose that* $f \pitchfork Z$. *Then* $f^{-1}Z$ *is a submanifold of* X *with dimension* $\dim X + \dim Z - \dim Y$.

Proof. We work locally. Let the dimensions of X, Y, Z be k, n, m respectively. Let $V \subseteq \mathbb{R}^n$ be a cube centered at the origin which represents a local parametrisation of Y, such that the zero slice

$$W \triangleq \{(y^1, \cdots, y^n) \in V : y^{m+1} = \cdots = y^n = 0\}$$

defines a local parametrisation of Z. Let $U \triangleq f^{-1}V$ and $G \triangleq \pi \circ f$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-m}$ denotes the projection onto the vertical component.



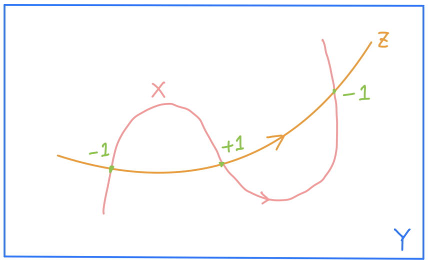
We want to show that $f^{-1}W = G^{-1}(\{0\})$ is a $(k+m-n)$-dimensional submanifold of U. To this end, it suffices to show that $(dG)_x$ is surjective for all $x \in f^{-1}W$. Indeed, let $\xi \in T_0\mathbb{R}^{n-m} \cong T_{f(x)}\mathbb{R}^{n-m}$. By the transversality property, one can write $\xi = v + w$ where $v = (df)_x(u)$ for some $u \in T_xX$ and $w \in T_{f(x)}W$. It follows that $\xi = (dG)_x(u)$ and thus $(dG)_x$ is surjective. □

Of particular interest is the situation where X is compact and $\dim X + \dim Z = \dim Y$, which will always be assumed in the sequel unless otherwise stated. Suppose that $f \pitchfork Z$. In this case, $(df)_x$ is necessarily injective for every $x \in f^{-1}Z$. In addition, it follows from Proposition 2.1 that $f^{-1}Z$ is a 0-dimensional submanifold, hence a finite collection of points in X.

Definition 2.2. Let V, W be real and oriented finite dimensional vector spaces. The *direct sum orientation* of $V \oplus W$ is defined by claiming that $\{v_1, \cdots, v_n, w_1, \cdots, w_m\}$ is a positively oriented basis if $\{v_1, \cdots, v_n\}$ is a positive basis of V and $\{w_1, \cdots, w_m\}$ is a positive basis of W.

Definition 2.3. Suppose that $f \pitchfork Z$. For each $x \in f^{-1}Z$, the *orientation number* of f with respect to Z at x is defined to be 1 if the direct sum orientation of $\mathrm{Im}(df)_x \oplus T_{f(x)}Z$ coincides with the orientation of $T_{f(x)}Y$. It is defined to be -1 otherwise. Here the orientation of the subspace $\mathrm{Im}(df)_x$ is the one induced from T_xX. The *intersection number* of f with respect to Z is the integer defined by

$$I(f, Z) \triangleq \sum_{x \in f^{-1}Z} i_x(f).$$



2.3 Intersection number of a smooth map

It is useful to have a notion of intersection number $I(f, Z)$ where f is only assumed to be a smooth map. The basic idea is to "wiggle" f into a map g which intersects Z transversally and use the intersection number of g to define $I(f, Z)$. This definition turns out to be independent of the choice of g due to the homotopy invariance of the intersection number. To make this idea precise, we need to recall some general results from intersection theory. The reader is referred to [GP74] for the detailed proofs.

Let X, Y, Z be oriented manifolds where X is compact and Z is a closed, embedded submanifold of Y. Only X is allowed to have boundary. The follow result asserts that one can always "wiggle" a smooth map to intersect a manifold transversally.

Theorem 2.2. *(i) (Transversal Homotopy Theorem) Let $f : X \to Y$ be a smooth map. Then there exists $g : X \to Y$ such that $g \simeq f$ and $g \pitchfork Z$.*

(ii) (Homotopy Extension Theorem) Let $f : X \to Y$ be a smooth map. Suppose that $f \pitchfork Z$ on C and $\partial f \pitchfork Z$ on $\partial X \cap C$ for some closed subset $C \subseteq X$. Then there exists a smooth map $g : X \to Y$ homotopic to f, such that $g \pitchfork Z$, $\partial g \pitchfork Z$ and $g = f$ on a neighbourhood of C.

From now on, we assume that $\dim X + \dim Z = \dim Y$. The next result is the key to the homotopy invariance of the intersection number.

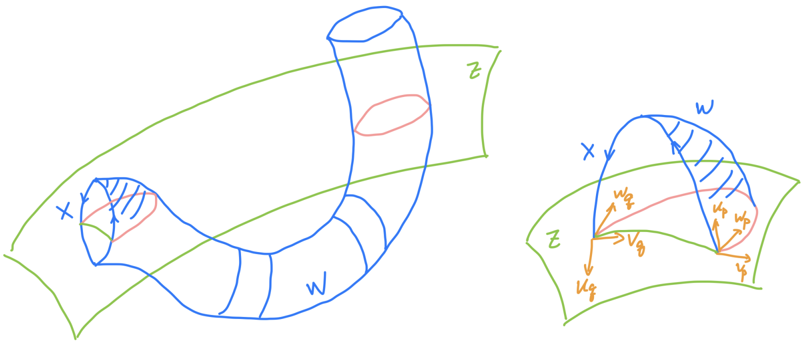
Proposition 2.2. *Suppose that $X = \partial W$ for some oriented manifold W whose induced boundary orientation is consistent with the orientation of X. Let $f : X \to Y$ be a smooth map such that $f \pitchfork Z$. Suppose that f admits a smooth extension $F : W \to Y$. Then $I(f, Z) = 0$.*

Sketche of proof. By the Homotopy Extension Theorem (see Theorem 2.2 (ii)), one may assume WLOG that $F \pitchfork Z$. According to Proposition 2.1, $F^{-1}Z$ is a compact, one-dimensional submanifold of W (possibly disconnected and with boundary). As a consequence, it is a finite disjoint union of objects of the the following two kinds:

(a) a closed loop contained in the interior of W;

(b) an arc with two distinct endpoints on the boundary X.

Type-(a) loops will not contribute to the intersection number $I(f, Z)$. For an arc γ from Type-(b), let p, q be its endpoints. The main observation is that $i_p(f)$ and $i_q(f)$ have opposite signs. The precise proof of this fact requires homological methods which will not be discussed here. Nonetheless, the intuition is easily seen from the figure below.



Here $\dim X = 1$, $\dim Z = 2$ and $Y = \mathbb{R}^3$. The orientation of $\mathrm{Im}(df)_p \oplus T_{f(p)}Z$ is determined by the basis $\{u_p, v_p, w_p\}$ while the corresponding orientation at q is determined by $\{u_q, v_q, w_q\}$. It is clear that

$$\text{orientation}\{u_p, v_p, w_p\} = -\text{orientation}\{u_q, v_q, w_q\},$$

because at point q the orientation of $\mathrm{Im}(df)$ gets flipped while the Z-orientations $\{v_p, w_p\}$ and $\{v_q, w_q\}$ are consistent. It follows that the intersection numbers of f arising from Type-(b) arcs are all cancelled in pairs. One therefore concludes that $I(f, Z) = 0$.

$\square$

Corollary 2.1 (Homotopy Invariance of Intersection Number)**.** *Suppose that* $f_i \pitchfork Z$ $(i = 0, 1)$ *and* $f_0 \simeq f_1$. *Then* $I(f_0, Z) = I(f_1, Z)$.

Proof. Let $W \triangleq X \times [0, 1]$ and we orient W in the way that the induced orientation on the bottom boundary $X \times \{0\}$ is the given X-orientation. Note that the top boundary $X \times \{1\}$ has an opposite orientation. Let $F : W \to Y$ be a smooth homotopy between f_0 and f_1. It follows from Proposition 2.2 that

$$I(f_0, Z) - I(f_1, Z) = I(F|_{\partial W}, Z) = 0.$$

Therefore, the intersection number is homotopy invariant. $\square$

Now we can define the intersection number for a general smooth map.

Definition 2.4. Let $f : X \to Y$ be a smooth map. The *intersection number* of f with respect to Z is defined to be $I(f, Z) \triangleq I(g, Z)$ where $g : X \to Y$ is any smooth map such that $g \simeq f$ and $g \pitchfork Z$. If X is itself a closed, embedded submanifold of Y, we set $I(X, Z) \triangleq I(\iota, Z)$ where $\iota : X \to Y$ is the inclusion.

As a special example, we give the definition of the number $I(\Delta, \Delta)$ which appears in the first major step for the Poincaré-Hopf theorem.

Definition 2.5. Let M be a closed and oriented manifold. Let $\Delta \triangleq \{(x,x) : x \in M\}$ denote the diagonal of the product space $M \times M$. We orient both Δ and $M \times M$ in the obvious way in accordance with the orientation of M. The number $I(\Delta, \Delta)$, which is defined by taking $X = \Delta = Z$ and $Y = M \times M$ in Definition 2.4, is called the *self-intersection number of the diagonal inside* $M \times M$.

2.4 Degree of a smooth map

An important example of intersection number is the degree of a smooth map. Let X, Y be closed, oriented manifolds of the same dimension and assume that Y is connected. Let $f : X \to Y$ be a given smooth map.

Definition 2.6. The *degree* of f is the integer defined by $I(f, \{y\})$ where y is any given point in Y. It is denoted as $\deg f$.

Lemma 2.1. *The number* $\deg f$ *is well-defined, i.e. it is independent of the choice of* $y \in Y$.

Proof. Let $y \in Y$ be given fixed. According to the Transversal Homotopy Theorem (Theorem 2.2 (i)), one can assume WLOG that f intersects $Z = \{y\}$ transversally. This means that $(df)_x : T_xX \to T_yY$ is an isomorphism for all $x \in f^{-1}\{y\}$. By the compactness of X, one knows that $f^{-1}\{y\}$ is a finite set, say $\{x_1, \cdots, x_k\}$. As a result, there exist neighbourhoods U_i of x_i and V of y such that $f^{-1}V$ is the disjoint union of the U_i's and $f|_{U_i} : U_i \to V$ is a diffeomorphism. By continuity, the intersection number $I(f, \{y\})$ (which is the sum of the orientation numbers $i_{x_j}(f)$) remains constant when one changes y to any $z \in V$. In particular, the function $y \mapsto I(f, \{y\})$ is locally constant. It follows from the connectedness of Y that it is also globally constant. $\square$

Suppose that $f \pitchfork \{y\}$. In the above proof, one has seen that

$$\deg f = \sum_{x \in f^{-1}(y)} \mathrm{sgn}\big((df)_x\big)$$

where $\mathrm{sgn}((df)_x) = 1$ if $(df)_x : T_xX \to T_{f(x)}Y$ is orientation preserving and $\mathrm{sgn}((df)_x) = -1$ otherwise. In addition, the number $\deg f$ depends only on the homotopy class of f. The following result provides an alternative characterisation of the degree, which is often taken as an equivalent definition. Let $n \triangleq \dim X = \dim Y$.

Proposition 2.3. *One has*

$$\int_X f^*\alpha = \deg f \times \int_Y \alpha \tag{2.3}$$

for any n-form α on Y.

Proof. We continue to use the notation from the proof of Lemma 2.1, in particular, the representative f as well as the stack of diffeomorphisms $f|_{U_i} : U_i \to V$. Recall that $x_i = f^{-1}\{y\} \cap U_i$. The proof consists of the following steps.

(i) *The formula (2.3) holds if α is supported in V.* Indeed, on each U_j one has

$$\int_{U_j} f|_{U_j}^* \alpha = \begin{cases} \int_V \alpha, & \text{if } f|_{U_i} \text{ is orientation-preserving;} \\ -\int_V \alpha, & \text{if } f|_{U_i} \text{ is orientation-reversing.} \end{cases}$$

Since $f^*\alpha$ is supported in $\cup_i U_i$, it follows that

$$\int_X f^*\alpha = \sum_{j=1}^{k} \int_{U_j} f|_{U_j}^* \alpha = \sum_{j=1}^{k} i_{x_j}(f) \times \int_V \alpha = \deg f \times \int_Y \alpha.$$

(ii) *Let M, N be closed, oriented manifolds with the same dimension. The integral $\int_M g^*\alpha$ (α is any top form on N) depends only on the homotopy class of g.* To see this, let $G : M \times [0,1] \to N$ be a homotopy between g_0 and g_1. Since α is a top form, one has $d\alpha = 0$. It follows from Stokes' theorem that

$$\begin{aligned} 0 &= \int_{M\times[0,1]} G^* d\alpha = \int_{M\times[0,1]} d(G^*\alpha) \\ &= \int_{M\times\{0,1\}} G^*\alpha = \pm\Big(\int_M g_1^*\alpha - \int_M g_0^*\alpha\Big). \end{aligned}$$

(iii) *Let $\varphi : Y \to Y$ be a diffeomorphism and $\varphi \simeq \mathrm{id}$. Then*

$$\int_X f^*\varphi^*\alpha = \int_X f^*\alpha, \quad \int_Y \varphi^*\alpha = \int_Y \alpha.$$

This follows immediately from (ii). In particular, the formula (2.3) is valid for all α supported in $\varphi(V)$.

(iv) *The family*

$$\mathcal{U} \triangleq \{\varphi(V) : \varphi \textit{ is a diffeomorphism over } Y,\ \varphi \simeq \mathrm{id}\}$$

is an open cover of Y. Let $z \in Y$. We want to show that $z = \varphi(y)$ for some diffeomorphism φ that is homotopy equivalent to the identity. A natural way of constructing such a φ is through the flow of diffeomorphisms induced by a vector field (see Appendix B.1). Suppose that z, y are close enough so that they are contained in a coordinate chart that is diffeomorphic to an open ball B in $\mathbb{R}^n$. In this case, it is elementary to construct a vector field W (e.g. supported in B) such that its induced flow of diffeomorphisms $\varphi_t : Y \to Y$ satisfies $\varphi_1(y) = z$. For instance, one can simply flow from y to z along the line segment joining them.

Obviously, $\varphi_1 \simeq \varphi_0 = \mathrm{id}$. For a general position $z \in Y$, by path-connectedness one can join y to z through a finite sequence of points $z_1, \cdots, z_r$ such that z_i, z_{i+1} satisfy the above property for each i. One can then flow from y to z through the chain $y \to z_1 \to \cdots \to z_r \to z$.

(v) *Proof of (2.3) for general α.* Choose a partition of unity $\{\psi_j\}$ subordinated to the open cover $\mathcal{U}$ defined previously (in particular, $\mathrm{supp}\psi_j \subseteq \varphi(V)$ for some φ). Let α be an n-form on Y. Since $\psi_j\alpha$ is supported on some $\varphi(V)$, one knows from Step (iii) that

$$\int_X f^*(\psi_j\alpha) = \deg f \times \int_Y \psi_j\alpha.$$

Therefore, one has

$$\int_X f^*\alpha = \sum_j \int_X \psi_j(f) \cdot f^*\alpha = \deg f \times \sum_j \int_Y \psi_j\alpha = \deg f \times \int_Y \alpha.$$

The proof of the formula (2.3) is thus complete. □

A simple application of the degree is that

$$\deg f \neq 0 \implies f \text{ surjective}.$$

Indeed, if f is not surjective, one can choose a top form supported outside the image of f whose integral over Y is nonzero. The formula (2.3) implies that $\deg f = 0$.

Example 2.1. Let A be an $n \times n$ invertible matrix. Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ denote the unit sphere which is equipped with the canonical orientation induced from $\mathbb{R}^n$. Then one has

$$\deg\left[x \mapsto \frac{Ax}{|Ax|} : S^{n-1} \to S^{n-1}\right] = \begin{cases} 1, & \det A > 0; \\ -1, & \det A < 0. \end{cases} \tag{2.4}$$

Indeed, suppose that $\det A > 0$. Then there exists a smooth curve $A_t \in \mathrm{GL}_n^+(\mathbb{R})$ such that $A_0 = A$ and $A_1 = \mathrm{id}$. It follows from the homotopy invariance of the degree that

$$\deg\left[x \mapsto \frac{Ax}{|Ax|} : S^{n-1} \to S^{n-1}\right] = \deg\left[x \mapsto \frac{x}{|x|} : S^{n-1} \to S^{n-1}\right] = 1.$$

If $\det A < 0$, one connects A to $\mathrm{diag}(-1, 1, \cdots, 1)$ through $\mathrm{GL}_n^-(\mathbb{R})$ and the same argument yields the other scenario of (2.4).

2.5 Lefschetz maps and the Lefschetz number

Fixed points of a smooth map can be studied from the perspective of intersection theory. Suppose that $f : M \to M$ is a given smooth map on a closed, oriented n-manifold M. Let Δ denote the diagonal of $M \times M$ and let

$$\Gamma(f) \triangleq \big\{(x, f(x)) : x \in M\big\}$$

be the graph of f. Note that both Δ and $\Gamma(f)$ are closed, n-dimensional submanifolds of $M \times M$. The orientations on the spaces $M \times M$, Δ and $\Gamma(f)$ are chosen to be the natural ones induced from M.

The basic observation is that $x \in M$ is a fixed point of f (i.e. $f(x) = x$) if and only if (x, x) is an intersection point between $\Gamma(f)$ and Δ. It is thus natural to first consider the situation of transversal intersection.

Definition 2.7. Let $x \in M$ be a fixed point of f. We say that x is a *Lefschetz fixed point* if $\Gamma(f)$ (more precisely, the inclusion map $\mathrm{id} \times f : M \to M \times M$) intersects Δ transversally at (x, x), i.e. if

$$\mathrm{Im}(\mathrm{id} \times (df)_x) \oplus T_{x,x}\Delta = T_{x,x}(M \times M).$$

The map f is said to be *Lefschetz* if $\Gamma(f) \pitchfork \Delta$, i.e. every fixed point of f is Lefschetz.

According to Proposition 2.1, the set of fixed points of a Lefschetz map is a 0-dimensional submanifold and thus a finite subset of M.

Definition 2.8. Let f be a Lefschetz map. The *local Lefschetz number* of f at a fixed point x, denoted as $L_x(f)$, is the orientation number of $\Gamma(f)$ with respect to Δ at (x, x). The *global Lefschetz number* of f, denoted as $L(f)$, is the intersection number $I(\Gamma(f), \Delta)$. In other words,

$$L(f) = \sum_{x:f(x)=x} L_x(f),$$

where $L_x(f) = 1$ if the direct sum orientation of $\mathrm{Im}(\mathrm{id} \times (df)_x) \oplus T_{x,x}\Delta$ coincides with the product orientation $T_{x,x}(M \times M)$ and $L_x(f) = -1$ otherwise.

Below is a simple criterion for Lefschetz fixed points and a simple way of computing the local Lefschetz number.

Lemma 2.2. *A fixed point $x \in M$ of f is Lefschetz if and only if $\mathrm{id} - (df)_x : T_xM \to T_xM$ is a linear isomorphism. In this case, one has*

$$L_x(f) = \begin{cases} 1, & \text{if } \mathrm{id} - (df)_x \text{ preserves orientation;} \\ -1 & \text{if } \mathrm{id} - (df)_x \text{ reverses orientation.} \end{cases} \tag{2.5}$$

Proof. (i) By comparing dimensions, it is easily seen that a fixed point x is Lefschetz if and only if

$$\mathrm{Im}(\mathrm{id} \times (df)_x) \cap T_{x,x}\Delta = \{0\}.$$

Elements (v, w) in $\mathrm{Im}(\mathrm{id} \times (df)_x) \cap T_{x,x}\Delta$ are of the form

$$(v, w) = (v, (df)_x v) = (v, v) \iff v \in \mathrm{Ker}(\mathrm{id} - (df)_x).$$

Therefore, x is Lefschetz if and only if $\mathrm{Ker}(\mathrm{id} - (df)_x) = \{0\}$.

(ii) Let $\{v_1, \cdots, v_m\}$ be an oriented basis of T_xM. The family

$$\mathcal{B} \triangleq \{(v_1, (df)_x v_1), \cdots, (v_m, (df)_x v_m), (v_1, v_1), \cdots, (v_m, v_m)\}$$

defines a basis of $T_{x,x}(M \times M)$ and the orientation of $\mathcal{B}$ is the direct sum orientation of

$$\mathrm{Im}(\mathrm{id} \times (df)_x) \oplus T_{x,x}\Delta.$$

One can use Gaussian elimination to turn $\mathcal{B}$ into a new basis without changing its orientation; more precisely,

$$\begin{aligned}\mathcal{B} \to &\{(v_1, (df)_x v_1), \cdots, (v_m, (df)_x v_m),\\ &\quad (0, (\mathrm{id} - (df)_x)v_1), \cdots, (0, (\mathrm{id} - (df)_x)v_m)\}\\ \to &\{(v_1, 0), \cdots, (v_m, 0), (0, (\mathrm{id} - (df)_x)v_1), \cdots, (0, (\mathrm{id} - (df)_x)v_m)\} =: \mathcal{B}'.\end{aligned}$$

The first arrow follows by subtracting the $(k+m)$-th component by the k-th component of $\mathcal{B}$, and the second arrow follows by eliminating $(df)_x v_k$ in the k-th component using the isomorphism $(\mathrm{id} - (df)_x v_k)$ from the $(k+m)$-th component. Since $\mathcal{B}'$ is block diagonal, its orientation coincides with the orientation of $T_{x,x}(M \times M)$ if and only if $\mathrm{id} - (df)_x$ is orientation preserving. The formula (2.5) thus follows. □

2.6 Computing the Lefschetz number through mapping degree

Since the global Lefschetz number $L(f)$ is defined through the intersection number, it is not necessary to assume that f is a Lefschetz map.

Definition 2.9. The *global Lefschetz number* of a smooth map $f : M \to M$ is defined to be $L(f) \triangleq I(\mathrm{id} \times f, \Delta)$.

Note that $L(f)$ is well-defined according to Theorem 2.2. Such an extension is useful because one does not always have a Lefschetz map. In our study, we will be particularly interested in the following situation where f only has isolated fixed points (but they need not be Lefschetz).

Example 2.2. Let V be a smooth vector field on a closed manifold M with only isolated zeros. Let $f_t : M \to M$ be the flow of diffeomorphisms induced by V. When t is small, the fixed points of f_t are precisely the zeros of V (they are thus isolated by assumption). However, f_t needs not be Lefschetz in general. For instance, consider the vector field $V(x) = \sin^2 \pi x$ on the circle $M = \mathbb{R}/\mathbb{Z}$. The point $0 = 1$ is the unique zero of V. We claim that the graph of f_t is tangential to Δ at $(0, 0)$ (in particular, the intersection is not transversal). To see this, one notes that the function $t \mapsto \partial_x f_t(x)$ satisfies the ODE

$$\frac{d}{dt}\partial_x f_t(x) = 2\pi\big(\sin \pi f_t(x)\big) \cos(\pi f_t(x))\partial_x f_t(x),\ \partial_x f_0(x) = 1.$$

Since $f_t(0) = 0$ for all t, it follows that $\partial_x f_t(0) = 1$ for all t. As a result, the tangent vector of the graph $(x, f_t(x))$ at $(0, 0)$ is the vector $(1, 1)$.

We want to have a general method of computing $L(f)$ when f is only assumed to have isolated fixed points. Such a method is inspired by the following fact. Let $x \in M$ be a Lefschetz fixed point of f. Let U be a positively oriented coordinate chart around x which does not contain other fixed points of f. We may therefore assume WLOG that $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ where $0 \in U$, $f(0) = 0$ is the unique fixed point in U which is Lefschetz.

Lemma 2.3. *Let B_ε be the ε-ball centered at 0 whose closure is contained in U. Then one has*

$$L_0(f) = \deg\Big[x \mapsto \frac{x - f(x)}{|x - f(x)|} : \partial B_\varepsilon \to S^{n-1}\Big].$$

Here both spheres ∂B_ε and S^{n-1} are oriented in the standard way from the ambient Euclidean space (see Example C.1).

Proof. By the homotopy invariance of the degree, one may take ε as small as needed. Since $f(0) = 0$, one can write

$$f(x) = (df)_0(x) + R(x)$$

where $R(x)/|x| \to 0$ as $x \to 0$. Define

$$f_t(x) \triangleq (df)_0(x) + tR(x), \quad t \in [0, 1], x \in \partial B_\varepsilon.$$

Note that

$$\begin{aligned} |x - f_t(x)| &\geqslant |x - (df)_0(x)| - |R(x)| \\ &= |x| \cdot \big[\big|(\mathrm{id} - (df)_0)\big(x/|x|\big)\big| - |R(x)|/|x|\big]. \end{aligned}$$

Let

$$c \triangleq \inf_{y:|y|=1} \big|(\mathrm{id} - (df)_0)(y)\big|,$$

which is strictly positive since 0 is Lefschetz. By choosing ε small enough so that $|R(x)|/|x| < c/2$ for all $x \in \partial B_\varepsilon$, it follows that

$$|x - f_t(x)| \geqslant \frac{c\varepsilon}{2} \quad \forall x \in \partial B_\varepsilon.$$

In particular, the map

$$g_t(x) \triangleq \frac{x - f_t(x)}{|x - f_t(x)|}, \quad t \in [0,1], x \in \partial B_\varepsilon$$

is well-defined. As a result, one concludes from Example 2.1 and Lemma 2.2 that

$$\deg g_1 = \deg g_0 = \text{sign of } \det\big(\mathrm{id} - (df)_0\big) = L_0(f).$$

The result of the lemma thus follows. □

Lemma 2.3 suggests the following definition in general.

Definition 2.10. Let $x \in M$ be an isolated fixed point of a smooth map $f : M \to M$. Choose any positively oriented chart $(U, \varphi; y^i)$ around x which does not contain other fixed points and satisfies $\varphi(x) = 0$. Fix a small $\varepsilon > 0$ such that

$$B_\varepsilon \triangleq \{\mathbf{y} = (y^1, \cdots, y^n) : |\mathbf{y}| < \varepsilon\} \Subset \varphi(U)$$

and $f(\varphi^{-1}B_\varepsilon) \subseteq U$. The *local Lefschetz number of* f at x is defined by

$$L_x(f) = \deg\Big[\mathbf{y} \mapsto \frac{\mathbf{y} - (\varphi f \varphi^{-1})(\mathbf{y})}{|\mathbf{y} - (\varphi f \varphi^{-1})(\mathbf{y})|} : \partial B_\varepsilon \to S^{n-1}\Big], \tag{2.6}$$

where both ∂B_ε and S^{n-1} are oriented in the standard way from the ambient Euclidean space $\mathbb{R}^n$.

By using the homotopy invariance of the mapping degree, one can show that the number $L_x(f)$ does not depend on ε nor on the coordinate chart U. We are now ready to present the formula for computing $L(f)$.

Theorem 2.3. *Let $f : M \to M$ be a smooth map over a closed, oriented manifold M which only possesses isolated fixed points. Then one has*

$$L(f) = \sum_{x : f(x) = x} L_x(f) \tag{2.7}$$

where $L_x(f)$ is the local Lefschetz number defined by (2.6).

The key idea of proving Theorem 2.3 is to "wiggle" f into a Lefschetz map in a tractable way. More specifically, one can "split" each fixed point x of f into a bunch of Lefschetz fixed points x_i of a Lefschetz map $g \simeq f$ which agrees with f outside a small neighbourhood of x. It is then seen that $L_x(f)$ equals the sum of the local Lefschetz numbers of g at those x_i's. The formula thus follows from the Lefschetz case and the homotopy invariance of the global Lefschetz number.

We first make precise the splitting of a fixed point.

Lemma 2.4. *Let U be an open subset in which $x \in U$ is the unique fixed point of f. Let $K \subseteq U$ be a compact neighbourhood of x. Then there exists a smooth homotopy $f_t : M \to M$ $(0 \leqslant t \leqslant 1)$ such that the following properties hold true.*

(i) $f_0 = f$.

(ii) $f_t = f$ outside K for all $t \in [0, 1]$.

(iii) f_1 has only Lefschetz fixed points in U (they are thus all in K).

Proof. Since the statement is essentially local, one may assume WLOG that $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ and x is the origin. Fix an open neighbourhood $0 \in V \Subset K$. Choose a bump function $\phi \in C^\infty$ such that $\phi = 1$ on V and $\phi = 0$ on K^c. Let $a \in \mathbb{R}^n$ be a vector which will be chosen later on. We define

$$f_t(y) \triangleq f(y) + t\phi(y)a.$$

It is clear that $f_0 = f$ and $f_t = f$ outside K for all t.

Let

$$c \triangleq \min_{K \setminus V} |f(y) - y|,$$

which is strictly positive by assumption. One has

$$|f_t(y) - y| \geqslant |f(y) - y| - |a| \geqslant c - |a|$$

for all $y \in K \setminus V$ and $t \in [0, 1]$. We require a to satisfy $|a| < c/2$. This ensures that

$$|f_t(y) - y| > 0$$

for all $y \in K \setminus V$ and $t \in [0, 1]$. In particular, all fixed points of f_t in U are inside V.

To make f_1 Lefschetz on U, one first observes that

$$\mathrm{id} - (df_1)_y = \mathrm{id} - (df)_y \tag{2.8}$$

for all $y \in V$ (because $\phi = 1$ on V). In addition, y is a fixed point of f_1 in V if and only if $y - f(y) = a$. We now choose a to be a regular value of the map $(\mathrm{id} - f)|_V$ such that $|a| < c/2$. This is possible due to Sard's theorem (see Theorem B.6). By the definition of regular value and the relation (2.8), one knows that $\mathrm{id} - (df_1)_y$ is a linear isomorphism for all fixed points y of f_1 in V. As a consequence, all fixed points of f_1 in V (thus in U) are Lefschetz. $\square$

Next, we compute the local Lefschetz number of f at x through the above splitting.

Lemma 2.5. *Fix $B_\varepsilon \subseteq \varphi(U)$ as in Definition 2.10. Let f_t be the homotopy given by Lemma 2.4, so that f_1 has only Lefschetz fixed points $\{x_1, \cdots, x_k\}$ in U and they are all inside $\varphi^{-1}(B_\varepsilon)$. Then one has*

$$L_x(f) = \sum_{i=1}^{k} L_{x_i}(f_1).$$

Proof. Let B_i be a small Euclidean ball centered at $\varphi(x_i)$ $(i = 1, \cdots, k)$. Consider the manifold $W \triangleq \overline{B_\varepsilon} \setminus \cup_i B_i$ with boundary $\partial W = \cup_i \partial B_i \cup \partial B_\varepsilon$ and induced boundary orientation from W. Define the function

$$G : W \to S^{n-1},\ G(\mathbf{y}) \triangleq \frac{\mathbf{y} - (\varphi f_1 \varphi^{-1})(\mathbf{y})}{|\mathbf{y} - (\varphi f_1 \varphi^{-1})(\mathbf{y})|}.$$

This is well-defined since there are no fixed points of $f_1(y)$ in W. Let $\xi \in S^{n-1}$ be a regular value of $G|_{\partial W}$ so that $G|_{\partial W}$ intersects $\{\xi\}$ transversally in S^{n-1}. Since G is a smooth extension of $G|_{\partial W}$ to W, one knows from Proposition 2.2 and the definition of mapping degree that

$$\deg \left[G|_{\partial W} : \partial W \to S^{n-1} \right] = I(G|_{\partial W}, \{\xi\}) = 0. \tag{2.9}$$

On the other hand, note that $f = f_1$ on ∂B_ε by Lemma 2.4. It follows from the definition of $L_x f$, the relation (2.9) and Lemma 2.3 that

$$\begin{aligned} L_x f &= \deg \left[\mathbf{y} \mapsto \frac{\mathbf{y} - (\varphi f_1 \varphi^{-1})(\mathbf{y})}{|\mathbf{y} - (\varphi f_1 \varphi^{-1})(\mathbf{y})|} : \partial B_\varepsilon \to S^{n-1} \right] \\ &= \sum_{i=1}^{k} \deg \left[\mathbf{y} \mapsto \frac{\mathbf{y} - (\varphi f_1 \varphi^{-1})(\mathbf{y})}{|\mathbf{y} - (\varphi f_1 \varphi^{-1})(\mathbf{y})|} : \partial B_i \to S^{n-1} \right] = \sum_{i=1}^{k} L_{x_i}(f_1). \end{aligned}$$

This proves the result of the lemma. □

By applying the above two lemmas, one can now easily establish the formula (2.7).

Proof of Theorem 2.3. Let $x^{(1)}, \cdots, x^{(r)}$ be all the isolated fixed points of f. According to Lemma 2.4, one can deform f into a Lefschetz map f_1 which only has Lefschetz fixed points $x_1^{(i)}, \cdots, x_{k_i}^{(i)}$ near $x^{(i)}$ and coincides with f outside a neighbourhood of $\{x^{(1)}, \cdots, x^{(r)}\}$. It follows from the homotopy invariance of intersection number and Lemma 2.5 that

$$L(f) = L(f_1) = \sum_{i=1}^{r} \sum_{j=1}^{k_i} L_{x_j^{(i)}}(f_1) = \sum_{i=1}^{r} L_{x^{(i)}}(f).$$

The proof of Theorem 2.3 is thus complete.

□

2.7 Index of a vector field

We now define the index of a vector field at an isolated zero, which is a key quantity appearing in the Poincaré-Hopf theorem.

Definition 2.11. Let V be a smooth vector field on an oriented manifold M. Let $x \in M$ be an isolated zero of V (i.e. $V(x) = 0$ and $V(y) \neq 0$ for all y near x). Choose any positively oriented chart $(U, \varphi; y^i)$ around x such that $\varphi(x) = 0$. Fix a small $\varepsilon > 0$ such that $B_\varepsilon \Subset \varphi(U)$ and $V(y) \neq 0$ for all $y \in \varphi^{-1}(\overline{B_\varepsilon})\backslash\{0\}$. The *index* of V at x is defined by

$$\mathrm{ind}_x(V) \triangleq \deg\Big[\mathbf{y} \mapsto \Big(\frac{V^i(\mathbf{y})}{\sqrt{V^1(\mathbf{y})^2 + \cdots + V^n(\mathbf{y})^2}}\Big)_{1\leqslant i\leqslant n} : \partial B_\varepsilon \to S^{n-1}\Big].$$

Here we write $V(y) = V^i(\mathbf{y})\partial_{y^i}$ ($\mathbf{y} = \varphi(y)$) on U. Both spheres ∂B_ε and S^{n-1} are oriented in the standard way from the ambient Euclidean space $\mathbb{R}^n$.

Similar to the local Lefschetz number, the index $\mathrm{ind}_x(V)$ does not depend on ε or the choice of the local chart U. Now we can complete the first major step for the proof of Theorem 2.1.

Theorem 2.4 (Step 1 for Poincaré-Hopf)**.** *Let V be a smooth vector field on a closed, oriented n-manifold M which has only isolated zeros $\{x_1, \cdots, x_k\}$. Then one has*

$$\sum_{i=1}^k \mathrm{ind}_{x_i}(V) = (-1)^n I(\Delta, \Delta). \tag{2.10}$$

Remark 2.3. By considering the diffeomorphism $(x, y) \mapsto (y, x)$ on $M \times M$, one can show that $I(\Delta, \Delta) = 0$ when n is odd. The proof of this fact is left as an exercise. As a result, the formula (2.10) can be rewritten as

$$\sum_{i=1}^k \mathrm{ind}_{x_i}(V) = I(\Delta, \Delta).$$

This is consistent with what we stated at the beginning of this chapter.

Proof. Let $\{f_t\}$ be the flow of diffeomorphisms induced by V. Then $f_t \simeq \mathrm{id}$. In addition, when t is small, fixed points of f_t are precisely the zeros of V, i.e.

$\{x_1, \cdots, x_k\}$, and they are all isolated fixed points by assumption. According to Theorem 2.3, one has

$$\begin{aligned} I(\Delta, \Delta) = I(\Gamma(f_t), \Delta) &= \sum_{i=1}^{k} L_{x_i}(f_t) \\ &= \sum_{i=1}^{k} \deg\left[y \mapsto \frac{x - f_t(x)}{|x - f_t(x)|} : \partial B_i \to S^{n-1}\right]. \end{aligned} \tag{2.11}$$

Here B_i is a small ball around x_i as before and we have omitted the reference to the local coordinate map φ for simplicity. On the other hand, by the definition of the flow $\{f_t\}$ one can write

$$f_t(x) = x + tV(x) + t^2 R(t, x)$$

for some smooth function $R(t, x)$. It follows that

$$\frac{x - f_t(x)}{|x - f_t(x)|} = -\frac{V(x) + tR(t, x)}{|V(x) + tR(t, x)|} \simeq -\frac{V(x)}{|V(x)|}$$

on each B_i for small t. Therefore,

$$\begin{aligned} &\deg\left[x \mapsto \frac{x - f_t(x)}{|x - f_t(x)|} : \partial B_i \to S^{n-1}\right] \\ &= \deg\left[x \mapsto -\frac{V(x)}{|V(x)|} : \partial B_i \to S^{n-1}\right] \\ &= (-1)^n \mathrm{ind}_{x_i}(V). \end{aligned} \tag{2.12}$$

The result follows by substituting the relation (2.12) into (2.11). □

We conclude this chapter with the existence of a smooth vector field on a closed manifold which has only isolated zeros. Recall that a critical point p of a function $f : M \to \mathbb{R}$ (i.e. $df = 0$ at p) is said to be *nondegenerate* if the matrix $\left(\partial^2_{x^i x^j} f(p)\right)_{1 \leqslant i,j \leqslant n}$ is invertible, where $(U; x^i)$ is any local chart around p (this definition does not depend on the choice of the chart). A *Morse function* is a smooth function whose critical points are nondegenerate. It is clear that critical points of a Morse function are all isolated (thus they form a finite set if M is compact).

Proposition 2.4. *Let M be a closed manifold. Then there exists a smooth vector field V on M which has finitely many zeros and they are all isolated.*

Proof. According to [AD14, Proposition 1.2.4], Morse functions on M always exist (in fact, Morse functions are dense in the space of continuous functions under the uniform topology; see [AD14, Theorem 1.2.5]). Fix a Morse function f. Let g be a Riemannian metric on M (the existence of a Riemannian metric is standard; for instance, one can use a partition of unity to construct one). The gradient vector field ∇f with respect to g is a smooth vector field whose zeros are precisely the critical points of f. $\square$

Remark 2.4. The zeros of the vector field V constructed in the above proof are in fact *nondegenerate*, in the sense that the Jacobian of V at each zero, viewed as a linear transform over $\mathbb{R}^n$ under a local chart, is nondegenerate.

Remark 2.5. A more "explicit" way of constructing a Morse function is the following. Let f be an arbitrary smooth function. Consider a perturbation of f defined by

$$g_a(x) \triangleq f(x) + \sum_{i=1}^{N} a_i \psi_i(x),$$

where $a = (a_1, \cdots, a_N) \in \mathbb{R}^N$ and $\psi_i \in C^\infty(M)$. Suppose that N and the ψ_i's are chosen properly so that

$$\mathrm{Span}\big\{d\psi_1(x), \cdots, d\psi_N(x)\big\} = T_x^* M$$

for all $x \in M$. One can then show that (e.g. by using Sard's theorem) for a generic vector a, the function g_a is a Morse function.

Remark 2.6. Here is another way of constructing a vector field with isolated zeros without using a Morse function. Choose a smooth triangulation $h : |K| \to M$ (see Definition 1.4) which always exists according to [Mun66]. For each simplex $(s) = (v_0, \cdots, v_k) \in K$, let $x_s \triangleq h(b_s)$ where b_s is the *barycenter* of (s), i.e. $b_s = \frac{1}{k+1}\sum_{i=0}^{k} v_i$. We define a vector field V which vanishes at x_s for every $(s) \in K$. Near each x_s, the vector field V flows inward from the barycenters of the simplices having (s) as a face and flows outward towards the barycenters of the faces of (s). Locally, the flow line directions emitting from (respectively, entering) the barycenter x_s of a k-simplex (s) form a $(k-1)$-dimensional (respectively, $(n-k-1)$-dimensional with $n = \dim M$) submanifold (cf. Section 2.1 and Figure 2.1).

3 The Lefschetz fixed point formula

The goal of this chapter is to prove the following result, which yields the conclusion of Step 2 for the Poincaré-Hopf theorem.

Theorem 3.1. *Let M be a closed, oriented manifold and let $\Delta \triangleq \{(x,x) : x \in M\}$ be the diagonal. Then the self-intersection number of Δ inside $M \times M$ is equal to the Euler characteristic of M, i.e. $I(\Delta, \Delta) = \chi(M)$.*

Theorem 3.1 together with Theorem 2.4 completes the proof of the Poincaré-Hopf theorem. The heart of this chapter is the proof of a fundamental result in algebraic topology, which is known as the *Lefschetz fixed point formula*. This formula relates the intersection number of a smooth map with its induced traces on homology groups. Theorem 3.1 will be seen as a direct corollary of this formula.

3.1 Main theorem and idea of proof

We begin by stating the core theorem of this chapter.

Theorem 3.2 (The Lefschetz Fixed Point Formula)**.** *Let M be a closed, oriented n-manifold and let $f : M \to M$ be a smooth map. Then one has*

$$I(\mathrm{id} \times f, \Delta) = \sum_{k=0}^{n} (-1)^k \mathrm{Tr}\big[f^* : H^k(M;\mathbb{R}) \to H^k(M;\mathbb{R})\big]. \tag{3.1}$$

Here f^* denotes the induced homomorphism on the singular cohomology groups $H^k(M;\mathbb{R})$.

Remark 3.1. From the definition of $I(f, \Delta)$ (see Definition 2.3 and Definition 2.4), the above result shows that if the RHS of (3.1) is nonzero, the map f must have at least one fixed point.

The formula (3.1) immediately concludes Step 2 for the Poincaré-Hopf index theorem.

Proof of Step 2 for Poincaré-Hopf. Let us take $f = \mathrm{id}$ in the formula (3.1). The LHS becomes $I(\Delta, \Delta)$ and the RHS is precisely $\chi(M)$; since $f^* = \mathrm{id}$ one has

$$\mathrm{Tr}\big[f^* : H^k(M;\mathbb{R}) \to H^k(M;\mathbb{R})\big] = \dim H^k(M;\mathbb{R}).$$

This completes the entire proof of the Poincaré-Hopf index theorem.

□

Now our task becomes proving the Lefschetz fixed point formula (3.1). The approach we take here, which has a topological flavour, is largely inspired by [MS74]. Before developing the main topological tools, we first explain the essential idea. There are two key insights.

(i) *Submanifolds are represented by cohomology classes.* Let X be a closed, oriented n-manifold. The orientation of X is represented by a top homology class $[X] \in H_n(X)$ (the fundamental class). The cap product operator (see Section 3.3.1 below)

$$\cap[X] : H^{n-k}(X) \to H_k(X), \ c \mapsto c \cap [X] \tag{3.2}$$

provides an isomorphism between $H^{n-k}(X)$ and $H_k(X)$ for any $0 \leqslant k \leqslant n$. This is known as the *Poincaré duality*. Now suppose that A is a closed, oriented k-submanifold of M. The fundamental class of A can be viewed as a homology class $[A] \in H_k(X)$ under inclusion. According to the Poincaré duality (3.2), it corresponds to a cohomology class $D([A]) \in H^{n-k}(X)$. This provides a topological way (cohomology classes) of representing geometric objects (submanifolds).

(ii) *Intersection of submanifolds are represented by the cohomology cup product.* Let A, B be closed, oriented submanifolds of X. Suppose that A intersects B transversally so that $A \cap B$ is also an oriented submanifold of X. The *intersection product formula* asserts that

$$D([A \cap B]) = D([A]) \cup D([B])$$

where $D(\cdot)$ is the aforementioned Poincaré dual and $\cup$ is the cohomology cup product. This provides a topological interpretation of geometric intersection.

We now return to the discussion of the Lefschetz fixed point formula. The main idea (similar to what we did in Section 2) is to perform intersections in the product space. Let $f : M \to M$ be a smooth map. Since both sides of (3.1) are homotopy-invariant, one may assume that f is a Lefschetz map (cf. Theorem 2.2 and Corollary 2.1). We consider the ambient space $X = M \times M$ together with submanifolds $A = \Gamma(f)$ (the graph of f in X) and $B = \Delta$ (the diagonal embedding of M). Note that $\Gamma(f)$ intersects Δ transversally since f is Lefschetz. The intersection product formula yields that

$$D([\Gamma(f) \cap \Delta]) = D([\Gamma(f)]) \cup D([\Delta]) \tag{3.3}$$

at the level of cohomology. It will be easily seen that the LHS of (3.3) corresponds to the intersection number $I(f, \Delta)$. For the RHS, the crucial point is that the Poincaré dual $D([\Delta]) \in H^n(M \times M)$ corresponds to the Thom class of the normal bundle over Δ inside X. This requires an in-depth study of the so-called *Thom isomorphism theorem*. Using topological tools, one can further express $D([\Delta])$ in terms of a cohomology basis over M:

$$D([\Delta]) = \sum_i (-1)^{\deg b_i} b_i \times b_i^{\#}. \tag{3.4}$$

Here $\{b_i\}$ is a basis of $H^*(M;\mathbb{R})$ and $\{b_i^{\#}\}$ is its dual in the sense that

$$\langle b_i \cup b_j^{\#}, [M]\rangle = \delta_{ij}.$$

By using the representation (3.4), one can easily compute the cup product on the RHS of (3.3). This yields the RHS of the Lefschetz formula (3.1).

In what follows, we will develop the main topological tools towards proving Theorem 3.2 in the spirit of the above discussion.

Convention. If we only write $H_k(M)$ or $H^k(M)$, that means the underlying coefficient ring is $\mathbb{Z}$. Otherwise, we will write $H_k(M;\Lambda)$ or $H^k(M;\Lambda)$ for a general coefficient ring Λ. The main examples of Λ are either $\mathbb{Z}$ or one of the fields $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/2\mathbb{Z}$ (the last one is related to the nonorientable case).

3.2 Orientation and the fundamental class

Recall from Definition B.8 (as well as the paragraph following that definition) that an n-manifold M is *orientable* if it admits a nowhere vanishing n-form ω. This is equivalent to saying that there is an atlas over M whose transition functions all have positive Jacobian determinants. The aim of this section is to study orientability from an equivalent homological perspective.

We first prove a technical lemma which will be useful in the sequel. For $x \in K \subseteq L \subseteq M$, we use

$$\rho_x : H_i(M, M\backslash K) \to H_i(M, M\backslash\{x\}), \quad \rho_K : H_i(M, M\backslash L) \to H_i(M, M\backslash K)$$

to denote the inclusion homomorphisms for the relative homology groups.

Lemma 3.1. *Let K be a nonempty compact subset of M. Then the following statements hold true.*

(i) $H_i(M, M\backslash K) = 0$ for all $i > n$.
(ii) Let $\alpha \in H_n(M, M\backslash K)$. Then $\alpha = 0$ if and only if $\rho_x(\alpha) \in H_n(M, M\backslash\{x\})$ is zero for all $x \in K$.

Proof. We divide the argument into the following steps.

(i) $\underline{M = \mathbb{R}^n,\ K \text{ is compact and convex}}$.

Fix $x \in K$. It is not hard to show that the inclusion $\iota : \mathbb{R}^n \backslash K \to \mathbb{R}^n \backslash \{x\}$ is a homotopy equivalence. Therefore, one has

$$\rho_x : H_i(\mathbb{R}^n, \mathbb{R}^n \backslash K) \xrightarrow{\cong} H_i(\mathbb{R}^n, \mathbb{R}^n \backslash \{x\}) \cong \begin{cases} 0, & i \neq n; \\ \mathbb{Z}, & i = n, \end{cases} \tag{3.5}$$

where the second isomorphism follows from (A.18). The two claims of the lemma are an immediate consequence of (3.5) in this case.

(ii) *Suppose that the lemma is true for K_1, K_2 and $K_1 \cap K_2$ (with a general M). Then it is also true for $K_1 \cup K_2$.*

According to (A.16), there is a long exact sequence

$$\begin{aligned} \cdots \to H_{i+1}(M, M\backslash(K_1 \cap K_2)) &\to H_i(M, M\backslash(K_1 \cup K_2)) \\ &\xrightarrow{\iota_*} H_i(M, M\backslash K_1) \oplus H_i(M, M\backslash K_2) \to H_i(M, M\backslash(K_1 \cap K_2)) \\ &\to \cdots \to H_0(M, M\backslash K_1) \oplus H_0(M, M\backslash K_2) \to H_0(M, M\backslash(K_1 \cap K_2)) \to 0. \end{aligned} \tag{3.6}$$

Suppose $i > n$. One has from (3.6) and the assumption that

$$0 \to H_i(M, M\backslash(K_1 \cup K_2)) \to 0.$$

In particular, $H_i(M, M\backslash(K_1 \cup K_2)) = 0$ which proves the first claim of the lemma in this case. For the second claim, let $\alpha \in H_n(M, M\backslash(K_1 \cup K_2))$ and assume that $\rho_x(\alpha) = 0$ for all $x \in K_1 \cup K_2$. Let

$$(\alpha_1, \alpha_2) \triangleq \iota_*(\alpha) \in H_n(M, M\backslash K_1) \oplus H_n(M, M\backslash K_2),$$

where ι_* is the canonical inclusion map appearing in the sequence (3.6). Fix $x \in K_1$. The following commutative diagram

$$\begin{tikzcd} H_n(M, M \setminus (K_1 \cup K_2)) \arrow[rr] \arrow[dr, "\rho_x"'] & & H_n(M, M \setminus K_1) \arrow[dl, "\rho_x"] \\ & H_n(M, M \setminus \{x\}) & \end{tikzcd}$$

shows that $\rho_x(\alpha_1) = 0$. Since this is true for all $x \in K_1$, it follows from the assumption for the K_1-case that $\alpha_1 = 0$. Similarly, one also has $\alpha_2 = 0$. On the other hand, since $H_{n+1}(M, M\backslash(K_1 \cap K_2)) = 0$, one knows from (3.6) that ι_* is injective. Therefore, $\alpha = 0$. This proves the second claim of the lemma in this case.

(iii) *$M = \mathbb{R}^n$, $K = K_1 \cup \cdots \cup K_r$ where each K_i is compact and convex.*

This case follows from Steps (i), (ii) and induction.

(iv) *$M = \mathbb{R}^n$, K is compact.*

For the first claim, let $\alpha \in H_i(\mathbb{R}^n, \mathbb{R}^n \backslash K)$ with $i > n$. Write $\alpha = [\sigma]$ where $\partial\sigma \in C_{i-1}(\mathbb{R}^n \backslash K)$. By the compactness of K, one can draw closed Euclidean balls $\bar{B}_1, \cdots, \bar{B}_r$ such that

$$K \subseteq \bigcup_{j=1}^{r} \bar{B}_j =: L$$

and $\partial\sigma \in C_{i-1}(\mathbb{R}^n \backslash L)$. It follows that σ represents an element $\alpha' \in H_i(\mathbb{R}^n, \mathbb{R}^n \backslash L)$ and one has $\alpha = \rho_K(\alpha')$. Step (iii) shows that $H_i(\mathbb{R}^n, \mathbb{R}^n \backslash L) = 0$ and thus $\alpha' = 0$. As a result, one has $\alpha = 0$.

For the second claim, let $\alpha \in H_n(\mathbb{R}^n, \mathbb{R}^n \backslash K)$ and assume that $\rho_x(\alpha) = 0$ for all $x \in K$. As before, one has $\alpha = \rho_K(\alpha')$ for some $\alpha' \in H_n(\mathbb{R}^n, \mathbb{R}^n \backslash L)$. Define $(\alpha_1, \cdots, \alpha_r) \triangleq i_*(\alpha')$ where

$$i_* : H_n(\mathbb{R}^n, \mathbb{R}^n \backslash L) \to \bigoplus_{j=1}^{r} H_n(\mathbb{R}^n, \mathbb{R}^n \backslash \bar{B}_j)$$

is the inclusion homomorphism. Pick a point $x_j \in K \cap \bar{B}_j$ for each $j = 1, \cdots, r$. The following commutative diagram

$$\begin{tikzcd}
\alpha' \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus L) \arrow[r] \arrow[d] \arrow[dr] & \alpha \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus K) \arrow[d] \\
\alpha_j \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \bar{B}_j) \arrow[r, "\cong"] & 0 \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x_j\})
\end{tikzcd}$$

shows that $\alpha_j = 0$ for every j. Therefore, $i_*(\alpha) = 0$. Since i_* is injective (for a similar reason as in Step (ii)), it follows that $\alpha = 0$.

(v) *M is general, K is contained in an open set U which is diffeomorphic to $\mathbb{R}^n$.*

For any $x \in K$, one has the commutative diagram

$$\begin{tikzcd}
H_*(M, M \setminus K) \arrow[r] \arrow[d, "\rho_x"] & H_*(U, U \setminus K) \arrow[d, "\rho_x"'] \\
H_*(M, M \setminus \{x\}) \arrow[r, "\cong"] & H_*(U, U \setminus \{x\}),
\end{tikzcd}$$

where the horizontal isomorphisms are a consequence of excision. Both claims of the lemma follow from the above diagram and Step (iv) in this case.

(v) *M is general, K is compact.*

Write $K = K_1 \cup \cdots \cup K_r$ where each K_j is contained in some open set diffeomorphic to $\mathbb{R}^n$ (also true for finite intersections among them). The result of the lemma follows from Steps (ii), (v) and induction. $\square$

We now define the notion of orientation from the homological viewpoint. Note that $H_n(M, M\backslash\{x\}) \cong \mathbb{Z}$ for any $x \in M$, which follows easily from excision. In particular, the group $H_n(M, M\backslash\{x\})$ has precisely two generators.

Definition 3.1. A manifold M is said to be *orientable* if there exists an assignment (an *orientation*)

$$x \mapsto \mu_x : \text{a generator of } H_n(M, M\backslash\{x\}),$$

such that for any $x \in M$, there exists a neighbourhood N of x and $\mu \in H_n(M, M\backslash N)$ satisfying $\mu_y = \rho_y(\mu)$ for all $y \in N$. A manifold equipped with an orientation is called an *oriented manifold.*

Remark 3.2. It will be proved in Proposition 3.4 that the above definition is equivalent to the more familiar Definition B.8 from the differential viewpoint.

Theorem 3.3. *Let M be an oriented n-manifold with orientation $\{\mu_x : x \in M\}$. Let K be a compact subset of M. Then there exists a unique $\mu_K \in H_n(M, M\backslash K)$ such that $\mu_x = \rho_x(\mu_K)$ for all $x \in K$.*

Proof. (i) Uniqueness follows from the second claim of Lemma 3.1.

(ii) Suppose that the claim is true for K_j with $\mu_{K_j} \in H_n(M, M\backslash K_j)$ $(j = 1, 2)$. Consider the long exact sequence of relative homology groups which we used in Step (ii) of the proof of Lemma 3.1. Uniqueness implies that

$$j_*(\mu_{K_1}, \mu_{K_2}) = \rho_{K_1\cap K_2}(\mu_{K_1}) - \rho_{K_1\cap K_2}(\mu_{K_2}) = 0.$$

It follows from exactness that $(\mu_{K_1}, \mu_{K_2}) \in \mathrm{Im}(i_*)$, i.e. $(\mu_{K_1}, \mu_{K_2}) = i_*(\mu_K)$ for some $\mu_{K_1\cup K_2} \in H_n(M, M\backslash(K_1 \cup K_2))$. Therefore, the claim is true for $K_1 \cup K_2$ with the (unique) class $\mu_{K_1\cup K_2}$.

(iii) Let K be a given compact set. By the definition of orientation, for each $x \in K$ there exists a neighbourhood N_x of x and $\mu_{N_x} \in H_n(M, M\backslash N_x)$ such that $\rho_y(\mu_{N_x}) = \mu_y$ for all $y \in N_x$. By reducing N_x if necessary, we assume that N_x is compact. Due to the compactness of K, one can write

$$K \subseteq \bigcup_{j=1}^{r} N_{x_j} =: L$$

for some $r \geqslant 1$ and $x_1, \cdots, x_r \in K$. Part (ii) and induction implies that the claim is true for L with a unique class μ_L. By setting $\mu_K \triangleq \rho_K(\mu_L)$, one obtains the result for the compact set K. $\square$

Suppose further that M is compact. Taking $K = M$ in the above theorem leads to the following important concept.

Definition 3.2. Let M be a closed, oriented n-manifold. The homology class $\mu_M \in H_n(M)$ given by Theorem 3.3 with $K = M$ is called the *fundamental class* of M. We also denote μ_M as $[M]$.

Remark 3.3 (The nonorientable case). The above discussion naturally applies to the situation when $\Lambda = \mathbb{Z}_2$. In this case, $H_n(M, M\backslash\{x\}; \mathbb{Z}_2)$ has a unique generator (the unique nonzero element) for every $x \in M$. The property stated in Definition 3.1 is always valid; given $x \in M$ one can choose a small neighbourhood N so that $H_n(M, M\backslash N; \mathbb{Z}_2) \cong \mathbb{Z}_2$ (by excision) and simply take μ_N to be the unique nonzero element of $H_n(M, M\backslash N; \mathbb{Z}_2)$. Theorem 3.3 also applies without any orientability assumption. For a closed manifold M, its *fundamental class over* $\mathbb{Z}_2$ is thus always well-defined. This is useful when M is not orientable. If M is oriented with fundamental class $[M]$, the corresponding fundamental class over $\mathbb{Z}_2$ is just the image of $[M]$ under the homomorphism $H_n(M) \to H_n(M; \mathbb{Z}_2)$ induced by $\mathbf{1}_{\mathbb{Z}} \mapsto \mathbf{1}_{\mathbb{Z}_2}$.

3.3 The Poincaré duality theorem

In this section, we prove an important duality result in algebraic topology. Let M be a closed, oriented n-manifold with fundamental class $[M]$. The *Poincaré duality theorem* provides a natural isomorphism between $H^i(M)$ and $H_{n-i}(M)$. The precise construction of this isomorphism requires the notion of cohomology cap product which we first define.

3.3.1 Cohomology cap product

Let X be a topological space. The *cap product* is the bilinear operator

$$\cap : C^i(X) \times C_n(X) \to C_{n-i}(X)$$

characterised by the duality relation

$$\langle \varphi, \psi \cap \sigma \rangle = \langle \varphi \cup \psi, \sigma \rangle$$

for all $\varphi \in C^{n-i}(X), \psi \in C^i(X)$ and $\sigma \in C_n(X)$. Here $\cup$ is the cup product (see Definition A.11) and $\langle \cdot, \cdot \rangle$ denotes the pairing between singular cochains and chains. Geometrically, $\psi \cap \sigma$ means that ψ eats the back i-face of σ leaving its front $(n-i)$-face as an $(n-i)$-simplex (in the case when σ is a singular n simplex). This is described by the following result.

Proposition 3.1. *Let $\sigma = [v_0, \cdots, v_n]$ be a singular n-simplex in X and let $\psi \in C^i(X)$. Then one has*

$$\psi \cap \sigma = \langle \psi, \sigma|_{[v_{n-i}, \cdots, v_n]} \rangle \sigma|_{[v_0, \cdots, v_{n-i}]}.$$

Proof. By the definition of the cap and cup products, one has

$$\begin{aligned}\langle\varphi, \psi\cap\sigma\rangle &= \langle\varphi\cup\psi, \sigma\rangle = \langle\varphi, \sigma|_{[v_0,\cdots,v_{n-i}]}\rangle\langle\psi, \sigma|_{[v_{n-i},\cdots,v_n]}\rangle \\ &= \big\langle\varphi, \langle\psi, \sigma|_{[v_{n-i},\cdots,v_n]}\rangle\sigma|_{[v_0,\cdots,v_{n-i}]}\big\rangle\end{aligned}$$

for any $\varphi\in C^{n-i}(X)$. □

The cap product satisfies the following basic properties. Recall that ∂ is the boundary map on singular chains and δ (defined as the dual of ∂) is the coboundary map on singular cochains (see Definition A.7 and Definition A.9).

Proposition 3.2. *(i)* $(\varphi\cup\psi)\cap\sigma = \varphi\cap(\psi\cap\sigma)$.

(ii) $1\cap\sigma = \sigma$.

(iii) $\partial(\psi\cap\sigma) = \psi\cap\partial\sigma + (-1)^{\deg\sigma-\deg\psi}\delta\psi\cap\sigma$.

(iv) (Naturality of cap product) $f_*(f^*\psi\cap\sigma) = \psi\cap f_*\sigma$ *where* $f: X\to Y$ *is a continuous map,* ψ *is a singular cochain in* Y *and* σ *is a singular chain in* X.

Proof. (i) One has

$$\langle\zeta, (\varphi\cup\psi)\cap\sigma\rangle = \langle\zeta\cup\varphi\cup\psi, \sigma\rangle = \langle\zeta\cup\varphi, \psi\cap\sigma\rangle = \langle\zeta, \varphi\cap(\psi\cap\sigma)\rangle$$

for any cochain ζ (with suitable degree).

(ii) Obvious.

(iii) Write $i \triangleq \deg\psi$ and $n\triangleq\deg\sigma$. Let $\varphi\in C^{n-i-1}(X)$. According to the ∂-δ duality and the Leibniz rule (A.4), one finds that

$$\begin{aligned}\langle\varphi, \partial(\psi\cap\sigma)\rangle &= \langle\delta\varphi, \psi\cap\sigma\rangle = \langle\delta\varphi\cup\psi, \sigma\rangle \\ &= \langle\delta(\varphi\cup\psi), \sigma\rangle - (-1)^{n-i-1}\langle\varphi, \delta\psi\cap\sigma\rangle \\ &= \langle\varphi\cup\psi, \partial\sigma\rangle + (-1)^{n-i}\langle\varphi, \delta\psi\cap\sigma\rangle \\ &= \langle\varphi, \psi\cap\partial\sigma + (-1)^{n-i}\delta\psi\cap\sigma\rangle.\end{aligned}$$

The result thus follows.

(iv) One has

$$\begin{aligned}\langle\varphi, f_*(f^*\psi\cap\sigma)\rangle &= \langle f^*\varphi, f^*\psi\cap\sigma\rangle = \langle f^*\varphi\cup f^*\psi, \sigma\rangle \\ &= \langle f^*(\varphi\cup\psi), \sigma\rangle = \langle\varphi\cup\psi, f_*\sigma\rangle = \langle\varphi, \psi\cap f_*\sigma\rangle\end{aligned}$$

for any singular cochain φ in Y. □

The third property of Proposition 3.2 shows that the cap product descends to (co)homology groups, namely, one has a well-defined cap product operator

$$\cap : H^i(X) \times H_n(X) \to H_{n-i}(X).$$

The cap product also extends naturally to a pair (X, A) (A is a subset of X):

$$\cap : C^i(X, A) \times C_n(X, A) \to C_{n-i}(X). \tag{3.7}$$

Indeed, let $\psi \in C^i(X, A)$ (i.e. $\psi \in C^i(X)$ annihilates singular i-chains in A) and let $[\sigma] \in C_n(X, A)$. One defines $\psi \cap [\sigma] \triangleq \psi \cap \sigma$ where σ is any representative of $[\sigma]$. This is well-defined since different representatives differ by a singular n-chain in A, whose back i-face (as a singular i-chain in A) is annihilated by ψ. In a similar fashion, the relative cap product (3.7) also descends to relative (co)homology groups:

$$\cap : H^i(X, A) \times H_n(X, A) \to H_{n-i}(X). \tag{3.8}$$

This will be useful in our later study of the Poincaré duality.

3.3.2 Cohomology with compact support

The Poincaré duality theorem also requires the notion of singular cohomology with compact support.

Definition 3.3. A cochain $a \in C^i(X)$ is said to *have compact support* if there exists a compact set $K \subseteq X$ such that $a \in C^i(X, X\backslash K)$ (i.e. a annihilates singular i-chains in $X\backslash K$). The $\mathbb{Z}$-submodule of i-cochains with compact support is denoted as $C_c^i(X)$. The cohomology of the complex $C_c^*(X)$, which is denoted as $H_c^*(X)$, is called the *singular cohomology of X with compact support*.

Remark 3.4. The space $H_c^i(X)$ is isomorphic to the direct limit of $H^i(X, X\backslash K)$, where the limit is taken over compact subsets of X (see [MS74, Spa66]). As a result, an element $a \in H_c^i(X)$ is represented by a relative cohomology class $a' \in H^i(X, X\backslash K)$ with some compact set K. Two representations $a' \in H^i(X, X\backslash K)$ and $a'' \in H^i(X, X\backslash L)$ are identified if their images in $H^i(X, X\backslash(K \cup L))$ coincide. If X is compact, the groups $H_c^i(X)$ and $H^i(X)$ are the same thing.

Let M be an oriented n-manifold. There is a notion of integration over M in the algebraic topology sense. Let $a \in H_c^n(M)$ which is represented by some $a' \in H^n(M, M\backslash K)$. Let $\mu_K \in H_n(M, M\backslash K)$ be the relative homology class given by Theorem 3.3. Using the natural pairing $\langle \cdot, \cdot \rangle$ between (relative) cohomology and homology (see Appendix A.3), one can define a number $\langle a', \mu_K \rangle \in \mathbb{Z}$. It is routine to check that this integer does not depend on the choice of the representative a'. We denote it as $I_M(a)$. If M is compact, one has $I_M(a) = \langle a, [M] \rangle$ where $[M]$ is the fundamental class of M.

Definition 3.4. The number $I_M(a)$ is called the *integral* of a over M.

3.3.3 The Poincaré duality

By using the cap product, we now proceed to establish the Poincaré duality. Let M be a closed, oriented n-manifold with fundamental class $[M]$. The main theorem is stated as follows.

Theorem 3.4 (The Poincaré Duality)**.** *The homomorphism*

$$\cap[M] : H^i(M) \to H_{n-i}(M), a \mapsto a \cap [M] \tag{3.9}$$

defined by taking cap product with $[M]$ *is an isomorphism for every* $0 \leqslant i \leqslant n$*.*

Proof. We shall consider a more general situation. Suppose that M is an oriented n-manifold (without assuming compactness). We generalise the duality map (3.9) in the following way. Let $a \in H_c^i(M)$ which is represented by some $a' \in H^i(M, M\backslash K)$. Let $\mu_K \in H_n(M, M\backslash K)$ be relative homology class given by Theorem 3.3. We then define (see (3.8))

$$P_M^i(a) \triangleq a' \cap \mu_K \in H_{n-i}(M).$$

By using the characterising property of μ_K (cf. Theorem 3.3), one can check that $P_M^i(a)$ does not depend on the representative a'. It follows that there is a well-defined homomorphism

$$P_M^i : H_c^i(M) \to H_{n-i}(M). \tag{3.10}$$

When M is compact, the map P_M^i is just the earlier $\cap[M]$ defined by (3.9). With the above preparation, we now prove the theorem in the following steps.

(i) *P_M^i is an isomorphism for all i if $M = \mathbb{R}^n$*.

Let $\bar{B}$ be any closed ball. Similar to (3.5), one has

$$H^i(\mathbb{R}^n, \mathbb{R}^n\backslash\bar{B}) \cong \begin{cases} \mathbb{Z}, & \text{if } i = n; \\ 0, & \text{otherwise.} \end{cases}$$

By taking direct limit, it follows that

$$H_c^i(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } i = n; \\ 0, & \text{otherwise.} \end{cases}$$

The claim holds trivially if $i \neq n$ because both sides of (3.10) are zero.

To prove that $P_{\mathbb{R}^n}^n$ is an isomorphism, it suffices to show that $P_{\mathbb{R}^n}^n$ maps a generator of $H_c^n(\mathbb{R}^n)$ to a generator of $H_0(\mathbb{R}^n)$. Let a be the unique element of $H_c^n(\mathbb{R}^n)$ such that $\langle a', \mu_{\bar{B}} \rangle = 1 \in \mathbb{Z}$ where $a' \in H^n(\mathbb{R}^n, \mathbb{R}^n\backslash\bar{B})$ is any representative of a with some closed ball $\bar{B}$. By definition, one has

$$\begin{aligned} \langle \mathbf{1}_{H^0(\mathbb{R}^n)}, P_{\mathbb{R}^n}^n(a) \rangle &= \langle \mathbf{1}_{H^0(\mathbb{R}^n)}, a' \cap \mu_{\bar{B}} \rangle \\ &= \langle \mathbf{1}_{H^0(\mathbb{R}^n)} \cup a', \mu_{\bar{B}} \rangle = \langle a', \mu_{\bar{B}} \rangle = 1. \end{aligned}$$

Therefore, $P^n_{\mathbb{R}^n}(a)$ is a generator of $H_0(\mathbb{R}^n)$.

(ii) *Let $M = U \cup V$ where U, V are open subsets. Suppose that $P^i_U, P^i_V, P^i_{U\cap V}$ are isomorphisms for all i. Then P^i_M is an isomorphism for all i.*

The main observation is that there is a Mayer-Vietoris sequence

$$\cdots \to H^i_c(U\cap V) \to H^i_c(U)\oplus H^i_c(V) \to H^i_c(M) \to H^{i+1}_c(U\cap V) \to \cdots \tag{3.11}$$

for the compactly supported cohomology. Presume that such a sequence is valid. By applying the Five Lemma (see Lemma A.2) to the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^i_c(U\cap V) \arrow[r] \arrow[d, "P^i_{U\cap V}"] & H^i_c(U)\oplus H^i_c(V) \arrow[r] \arrow[d, "P^i_U\oplus P^i_V"] & H^i_c(M) \arrow[r] \arrow[d, "P^i_M"] & H^{i+1}_c(U\cap V) \arrow[r] \arrow[d, "P^{i+1}_{U\cap V}"] & \cdots \\
\cdots \arrow[r] & H_{n-i}(U\cap V) \arrow[r] & H_{n-i}(U)\oplus H_{n-i}(V) \arrow[r] & H_{n-i}(M) \arrow[r] & H_{n-i-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

where the bottom row is the usual Mayer-Vietoris sequence for homology (see (A.15)), one concludes that the vertical homomorphism in the middle is an isomorphism (since the other four are). Therefore, the theorem is true for $U \cup V$.

To justify the long exact sequence (3.11), one uses the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(M, M\setminus(K\cap L)) \arrow[r] \arrow[d, "\cong"] & H^i(M, M\setminus K)\oplus H^i(M, M\setminus L) \arrow[r] \arrow[d, "\cong"] & H^i(M, M\setminus(K\cup L)) \arrow[r] \arrow[d, "\mathrm{id}"] & \cdots \\
\cdots \arrow[r] & H^i(U\cap V, U\cap V\setminus(K\cap L)) \arrow[r] & H^i(U, U\setminus K)\oplus H^i(V, V\setminus L) \arrow[r] & H^i(M, M\setminus(K\cup L)) \arrow[r] & \cdots
\end{tikzcd}$$

Here K (respectively, L) is a compact subset of U (respectively, V). The two rows are the usual Mayer-Vietoris sequences for relative cohomology. The vertical isomorphisms are obtained by excision. The desired long exact sequence follows by taking direct limits over compact K's within U and compact L's within V respectively.

(iii) *Suppose that M is a closed, oriented n-manifold. Then P^i_M is an isomorphism for all i.*

Recall from Theorem B.3 that there exists a finite good cover of M, say $M = \cup_{j=1}^k U_j$, where each U_j is open and any of their (nonempty) intersections is diffeomorphic to $\mathbb{R}^n$. It follows from Steps (i), (ii) and induction that P^i_M is an isomorphism for every i. This completes the proof of the Poincaré duality theorem.

□

Definition 3.5. The cohomology class corresponding to $\alpha \in H_{n-i}(M)$ under the isomorphism (3.9) is called the *Poincaré dual* of α.

Remark 3.5. With some extra technical effort, one can show that the homomorphism P^i_M defined by (3.10) is an isomorphism even in the noncompact case (see [MS74]).

We briefly mention a useful extension of the Poincaré duality theorem to the case with boundary. Let M be a compact, oriented n-manifold with boundary. This means a preferred generator $\mu_x \in H_n(M, M\backslash\{x\})$ is chosen for every $x \in M\backslash\partial M$ such that the condition in Definition 3.1 holds. Note that as a topological space, M contains both the interior and the boundary sets. In this case, there is a unique relative homology class $[M, \partial M] \in H_n(M, \partial M)$ such that $\rho_x([M, \partial M]) = \mu_x$ for every $x \in M\backslash\partial M$. This induces an orientation on ∂M in a natural way; the resulting fundamental class of ∂M is the image of $[M, \partial M]$ under the connecting homomorphism in the long exact sequence for the pair $(M, \partial M)$ (see (A.7)). The Poincaré duality theorem in this setting asserts that both of the following homomorphisms

$$\begin{aligned} \cap[M, \partial M] :& H^i(M, \partial M) \to H_{n-i}(M), \\ \cap[M, \partial M] :& H^i(M) \to H_{n-i}(M, \partial M) \end{aligned} \tag{3.12}$$

are isomorphisms. The reader is referred to [Bre93] for the details.

Remark 3.6. The Poincaré duality theorem has an obvious extension to the case when the coefficient ring Λ is a field. In this case, the corresponding fundamental class $[M]_\Lambda$ is the image of $[M]$ under the homomorphism $H_n(M) \to H_n(M; \Lambda)$ induced by $\mathbf{1}_\mathbb{Z} \mapsto \mathbf{1}_\Lambda$.

3.4 The Thom isomorphism theorem

Here comes a basic question: *how can one construct the Poincaré dual of a homology class* $\alpha \in H_{n-i}(M)$? A typical situation is when α is represented by a submanifold, i.e. $\alpha = [S]$ where S is a closed, oriented $(n-i)$-submanifold of M with fundamental class $[S]$ (viewed as a homology class in M). To understand this question, we need to spend some time discussing another fundamental result: the *Thom isomorphism*.

3.4.1 Canonical generator of $H^r(\mathbb{R}^r, \mathbb{R}^r_0)$

We first recall a standard computation in $\mathbb{R}^r$ which will play a basic role in the sequel. Let $\mathbb{R}^r_0 \triangleq \mathbb{R}^r\backslash\{0\}$. It is well known that $H^r(\mathbb{R}^r, \mathbb{R}^r_0) \cong \mathbb{Z}$ (see (A.18)). A canonical generator of $H^r(\mathbb{R}^r, \mathbb{R}^r_0)$ can be obtained in the following way.

First of all, consider the triple $(\mathbb{R}, \mathbb{R}_0, \mathbb{R}_-)$ where $\mathbb{R}_- \triangleq (-\infty, 0)$. The long exact sequence (A.10) of relative cohomology applied to this triple yields the exact sequence

$$0 = H^0(\mathbb{R}, \mathbb{R}_-) \to H^0(\mathbb{R}_0, \mathbb{R}_-) \xrightarrow{\delta} H^1(\mathbb{R}, \mathbb{R}_0) \to H^1(\mathbb{R}, \mathbb{R}_-) = 0,$$

where δ is the connecting homomorphism. In particular, δ an isomorphism in this case. There is a distinguished generator $\mathfrak{a}$ of $H^0(\mathbb{R}_0, \mathbb{R}_-)$; the evaluation of $\mathfrak{a}$ on 0-simplices in $\mathbb{R}_+ \triangleq (0, \infty)$ is identically equal to 1 and it annihilates $\mathbb{R}_-$. We set $\mathfrak{e} \triangleq \delta(\mathfrak{a})$. Note that $\mathfrak{e}$ is a generator of $H^1(\mathbb{R}, \mathbb{R}_0)$.

One can now apply the Künneth theorem to see that

$$\underbrace{H^1(\mathbb{R}, \mathbb{R}_0) \otimes \cdots \otimes H^1(\mathbb{R}, \mathbb{R}_0)}_{r} \cong H^r(\mathbb{R}^r, \mathbb{R}_0^r).$$

Under the above isomorphism, the class

$$\mathfrak{e}^r \triangleq \underbrace{\mathfrak{e} \times \cdots \times \mathfrak{e}}_{r} \tag{3.13}$$

is a generator of $H^r(\mathbb{R}^r, \mathbb{R}_0^r)$ (here $\times$ is the cohomology cross product; see Definition A.13). This is referred to as the *canonical generator*.

Remark 3.7. In the above construction, a preferred orientation of $\mathbb{R}^r$ (the standard orientation) is implicitly chosen when one decides to use the triple $(\mathbb{R}, \mathbb{R}_0, \mathbb{R}_-)$ rather than $(\mathbb{R}, \mathbb{R}_0, \mathbb{R}_+)$. In fact, the latter flips the sign of $\mathfrak{e}$, which can be interpreted as a change of orientation.

3.4.2 Orientation of a real vector bundle

Recall from Definition B.16 that a real vector bundle $(E, M, \pi; V)$ is *orientable* if it admits a local trivialisation whose transition functions all have positive determinant. In particular, one can choose an orientation of the vector space E_x (the fiber at x) in a globally consistent manner. The aim of this subsection is to study orientability of a real vector bundle from an equivalent *(co)homological* perspective.

Let V be an r-dimensional real vector space V which is equipped with the standard Euclidean topology. Let $V_0 \triangleq V \backslash \{0\}$. Recall from (A.18) and its cohomology version that

$$H_r(V, V_0) \cong \mathbb{Z} \cong H^r(V, V_0). \tag{3.14}$$

In particular, there are precisely two generators of $H_r(V, V_0)$ (same for $H^r(V, V_0)$).

Lemma 3.2. *The following three objects are equivalent:*

(i) an orientation of V, i.e. an equivalence class of bases of V (two bases are equivalent if their transformation matrix has positive determinant);

(ii) a generator of $H_r(V, V_0)$;

(iii) a generator of $H^r(V, V_0)$.

Proof. An orientation of V is represented by a basis $\{\varepsilon_1, \cdots, \varepsilon_r\}$. This provides an isomorphism between $\mathbb{R}^r$ and V. Let u denote the image of the standard generator $\mathfrak{e}^r$ of $H^r(\mathbb{R}^r, \mathbb{R}^r_0)$ (see (3.13)). Then u is a generator of $H^r(V, V_0)$ which is independent of the basis representing the given orientation of V (left as an exercise). Through the duality $H^r(V, V_0) \cong \mathrm{Hom}_{\mathbb{Z}}(H_r(V, V_0), \mathbb{Z})$, there exists a unique class $a \in H_r(V, V_0)$ such that $\langle u, a\rangle = 1$. This a is a generator of $H_r(V, V_0)$. It is represented by any r-simplex $[v_0, \cdots, v_r] \subseteq V$ whose center is the origin and $\{v_1 - v_0, \cdots, v_r - v_0\}$ form a positively oriented basis. Flipping the orientation of V changes both $u \mapsto -u$ and $a \mapsto -a$. □

We now move to a real vector bundle $(E, M, \pi; \mathbb{R}^r)$ and define its orientability from the cohomological viewpoint.

Definition 3.6. The bundle E is said to be *orientable* if there exists an assignment $x \mapsto u_x \in H^r(E_x, (E_x)_0)$ (an *orientation*) such that the following properties hold true.

(i) u_x is a generator of $H^r(E_x, (E_x)_0)$ for every $x \in M$.

(ii) For any $x \in M$, there exists a neighbourhood N of x and a relative cohomology class $u_N \in H^r(\pi^{-1}N, (\pi^{-1}N)_0)$ such that $u_N|_y = u_y$ for all $y \in N$. Here $(\pi^{-1}N)_0$ is the space of nonzero elements of $\pi^{-1}N$ and

$$\cdot|_y : H^r(\pi^{-1}N, (\pi^{-1}N)_0) \to H^r(E_y, (E_y)_0)$$

is the standard restriction homomorphism.

A real vector bundle equipped with an orientation is called an *oriented vector bundle.*

Proposition 3.3. *Definition 3.6 and Definition B.16 are equivalent.*

Proof. Suppose that E is oriented in the sense of Definition 3.6. One can obtain an oriented local trivialisation in the following way. Given $x \in M$, pick an arbitrary local trivialisation $\psi : \pi^{-1}N \to N \times \mathbb{R}^r$. We may assume that N is a contractible neighbourhood of x and is chosen small enough so that a relative cohomology class u_N is also defined according to Definition 3.6 (ii). It is clear that $(\psi^{-1})^* u_N = \pm 1 \times \mathfrak{e}^r$. If it is a plus sign, one keeps ψ as it is. Otherwise, one applies an orientation reversing automorphism of $\mathbb{R}^r$ to get a new trivialisation $\hat{\psi}$ satisfying $(\hat{\psi}^{-1})^* u_N = 1 \times \mathfrak{e}^r$. By varying x, one obtains an oriented local trivialisation of the bundle E whose transition functions all have positive determinants.

Conversely, suppose that $\{U_\alpha, \psi_\alpha\}$ is a given local trivialisation whose transition functions have positive determinants. Let $x \in M$ and assume that $x \in U_\alpha$. Using the linear isomorphism $\psi_\alpha|_{E_x} : E_x \to \mathbb{R}^r$, one can define $u_x \triangleq (\psi_\alpha|_{E_x})^* \mathfrak{e}^r$. It follows that u_x is a generator of $H^r(E_x, (E_x)_0)$ which is independent of the choice of U_α containing x. To justify Condition (ii) of Definition 3.6, one simply defines

$$u_\alpha \triangleq (\psi_\alpha^{-1})^*(1 \times \mathfrak{e}^r) \in H^r(\pi^{-1}U_\alpha, (\pi^{-1}U_\alpha)_0).$$

Then $u_\alpha|_y = u_y$ for any $y \in U_\alpha$. □

Restricting to the tangent bundle, all notions of orientability we have seen so far are equivalent.

Proposition 3.4. *Let M be an n-manifold. The following statements are equivalent.*

(i) M is orientable in the sense of Definition B.8.

(ii) M is orientable in the sense of Definition 3.1.

(iii) TM is orientable.

Proof. $\underline{\text{(i)} \Longrightarrow \text{(iii)}}$. Suppose that there exists an open cover $\mathcal{U} = \{(U_\alpha; x_\alpha^i)\}$ of M by coordinate charts whose Jacobians all have positive determinants. Each chart $(U_\alpha; x_\alpha^i)$ gives rise to a trivialisation of TU_α with respect to the local frame field $\{\partial_{x_\alpha^1}, \cdots, \partial_{x_\alpha^n}\}$. The transition functions of this local trivialisation is just the Jacobians of the change of coordinates from $\mathcal{U}$. Therefore, they all have positive determinants. This shows that TM is orientable in the sense of Definition B.16.

$\underline{\text{(iii)} \Longrightarrow \text{(i)}}$. Suppose that $\{(U_\alpha; e_{1,\alpha}, \cdots, e_{n,\alpha})\}$ is a local trivialisation of TM, where $\{e_{1,\alpha}, \cdots, e_{n,\alpha}\}$ is a local frame field on U_α, and the transition functions all have positive determinants. Let $\omega_\alpha \triangleq \eta_\alpha^1 \wedge \cdots \wedge \eta_\alpha^n$ where $\{\eta_\alpha^i\}$ is the coframe field dual to $\{e_{i,\alpha}\}$. Let $\{\varphi_\alpha\}$ be a partition of unity subordinated to $\{U_\alpha\}$ with the same index (see Theorem B.2 (ii)). Define

$$\omega \triangleq \sum_\alpha \varphi_\alpha \omega_\alpha.$$

It is easily seen that ω is a smooth n-form on M. In addition, by writing

$$\omega = \sum_\alpha \varphi_\alpha \Big(\sum_\beta \det \rho_{\alpha\beta} \omega_\beta \Big)$$

where $\rho_{\alpha\beta}$ is the transition function and noting that $\det \rho_{\alpha\beta} > 0$ whenever $U_\alpha \cap U_\beta \neq \emptyset$, one has $\omega(x) \neq 0$ for every $x \in M$.

$\underline{\text{(ii)} \Longleftrightarrow \text{(iii)}}$. Let $x \in M$ and let U be a small open neighbourhood of x. There are isomorphisms

$$H_n(M, M \backslash \{x\}) \cong H_n(U, U \backslash \{x\}) \cong H_n(T_xM, (T_xM)_0). \tag{3.15}$$

The first isomorphism follows from excision on M. The second isomorphism can be obtained by putting a Riemannian metric on M, using the exponential map (as a local diffeomorphism around x) and applying excision on T_xM. There is also a canonical isomorphism

$$H^n(T_xM, (T_xM)_0) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(T_xM, (T_xM)_0), \mathbb{Z}). \tag{3.16}$$

Using the isomorphisms (3.15, 3.16), there is a natural correspondence between generators of $H_n(M, M\backslash\{x\})$ and of $H^n(T_xM, (T_xM)_0)$. Indeed, let μ_x be a generator of $H_n(M, M\backslash\{x\})$. One then defines u_x to be the unique element in $H^n(M, M\backslash\{x\})$ such that $\langle u_x, \mu_x\rangle = 1$ through the duality (3.16). Vice versa. By varying x, one thus obtains a correspondence between an orientation of M (in the sense of Definition 3.1) and an orientation of TM (in the sense of Definition 3.6). The local classes μ_N and u_N in the underlying definitions also correspond in a similar way; indeed, when N is small one has the canonical isomorphisms

$$H_n(M, M\backslash N) \cong H_n(M, M\backslash\{x\}),\ \ \mu_N \longleftrightarrow \mu_x,$$

$$H^n(TN, (TN)_0) \cong H^n(T_xM, (T_xM)_0),\ \ u_N \longleftrightarrow u_x.$$

This proves the equivalence between an orientation on M and an orientation on TM in the algebraic topology sense. □

3.4.3 The Thom class and Thom isomorphism

Let $(E, M, \pi; \mathbb{R}^r)$ be a real, oriented vector bundle over a closed manifold M. Let $E_0 \triangleq \{e \in E : e \neq 0\}$. The goal of this subsection is to construct a relative cohomology class $u \in H^r(E, E_0)$, such that the cohomology cup product

$$a \mapsto a \cup u$$

induces an isomorphism between $H^j(E)$ and $H^{j+r}(E, E_0)$ for every j.

We work over a general coefficient ring Λ with unit $\mathbf{1}_\Lambda$. Let $\{u_x : x \in M\}$ be the given orientation of E. By abuse of notation, the image of u_x in $H^r(E_x, (E_x)_0; \Lambda)$ induced by $\mathbf{1}_{\mathbb{Z}} \mapsto \mathbf{1}_\Lambda$ is still denoted as u_x. The main theorem is stated as follows.

Theorem 3.5 (The Thom Isomorphism)**.** *There exists a unique relative cohomology class $u \in H^r(E, E_0; \Lambda)$ such that $u|_x = u_x$ for all $x \in M$. In addition, the map*

$$\cup u : H^j(E; \Lambda) \to H^{j+r}(E, E_0; \Lambda),\ \ a \mapsto a \cup u \tag{3.17}$$

is an isomorphism for every $j \in \mathbb{Z}$.

Remark 3.8. When $j < 0$, one has $H^j(E; \Lambda) = 0$. In this case, the isomorphism (3.17) is understood as the statement that $H^{j+r}(E, E_0; \Lambda) = 0$.

Proof. (i) *$E = M \times \mathbb{R}^r$ is the trivial bundle.*

Assume that E is oriented by $\mathbb{R}^r$, i.e. $u_x = i_x^*(\mathfrak{e}^r)$ where $\mathfrak{e}^r$ is the canonical generator of $H^r(\mathbb{R}^r, \mathbb{R}^r_0; \Lambda)$ and $i_x : \{x\} \times \mathbb{R}^r \to \mathbb{R}^r$ is the standard identification. Let $\pi_2 : E \to \mathbb{R}^r$ denote the projection onto the second component. Define $u \triangleq \pi_2^* \mathfrak{e}^r$. Then $u|_x = u_x$ for all $x \in M$. To see that $\cup u$ is an isomorphism, one considers the following commutative diagram

$$\begin{tikzcd}
H^j(E) \arrow[d] \arrow[dr, "\cup \mathfrak{e}^r"] & \\
H^j(M) \arrow[r, "\times \mathfrak{e}^r"] & H^{j+r}(M \times \mathbb{R}^r,\ M \times \mathbb{R}^r_0).
\end{tikzcd}$$

The bottom map is an isomorphism due to the Künneth theorem (see Theorem A.2). The vertical map is an isomorphism because M is a deformation retract of E (contracting the entire fiber to the base point). Therefore, $\cup u$ must also be an isomorphism.

(ii) *Suppose that $M = M' \cup M''$ and the theorem is true for $E|_{M'}$, $E|_{M''}$, $E|_{M' \cap M''}$ with classes u', u'', $u^\cap$ respectively. Then the result holds for M.*

For simplicity, we write $E' \triangleq E|_{M'}$, $E'' \triangleq E|_{M''}$ and $E^\cap \triangleq E|_{M' \cap M''}$. Consider the Mayer-Vietoris sequence

$$\begin{aligned}
\cdots \to 0 = H^{r-1}(E^\cap, E^\cap_0; \Lambda) &\to H^r(E, E_0; \Lambda) \\
&\xrightarrow{i^*} H^r(E', E'_0; \Lambda) \oplus H^r(E'', E''_0; \Lambda) \xrightarrow{j^*} H^r(E^\cap, E^\cap_0; \Lambda) \to \cdots
\end{aligned}$$

for the relative cohomology (see (A.17)), where

$$i^*(v) \triangleq (v|_{E'}, v|_{E''}),\ \ j^*(v', v'') \triangleq v''|_{E^\cap} - v'|_{E^\cap}.$$

Since $j^*(u', u'') = 0$, one has by exactness that $(u', u'') \in \mathrm{Im}(i^*)$, i.e. $(u', u'') = i^*(u)$ for some $u \in H^r(E, E_0; \Lambda)$. Clearly, $u|_x = u_x$ for every x. The uniqueness of u follows from the injectivity of i^*.

The isomorphism property of $\cup u$ follows from the Five Lemma applied to the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^{j-1}(E^\cap) \arrow[r] \arrow[d, "\cup u^\cap"] & H^j(E) \arrow[r] \arrow[d, "\cup u"] & H^j(E') \oplus H^j(E'') \arrow[r] \arrow[d, "\cup u' \oplus \cup u''"] & H^j(E^\cap) \arrow[r] \arrow[d, "\cup u^\cap"] & \cdots \\
\cdots \arrow[r] & H^{j-1+r}(E^\cap, E^\cap_0) \arrow[r] & H^{j+r}(E, E_0) \arrow[r] & H^{j+r}(E', E'_0) \oplus H^{j+r}(E'', E''_0) \arrow[r] & H^{j+r}(E^\cap, E^\cap_0) \arrow[r] & \cdots,
\end{tikzcd}$$

noting that $\cup u^\cap$ and $\cup u' \oplus \cup u''$ are both isomorphisms.

(iii) Since M is closed, there is a trivialisation of E consisting of at most finitely many members. The result thus follows from the previous two steps and induction.

□

Definition 3.7. The relative cohomology class $u \in H^r(E, E_0)$ is known as the *Thom class* of E. The isomorphism (3.17) is called the *Thom isomorphism.*

Remark 3.9. The Thom isomorphism theorem remains valid over a noncompact base manifold M with coefficient ring Λ being a principal ideal domain. The extension to this case requires additional tools from homological algebra. The reader is referred to [MS74] for the details.

3.5 The Poincaré dual of a submanifold

We now return to the question raised at the beginning of Section 3.4 about constructing the Poincaré dual of a submanifold. Let X be a closed, oriented n-manifold. We use $D : H_*(X) \to H^{n-*}(X)$ to denote the inverse of the Poincaré duality (3.9) (the *Poincaré dual*). In other words, one has

$$D(a) \cap [X] = a \tag{3.18}$$

for any homology class a in some $H_{n-i}(X)$.

A particular situation is the case when the homology class a comes from a submanifold of X. Let A be a closed, oriented $(n-i)$-manifold that is closed embedded as a submanifold of X. The aim of this section is to understand the Poincaré dual of the homology class $[A]$ in X. Stated in vague terms, the main theorem we shall prove in this section is that *the Poincaré dual of* $[A]$ *in* X *is the (absolute) Thom class of the normal bundle of* A *in* X. See Proposition 3.5 and Corollary 3.1 for the precise formulation. This is a nice result as it provides a cohomological description (the Poincaré dual as a Thom class) of a geometric object (the submanifold).

3.5.1 Normal bundle and tubular neighbourhood

We begin by defining the normal bundle. For simplicity, we will just assume that A is an actual subset of X.

Definition 3.8. The *normal bundle of* A *in* X, denoted as N_A^X, is the real vector bundle over A whose fiber at each $x \in A$ is the quotient space $N_x \triangleq T_xX/T_xA$.

We will fix an orientation of N_A^X in the following way. Let $x \in A$. Pick any complementary subspace V of T_xA in T_xX (i.e. $T_xX = T_xA \oplus V$). The quotient map $T_xX \to N_x$ restricts to an isomorphism between V and N_x. A basis $\{f_1, \cdots, f_i\}$ of N_x is declared to be *positive* if

$$\{e_1, \cdots, e_{n-i}, f'_1, \cdots, f'_i\}$$

is a positive basis of T_xX whenever $\{e_1, \cdots, e_{n-i}\}$ is a positive basis of T_xA. Here f'_j is the element in V corresponding to f_j under the aforementioned isomorphism.

We let the reader check that the above orientation of N_A^X at x does not depend on the choice of the splitting factor V. By varying $x \in A$, one obtains a well-defined orientation on N_A^X, which is canonically induced from the ones on A and X. Heuristically, this orientation is determined by the following condition:

$$\operatorname{or}(X) = \operatorname{or}(A) \oplus \operatorname{or}(N_A^X), \quad (\text{"base}\rightarrow\text{fiber" orientation}) \tag{3.19}$$

where $\operatorname{or}(\cdot)$ means orientation.

Sometimes it is more convenient to work with a homotopy equivalent version of the normal bundle: a tubular neighbourhood.

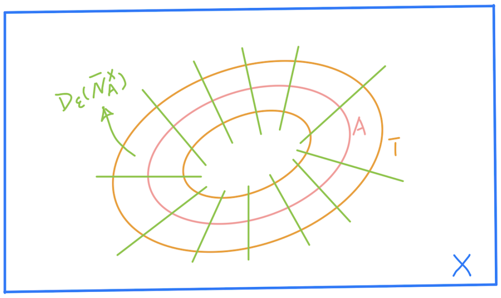
Definition 3.9. A *tubular neighbourhood of A in X* is an open neighbourhood U of A in X that is diffeomorphic to an open neighbourhood of the zero section of N_A^X.

Theorem 3.6 (Tubular Neighbourhood Theorem)**.** *A tubular neighbourhood exists.*

Proof. Fix a Riemannian metric g on X. Consider the bundle $\bar{N}_A^X$ whose fiber $\bar{N}_x$ at $x \in A$ is the orthogonal complement of T_xA in T_xX with respect to the given metric. It is easily seen that N_A^X and $\bar{N}_A^X$ are isomorphic as vector bundles. By the compactness of A, there exists $\varepsilon > 0$ such that the exponential map

$$\exp : D_\varepsilon(\bar{N}_A^X) \to X \tag{3.20}$$

maps the ε-disk bundle $D_\varepsilon(\bar{N}_A^X) \triangleq \left\{v \in \bar{N}_A^X : \|v\|_g < \varepsilon\right\}$ diffeomorphically onto an open neighbourhood of A in X.



This is by definition a tubular neighbourhood of A in X. $\square$

Remark 3.10. From the above proof, it is clear that the tubular neighbourhood can be chosen to be diffeomorphic to N_A^X.

We will use N to denote the normal bundle N_A^X and use T to denote the closure of the tubular neighbourhood of A constructed in the previous proof. Note that T is a (closed) disk bundle over A whose fibers are closed disks with spherical boundary. In addition, as a submanifold T carries an induced orientation from X . This orientation is consistent with (3.19).

There are obvious identifications

$$H^i(X, X\backslash A) \cong H^i(T, \partial T) \cong H^i(N, N_0), \tag{3.21}$$

which will be used frequently in the sequel. The first isomorphism of (3.21) follows from the relations $H^i(X, X\backslash A) \cong H^i(T, T\backslash A)$ (excision) and $H^i(T, T\backslash A) \cong H^i(T, \partial T)$ (∂T is a deformation retract of $T\backslash A$). The second isomorphism of (3.21) is obtained in the same way. Let $u_N \in H^i(N, N_0)$ be the Thom class of N with respect to the orientation (3.19). We denote $u_T \in H^i(T, \partial T)$ and $u_A^{X, X\backslash A} \in H^i(X, X\backslash A)$ as the counterparts of u_N under the isomorphisms (3.21). We also denote $u_A^X \triangleq j^*(u_A^{X, X\backslash A})$ where $j^* : H^i(X, X\backslash A) \to H^i(X)$ is the natural map (the *absolute Thom class*).

3.5.2 The Thom class and Poincaré dual

Recall that T inherits an orientation from X which is consistent with (3.19). In particular, there is a well-defined (relative) fundamental class $[T, \partial T] \in H_n(T, \partial T)$ and a Poincaré duality map (3.12). We use P_X (respectively, P_T) to denote the map (3.9) (respectively, (3.12)). We also denote $D_X \triangleq P_X^{-1}$ (respectively, $D_T \triangleq P_T^{-1}$) as the Poincaré dual. The main result of this subsection is stated as follows.

Proposition 3.5. *The Poincaré dual of $[A]$ in T is equal to the Thom class u_T, i.e. $D_T\big((\iota_A^T)_*([A])\big) = u_T$, where $\iota_A^T : A \to T$ is the inclusion map.*

Proof. Let $v_T \triangleq D_T\big((\iota_A^T)_*([A])\big)$. For each $x \in A$, our aim is to show that $\tau \triangleq v_T|_x$ is the positive generator of $H^i(T_x, \partial T_x)$. Let B be a small closed disk around x so that $T|_B \cong B \times D^i =: W$ where D^i is the closed i-disk. Let $w \triangleq v_T|_W \in H^i(W, B \times \partial D^i)$. The relation $v_T \cap [T, \partial T] = [A]$ descends to the local relation that

$$w \cap [W, \partial W] = [B, \partial B]. \tag{3.22}$$

On the other hand, noting that

$$H^i(W, B \times \partial D^i) \cong H^0(B) \otimes H^i(D^i, \partial D^i),$$

one can write $w = 1_B \otimes \tau$. Since $[W, \partial W] = [B, \partial B] \times [D^i, \partial D^i]$, the relation (3.22) implies that $\langle \tau, [D^i, \partial D^i] \rangle = 1$. As a consequence, τ is the positive generator of $H^i(T_x, \partial T_x)$. $\square$

The following result is a consequence of Proposition 3.5. It justifies the earlier claim that the Poincaré dual of a submanifold is the absolute Thom class of the normal bundle.

Corollary 3.1. *The Poincaré dual of* $[A]$ *in* X *is equal to* u_A^X*, i.e.* $D_X((\iota_A^X)_*([A])) = u_A^X$*, where* $\iota_A^X : A \to X$ *is the inclusion.*

Proof. This follows immediately from the commutative diagram

$$\begin{tikzcd}
H_{n-i}(A) \arrow[r, "\iota_A^T"] \arrow[dr, "\iota_A^X"'] & H_{n-i}(T) \arrow[r, "D_T"] \arrow[d, "\iota_T^X"] & H^i(T, \partial T) \arrow[r, "\cong", "(3.21)"'] & H^i(X, X \setminus A) \arrow[d] \\
 & H_{n-i}(X) \arrow[rr] & & H^i(X),
\end{tikzcd}$$

noting that D_T sends $(\iota_A^T)_*([A])$ to u_T by Proposition 3.5. ☐

3.6 Intersection product formula

In this section, we give a cohomological description of intersection: the *intersection product formula*. This plays an essential role in the later proof of the Lefschetz fixed point formula.

Let X be a closed, connected and oriented n-manifold. Let A, B be closed, oriented and embedded submanifolds of X with dimensions $n-i$, $n-j$ respectively. We assume that A intersects B transversally (see Definition 2.1 with $f = \mathrm{id}$). In particular, $i + j \leqslant n$ and $A \cap B$ is a submanifold of X with dimension $n - i - j$.

We orient $A \cap B$ in the following way. Let $x \in A \cap B$. A basis $\mathcal{U}$ of $T_x(A \cap B)$ is declared to be *positive* if the following property holds true:

- Extend $\mathcal{U}$ to a family $\mathcal{B} = \{\mathcal{V}, \mathcal{U}, \mathcal{W}\} \subseteq T_xX$ such that $\mathcal{B}$ is a basis of T_xX, $\{\mathcal{V}, \mathcal{U}\}$ is a basis of T_xA and $\{\mathcal{U}, \mathcal{W}\}$ is a basis of T_xB. If $\{\mathcal{V}, \mathcal{U}\}$ and $\{\mathcal{U}, \mathcal{W}\}$ are both positively oriented, then $\mathcal{B}$ is also positively oriented.

We leave it as an exercise to show that the above orientation of $T_x(A \cap B)$ is well-defined. By varying x in $A \cap B$, one obtains an orientation of $A \cap B$.

Let $[A], [B], [A \cap B]$ be the fundamental classes of $A, B, A \cap B$ respectively. Here we view them as homology classes in X without writing the inclusion homomorphisms for simplicity. Recall that $D_X = P_X^{-1} : H_{n-*}(X) \to H^*(X)$ is the Poincaré dual (inverse of (3.9)).

Definition 3.10. The *intersection product* between $[A]$ and $[B]$ is the homology class $[A] \cdot [B] \in H_{n-i-j}(X)$ defined by the relation

$$D_X([A] \cdot [B]) = D_X([A]) \cup D_X([B]),$$

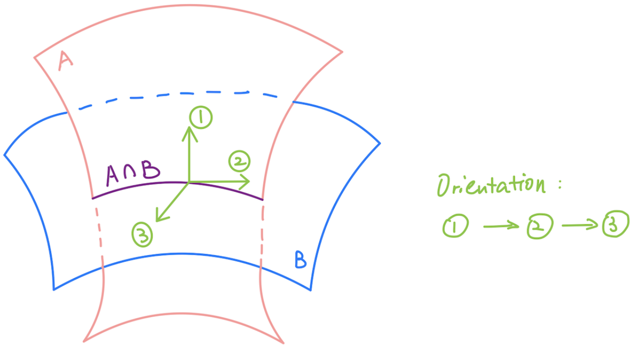
where $\cup$ is the cohomology cup product.

Remark 3.11. If $i+j=n$, the intersection $A \cap B$ is just a finite set of discrete points. In this case, $[A \cap B] = I(A,B)[\mathrm{pt}] \in H_0(X)$, where $I(A,B)$ is the intersection number of A with respect to B (see Definition 2.3) and $[\mathrm{pt}]$ is the homology class defined by any 0-simplex (a point) in X.

The main result of this section is stated as follows. It provides a cohomological description of geometric intersection.

Theorem 3.7 (Intersection Product Formula)**.** $[A \cap B] = [A] \cdot [B]$.

Proof. Let N_A^X be the normal bundle of A in X which is oriented by the convention (3.19). Similarly, let $N_{A\cap B}^B$ be the normal bundle of $A \cap B$ in B, whose orientation is induced from the orientations of $A \cap B$ and B through the same convention (3.19). Let $N_A^X|_{A\cap B}$ be the bundle obtained by restricting N_A^X to $A \cap B$ (the pullback of N_A^X by the inclusion $\iota_{A\cap B}^A : A \cap B \to A$; cf. (B.18)).



Under the above orientation conventions, it is apparent that

$$N_A^X|_{A\cap B} \cong N_{A\cap B}^B \tag{3.23}$$

as oriented vector bundles (namely, there is an orientation-preserving bundle isomorphism between them).

Let $u_A^{X,X\backslash A} \in H^i(X, X\backslash A)$ and $u_A^X \in H^i(X)$ denote the relative and absolute Thom classes (see the discussion at the end of Section 3.5.1). The next crucial observation is that

$$u_{A\cap B}^{B,B\backslash A\cap B} = r_1\big(u_A^{X,X\backslash A}\big),\ u_{A\cap B}^B = r_2\big(u_A^X\big), \tag{3.24}$$

where

$$r_1 : H^i(X, X\backslash A) \to H^i(B, B\backslash A \cap B), \quad r_2 : H^i(X) \to H^i(B)$$

are the restriction homomorphisms. The first relation of (3.24) follows from the characterisation of Thom class and the fact that both sides restrict to the positive generator of the i-th relative cohomology group of $(N_A^X|_{A\cap B})_x \cong (N_{A\cap B}^B)_x$ at every $x \in A \cap B$; the latter claim is an immediate consequence of (3.23). The second relation of (3.24) follows from the first one and the commutative diagram

$$\begin{tikzcd}
u_A^{X,X\setminus A} \in H^i(X, X \setminus A) \arrow[r] \arrow[d, "r_1"] & u_A^X \in H^i(X) \arrow[d, "r_2"'] \\
u_{A\cap B}^{B,B\setminus A\cap B} \in H^i(B, B \setminus A \cap B) \arrow[r] & u_{A\cap B}^B \in H^i(B).
\end{tikzcd}$$

Using the second relation of (3.24) and duality, the proof of the intersection product formula now becomes a matter of algebraic manipulation. Indeed, by Definition 3.10 of the intersection product, it suffices to prove that

$$(\iota_{A\cap B}^X)_*[A \cap B] = (u_A^X \cup u_B^X) \cap [X], \tag{3.25}$$

By Corollary 3.1, one has

$$\begin{aligned}
(\iota_{A\cap B}^X)_*[A \cap B] &= (\iota_B^X)_*(\iota_{A\cap B}^B)_*[A \cap B] \\
&= (\iota_B^X)_*\big(u_{A\cap B}^B \cap [B]\big) = (\iota_B^X)_*\Big(r_2\big(u_A^X\big) \cap [B]\Big) \\
&= u_A^X \cap \big((\iota_B^X)_*[B]\big) = u_A^X \cap \big(u_B^X \cap [X]\big) \\
&= (u_A^X \cup u_B^X) \cap [X].
\end{aligned}$$

The third equality follows from (3.24), the fourth equality follows from the naturality of cap product (see Proposition 3.2 (iv)) and the last equality follows from Proposition 3.2 (i). This proves the relation (3.25), which therefore completes the proof of the theorem. □

3.7 The Lefschetz fixed point formula

In this section, we develop the proof of Theorem 3.2 following the strategy outlined in Section 3.1. Let M be a closed, oriented n-manifold. The essential idea is to apply intersection theory to the pair $(\Gamma(f), \Delta)$, where

$$\Gamma(f) \triangleq \{(x, f(x)) : x \in M\}$$

is the graph of f and

$$\Delta \triangleq \{(x, x) : x \in M\}$$

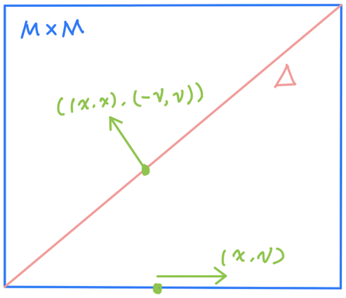
is the diagonal in $M \times M$.

3.7.1 The diagonal class

Let N be the normal bundle of Δ in $M \times M$ (see Definition 3.8). There is a canonical isomorphism $N \cong TM$ which is defined by

$$T_xM \ni (x,v) \mapsto \big((x,x),[(-v,v)]\big) \in N_x = \frac{T_xM \oplus T_xM}{T_{(x,x)}\Delta}. \tag{3.26}$$

We orient N by TM (which is in turn oriented by M itself) through this isomorphism. Note that this orientation is consistent with (3.19), where both Δ and $M \times M$ are oriented by M.



Let

$$u' \in H^n(M \times M, (M \times M)\backslash\Delta) \cong H^n(N, N_0)$$

be the Thom class of N with respect to the above orientation. For each $x \in M$, let $\mu_x \in H_n(M, M\backslash\{x\})$ be the preferred generator (coming from the orientation of M) and let $u_x \in H^n(M, M\backslash\{x\})$ be its dual, i.e. $\langle u_x, \mu_x\rangle = 1$. Define $j_x : M \to M \times M$ by $j_x(y) \triangleq (x,y)$ (the vertical slice at x).

Lemma 3.3. *u' is the unique class in $H^n(M\times M, (M\times M)\backslash\Delta)$ such that $j_x^* u' = u_x$ for every $x \in M$.*

Proof. Pick a Riemannian metric on M. Fix $x \in M$ and let B_ε be the closed ε-ball on T_xM (ε small). Consider the following two maps:

$$\varphi_1 : B_\varepsilon \to M \times M,\ \varphi_1(v) \triangleq \big(\exp_x(-v), \exp_x(v)\big),$$
$$\varphi_2 : B_\varepsilon \to M \times M,\ \varphi_2(v) \triangleq \big(x, \exp_x(v)\big).$$

Note that $\varphi_1 \simeq \varphi_2$ through the homotopy

$$v \mapsto \big(\exp_x(-tv), \exp_x(v)\big)_{0\leqslant t\leqslant 1}.$$

As a result, $\varphi_1^* = \varphi_2^*$ on $H^n(M\times M, (M\times M)\backslash\Delta)$. Since u' is the Thom class, it restricts to the local orientation class of the normal bundle N at x, which corresponds to u_x under the isomorphisms

$$H^n(M \times M, (M \times M)\backslash\{(x,x)\}) \cong H^n(N_x, (N_x)_0). \tag{3.27}$$

By identifying $H^n(B_\varepsilon, B_\varepsilon\backslash\{0\})$ with (3.27), one has $\varphi_1^* u' = u_x$. On the other hand, under the same identification one has $\varphi_2^* u' = j_x^* u'$. Therefore, $j_x^* u' = u_x$. Uniqueness follows from the characterisation of the Thom class. ☐

Definition 3.11. The *diagonal class* $u'' \in H^n(M \times M)$ is the image of u' under the inclusion homomorphism $H^n(M \times M, (M \times M)\backslash\Delta) \to H^n(M \times M)$.

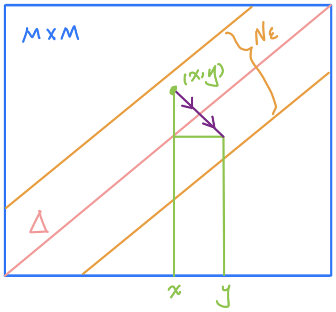
The following result shows that the diagonal class u'' "concentrates" at the diagonal Δ in an algebraic sense.

Lemma 3.4. *One has $(a \times 1) \cup u'' = (1 \times a) \cup u''$ for all $a \in H^*(M)$, where $\times$ is the cohomology cross product.*

Proof. Let $p_i : M \times M \to M$ denote the projection onto the i-th component ($i = 1, 2$). Then one has

$$p_1^* a = a \times 1,\ \ p_2^* a = 1 \times a$$

for any $a \in H^k(M)$. Let N_ε be a tubular neighbourhood of Δ in $M \times M$ (ε small). It is easy to show that $p_1|_{N\varepsilon} \simeq p_2|_{N_\varepsilon}$.



As a result,

$$p_1^* a|_{N_\varepsilon} = p_2^* a|_{N_\varepsilon} \in H^k(N_\varepsilon).$$

Following the arrows in the commutative diagram

$$\begin{tikzcd}
H^k(M \times M) \arrow[r, "\iota^*"] \arrow[d, "\cup u'"] & H^k(N_\varepsilon) \arrow[d, "\cup u'"'] \\
H^k(M \times M, M \times M \setminus \Delta) \arrow[r, "\cong"] & H^k(N_\varepsilon, N_\varepsilon \setminus \Delta)
\end{tikzcd}$$

where $\iota : N_\varepsilon \to M \times M$ denotes the inclusion, one finds that

$$(p_1^* a) \cup u' = (p_2^* a) \cup u'. \tag{3.28}$$

The result follows by applying the inclusion homomorphism $H^*(M \times M, (M \times M)\backslash\Delta) \to H^*(M \times M)$ to the relation (3.28). $\square$

3.7.2 Slant product and dual cohomology basis

In this subsection, we work over a field Λ. Our aim is to prove the following result, which provides a representation of u'' in terms of a cohomology basis. Note that $H^k(M;\Lambda)$ is finite dimensional for every k, since M is assumed to be closed.

Proposition 3.6. *Let $\{b_1, \cdots, b_r\}$ be a basis of the cohomology ring $H^*(M;\Lambda)$. Then there exists a dual basis $\{b_1^\#, \cdots, b_r^\#\}$ in the sense that*

$$\langle b_i \cup b_j^\#, [M] \rangle = \delta_{ij} \tag{3.29}$$

for all i, j. In addition, one has

$$u'' = \sum_{i=1}^{r} (-1)^{\deg b_i} b_i \times b_i^\#, \tag{3.30}$$

where $\times$ is the cohomology cross product.

Remark 3.12. By abuse of notation, we used the same u'' to denote the image of u'' (over $\mathbb{Z}$) under the homomorphism $H^n(M \times M) \to H^n(M \times M;\Lambda)$ induced by $\mathbf{1}_\mathbb{Z} \mapsto \mathbf{1}_\Lambda$.

The proof of Proposition 3.6 relies on the notion of slant product, which we shall first define. Let X, Y be given topological spaces. Recall that the Künneth theorem provides a canonical isomorphism

$$H^k(X \times Y;\Lambda) \cong \bigoplus_{i+j=k} H^i(X;\Lambda) \otimes H^j(Y;\Lambda). \tag{3.31}$$

Definition 3.12. The *slant product* is the map

$$/ : H^k(X \times Y;\Lambda) \otimes H_j(Y;\Lambda) \to H^{k-j}(X;\Lambda)$$

induced by the relation

$$(a \otimes b)/\beta \mapsto \langle b, \beta \rangle a, \quad a \in H^{k-*}(X;\Lambda), b \in H^*(Y;\Lambda), \beta \in H_j(Y;\Lambda)$$

under the Künneth isomorphism (3.31).

Lemma 3.5. *The slant product is left $H^*(X)$-linear, i.e.*

$$\big((a\times 1)\cup b\big)/\beta = a\cup (b/\beta) \tag{3.32}$$

for any $a\in H^*(X;\Lambda)$, $b\in H^*(X\times Y;\Lambda)$ and $\beta\in H_*(Y;\Lambda)$.

Proof. It suffices to consider $b=b'\times b''$ with $b'\in H^*(X;\Lambda)$ and $b''\in H^*(Y;\Lambda)$. In this case, by using the relation (A.20) one finds that

$$\big((a\times 1)\cup b\big)/\beta = \langle b'',\beta\rangle(a\cup b') = a\cup\big(\langle b'',\beta\rangle b'\big) = a\cup (b/\beta).$$

The relation (3.32) thus follows. □

The slant product provides a very simple relation between the diagonal class u'' and the fundamental class $[M]$.

Lemma 3.6. $u''/[M]=1\in H^0(M;\Lambda)$.

Proof. Fix any $x\in M$. We want to show that $(u''/[M])|_x = 1\in\Lambda$. Consider the map

$$J_x:\{x\}\times M\to M\times M,\ J_x(x,y)\triangleq (x,y).$$

Using the commutative diagram

$$\begin{tikzcd}
H^n(M\times M) \arrow[r, "\cdot/{[M]}"] \arrow[d, "J_x^*"] & H^0(M) \arrow[d] \\
H^n(\{x\}\times M) \arrow[r, "\cdot/{[M]}"] & H^0(\{x\})
\end{tikzcd}$$

and noting that $J_x^*u''=1\times j_x^*u''$, it suffices to show that

$$\langle j_x^*u'',[M]\rangle = 1. \tag{3.33}$$

To this end, consider the following commutative diagram

$$\begin{tikzcd}
H^n(M\times M, M\times M\setminus\Delta) \arrow[r, "j_x^*"] \arrow[d] & H^n(M, M\setminus\{x\}) \arrow[d] \\
H^n(M\times M) \arrow[r, "j_x^*"] & H^n(M)
\end{tikzcd}$$

where the vertical arrows are the inclusion homomorphisms. According to Lemma 3.3, one has

$$\langle j_x^*u'',[M]\rangle = \langle \rho_x^*u_x,[M]\rangle = \langle u_x,\rho_x([M])\rangle = \langle u_x,\mu_x\rangle = 1.$$

Here $\rho_x: H_n(M)\to H_n(M,M\backslash\{x\})$ is the inclusion homomorphism and $u_x\in H^n(M,M\backslash\{x\})$ (respectively, $\mu_x\in H_n(M,M\backslash\{x\})$) is the local orientation cohomology (respectively, homology) class at x. The claim (3.33) thus follows. □

We are now in a position to prove Proposition 3.6.

Proof of Proposition 3.6. By using the Künneth isomorphism

$$H^n(M \times M; \Lambda) = \bigoplus_{k=0}^{n} H^k(M; \Lambda) \otimes H^{n-k}(M; \Lambda),$$

one can write

$$u'' = b_1 \times c_1 + \cdots + b_r \times c_r, \tag{3.34}$$

with some $c_i \in H^{n-\deg b_i}(M; \Lambda)$.

Let $a \in H^*(M; \Lambda)$ be any given cohomology class. Recall from Lemma 3.4 that

$$\big((a \times 1) \cup u''\big)/[M] = \big((1 \times a) \cup u''\big)/[M]. \tag{3.35}$$

By Lemma 3.5 and Lemma 3.6, the LHS is just a. The RHS equals

$$\sum_i (-1)^{\deg a \deg b_i} (b_i \times (a \cup c_i))/[M] = \sum_i (-1)^{\deg a \deg b_i} \langle a \cup c_i, [M] \rangle b_i.$$

We now choose $a = b_j$. The relation (3.35) thus becomes

$$b_j = \sum_i (-1)^{\deg a \deg b_i} \langle b_j \cup c_i, [M] \rangle b_i.$$

Since $\{b_i\}$ is a cohomology basis, one finds that

$$(-1)^{\deg b_j \deg b_i} \langle b_j \cup c_i, [M] \rangle = \delta_{ij}.$$

The relation (3.29) follows by defining

$$b_i^{\#} \triangleq (-1)^{\deg b_i} c_i.$$

It is clear from (3.29) that the family $\{b_i^{\#}\}$ is linearly independent and thus provides a basis of $H^*(M; \Lambda)$. The relation (3.30) is also immediate from (3.34) and the definition of $b_i^{\#}$.

□

As an application of Proposition 3.6, one can obtain another proof of the Poincaré duality theorem which is now stated in terms of cohomology.

Corollary 3.2 (The Poincaré Duality)**.** *There is a canonical isomorphism*

$$\hat{P}_M : H^k(M; \Lambda) \to \mathrm{Hom}_\Lambda(H^{n-k}(M; \Lambda), \Lambda), \tag{3.36}$$

$$\hat{P}_M(a)(b) \triangleq \langle b \cup a, [M] \rangle, \ a \in H^k(M; \Lambda), b \in H^{n-k}(M; \Lambda)$$

for every $k = 0, 1, \cdots, n$.

Proof. Let $\{b_i\}$ and $\{b_i^{\#}\}$ be the bases given by Proposition 3.6. We claim that the family

$$\{b_i^{\#} : \deg b_i = k\} \tag{3.37}$$

is a basis of $H^{n-k}(M;\Lambda)$. Indeed, suppose that

$$\sum_{i:\deg b_i = k} \lambda_i b_i^{\#} = 0$$

with some $\lambda_i \in \Lambda$. By taking cup product with b_j ($\deg b_j = k$) and using the relation (3.29), one finds that $\lambda_i = 0$. As a result, the family (3.37) is linear independent. In particular,

$$\dim H^k(M;\Lambda) \leqslant \dim H^{n-k}(M;\Lambda).$$

By interchanging the roles of k and $n-k$, one obtains the reversed inequality as well. Therefore,

$$\dim H^k(M;\Lambda) = \dim H^{n-k}(M;\Lambda)$$

and the family (3.37) is a basis of $H^{n-k}(M;\Lambda)$.

By using the relation (3.29) again, it is easily seen that the family

$$\left\{\hat{P}_M(b_i) : \deg b_i = k\right\}$$

is dual to (3.37), i.e. $\hat{P}_M(b_i)(b_j^{\#}) = \delta_{ij}$. As a result, it is a basis of $\mathrm{Hom}_\Lambda(H^{n-k}(M;\Lambda),\Lambda)$. It follows that $b_i \mapsto \hat{P}_M(b_i)$ is a basis correspondence and the map $\hat{P}_M$ is therefore an isomorphism. □

Remark 3.13. Under the natural isomorphism $H_{n-k}(M;\Lambda) \cong \mathrm{Hom}_\Lambda(H^{n-k}(M;\Lambda),\Lambda)$ (which is valid since Λ is a field), the two maps P_M and $\hat{P}_M$ are identical.

3.7.3 The Euler class and Euler characteristic

In this subsection, we introduce the (topological) Euler class and investigate its relation with the Euler characteristic.

Definition 3.13. Let $(E, M, \pi; V)$ be a real, oriented vector bundle of rank r over a closed manifold M. The *Euler class* of E, denoted as $e(E) \in H^r(M)$, is the image of the Thom class u of E under the homomorphism

$$H^r(E, E_0) \xrightarrow{j^*} H^r(E) \xrightarrow{\cong} H^r(M), \ u \mapsto e(E).$$

Here the first map j^* is the inclusion homomorphism. The second map is the pullback by the zero section, which is an isomorphism since M is a deformation retract of E.

Remark 3.14. If Λ is a field, the image of $e(E)$ under $\mathbf{1}_{\mathbb{Z}} \mapsto \mathbf{1}_{\Lambda}$ is still denoted as $e(E) \in H^r(M;\Lambda)$.

We now establish a basic result which asserts that the natural pairing between the Euler class of TM and the fundamental class of M yields the Euler characteristic.

Proposition 3.7. *One has*

$$e(TM) = \Delta^*(u'') \text{ and } \langle e(TM), [M]\rangle = \chi(M), \tag{3.38}$$

where $\Delta : M \to M \times M$ $(\Delta(x) \triangleq (x,x))$ *is the diagonal embedding.*

Proof. The first relation in (3.38) is a direct consequence of the following commutative diagram

$$\begin{tikzcd}
H^n(M \times M, M \times M \setminus \Delta) \arrow[r] \arrow[d, "\cong"] & H^n(M \times M) \arrow[r, "\Delta^*"] & H^n(M) \arrow[d, "\pi^*"] \\
H^n(N, N_0) \arrow[r] & H^n(TM, (TM)_0) \arrow[r] & H^n(TM).
\end{tikzcd}$$

Here N is the normal bundle of Δ in $M \times M$ (recall from (3.26) that $N \cong TM$) and $\pi : TM \to M$ is the bundle projection. To prove the second relation, one recalls from Proposition 3.6 that

$$u'' = \sum_i (-1)^{\deg b_i} b_i \times b_i^{\#}$$

over a field, say $\Lambda = \mathbb{R}$. By using the first relation in (3.38) (more precisely, the same induced relation over $\mathbb{R}$), one has

$$e(TM) = \sum_i (-1)^{\deg b_i} b_i \cup b_i^{\#}.$$

It follows from the duality relation (3.29) that

$$\langle e(TM), [M]\rangle = \sum_i (-1)^{\deg b_i} \langle b_i \cup b_i^{\#}, [M]\rangle = \sum_{k=0}^{n} (-1)^k \dim H^k(M;\mathbb{R}).$$

Note that the above equation is a numerical identity and the RHS is an integer. Therefore, the same identity also holds over $\mathbb{Z}$. This completes the proof of the second relation in (3.38). □

3.7.4 Proof of Theorem 3.2

Finally, we are in a position to prove the Lefschetz fixed point formula, which therefore also completes the proof of the Poincaré-Hopf index formula. Here the coefficient ring is assumed to be the real field $\mathbb{R}$.

Proof of Theorem 3.2. Since both sides of (3.1) are homotopy invariant, one may assume without loss of generality that the graph $\Gamma(f)$ of f intersects Δ transversally. According to the intersection product formula (Theorem 3.7), one has

$$[\Gamma(f) \cap \Delta] = [\Gamma(f)] \cdot [\Delta], \tag{3.39}$$

where both $\Gamma(f)$ and Δ are oriented by M in the obvious way. The LHS of (3.39) is just $I(\mathrm{id} \times f, \Delta)[\mathrm{pt}]$ by Remark 3.11. Now we compute the RHS.

Let $D : H_{2n-*}(M \times M; \mathbb{R}) \to H^*(M \times M; \mathbb{R})$ denote the Poincaré dual map in $M \times M$. According to Corollary 3.1, one has $D([\Delta]) = u''$. Denote $v \triangleq D(\Gamma(f))$. It follows from Definition 3.10 of the intersection product that

$$\begin{aligned}
[\Gamma(f)] \cdot [\Delta] &= \langle v \cup u'', [M \times M] \rangle [\mathrm{pt}] \\
&= (-1)^n \langle u'' \cup v, [M \times M] \rangle [\mathrm{pt}] \\
&= (-1)^n \langle u'', v \cap [M \times M] \rangle [\mathrm{pt}] \\
&= (-1)^n \langle u'', [\Gamma(f)] \rangle [\mathrm{pt}].
\end{aligned} \tag{3.40}$$

Let $F : M \to M \times M$ be defined by $F(x) \triangleq (x, f(x))$. Note that $F_*([M]) = [\Gamma(f)]$. By using the representation (3.30) of u'' in terms of a cohomology basis $\{b_i\}$ and its dual $\{b_i^{\#}\}$, one finds that

$$\begin{aligned}
\langle u'', [\Gamma(f)] \rangle &= \Big\langle \sum_i (-1)^{\deg b_i} b_i \times b_i^{\#}, F_*([M]) \Big\rangle \\
&= \Big\langle \sum_i (-1)^{\deg b_i} F^*(b_i \times b_i^{\#}), [M] \Big\rangle \\
&= \Big\langle \sum_i (-1)^{\deg b_i} b_i \cup f^* b_i^{\#}, [M] \Big\rangle,
\end{aligned} \tag{3.41}$$

where the last equality follows from the simple relation that $F^*(b_i \times b_i^{\#}) = b_i \cup f^* b_i^{\#}$ (left as an exercise).

Since $\{b_i^{\#}\}$ is a cohomology basis, one can write $f^* b_i^{\#} = \sum_j \lambda_{ij} b_j^{\#}$ uniquely with some $\lambda_{ij} \in \mathbb{R}$. It follows from the relation (3.29) that

$$\begin{aligned}
&\Big\langle \sum_i (-1)^{\deg b_i} b_i \cup f^* b_i^{\#}, [M] \Big\rangle \\
&= \sum_{i,j} (-1)^{\deg b_i} \langle b_i \cup \lambda_{ij} b_j^{\#}, [M] \rangle = \sum_i (-1)^{\deg b_i} \lambda_{ii} \\
&= \sum_{k=0}^{n} (-1)^k \operatorname{Tr}\big[f^* : H^{n-k}(M;\mathbb{R}) \to H^{n-k}(M;\mathbb{R}) \big]. \qquad (3.42)
\end{aligned}$$

By substituting (3.42) back into (3.41, 3.40) and changing the index k to $n-k$, one concludes that

$$[\Gamma(f)] \cdot [\Delta] = \sum_{k=0}^{n} (-1)^k \operatorname{Tr}\big[f^* : H^k(M;\mathbb{R}) \to H^k(M;\mathbb{R}) \big] \times [\mathrm{pt}].$$

The Lefschetz fixed point formula (3.1) follows by equating the LHS with $[\Gamma(f) \cap \Delta] = I(\mathrm{id} \times f, \Delta)[\mathrm{pt}]$.

$\square$

4 The differential viewpoint

In this chapter, we establish the differential counterparts of various topological results (Poincaré duality, Thom isomorphism, intersection product formula) in the previous chapter using the de Rham cohomology. Such correspondence is not surprising in view of the de Rham theorem, which will be addressed in Section 4.7.

4.1 The Poincaré lemma

We first introduce another important type of cohomology, which is the differential counterpart of singular cohomology with compact support (cf. Definition 3.3).

Definition 4.1. Let M be a manifold. We use $\Omega_c^*(M)$ to denote the complex of differential forms on M *with compact support*. The exterior derivative operator d is well defined on $\Omega_c^*(M)$. The resulting cohomology $H_c^*(M)$ is known as the *de Rham cohomology of M with compact support* (or *compactly supported cohomology* in short).

In this section, we compute the two types of de Rham cohomology of $\mathbb{R}^n$. This is known as *the Poincaré lemma*.

Theorem 4.1. *One has*

$$H^p(\mathbb{R}^n) \cong \begin{cases} \mathbb{R}, & p = 0; \\ 0, & p \neq 0, \end{cases} \quad H_c^p(\mathbb{R}^n) \cong \begin{cases} \mathbb{R}, & p = n; \\ 0, & p \neq n. \end{cases}$$

4.1.1 One-dimensional case

Theorem 4.1 will be proved by induction on the dimension n. In this subsection, we first handle the one-dimensional case, which is a rather elementary matter.

Proof of Theorem 4.1 for $\mathbb{R}^1$. (i) (The de Rham cohomology) If f is a smooth function on $\mathbb{R}^1$, one has $df = 0 \iff f =$ constant. This proves $H^0(\mathbb{R}^1) \cong \mathbb{R}$. If $\omega = g(x)dx$ is a 1-form on $\mathbb{R}^1$, one can write $\omega = df$, where $f(x) \triangleq \int_0^x g(y)dy$. This proves $H^1(\mathbb{R}^1) = 0$.

(ii) (Compactly supported cohomology) If f is a closed 0-form, it has to be a constant function, hence being identically zero since it has compact support by assumption. Therefore, $H_c^0(\mathbb{R}^1) = 0$.

For the first order cohomology, consider the integration map

$$\int_{\mathbb{R}^1} : \Omega_c^1(\mathbb{R}^1) \to \mathbb{R}, \ \omega \mapsto \int_{\mathbb{R}^1} \omega.$$

It is obvious that $\int_{\mathbb{R}^1}$ is surjective. We claim that

$$\ker \int_{\mathbb{R}^1} = \{df : f \in C_c^\infty(\mathbb{R}^1)\}. \tag{4.1}$$

Indeed, suppose that $\omega = df$ for some $f \in C_c^\infty(\mathbb{R}^1)$. Then

$$\int_{\mathbb{R}^1} \omega = \int_{\mathbb{R}^1} df = f(\infty) - f(-\infty) = 0.$$

Conversely, suppose that $\omega \in \ker \int_{\mathbb{R}^1}$. Write $\omega = g(x)dx$ for some $g \in C_c^\infty(\mathbb{R}^1)$ and define $f(x) \triangleq \int_{-\infty}^x g(y)dy$. Then $f \in C_c^\infty(\mathbb{R}^1)$ (because $\int_{\mathbb{R}^1} \omega = 0$) and $\omega = df$. Therefore, the claim (4.1) holds. It follows that $H_c^1(\mathbb{R}^1) \cong \mathbb{R}$.

□

4.1.2 Proof of Theorem 4.1: de Rham cohomology

We now prove the first part of the Poincaré lemma. The main strategy is to prove that

$$H^*(\mathbb{R}^n \times \mathbb{R}^1) \cong H^*(\mathbb{R}^n). \tag{4.2}$$

Once this is established, the result follows immediately from the one-dimensional case and induction.

To prove the isomorphism (4.2), we consider the following two maps:

$$\pi : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n, \ \pi(x,t) \triangleq x$$

$$s : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^1, \ s(x) \triangleq (x,0).$$

Apparently, $s^*\pi^* = \mathrm{id} : H^*(\mathbb{R}^n) \to H^*(\mathbb{R}^n)$ because $\pi s = \mathrm{id}$. It remains to prove the following result.

Lemma 4.1. $\pi^* s^* = \mathrm{id} : H^*(\mathbb{R}^n \times \mathbb{R}^1) \to H^*(\mathbb{R}^n \times \mathbb{R}^1)$.

Proof. The key point is to construct a linear operator

$$K_p : \Omega^p(\mathbb{R}^n \times \mathbb{R}^1) \to \Omega^{p-1}(\mathbb{R}^n \times \mathbb{R}^1)$$

for each degree p, which satisfies the relation

$$\mathrm{id} - \pi^* s^* = \pm(dK_p \pm K_{p+1}d) \tag{4.3}$$

on $\Omega^p(\mathbb{R}^n \times \mathbb{R}^1)$. We leave it as an exercise to show that the relation (4.3) implies $\mathrm{id} = \pi^* s^*$ on cohomology.

To construct K_p, one first observes that any differential form on $\mathbb{R}^n \times \mathbb{R}^1$ is a linear combination of forms of the following two types:

$$\text{Type I} : (\pi^*\phi) \cdot f(x,t); \quad \text{Type II} : (\pi^*\phi) \wedge (f(x,t)dt),$$

where $\phi \in \Omega^*(\mathbb{R}^n)$ and $f(x,t)$ is a smooth function on $\mathbb{R}^n \times \mathbb{R}^1$. We define K_p by sending any Type I form to zero and sending any Type II form

$$(\pi^*\phi) \wedge (f(x,t)dt) \mapsto (\pi^*\phi) \cdot \int_0^t f(x,s)ds. \tag{4.4}$$

We leave it as an exercise to check that

$$\mathrm{id} - \pi^* s^* = (-1)^{p-1}(dK_p - K_{p+1}d)$$

on $\Omega^p(\mathbb{R}^n \times \mathbb{R}^1)$ for every degree p (consider the evaluation on Type I and Type II forms separately). This proves the desired relation (4.3), hence completing the proof of the first part of Theorem 4.1. $\square$

Remark 4.1. The family of operators $\{K_p\}$ is known as a *homotopy operator* between the two chain maps id and π^*s^*. The construction of a homotopy operator satisfying (4.3) is a standard homological algebra technique for proving that two different chain maps induce identical actions at the (co)homology level.

The following extension of Theorem 4.1 will be useful to us. A subset $U \subseteq \mathbb{R}^n$ is said to be *star-shaped* with center $x \in U$ if the closed line segment $\overline{xy}$ is contained in U for every $y \in U$.

Lemma 4.2. *Let U be a star-shaped domain (i.e. open and connected subset) in $\mathbb{R}^n$. Then*

$$H^p(U) \cong \begin{cases} \mathbb{R}, & p = 0; \\ 0, & p \neq 0. \end{cases}$$

Proof. The idea is very similar to the proof of Lemma 4.1. The key is to construct an operator $S_p : \Omega^p(U) \to \Omega^{p-1}(U)$ for every $p > 0$, such that

$$\mathrm{id} = \pm(dS_p \pm S_{p+1}d). \tag{4.5}$$

If such an operator exists, it is clear that every closed p-form is exact, hence yielding that $H^p(U) = 0$ for every $p > 0$. The fact that $H^0(U) \cong \mathbb{R}$ is obvious since closed 0-forms on a connected manifold are constant functions.

Assume WLOG that U is centered at the origin. To construct the operator S_p, one first defines (cf. (4.4))

$$\hat{S}_p : \Omega^p(U \times \mathbb{R}) \to \Omega^{p-1}(U)$$

by sending a Type I form to zero and sending a Type II form

$$(\pi^*\phi)\wedge(f(x,t)dt)\mapsto\phi\cdot\int_0^1 f(x,t)dt.$$

Next, let $\rho:\mathbb{R}\to\mathbb{R}$ be a smooth function such that

$$\rho(t)=0 \text{ for } t\leqslant 0;\ \rho(t)=1 \text{ for } t\geqslant 1;\ 0\leqslant\rho\leqslant 1.$$

Consider the map

$$T:U\times\mathbb{R}\to U,\ T(x,t)\triangleq\rho(t)x.$$

This is well-defined since U is star-shaped. One can check that the operator $S_p\triangleq\hat{S}_pT^*$ satisfies the relation (4.5) with suitable signs. Details are left as an exercise. □

4.1.3 Proof of Theorem 4.1: compactly supported cohomology

The second part of the Poincaré lemma is proved using a similar strategy. Note that compactly supported differential forms on $\mathbb{R}^n\times\mathbb{R}^1$ are linear combinations of forms of the following two types:

$$\text{Type I}:(\pi^*\phi)\cdot f(x,t);\quad \text{Type II}:(\pi^*\phi)\wedge(f(x,t)dt),$$

where $\phi\in\Omega^*(\mathbb{R}^n)$ and $f\in C_c^\infty(\mathbb{R}^n\times\mathbb{R}^1)$. First of all, we define the map

$$\pi_*:\Omega_c^*(\mathbb{R}^n\times\mathbb{R}^1)\to\Omega_c^{*-1}(\mathbb{R}^n)\tag{4.6}$$

by sending Type I forms to zero and sending Type II forms

$$(\pi^*\phi)\wedge(f(x,t)dt)\mapsto(\pi^*\phi)\cdot\int_{\mathbb{R}^1}f(x,t)dt.$$

Next, let $e=e(t)dt\in\Omega_c^1(\mathbb{R}^1)$ be a given fixed bump form such that $\int_{\mathbb{R}^1}e(t)dt=1$. We also define the map

$$e_*:\Omega_c^*(\mathbb{R}^n)\to\Omega_c^{*+1}(\mathbb{R}^n\times\mathbb{R}^1),\ e_*(\phi)\triangleq(\pi^*\phi)\wedge e.$$

Lemma 4.3. *Both π_* and e_* are chain maps, i.e. one has $d\pi_*=\pi_*d$ and $de_*=e_*d$.*

Proof. Left as an exercise. □

Note that $\pi_*e_*=\mathrm{id}$ by definition. It remains to show that

$$e_*\pi_*=\mathrm{id}:H_c^*(\mathbb{R}^n\times\mathbb{R}^1)\to H_c^*(\mathbb{R}^n\times\mathbb{R}^1).\tag{4.7}$$

Once this is proven, it follows that

$$\pi_* : H_c^{*+1}(\mathbb{R}^n \times \mathbb{R}^1) \to H_c^*(\mathbb{R}^n),$$

is an isomorphism whose inverse is e_*. The Poincaré lemma for compactly supported cohomology thus follows from the one-dimensional case and induction.

As in the de Rham cohomology case, the key point to proving the relation (4.7) is the construction of a linear operator

$$L_p : \Omega_c^p(\mathbb{R}^n \times \mathbb{R}^1) \to \Omega_c^{p-1}(\mathbb{R}^n \times \mathbb{R}^1)$$

satisfying $\mathrm{id} - e_*\pi_* = \pm(dL_p - L_{p+1}d)$ for every degree p (cf. (4.3)). We define L_p by sending any Type I form to zero and sending any Type II form

$$(\pi^*\phi) \wedge (f(x,t)dt) \mapsto (\pi^*\phi) \cdot \Big(\int_{-\infty}^{t} f(x,s)ds - A(t) \int_{-\infty}^{\infty} f(x,s)ds \Big),$$

where $A(t) \triangleq \int_{-\infty}^{t} e(s)ds$. In the same way as before, the following lemma shows that $\{L_p\}$ is indeed a homotopy operator between id and $e_*\pi_*$, thus yielding the relation (4.7) and completing the proof of Theorem 4.1.

Lemma 4.4. *One has*

$$\mathrm{id} - e_*\pi_* = (-1)^{p-1}(dL_p - L_{p+1}d) \tag{4.8}$$

on $\Omega_c^p(\mathbb{R}^n \times \mathbb{R}^1)$ *for every degree* p.

Proof. (i) Let $\omega = (\pi^*\phi)f(x,t)$ be a Type I p-form where $\phi \in \Omega^p(\mathbb{R}^n)$. By the definitions of π_* and L_p, one has $(\mathrm{id} - e_*\pi_*)\omega = \omega$ and

$$\begin{aligned}(dL_p - L_{p+1}d)\omega &= -L_{p+1}\Big[d\pi^*\phi \cdot f + (-1)^p(\pi^*\phi)\big(\frac{\partial f}{\partial x^i}dx^i + \frac{\partial f}{\partial t}dt\big)\Big] \\ &= (-1)^{p-1}(\pi^*\phi)\Big(\int_{-\infty}^{t} \frac{\partial f}{\partial s}ds - A(t)\int_{-\infty}^{\infty} \frac{\partial f}{\partial s}ds\Big) \\ &= (-1)^{p-1}(\pi^*\phi)f(x,t) = (-1)^{p-1}\omega.\end{aligned}$$

(ii) Let $\omega = (\pi^*\phi) \wedge (f(x,t)dt)$ be a Type II p-form where $\phi \in \Omega^{p-1}(\mathbb{R}^n)$. First of all, one has

$$(\mathrm{id} - e_*\pi_*)\omega = (\pi^*\phi) \wedge (fdt) - \Big(\int_{-\infty}^{\infty} fds\Big)(\pi^*\phi) \wedge e.$$

On the other hand,

$$\begin{aligned}
dL_p\omega =& d\Big[(\pi^*\phi)\Big(\int_{-\infty}^{t} f ds - A(t)\int_{-\infty}^{\infty} f ds\Big)\Big]\\
=&(d\pi^*\phi)\Big(\int_{-\infty}^{t} f ds - A(t)\int_{-\infty}^{\infty} f ds\Big)\\
&+(-1)^{p-1}(\pi^*\phi)\wedge\Big(dx^i\int_{-\infty}^{t}\frac{\partial f}{\partial x^i}ds + f dt\\
&-e(t)dt\int_{-\infty}^{\infty} f ds - A(t)\int_{-\infty}^{\infty}\frac{\partial f}{\partial x^i}ds\Big)
\end{aligned}$$

and

$$\begin{aligned}
L_{p+1}d\omega =& L_{p+1}\Big[(d\pi^*\phi)\wedge(f dt) + (-1)^{p-1}(\pi^*\phi)\wedge\frac{\partial f}{\partial x^i}dx^i\wedge dt\Big]\\
=&(d\pi^*\phi)\Big(\int_{-\infty}^{t} f ds - A(t)\int_{-\infty}^{\infty} f ds\Big)\\
&+(-1)^{p-1}(\pi^*\phi\wedge dx^i)\Big(\int_{-\infty}^{t}\frac{\partial f}{\partial x^i}ds - A(t)\int_{-\infty}^{\infty}\frac{\partial f}{\partial x^i}ds\Big).
\end{aligned}$$

It follows that

$$\begin{aligned}
(dL_p - L_{p+1}d)\omega &= (-1)^{p-1}(\pi^*\phi)\wedge\Big(f dt - e(t)dt\int_{-\infty}^{\infty} f ds\Big)\\
&= (-1)^{p-1}(\mathrm{id} - e_*\pi_*)\omega.
\end{aligned}$$

This proves the desired relation (4.8). ☐

Remark 4.2. The map π_* is essentially *integration along the fiber* by viewing $\mathbb{R}^n\times\mathbb{R}^1$ as a (trivial) vector bundle over $\mathbb{R}^n$ with fiber $\mathbb{R}^1$ (cf (4.13)).

We define the generator of $H_c^n(\mathbb{R}^n)$ to be the unique cohomology class $[\omega]\in H_c^n(\mathbb{R}^n)$ whose total integral with respect to the standard orientation of $\mathbb{R}^n$ is equal to one. The following corollary is immediate from the previous discussion.

Corollary 4.1. *Let* $e_1(t),\cdots,e_n(t)\in C_c^\infty(\mathbb{R}^1)$ *with* $\int_{\mathbb{R}^1} e_i(t)dt = 1$ *for every* i. *Then the form*

$$e_1(x^1)\cdots e_n(x^n)dx^1\wedge\cdots\wedge dx^n$$

is a representative of the generator of $H_c^n(\mathbb{R}^n)$.

4.2 The Poincaré duality revisited

In this section, we establish the differential counterpart of the Poincaré duality theorem (more specifically, of Corollary 3.2). Let M be an oriented n-manifold. For every $0 \leqslant k \leqslant n$, we define the linear operator

$$\hat{P}_M^{\mathrm{diff}} : H^k(M) \to H_c^{n-k}(M)^*, \quad \hat{P}_M^{\mathrm{diff}}([\omega])([\eta]) \triangleq \int_M \eta \wedge \omega \tag{4.9}$$

for $[\omega] \in H^k(M)$ and $[\eta] \in H_c^{n-k}(M)$.

The (differential) Poincaré duality theorem is stated as follows. Recall that a good cover of M is an open cover $\{U_\alpha\}$ such that any (nonempty) finite intersection $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ is diffeomorphic to $\mathbb{R}^n$.

Theorem 4.2. *Suppose that M has a finite good cover. Then $\hat{P}_M^{\mathrm{diff}}$ is an isomorphism for every k.*

Remark 4.3. Theorem 4.2 holds without assuming the existence of a finite good cover. However, we will only prove the theorem under this additional assumption for simplicity.

We first recall the Mayer-Vietoris sequences for the two types of de Rham cohomologies, in particular, the explicit description of the connecting homomorphisms which will be used in the proof of Theorem 4.3. Let U, V be open subsets of M with $U \cap V \neq \emptyset$. Let $\{\varphi_U, \varphi_V\}$ be a partition of unity subordinated to $\{U, V\}$ on $U \cup V$.

(i) (The de Rham cohomology) Recall from (B.14) that there is a long exact sequence

$$\cdots \to H^k(U \cup V) \xrightarrow{i^*} H^k(U) \oplus H^k(V) \xrightarrow{j^*} H^k(U \cap V) \xrightarrow{\delta^k} H^{k+1}(U \cup V) \to \cdots. \tag{4.10}$$

Here

$$i^*([\omega]) \triangleq ([\omega]|_U, [\omega]|_V), \quad j^*([\omega], [\tau]) \triangleq [\tau|_{U\cap V} - \omega|_{U \cap V}].$$

The connecting homomorphism δ^k is defined in the following way. Let $[\omega] \in H^k(U \cap V)$. The pair of forms

$$(-d(\varphi_V \omega), d(\varphi_U \omega)) \in \Omega^{k+1}(U) \oplus \Omega^{k+1}(V)$$

agree on $U \cap V$ and hence they patch to a global form $\tau \in \Omega^{k+1}(U \cup V)$. Then $\delta^k[\omega] \triangleq [\tau]$. Note that τ is supported in $U \cap V$.

(ii) (Compactly supported cohomology) There is also a long exact sequence for the compactly supported cohomology:

$$\cdots \to H_c^k(U \cap V) \xrightarrow{i_*} H_c^k(U) \oplus H_c^k(V) \xrightarrow{j_*} H_c^k(U \cup V) \xrightarrow{\delta_c^k} H_c^{k+1}(U \cap V) \to \cdots.$$

This is induced from the short exact sequence

$$0 \to \Omega_c^k(U \cap V) \xrightarrow{i_*} \Omega_c^k(U) \oplus \Omega_c^k(V) \xrightarrow{j_*} \Omega_c^k(U \cup V) \to 0.$$

Here

$$i_*([\omega]) \triangleq (-[\omega], [\omega]), \quad j_*([\omega], [\tau]) \triangleq [\omega + \tau].$$

The connecting homomorphism is explicitly given as follows. Let $[\omega] \in H_c^k(U \cup V)$. The form $\tau \triangleq d(\varphi_V \omega)$ is compactly supported in $U \cap V$. Then $\delta_c^k[\omega] \triangleq [\tau]$.

The following fact is a simple application of the Mayer-Vietoris sequences.

Proposition 4.1. *Suppose that M has a finite good cover. Then both $H^k(M)$ and $H_c^k(M)$ are finite dimensional for every k.*

Proof. We only the consider $H^*(M)$. The compactly supported case is treated in a similar way. Let U, V be open subsets of M such that $U, V, U \cap V$ are all diffeomorphic to $\mathbb{R}^n$ (in particular, they all have finite dimensional cohomology groups by the Poincaré lemma). The Mayer-Vietoris sequence (4.10) together with the Rank-Nullity theorem shows that

$$H^k(U \cup V) \cong \ker i^* \oplus \mathrm{Im} i^* \cong \mathrm{Im}\delta^{k-1} \oplus \mathrm{Im} i^*.$$

Since

$$\dim(\mathrm{Im}\delta^{k-1}) \leqslant \dim H^{k-1}(U \cap V) < \infty,$$
$$\dim(\mathrm{Im} i^*) \leqslant \dim H^k(U) + \dim H^k(V) < \infty,$$

one concludes that $H^k(U \cup V)$ is finite dimensional. The general case with more than two open subsets follows by induction on the cardinality of the cover, which is finite by assumption. □

We now proceed to prove Theorem 4.2. The argument is parallel to the singular case (cf. Section 3.3.3).

Proof of Theorem 4.2. (i) If $M = \mathbb{R}^n$, the result is immediate from the Poincaré lemma. Indeed, in this case one only needs to prove that the linear map

$$H^0(\mathbb{R}^n) \to H_c^n(\mathbb{R}^n)^*, \quad f \mapsto \left[\omega \mapsto \int_{\mathbb{R}^n} f\omega\right]$$

is an isomorphism. But this is obvious since they are both isomorphic to $\mathbb{R}$ by the Poincaré lemma and this map is clearly nonzero.

(ii) Let U, V be open subsets of M such that $U, V, U \cap V$ are all diffeomorphic to $\mathbb{R}^n$. Consider the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(U \cup V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{A}"] & H^k(U) \oplus H^k(V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{B}"] & H^k(U \cap V) \arrow[r, "\delta^k"] \arrow[d] \arrow[dr, phantom, "\mathrm{C}"] & H^{k+1}(U \cup V) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_c^{n-k}(U \cup V)^* \arrow[r] & H_c^{n-k}(U)^* \oplus H_c^{n-k}(V)^* \arrow[r] & H_c^{n-k}(U \cap V)^* \arrow[r, "(\delta_c^k)^*"] & H_c^{n-k-1}(U \cup V)^* \arrow[r] & \cdots
\end{tikzcd} \tag{4.11}$$

where the two rows come from the Mayer-Vietoris sequences and the vertical homomorphisms are the Poincaré duality map. It is straight forward to check that the squares A and B are both commutative. We claim that Square C is sign-commutative, i.e.

$$\int_{U\cap V}\omega\wedge\delta_c\tau=\pm\int_{U\cup V}(\delta\omega)\wedge\tau \tag{4.12}$$

for any $\omega\in\Omega^k(U\cap V)$ and $\tau\in\Omega_c^{n-k-1}(U\cup V)$, where δ,δ_c are the connecting homomorphisms in the previous long exact sequences (with superscripts omitted). Indeed, by the explicit expression of δ_c and the closedness of τ, one has

$$\begin{aligned}\int_{U\cap V}\delta_c\tau\wedge\omega&=\int_{U\cap V}d(\varphi_V\tau)\wedge\omega\\&=\int_{U\cap V}(d\varphi_V)\wedge\tau\wedge\omega=(-1)^{\deg\tau}\int_{U\cap V}\tau\wedge(d\varphi_V)\wedge\omega.\end{aligned}$$

In addition, by the explicit expression of δ and the closedness of ω (also noting that $\delta\omega$ is supported in $U\cap V$), one has

$$\int_{U\cup V}\tau\wedge(\delta\omega)=-\int_{U\cap V}\tau\wedge d(\varphi_V\omega)=-\int_{U\cap V}\tau\wedge(d\varphi_V)\wedge\omega.$$

The relation (4.12) thus follows.

Now the Poincaré duality for $U\cup V$ follows from the known cases for $U,V,U\cap V$ and the Five Lemma applied to the diagram (4.11).

(iii) As usual, for a manifold admitting a finite good cover, the result follows from above cases and induction on the cardinality of the good cover.

□

Corollary 4.2. *Suppose that M is a connected, oriented n-manifold which admits a finite good cover. Then the integration map*

$$\int_M:H_c^n(M)\to\mathbb{R},\quad[\omega]\mapsto\int_M\omega$$

is an isomorphism.

Proof. According to Theorem 4.2, one has $H_c^n(M)^*\cong H^0(M)\cong\mathbb{R}$. Therefore, $H_c^n(M)\cong\mathbb{R}$. The claim follows by simply observing that the integration map $\int_M$ is nonzero. □

4.3 The Thom isomorphism revisited

We discuss the Thom isomorphism from the perspective of de Rham cohomology. In contrast to the singular case (3.17) which requires the knowledge of the Thom class, it is much easier to write down the inverse isomorphism in the de Rham case, which is just integration along the fiber. In what follows, let $(E, M, \pi; \mathbb{R}^r)$ be a real, oriented vector bundle of rank r over a manifold M.

Definition 4.2. A differential form $\omega \in \Omega^*(E)$ is said to have *compact support along the fiber* if $(\pi^{-1}K) \cap \operatorname{supp}\omega$ is compact for any compact subset $K \subseteq M$. The complex of such differential forms is denoted as $\Omega^*_{\mathrm{cv}}(E)$. The cohomology of $(\Omega^*_{\mathrm{cv}}(E), d)$ is called the *de Rham cohomology of E with compact support along the fiber* and it is denoted as $H^*_{\mathrm{cv}}(E)$.

We now construct a linear operator

$$\pi_* : \Omega^{*+r}_{\mathrm{cv}}(E) \to \Omega^*(M) \tag{4.13}$$

which is a natural generalisation of (4.6). Over a positively oriented trivialisation $\psi : \pi^{-1}U \to U \times \mathbb{R}^r$, any form in $\Omega^*_{\mathrm{cv}}(E)$ is locally a linear combination of forms of the following two types:

$$\text{Type I} : (\pi^*\phi) \wedge f(x,t)dt^{i_1} \wedge \cdots \wedge dt^{i_l} \quad \text{with } l < r,$$

$$\text{Type II} : (\pi^*\phi) \wedge f(x,t)dt^1 \wedge \cdots \wedge dt^r,$$

where $\phi \in \Omega^*(M)$ and $f(x,t) \in C^\infty_{\mathrm{cv}}(E|_U)$. We define π_* by sending any Type I form to zero and sending any Type II form

$$(\pi^*\phi) \wedge f(x,t)dt^1 \wedge \cdots \wedge dt^r \mapsto \phi \cdot \int_{\mathbb{R}^r} f(x,t)dt^1 \wedge \cdots \wedge dt^r.$$

Lemma 4.5. *The operator π_* is well-defined (i.e. it does not depend on the trivialisation or the local representation of the form) and commutes with d.*

The operator π_* is known as the *integration map along the fiber*. According to Lemma 4.5, π_* descends to a homomorphism on corresponding cohomology groups. The (differential) Thom isomorphism theorem is stated as follows.

Theorem 4.3. *Let $(E, M, \pi; \mathbb{R}^r)$ be a real, oriented vector bundle of rank r over a closed manifold M. Then*

$$\pi_* : H^{j+r}_{\mathrm{cv}}(E) \to H^j(M)$$

is an isomorphism for every $j \geqslant 0$.

Proof. The proof is parallel to the singular case (cf. Theorem 3.5). We extend the claim to the situation where M has a finite good cover (see Appendix B.5; recall from Theorem B.3 that every compact manifold has a finite good cover). If $E = U \times \mathbb{R}^r$ where U is diffeomorphic to $\mathbb{R}^n$, the claim can be proved in exactly the same way as in Section 4.1.3 (the Poincaŕe lemma for compactly supported cohomology). Next, suppose that the claim is true for $E|_U, E|_V$ and $E|_{U\cap V}$. By applying the Five Lemma to the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_{\mathrm{cv}}^{j+r}(E|_{U\cup V}) \arrow[r] \arrow[d, "\pi_*"] & H_{\mathrm{cv}}^{j+r}(E|_U) \oplus H_{\mathrm{cv}}^{j+r}(E|_V) \arrow[r] \arrow[d, "\pi_*"] & H_{\mathrm{cv}}^{j+r}(E|_{U\cap V}) \arrow[r, "\delta_{\mathrm{cv}}"] \arrow[d, "\pi_*"] & H_{\mathrm{cv}}^{j+r+1}(E|_{U\cup V}) \arrow[r] \arrow[d, "\pi_*"] & \cdots \\
\cdots \arrow[r] & H^j(U\cup V) \arrow[r] & H^j(U)\oplus H^j(V) \arrow[r] & H^j(U\cap V) \arrow[r, "\delta"] & H^{j+1}(U\cup V) \arrow[r] & \cdots,
\end{tikzcd}$$

one concludes that the claim also holds for $E|_{U\cup V}$. Here the bottom row is the Mayer-Vietoris sequence (B.14) and the top row is defined similarly. Now let M be a manifold with a finite good cover. One can choose a finite good cover which is at the same time a positively oriented local trivialisation of E. The claim thus follows by induction on the number of the members in the cover as well as the previous steps. □

Under the assumption of Theorem 4.3, let $\mathcal{T} : H^*(M) \to H_{\mathrm{cv}}^{*+r}(E)$ be the inverse of π_*. Define

$$\Phi \triangleq \mathcal{T}([1]) \in H_{\mathrm{cv}}^r(E).$$

Then the integral of Φ along each fiber of E is equal to one. In addition, note that

$$\pi_*((\pi^*\omega)\wedge\eta) = \omega\wedge\pi_*\eta \tag{4.14}$$

for all $\omega \in \Omega^*(M)$ and $\eta \in \Omega_{\mathrm{cv}}^*(E)$. The relation (4.14) can be easily proved using local expressions, which is left as an exercise. By applying (4.14) to $\eta = \Phi$ (more precisely, any representative of Φ), one has

$$\pi_*((\pi^*\omega)\wedge\Phi) = \omega\wedge\pi_*\Phi = \omega$$

for all $\omega \in \Omega^*(M)$. As a result, the map $\mathcal{T}$ is explicitly given by

$$\mathcal{T}([\omega]) = (\pi^*[\omega])\wedge\Phi.$$

Proposition 4.2. *Φ is the unique cohomology class in $H_{\mathrm{cv}}^r(E)$ such that $\Phi|_x$ is the generator of $H_c^r(E_x)$ (the unique cohomology class whose total integral over E_x with respect to its given orientation is one) for every $x \in M$.*

Proof. Clearly, Φ has the claimed integral property by definition. Suppose that Φ' is another class satisfying the same property. According to the relation (4.14),

$$\pi_*(\pi^*\omega\wedge\Phi') = \omega\wedge\pi_*\Phi' = \omega$$

for all $\omega \in \Omega^*(M)$. It follows that $\mathcal{T}([\omega]) = (\pi^*[\omega])\wedge\Phi'$. Taking $[\omega] = [1]$, one concludes that

$$\Phi = \mathcal{T}([1]) = (\pi^*[1])\wedge\Phi' = \Phi'.$$

This proves the uniqueness of Φ. □

Definition 4.3. The map

$$\mathcal{T} = \pi_*^{-1} = \pi^*(\cdot) \wedge \Phi : H^*(M) \to H_{\text{cv}}^{*+r}(E)$$

is called the (*differential*) *Thom isomorphism.* The cohomology class Φ is called the (*differential*) *Thom class* of E.

Remark 4.4. Φ is the differential counterpart of the singular Thom class u in Theorem 3.5.

4.4 The Poincaré dual of a submanifold revisited

In this section, we reexamine the Poincaré dual of a submanifold from the differential perspective. We continue to use the same notation as in Section 3.5. In particular, we consider a closed, oriented $(n-i)$-submanifold A of a closed, oriented n-manifold X.

Definition 4.4. The (*differential*) *Poincaré dual* of A is the unique cohomology class $[\omega_A] \in H^i(X)$ such that

$$\int_X \eta \wedge \omega_A = \int_A (\iota_A^X)^* \eta$$

for all $[\eta] \in H^{n-i}(X)$, where $\iota_A^X : A \to X$ is the inclusion.

The Poincaré dual $[\omega_A]$ is well-defined due to Theorem 4.2. Our goal of this section is to show that $[\omega_A]$ coincides with the Thom class of the normal bundle of A in X.

Let N (respectively, T) be the normal bundle (respectively, a closed tubular neighbourhood) of A in X. We may choose T such that $\mathring{T}$ is diffeomorphic to N (see Theorem 3.6 and Remark 3.10). As in Section 3.5.1, we orient T by X and also orient N according to the convention

$$\text{or}(X) = \text{or}(A) \oplus \text{or}(N). \tag{4.15}$$

Let $\Phi \in H_{\text{cv}}^i(N)$ be the Thom class of N with respect to the above orientation. Since N is diffeomorphic to $\mathring{T}$, one can view Φ as a cohomology class in $H_c^i(\mathring{T})$, hence a class in $H^i(X)$ by zero extension outside $\mathring{T}$. This class is denoted as $\bar{\Phi}$.

The main result we shall prove is stated as follows, which is the differential counterpart of Corollary 3.1.

Proposition 4.3. *The Poincaré dual of A equals the Thom class of N, i.e.* $[\omega_A] = \bar{\Phi}$.

Proposition 4.3 follows from a product integration formula which we first discuss.

Definition 4.5. Let E be a real, oriented vector bundle of rank r over an oriented n-manifold M. Pick any positively oriented local trivialisation $\{(U_\alpha, \psi_\alpha)\}$ of E, where $\{(U_\alpha, \varphi_\alpha)\}$ is a positively oriented atlas of M. The *local product orientation* of E is defined by claiming that

$$\left\{\pi^{-1}(U_\alpha), (\varphi_\alpha \circ \pi, \rho_2 \circ \psi_\alpha)\right\}$$

is a positively oriented atlas of E (in the manifold sense). Here ρ_2 denotes the projection of $U_\alpha \times \mathbb{R}^r$ onto the $\mathbb{R}^r$-component.

Remark 4.5. For the normal bundle N (or the tubular neighbourhood T), the local product orientation is consistent with the convention (4.15).

Lemma 4.6 (Product Integration Formula)**.** *Let E be equipped with the local product orientation under the setting of Definition 4.5. Then one has*

$$\int_E (\pi^*\tau) \wedge \omega = \int_M \tau \wedge (\pi_*\omega)$$

for all $\tau \in \Omega^p_c(M)$ and $\omega \in \Omega^q_{\mathrm{cv}}(E)$ provided that $p + q = n + r$, where π_* is the integration map defined by (4.13).

Proof. Left as an exercise (Hint: work locally and consider Type I and Type II forms separately). ☐

Proof of Proposition 4.3. Let η be an arbitrary closed $(n-i)$-form on X. Since A is a deformation retract of T, one has $\pi^*(\iota_A^T)^* = \mathrm{id}$ on $H^*(\mathring{T})$. As a result, $(\iota_{\mathring{T}}^X)^*\eta = \pi^*(\iota_A^X)^*\eta + d\tau$ for some $\tau \in \Omega^{n-i-1}(\mathring{T})$. Since $\bar{\Phi}$ is compactly supported in $\mathring{T}$, one has

$$\int_X \eta \wedge \bar{\Phi} = \int_T \eta \wedge \bar{\Phi} = \int_T \pi^*(\iota_A^X)^*\eta \wedge \bar{\Phi} + \int_T d\tau \wedge \bar{\Phi}.$$

According to Stokes' theorem and the fact that $\bar{\Phi} = 0$ on ∂T,

$$\int_T d\tau \wedge \bar{\Phi} = \int_T d(\tau \wedge \bar{\Phi}) = 0.$$

In addition, according to Lemma 4.6 and the characterisation of the Thom class,

$$\int_T \pi^*(\iota_A^X)^*\eta \wedge \bar{\Phi} = \int_A (\iota_A^X)^*\eta \wedge \pi_*\bar{\Phi} = \int_A (\iota_A^X)^*\eta.$$

Therefore,

$$\int_X \eta \wedge \bar{\Phi} = \int_A (\iota_A^X)^*\eta.$$

The claim thus follows from the definition of $[\omega_A]$.

☐

4.5 Intersection product formula revisited

The differential counterpart of the intersection product formula (cf. Theorem 3.7) can be derived easily by using the Thom class. We assume the same set-up and notation as in Section 3.6. In particular, A, B are closed, oriented submanifolds of a closed, oriented n-manifold X with dimensions $n-i$, $n-j$ respectively. We assume that A intersects B transversally and orient the submanifold $A \cap B$ according to the convention described at the beginning of Section 3.6. Let $[\omega_A], [\omega_B], [\omega_{A\cap B}]$ be the differential Poincaré dual of $A, B, A\cap B$ respectively (see Definition 4.4). The following result is the differential counterpart of Theorem 3.7.

Theorem 4.4. $[\omega_{A\cap B}] = [\omega_A] \wedge [\omega_B]$.

Before proving Theorem 4.4, we first state a simple property of the Thom class.

Lemma 4.7. *Let E, F be real oriented vector bundles over M. We orient $E \oplus F$ by the direct sum orientation. Let $\Phi(E), \Phi(F), \Phi(E\oplus F)$ denote their Thom classes respectively. Then*

$$\Phi(E\oplus F) = \pi_1^*\Phi(E) \wedge \pi_2^*\Phi(F),$$

where $\pi_1 : E\oplus F \to E$ and $\pi_2 : E \oplus F \to F$ are the projections.

Proof. One has

$$\int_{E_x\oplus F_x} \big(\pi_1^*\Phi(E) \wedge \pi_2^*\Phi(F)\big)|_x = \Big(\int_{E_x} \Phi(E)|_x\Big) \times \Big(\int_{F_x} \Phi(F)|_x\Big) = 1.$$

The result follows from the characterisation of the Thom class (see Proposition 4.2). ☐

Proof of Theorem 4.4. Let $N_A, N_B, N_{A\cap B}$ be the normal bundles of $A, B, A\cap B$ in X respectively. Under the orientation convention (3.19) for normal bundles as well as the orientation convention for $A\cap B$, it is easily seen that

$$N_{A\cap B} \cong N_A \oplus N_B$$

as oriented vector bundles. The result thus follows from Lemma 4.7 as well as the identification between the Poincaré dual and Thom class (Proposition 4.3).

☐

4.6 The Euler class and global angular form: a geometric construction in rank two

Let $(E, M, \pi; \mathbb{R}^r)$ be an oriented vector bundle where M is a manifold admitting a finite good cover (so that Theorem 4.3 applies). The differential counterpart of the Euler class is defined as follows (cf. Definition 3.13).

Definition 4.6. The (*differential*) *Euler class* of E is defined by $e(E) \triangleq s_0^*\Phi \in H^r(M)$, where $s_0 : M \to E$ is the zero section and Φ is the Thom class of E.

In this section, we give a geometric construction of the Euler class in the case when $r = 2$. We assume additionally that E is equipped with a (fiberwise) Euclidean metric. A key insight is that the pullback of $e(E)$ to $E_0 \triangleq \{u \in E : u \neq 0\}$ is represented by an exact form $d\Psi$. This form Ψ has the essential property that the integral of its restriction to the unit sphere in $(E_0)_x$ is equal to -1 for every $x \in M$. By using these properties of Ψ, one can obtain a simple construction of the Thom class Φ.

Let $\{U_\alpha\}$ be a local trivialisation of E. Suppose that a local PONF is defined on each U_α, so that one can consider polar coordinates

$$(r_\alpha, \theta_\alpha) : (E_0)|_{U_\alpha} \to (0, \infty) \times \mathbb{R}_{\text{mod } 2\pi}$$

of elements in E_0 with respect to U_α. For any α, β with $U_\alpha \cap U_\beta \neq \emptyset$, there is a well-defined smooth function $\varphi_{\beta\alpha}$ (up to a constant multiple of 2π) such that

$$\theta_\beta - \theta_\alpha = \pi_0^*\varphi_{\beta\alpha}, \tag{4.16}$$

where $\pi_0^* : E_0 \to M$ is the projection map. More precisely, $\varphi_{\beta\alpha}$ is the counterclockwise rotation angle from the α-coordinate system to the β-coordinate system. If $\rho_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{SO}(2)$ are the transition functions of E, one has $\rho_{\beta\alpha} = e^{i\varphi_{\beta\alpha}}$.

Lemma 4.8. *There exists a* 1*-form* ξ_α *on* U_α*, such that*

$$\xi_\beta - \xi_\alpha = \frac{1}{2\pi} d\varphi_{\beta\alpha}$$

on any $U_\alpha \cap U_\beta \neq \emptyset$.

Proof. Let $\{\psi_\gamma\}$ be a partition of unity subordinated to $\{U_\gamma\}$. Define

$$\xi_\alpha \triangleq \frac{1}{2\pi} \sum_\gamma \psi_\gamma d\varphi_{\alpha\gamma}.$$

Then one has

$$\xi_\beta - \xi_\alpha = \frac{1}{2\pi} \sum_\gamma \psi_\gamma d(\varphi_{\beta\gamma} - \varphi_{\alpha\gamma}).$$

The main observation is that

$$\varphi_{\beta\gamma} = \varphi_{\beta\alpha} + \varphi_{\alpha\gamma} \pmod{2\pi},$$

and hence $d(\varphi_{\beta\gamma} - \varphi_{\alpha\gamma}) = d\varphi_{\beta\alpha}$. It follows that

$$\xi_\beta - \xi_\alpha = \frac{1}{2\pi} \sum_\gamma \psi_\gamma d\varphi_{\beta\alpha} = \frac{1}{2\pi} d\varphi_{\beta\alpha}.$$

This prove the claim of the lemma. □

Lemma 4.8 shows that $d\xi_\alpha = d\xi_\beta$ on $U_\alpha \cap U_\beta$. Therefore, they define a global 2-form $e \in \Omega^2(M)$. Note that e needs not be exact, since the ξ_α's may not patch to a global 1-form on M.

Remark 4.6. If E is trivial, one can choose $\varphi_{\beta\alpha}$ to be a constant function. In this case, $e = 0$. As a result, the form e measures the nontriviality (twisting) of the bundle E to some extent. As we will see, e is a representative of the Euler class.

Next, one observes from the relation (4.16) and Lemma 4.8 that

$$\pi_0^*\xi_\beta - \pi_0^*\xi_\alpha = \frac{1}{2\pi} d\pi_0^*\varphi_{\beta\alpha} = \frac{1}{2\pi}(d\theta_\beta - d\theta_\alpha).$$

Equivalently, one has

$$-\frac{1}{2\pi} d\theta_\alpha + \pi_0^*\xi_\alpha = -\frac{1}{2\pi} d\theta_\beta + \pi_0^*\xi_\beta.$$

As a consequence, the forms $-\frac{1}{2\pi}d\theta_\alpha + \pi_0^*\xi_\alpha$ patch to a global 1-form $\Psi \in \Omega^1(E_0)$ which is called the *global angular form*.

Lemma 4.9. *One has $d\Psi = \pi_0^* e$. In addition,*

$$\int_{S(E_x)} i_x^*\Psi = -1$$

for every $x \in M$. Here $S(E_x)$ is the unit circle on the fiber E_x, which is oriented according to the convention that

$$\mathrm{or}(E_x) = \text{“}\textit{outer normal}\text{”} \oplus \mathrm{or}(S(E_x)).$$

The map $i_x : S(E_x) \to E_0$ is the inclusion.

Proof. The first claim follows immediately from the definitions of e and Ψ. For the second claim, let $x \in U_\alpha$. Noting that $i_x^*\pi_0^*\xi_\alpha = 0$, one has

$$\int_{S(E_x)} i_x\Psi = -\frac{1}{2\pi}\int_0^{2\pi} d\theta_\alpha = -1.$$

The result of the lemma thus follows. □

One can easily construct a representative of the Thom class by using the global angular form Ψ. Let

$$r : E \to [0,\infty), \ \ r(u) \triangleq |u|_E$$

be the radial function defined by the metric of E.

Proposition 4.4. *Let* $\eta : [0,\infty) \to [0,\infty)$ *be a smooth function such that* $\eta = 1$ *on* $[0, 1/2]$ *and* $\eta = 0$ *on* $[1,\infty)$*. Then the form*

$$d(\eta(r)\Psi) = \eta'(r)dr \wedge \Psi + \eta(r)\pi_0^* e \tag{4.17}$$

extends smoothly to a global 2*-form on* E *with compact support along the fiber. In addition,* ϕ *is closed and* $[\phi]$ *is the Thom class of* E*.*

Proof. The form $\eta'(r)dr \wedge \Psi$ is well-defined near the zero section because $\eta' = 0$ near the origin. The form $\eta(r)\pi_0^* e$ naturally extends $\eta(r)\pi^* e$ on E. This shows that ϕ is well-defined. The support property of ϕ is obvious. The closedness of ϕ follows from the relation $d\Psi = \pi_0^* e$. Finally, one has

$$\int_{E_x} i_x^* \phi = \Big(\int_0^\infty \eta'(r)dr \Big) \wedge \Big(\int_{S(E_x)} \Psi \Big) = 1$$

for every $x \in M$. It follows from Proposition 4.2 that ϕ is a representative of the Thom class. □

Corollary 4.3. *The form* e *is a representative of the Euler class* $e(E)$*.*

Proof. It is clear from the construction (4.17) of ϕ that

$$s_0^* \phi = s_0^* \pi^* e = e.$$

Since $[\phi]$ is the Thom class, it follows from the definition of the Euler class $e(E)$ that $[e] = e(E)$. □

Remark 4.7. The extension of the above construction to the higher rank case will be treated in Section 7.4 using the Chern-Weil theory. This is the heart of the proof of CGB from the Chern-Weil perspective.

4.7 The de Rham theorem

In these notes, we have been switching between singular and de Rham cohomologies freely. For instance, we used singular cohomology theory to address the Lefschetz fixed point formula while we will mostly work with the de Rham cohomology when we develop various geometric-analytic approaches later on. The aim of this section is to establish a fundamental theorem of de Rham, which asserts that for a closed manifold the singular cohomology with real coefficients is canonically isomorphic to the de Rham cohomology. This theorem justifies the equivalent usage of the two cohomology theories. The main theorem we shall prove is stated in Theorem 4.5.

The modern approach to the de Rham theorem is based on sheaf theory (cf. [War83]). In this section, we present an elementary proof of this theorem following the excellent exposition of [ST67]. A nice benefit of this approach is that one gains a concrete and explicit understanding about this isomorphism. A small disadvantage is that one has to work with the simplicial cohomology instead (which in turn relies on the combinatorial structure of a triangulation). The isomorphism between the simplicial and singular cohomologies follows from standard algebraic topology argument and will be addressed in Appendix A.7.

4.7.1 Simplicial homology and cohomology

We will define the simplicial (co)homology of a closed manifold M in terms of a triangulation of M as a simplicial complex (cf. Definition 1.4). We first introduce the notion of orientation for a simplex (cf. Definition 1.2).

Definition 4.7. Let $s = (v_0, v_1, \cdots, v_k)$ be an open k-simplex. We identify two representations (orderings)

$$(v_{i_0}, \cdots, v_{i_k}) \sim (v_{j_0}, \cdots, v_{j_k})$$

of s if $(i_0, \cdots, i_k)$ is an even permutation of $(j_0, \cdots, j_k)$. An *orientation* of s is a choice of one of the two equivalence classes. An *oriented k-simplex* is an open k-simplex s together with a given orientation. We denote it as $\langle s \rangle = \langle v_0, \cdots, v_k \rangle$ if the underlying orientation is given by the equivalence class of the ordering $(v_0, \cdots, v_k)$.

Let K be a simplicial complex in the sense of Definition 1.3. As a standard route, in order to define the simplicial homology, we shall construct a (simplicial) chain complex together with a boundary map. These are defined as follows.

Definition 4.8. For each $k \geqslant 0$, we define the *k-th simplicial chain group* of K over $\mathbb{R}$ to be the real vector space generated by the (*simplicial*) *k-chains*, namely,

$$C_k^{\mathrm{sim}}(K; \mathbb{R}) \triangleq \mathrm{Span}\big\{\langle s \rangle : s \in K, \dim s = k\big\}.$$

In other words, $\{\langle s \rangle : s \in K, \dim s = k\}$ is a basis of $C_k^{\mathrm{sim}}(K; \mathbb{R})$ and every element of $C_k^{\mathrm{sim}}(K; \mathbb{R})$ is uniquely expressed as

$$c = \sum_{s \in K : \dim s = k} \lambda_s \langle s \rangle,$$

where all but finitely many $\lambda_s \in \mathbb{R}$ are zero.

Remark 4.8. We do not specify a preferred orientation of each s in the above definition. If $s = (v_0, v_1, \cdots, v_k)$, one has $\langle v_0, v_1, v_2, \cdots, v_k \rangle = -\langle v_1, v_0, v_2, \cdots, v_k \rangle$.

Definition 4.9. We define the *boundary map* $\partial_k^{\mathrm{sim}} : C_k^{\mathrm{sim}}(K;\mathbb{R}) \to C_{k-1}^{\mathrm{sim}}(K;\mathbb{R})$ ($k \geqslant 0$) to be the unique linear map satisfying

$$\partial_k^{\mathrm{sim}}\langle v_0,\cdots,v_k\rangle = \sum_{j=0}^{k}(-1)^j\langle v_0,\cdots,\hat{v}_j,\cdots,v_k\rangle$$

for any k-simplex $(v_0,\cdots,v_k)$ in K. As a convention, we set $C_{-1}^{\mathrm{sim}}(K;\mathbb{R}) \triangleq \{0\}$ and thus $\partial_0^{\mathrm{sim}}c = 0$ for all $c \in C_0(K;\mathbb{R})$.

It is elementary to check that $(\partial^{\mathrm{sim}})^2 = 0$. As a result, one obtains a chain complex of real vector spaces

$$\cdots \xrightarrow{\partial^{\mathrm{sim}}} C_k^{\mathrm{sim}}(K;\mathbb{R}) \xrightarrow{\partial^{\mathrm{sim}}} C_{k-1}^{\mathrm{sim}}(K;\mathbb{R}) \xrightarrow{\partial^{\mathrm{sim}}} \cdots \xrightarrow{\partial^{\mathrm{sim}}} C_1^{\mathrm{sim}}(K;\mathbb{R}) \xrightarrow{\partial^{\mathrm{sim}}} C_0^{\mathrm{sim}}(K;\mathbb{R}) \to 0.$$

As usual, one can construct the homology groups associated with this chain complex.

Definition 4.10. For each $k \geqslant 0$, we define

$$Z_k^{\mathrm{sim}}(K;\mathbb{R}) \triangleq \big\{c \in C_k^{\mathrm{sim}}(K;\mathbb{R}) : \partial c = 0\big\},$$
$$B_k^{\mathrm{sim}}(K;\mathbb{R}) \triangleq \big\{\partial^{\mathrm{sim}}c : c \in C_{k+1}^{\mathrm{sim}}(K;\mathbb{R})\big\}.$$

Elements of $Z_k^{\mathrm{sim}}(K;\mathbb{R})$ (respectively, $B_k^{\mathrm{sim}}(K;\mathbb{R})$) are called (*simplicial*) *k-cycles* (respectively, *k-boundaries*). It is clear that $B_k^{\mathrm{sim}}(K;\mathbb{R})$ is a subspace of $Z_k^{\mathrm{sim}}(K;\mathbb{R})$. The quotient space

$$H_k^{\mathrm{sim}}(K;\mathbb{R}) \triangleq \frac{Z_k^{\mathrm{sim}}(K;\mathbb{R})}{B_k^{\mathrm{sim}}(K;\mathbb{R})}$$

is called the k-th *simplicial homology group* of K with real coefficients.

Remark 4.9. Since K consists of finitely many simplices, the spaces $Z_k^{\mathrm{sim}}, B_k^{\mathrm{sim}}, H_k^{\mathrm{sim}}$ are all finite dimensional. In addition, C_k^{sim} (hence H_k^{sim}) is zero if $k > \dim K$.

Example 4.1. The following computations are left as an exercise.

(i) K is the boundary of a triangle:

$$H_0^{\mathrm{sim}} \cong H_1^{\mathrm{sim}} \cong \mathbb{R},\ H_k^{\mathrm{sim}} = 0 \text{ for all } k \geqslant 2.$$

(ii) K is a (closed) solid triangle:

$$H_0^{\mathrm{sim}} \cong \mathbb{R},\ H_k^{\mathrm{sim}} = 0 \text{ for all } k \geqslant 1.$$

(iii) K is the boundary of a tetrahedron:

$$H_0^{\mathrm{sim}} \cong H_2^{\mathrm{sim}} \cong \mathbb{R},\ H_1^{\mathrm{sim}} = H_k^{\mathrm{sim}} = 0 \text{ for all } k \geqslant 3.$$

A basic topological feature of a simplicial complex is the Euler characteristic.

Definition 4.11. The *(combinatorial) Euler characteristic* of a simplicial complex K is the integer defined by

$$\chi(K) \triangleq \sum_{k=0}^{\dim K} (-1)^k \beta_k,$$

where $\beta_k \triangleq \dim H_k^{\mathrm{sim}}(K;\mathbb{R})$ is known as the *k-th Betti number* of K.

Remark 4.10. Of course, $\chi(K)$ is just the (topological) Euler characteristic of the underlying topological space $|K|$. This is a consequence of the fact that the simplicial and singular homologies are isomorphic (see Theorem A.3).

Proposition 4.5. *One has*

$$\chi(K) = \sum_{k=0}^{\dim K} (-1)^k \alpha_k,$$

where α_k denotes the number of k-simplices in K.

Proof. Let $n \triangleq \dim K$. According to the Rank-Nullity Theorem, one has

$$\alpha_k = \dim Z_k^{\mathrm{sim}} + \dim B_{k-1}^{\mathrm{sim}}$$

for every $k = 0, 1, \cdots, n$ ($B_{-1}^{\mathrm{sim}} \triangleq \{0\}$ by convention). By the definition of $\chi(K)$, one has

$$\begin{aligned}
\chi(K) &= \sum_{k=0}^{n} (-1)^k \big(\dim Z_k^{\mathrm{sim}} - \dim B_k^{\mathrm{sim}}\big) \\
&= \sum_{k=0}^{n} (-1)^k \dim Z_k^{\mathrm{sim}} + \sum_{k=1}^{n} (-1)^k \dim B_{k-1}^{\mathrm{sim}} \ (\text{because } B_n^{\mathrm{sim}} = \{0\}) \\
&= \dim Z_0^{\mathrm{sim}} + \sum_{k=1}^{n} (-1)^k \big(\dim Z_k^{\mathrm{sim}} + \dim B_{k-1}^{\mathrm{sim}}\big) \\
&= \sum_{k=0}^{n} (-1)^k \alpha_k.
\end{aligned}$$

The result thus follows. □

By dualising the previous constructions, one is naturally led to the consideration of simplicial cohomology. As before, let K be a simplicial complex.

Definition 4.12. The space of (*simplicial*) *k-cochains* is defined by $C^k_{\mathrm{sim}}(K;\mathbb{R}) \triangleq C^{\mathrm{sim}}_k(K;\mathbb{R})^*$ (the vector space dual of $C^{\mathrm{sim}}_k(K;\mathbb{R})$). The *coboundary map* $\partial^*_{\mathrm{sim}} : C^k_{\mathrm{sim}}(K;\mathbb{R}) \to C^{k+1}_{\mathrm{sim}}(K;\mathbb{R})$ is the unique linear map satisfying

$$(\partial^*_{\mathrm{sim}}\varphi)(c) = \varphi(\partial^{\mathrm{sim}}c)$$

for all $\varphi \in C^k_{\mathrm{sim}}(K;\mathbb{R})$ and $c \in C^{\mathrm{sim}}_{k+1}(K;\mathbb{R})$.

It is clear that $(\partial^*_{\mathrm{sim}})^2 = 0$. One can therefore introduce the following definition.

Definition 4.13. The *k-th simplicial cohomology group* of K with real coefficients is the quotient space

$$H^k_{\mathrm{sim}}(K;\mathbb{R}) \triangleq \frac{Z^k_{\mathrm{sim}}(K;\mathbb{R})}{B^k_{\mathrm{sim}}(K;\mathbb{R})},$$

where

$$\begin{aligned} Z^k_{\mathrm{sim}}(K;\mathbb{R}) &\triangleq \big\{\varphi \in C^k_{\mathrm{sim}}(K;\mathbb{R}) : \partial^*_{\mathrm{sim}}\varphi = 0\big\}, \\ B^k_{\mathrm{sim}}(K;\mathbb{R}) &\triangleq \big\{\partial^*_{\mathrm{sim}}\varphi : \varphi \in C^{k-1}_{\mathrm{sim}}(K;\mathbb{R})\big\} \end{aligned}$$

are the space of (*simplicial*) *k-cocycles* and *k-coboundaries.*

Proposition 4.6. *$H^k_{\mathrm{sim}}(K;\mathbb{R})$ is canonically isomorphic to $H^{\mathrm{sim}}_k(K;\mathbb{R})^*$.*

Proof. Left as an exercise. □

Remark 4.11. One can also consider simplicial (co)homology over other coefficient rings. We choose to work with real coefficients because our aim is to establish its relation with the de Rham cohomology (over $\mathbb{R}$).

We conclude this subsection with a useful formula for the coboundary map $\partial^*_{\mathrm{sim}}$. Given a k-simplex $s \in K$, we define $\varphi_{\langle s\rangle} \in C^k_{\mathrm{sim}}(K;\mathbb{R})$ by

$$\varphi_{\langle s\rangle}(\langle t\rangle) \triangleq \begin{cases} \pm 1, & \text{if } \langle t\rangle = \pm\langle s\rangle; \\ 0, & \text{if } t \neq s \end{cases}$$

for any $\langle t\rangle \in C^{\mathrm{sim}}_k(K;\mathbb{R})$.

Lemma 4.10. *For any k-simplex $s = (v_0, \cdots, v_k) \in K$, one has*

$$\partial^*_{\mathrm{sim}}\varphi_{\langle v_0,\cdots,v_k\rangle} = \sum_v \varphi_{\langle v,v_0,\cdots,v_k\rangle}, \tag{4.18}$$

where the above summation is taken over all vertices $v \in K$ such that $(v, v_0, \cdots, v_k)$ is a $(k+1)$-simplex in K.

Proof. Let $\langle t\rangle = \langle w_0, \cdots, w_{k+1}\rangle$ by any oriented $(k+1)$-simplex. By the definition of $\partial^*_{\mathrm{sim}}$, one has

$$\begin{aligned}\big(\partial^*_{\mathrm{sim}}\varphi_{\langle v_0,\cdots,v_k\rangle}\big)(\langle t\rangle) &= \varphi_{\langle v_0,\cdots,v_k\rangle}(\partial^{\mathrm{sim}}\langle t\rangle) \\ &= \sum_{j=0}^{k+1}(-1)^j\varphi_{\langle v_0,\cdots,v_k\rangle}\big(\langle w_0,\cdots,\hat{w}_j,\cdots,w_{k+1}\rangle\big). \end{aligned} \tag{4.19}$$

The above expression is nonzero if and only if s is a face of t, say $t = (v, v_0, \cdots, v_k)$ for some additional vertex $v \in K$. If $\langle t\rangle = \langle v, v_0, \cdots, v_k\rangle$, the RHS of (4.19) is equal to 1 and the same is true for the RHS of (4.18) applied to $\langle t\rangle$. If $\langle t\rangle = -\langle v, v_0, \cdots, v_k\rangle$, both quantities are equal to -1. Therefore, the relation (4.18) holds. □

4.7.2 The de Rham isomorphism

Let M be a closed n-manifold. We fix a smooth triangulation (K, h) of M in the sense of Definition 1.4 (which always exists according to [Mun66]). In particular, $h : |K| \to M$ is a homeomorphism and for each $s \in K$, the map $h|_{[s]}$ extends to a smooth embedding $h_s : U \to M$ on a neighbourhood U of s in the subspace where s lives.

Our aim is to establish a canonical isomorphism between $H^k_{\mathrm{dR}}(M)$ (de Rham cohomology) and $H^k_{\mathrm{sim}}(K;\mathbb{R})$ (simplicial cohomology). From now on, we will omit the sub/superscript "sim" in all objects related to the simplicial cohomology. The reader should be aware that we are always working with the simplicial (rather than singular) setting in the following discussion.

To build a homomorphism $\bar{\mathcal{I}}_k : H^k_{\mathrm{dR}}(M) \to H^k(K;\mathbb{R})$, a standard route (from the perspective of homological algebra) is to construct a *chain map*

$$\mathcal{I}_* : (\Omega^*(M), d) \to (C^*(K;\mathbb{R}), \partial^*),$$

namely, a linear map $\mathcal{I}_k : \Omega^k(M) \to C^k(K;\mathbb{R})$ for each k satisfying

$$\partial^* \circ \mathcal{I}_k = \mathcal{I}_{k+1} \circ d. \tag{4.20}$$

Here $\Omega^*(M) = \{\Omega^k(M)\}_{0\leqslant k\leqslant n}$ is the de Rham complex of differential forms on M and d is the exterior derivative operator. Once this is achieved, it is immediate that $\mathcal{I}_k$ descends to a homomorphism $\bar{\mathcal{I}}_k$ on cohomology.

Construction of $\mathcal{I}_k$

The definition of $\mathcal{I}_k$ is nearly immediate. Let $\omega \in \Omega^k(M)$. We define $\mathcal{I}_k(\omega) \in C^k(K;\mathbb{R})$ by setting

$$\mathcal{I}_k(\omega)(\langle s\rangle) \triangleq \int_{\langle s\rangle} h_s^*\omega, \quad \langle s\rangle \in C_k(K;\mathbb{R}) \tag{4.21}$$

and extend the definition linearly. To be more precise, recall that $h_s : U \to M$ is a smooth embedding where U is a neighbourhood of $[s] = [v_0, \cdots, v_k]$ in the subspace S of $[s]$. We orient U by declaring that

$$\{v_1 - v_0, \cdots, v_k - v_0\} \tag{4.22}$$

is a positive basis of S. The integral on the RHS of (4.21) is then understood in the usual differential geometric sense (cf. Appendix B.6) as the integral of a k-form $h_s^*\omega$ on a regular domain $[s]$ in an oriented k-manifold U. If one writes $h_s^*\omega = g dx^1 \wedge \cdots \wedge dx^k$ for some $g \in C^\infty(U)$ where $(x^i)_{1\leqslant i\leqslant k}$ are the coordinates with respect to the basis (4.22), then

$$\int_{\langle s\rangle} h_s^*\omega = \int_{[s]} g dx^1 \cdots dx^k$$

as a Lebesgue integral.

Remark 4.12. The intuition behind $\mathcal{I}_k$ is very simple: the action of a k-form ω on any oriented k-simplex $\langle s\rangle$ is just defined to be "the integral of ω over $\langle s\rangle$".

Lemma 4.11. *$\mathcal{I}_* = \{\mathcal{I}_k\}_{0\leqslant k\leqslant n}$ is a chain map, i.e. it satisfies the relation (4.20) for every k.*

Proof. Let $\omega \in \Omega^k(M)$ and $\langle s\rangle \in C_{k+1}(K;\mathbb{R})$. By the definition of $\mathcal{I}_k$ and Stokes' theorem (see Theorem B.4), one has

$$\mathcal{I}_{k+1}(d\omega)(\langle s\rangle) = \int_{[s]} h_s^*(d\omega) = \int_{[s]} dh_s^*\omega = \int_{\partial[s]} h_s^*\omega = \mathcal{I}_k(\omega)(\partial\langle s\rangle).$$

This proves the relation (4.20). $\square$

As a consequence of Lemma 4.11, the map $\mathcal{I}_k$ descends to a homomorphism $\bar{\mathcal{I}}_k : H^k_{\mathrm{dR}}(M) \to H^k(K;\mathbb{R})$.

Theorem 4.5 (The de Rham Theorem)**.** *$\bar{\mathcal{I}}_k : H^k_{\mathrm{dR}}(M) \to H^k(K;\mathbb{R})$ is an isomorphism for every k.*

The proof of the de Rham theorem relies on the following two key lemmas: one for surjectivity and the other for injectivity.

Lemma 4.12 (Surjectivity Lemma)**.** *For each k, there exists a linear map $\alpha_k : C^k(K;\mathbb{R}) \to \Omega^k(M)$ such that $\mathcal{I}_k \circ \alpha_k = \mathrm{id}$ and $d \circ \alpha_k = \alpha_{k+1} \circ \partial^*$.*

Lemma 4.13 (Injectivity Lemma)**.** *Suppose that ω is a closed k-form and $\mathcal{I}_k(\omega) = \partial^*\varphi$ for some $\varphi \in C^{k-1}(K;\mathbb{R})$. Then there exists a $(k-1)$-form τ such that $d\tau = \omega$.*

We now complete the proof of the de Rham theorem, presuming the correctness of the above two lemmas. The proofs of these two lemmas will be given in the next two subsections.

Proof of Theorem 4.5. Let $[\varphi] \in H^k(K;\mathbb{R})$. Define $\omega \triangleq \alpha_k(\varphi) \in \Omega^k(M)$. Since $\partial^*\varphi = 0$, Lemma 4.12 shows that ω is closed and $\mathcal{I}_k(\omega) = \varphi$. Therefore, $\bar{\mathcal{I}}_k([\omega]) = [\varphi]$. This proves the surjectivity of $\bar{\mathcal{I}}_k$.

Let $\bar{\mathcal{I}}_k([\omega]) = 0$. Then one has $\mathcal{I}_k(\omega) = \partial^*\varphi$ for some $\varphi \in C^{k-1}(K;\mathbb{R})$. It follows from Lemma 4.13 that ω is exact (in particular, $[\omega] = 0$). This proves the injectivity of $\bar{\mathcal{I}}_k$.

□

Remark 4.13. Different triangulations of M induce isomorphic simplicial cohomology groups; in fact, they are all isomorphic to the singular cohomology (see Appendix A.7).

4.7.3 More on simplicial complex

Before proving the two key lemmas, we need to introduce a few more notions from simplicial complex.

Definition 4.14. Let $(s) = (v_0, \cdots, v_k)$ be an open k-simplex. For each $x \in (s)$, there is a unique vector $(b_0, \cdots, b_k)$ such that

$$b_i > 0,\ \sum_{i=0}^{k} b_i = 1 \text{ and } x = \sum_{i=0}^{k} b_i v_i.$$

The number b_i is called the *barycentric coordinate* of x with respect to the vertex v_i in (s).

Let K be a simplicial complex of dimension n, whose topological point set is $|K|$. For each $0 \leqslant r \leqslant n$, we use K^r to denote the *r-skeleton* of K, i.e. the subcomplex consisting of all simplices with dimension $\leqslant r$. Let $K^0 = \{v_1, \cdots, v_m\}$ denote the vertex set of K.

Definition 4.15. We define a coordinate function $\mathbf{b} = (b_1, \cdots, b_m) : |K| \to \mathbb{R}^m$ in the following way. Let $x \in |K|$. There is a unique $(s) = (v_{i_0}, \cdots, v_{i_k}) \in K$ such that $x \in (s)$. We then define $b_{i_l}(x)$ $(0 \leqslant l \leqslant k)$ to be the barycentric coordinate of x with respect to v_{i_l} in (s) and set $b_j(x) \triangleq 0$ if $j \notin \{i_0, \cdots, i_k\}$.

Lemma 4.14. *(i) The function $b_j : |K| \to \mathbb{R}$ is continuous for every j.*

(ii) $b_j \geqslant 0$ and $\sum_j b_j = 1$.

(iii) $x = \sum_j b_j(x) v_j$ for every $x \in |K|$.

(iv) Let $\{i_0, \cdots, i_k\} \subseteq \{0, \cdots, m\}$*. Then* $b_{i_0}(x) \cdots b_{i_k}(x) \neq 0$ *for some* $x \in |K|$ *if and only if* $(v_{i_0}, \cdots, v_{i_k}) \in K$*.*

Proof. Left as an exercise. $\square$

A benefit of working with a simplicial complex is that one can construct an open cover in a quite explicit way. Given $(s) = (v_{i_0}, \cdots, v_{i_k}) \in K$, we define the *star of* s by

$$\mathrm{St}(s) \triangleq \bigcup_{\substack{(t)\in K:\\ (s)\ \text{is a face of}\ (t)}} (t).$$

Lemma 4.15. *(i)* $\mathrm{St}(s)$ *is an open subset of* $|K|$*. In addition,* $x \in \mathrm{St}(s)$ *if and only if* $b_{i_0}(x) \cdots b_{i_k}(x) \neq 0$*.*

(ii) $\{\mathrm{St}(v) : v \in K^0\}$ *is an open cover of* $|K|$*.*

Proof. Left as an exercise. $\square$

4.7.4 Surjectivity lemma

We now formulate and prove a slightly stronger version of the surjectivity lemma. To ease notation, we will omit the homeomorphism h and simply identify $|K|$ with M. Recall that $\int_{\langle s\rangle} \omega = \mathcal{I}_k(\omega)(\langle s\rangle)$ for any $\langle s\rangle \in C_k(K;\mathbb{R})$ and $\omega \in \Omega^k(M)$.

Lemma 4.16. *For each* k*, there exists a linear map* $\alpha_k : C^k(K;\mathbb{R}) \to \Omega^k(M)$ *such that the following properties hold true.*

(i) $\mathcal{I}_k \circ \alpha_k = \mathrm{id}$.

(ii) $d \circ \alpha_k = \alpha_{k+1} \circ \partial^*$.

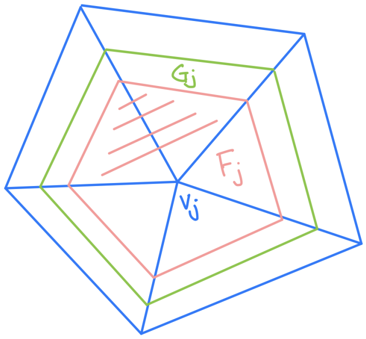
(iii) For each $\langle s\rangle \in C_k(K;\mathbb{R})$*,* $\alpha_k(\varphi_{\langle s\rangle})$ *is compactly supported in* $\mathrm{St}(\langle s\rangle)$*. In particular,* $\alpha_k(\varphi_{\langle s\rangle})_x = 0$ *for any* $x \in K^{k-1}$*.*

Proof. (i) Construction of partition of unity.

The main input we shall use to construct forms on M is a partition of unity. Recall that $\{v_1, \cdots, v_m\}$ is the vertex set of K. For each j, consider the sets

$$F_j \triangleq \left\{x \in |K| : b_j(x) \geqslant \frac{1}{n+1}\right\},\ G_j \triangleq \left\{x \in |K| : b_j(x) \geqslant \frac{1}{n+2}\right\},$$

where $n \triangleq \dim K$ and $b_j : |K| \to \mathbb{R}$ is the function introduced in Definition 4.15. It is clear that $F_j \subseteq G_j \subseteq \mathrm{St}(v_j)$.



We claim that $\{F_j : 1 \leqslant j \leqslant m\}$ is a cover of M. Indeed, let $x \in |K|$. Then $x \in (s) = (v_{i_0}, \cdots, v_{i_k})$ for some $(s) \in K$. Since

$$x = \sum_{l=0}^{k} b_{i_l}(x) v_{i_l}, \ \sum_{l=0}^{k} b_{i_l}(x) = 1,$$

there is some l such that

$$b_{i_l}(x) \geqslant \frac{1}{k+1} \geqslant \frac{1}{n+1}.$$

Therefore, $x \in F_{i_l}$.

Now pick a smooth function $0 \leqslant f_j \leqslant 1$ such that $f_j = 1$ on F_j and $f_j = 0$ on G_j^c. Define $g_j \triangleq \frac{f_j}{\sum_{l=1}^{m} f_l}$. Since $\{F_j\}$ is a cover of M, the function g_j is well-defined. It is clear that $\{g_j\}$ is a partition of unity subordinated to $\{\mathrm{St}(v_j) : 1 \leqslant j \leqslant m\}$ and g_j is compactly supported in $\mathrm{St}(v_j)$.

(ii) *Construction of α_k*.

It is enough to specify $\alpha_k(\varphi_{\langle s \rangle})$ for each $\langle s \rangle = \langle v_{i_0}, \cdots, v_{i_k} \rangle \in C_k(K; \mathbb{R})$. We define

$$\alpha_k(\varphi_{\langle s \rangle}) \triangleq k! \sum_{l=0}^{k} (-1)^l g_{i_l} dg_{i_0} \wedge \cdots \wedge \widehat{dg_{i_l}} \wedge \cdots \wedge dg_{i_k}.$$

According to Lemma 4.15 (i), one has

$$\mathrm{St}(s)^c \subseteq \bigcup_{l=0}^{k} G_{i_l}^c.$$

Since g_{i_l} vanishes on $G_{i_l}^c$ by construction, it follows that $\alpha_k(\varphi_{\langle s \rangle})$ is compactly supported in $\mathrm{St}(s)$.

(iii) $\underline{d \circ \alpha_k = \alpha_{k+1} \circ \partial^*}$.

Fix $\langle s \rangle = \langle v_{i_0}, \cdots, v_{i_k} \rangle$. By the definition of α_k, one has

$$d \circ \alpha_k(\varphi_{\langle s \rangle}) = (k+1)! dg_{i_0} \wedge \cdots \wedge dg_{i_k}. \tag{4.23}$$

On the other hand, according to Lemma 4.10,

$$\begin{aligned}\alpha_{k+1}\circ\partial^*(\varphi_{\langle s\rangle}) =&\alpha_{k+1}\big(\sum\nolimits' \varphi_{\langle v_j,v_{i_0},\cdots,v_{i_k}\rangle}\big)\\ =&(k+1)!\sum\nolimits_j' \big(g_j dg_{i_0}\wedge\cdots\wedge dg_{i_k}\\ &+\sum_{l=0}^{k}(-1)^{l+1}g_{i_l}dg_j\wedge dg_{i_0}\wedge\cdots\wedge\widehat{dg_{i_l}}\wedge\cdots\wedge dg_{i_k}\big),\end{aligned}\tag{4.24}$$

where $\sum_j'$ denotes the summation over all vertices $v_j\in K^0$ such that $(v_j,v_{i_0},\cdots,v_{i_k})$ is a $(k+1)$-simplex in K.

The main observation is that the above summation is the same is the one taken over all $j\notin\{i_0,\cdots,i_k\}$. Indeed, let $v_j\in K^0\backslash\{v_{i_0},\cdots,v_{i_k}\}$ and suppose that

$$(g_j dg_{i_0}\wedge\cdots\wedge dg_{i_k})_x\neq 0$$

for some $x\in|K|$. Then

$$x\in\mathrm{St}(v_j)\cap\mathrm{St}(v_{i_0})\cap\cdots\cap\mathrm{St}(v_{i_k}).$$

According to Lemma 4.15 (i), one has

$$b_j(x)b_{i_0}(x)\cdots b_{i_k}(x)\neq 0.$$

It follows from Lemma 4.14 (iv) that $(v_j,v_{i_0},\cdots,v_{i_k})\in K$. As a consequence, for those v_j's where $(v_j,v_{i_0},\cdots,v_{i_k})$ does not form a simplex, one must have

$$g_j dg_{i_0}\wedge\cdots\wedge dg_{i_k}\equiv 0.$$

Similarly, the form $g_{i_l}dg_j\wedge dg_{i_0}\wedge\cdots\wedge\widehat{dg_{i_l}}\wedge\cdots\wedge dg_{i_k}$ is also identically zero for the same v_j's.

Now one can rewrite the j-summation on the RHS of (4.24) as

$$\begin{aligned}&\sum_{j\notin\{i_0,\cdots,i_k\}}\big(g_j dg_{i_0}\wedge\cdots\wedge dg_{i_k}+\sum_{l=0}^{k}(-1)^{l+1}g_{i_l}dg_j\wedge dg_{i_0}\wedge\cdots\wedge\widehat{dg_{i_l}}\wedge\cdots\wedge dg_{i_k}\big)\\ =&\sum_{j\notin\{i_0,\cdots,i_k\}} g_j dg_{i_0}\wedge\cdots\wedge dg_{i_k}+\sum_{l=0}^{k}(-1)^{l+1}\sum_{j\neq i_l}g_{i_l}dg_j\wedge dg_{i_0}\wedge\cdots\wedge\widehat{dg_{i_l}}\wedge\cdots\wedge dg_{i_k}\\ =&\sum_{j\notin\{i_0,\cdots,i_k\}} g_j dg_{i_0}\wedge\cdots\wedge dg_{i_k}+\sum_{l=0}^{k}(-1)^{l}g_{i_l}dg_{i_l}\wedge dg_{i_0}\wedge\cdots\wedge\widehat{dg_{i_l}}\wedge\cdots\wedge dg_{i_k}\\ =&\sum_{j\notin\{i_0,\cdots,i_k\}} g_j dg_{i_0}\wedge\cdots\wedge dg_{i_k}+\sum_{l=0}^{k}g_{i_l}dg_{i_0}\wedge\cdots\wedge dg_{i_l}\wedge\cdots\wedge dg_{i_k}\\ =&\,dg_{i_0}\wedge\cdots\wedge dg_{i_k},\end{aligned}\tag{4.25}$$

where we used the fact that $\sum_j g_j = 1$ to reach the second and last equalities. The relation

$$d \circ \alpha_k(\varphi_{\langle s\rangle}) = \alpha_{k+1} \circ \partial^*(\varphi_{\langle s\rangle})$$

follows by substituting (4.25) into (4.24) and comparing the result with (4.23).

(iv) $\underline{\mathcal{I}_k \circ \alpha_k = \mathrm{id}}$.

We prove this claim by induction. For $k = 0$, one has by definition that $\alpha_0(\varphi_{\langle v_j\rangle}) = g_j$. It follows that

$$\int_{\langle v_k\rangle} \alpha_0(\varphi_{\langle v_j\rangle}) = g_j(v_k) = \delta_{jk} = \varphi_{\langle v_j\rangle}(\langle v_k\rangle)$$

for any $v_k \in K^0$. Therefore, $\mathcal{I}_0 \circ \alpha_0(\varphi_{\langle v_j\rangle}) = \varphi_{\langle v_j\rangle}$.

Suppose that the claim is true for α_{k-1}. We want to show that

$$\int_{\langle t\rangle} \alpha_k(\varphi_{\langle s\rangle}) = \begin{cases} 1, & \text{if } \langle t\rangle = \langle s\rangle; \\ 0, & \text{if } \langle t\rangle \neq \langle s\rangle \end{cases} \tag{4.26}$$

for any $\langle s\rangle, \langle t\rangle \in C_k(K;\mathbb{R})$. The case when $\langle t\rangle \neq \langle s\rangle$ is obvious because $\alpha_k(\varphi_{\langle s\rangle})$ is compactly supported in $\mathrm{St}(s)$ and hence vanishes identically on a neighbourhood of $[t]$. For the other case, write $\langle s\rangle = \langle v_{i_0}, \cdots, v_{i_k}\rangle$ and consider the simplex $\langle r\rangle \triangleq \langle v_{i_1}, \cdots, v_{i_k}\rangle$. According to the relation we proved in Step (iii) and the induction hypothesis,

$$\begin{aligned} \int_{\langle s\rangle} \alpha_k(\partial^* \varphi_{\langle r\rangle}) &= \int_{\langle s\rangle} d\big(\alpha_{k-1}(\varphi_{\langle r\rangle})\big) = \int_{\partial\langle s\rangle} \alpha_{k-1}(\varphi_{\langle r\rangle}) \\ &= \int_{\langle r\rangle} \alpha_{k-1}(\varphi_{\langle r\rangle}) = 1. \end{aligned}$$

On the other hand, by Lemma 4.10 and the second case of (4.26), one also has

$$\int_{\langle s\rangle} \alpha_k(\partial^* \varphi_{\langle r\rangle}) = \int_{\langle s\rangle} \alpha_k(\varphi_{\langle s\rangle}) + \sum_j{}' \int_{\langle s\rangle} \alpha_k(\varphi_{\langle v_j, v_{i_1}, \cdots, v_{i_k}\rangle}),$$

where the summation is taken over those vertices $v_j \neq v_{i_0}$ such that $(v_j, v_{i_1}, \cdots, v_{i_k})$ is a k-simplex of K. But none of these simplices would be equal to (s). As a result, the above $\sum_j'$-summation is zero. It follows that $\int_{\langle s\rangle} \alpha_k(\varphi_{\langle s\rangle}) = 1$, which proves the first case of (4.26). The induction step is therefore complete. $\square$

4.7.5 Injectivity lemma

The proof of the injectivity lemma relies crucially on the idea of extending closed / exact forms from lower to higher dimensional skeletons. We first summarise the key lemma that achieves this purpose.

Lemma 4.17. *(i) Let ω be a closed k-form defined near $\partial[s]$, where (s) is a given l-simplex. Suppose additionally that*

$$\int_{\partial\langle s\rangle} \omega = 0 \tag{4.27}$$

in the case when $l = k + 1$. Then there exists a closed k-form τ defined near $[s]$, such that $\tau = \omega$ near $\partial[s]$.

(ii) Let ω be a closed k-form defined near $[s]$, where (s) is a given l-simplex. Suppose that $\omega = d\tau$ near $\partial[s]$ for some $(k-1)$-form τ defined near $\partial[s]$. Suppose additionally that

$$\int_{\langle s\rangle} \omega = \int_{\partial\langle s\rangle} \tau \tag{4.28}$$

in the case when $l = k$. Then there exists an $(k-1)$-form τ' defined near $[s]$, such that $\tau' = \tau$ near $\partial[s]$ and $\omega = d\tau'$ near $[s]$.

Remark 4.14. The condition (4.27) is necessary because if such a τ exists, one must have

$$0 = \int_{\langle s\rangle} d\tau = \int_{\partial\langle s\rangle} \tau = \int_{\partial\langle s\rangle} \omega$$

by Stokes' theorem. In a similar way, the condition (4.28) is also necessary.

Proof. We use $(\mathbf{A})_k$ $(k \geqslant 0)$ to denote Property (i) and $(\mathbf{B})_k$ $(k \geqslant 1)$ for Property (ii) respectively. Our plan is to prove that

$$(\mathbf{A}_0) \implies (\mathbf{B})_1 \implies (\mathbf{A})_1 \implies (\mathbf{B})_2 \implies \cdots .$$

(i) $\underline{\text{Justification of } (\mathbf{A}_0)}$.

In this case, ω is a function defined near $\partial[s]$. If $l > 1$, by the closedness of ω it must be constant near $\partial[s]$. One just takes its obvious (constant) extension to a neighbourhood of $[s]$. If $l = 1$, say $\langle s\rangle = \langle v_{i_0}, v_{i_1}\rangle$. The condition (4.27) becomes $\omega(v_{i_0}) = \omega(v_{i_1})$. This allows one to extend ω constantly to a neighbourhood of $[s]$.

(ii) $\underline{(\mathbf{A})_{k-1} \implies (\mathbf{B})_k}$.

Let ω be a closed k-form defined near an l-simplex $[s]$ and suppose that $\omega = d\tau$ for some $(k-1)$-form τ near $\partial[s]$. Since $[s]$ is star-shaped, one knows from the Poincaré lemma (see Lemma 4.2) that $\omega = d\tau_1$ for some $(k-1)$-form τ_1 near $[s]$. Note that $\tau_1 - \tau$ is closed near $\partial[s]$. In the case when $l = k$, one also has

$$\int_{\partial\langle s\rangle} (\tau_1 - \tau) = \int_{\langle s\rangle} d\tau_1 - \int_{\partial\langle s\rangle} \tau = \int_{\langle s\rangle} \omega - \int_{\partial\langle s\rangle} \tau = 0$$

by assumption. It follows from Hypothesis $(\mathbf{A})_{k-1}$ that there exists a closed $(k-1)$-form σ defined near $[s]$ such that $\sigma = \tau_1 - \tau$ near $\partial[s]$. Setting $\tau' \triangleq \tau_1 - \sigma$, it is plain to see that $\tau' = \tau$ near $\partial[s]$ and

$$d\tau' = d\tau_1 - d\sigma = \omega - 0 = \omega$$

near $[s]$. This proves the claim $(\mathbf{B})_k$.

(iii) $\underline{(\mathbf{B})_k \implies (\mathbf{A})_k}$.

Let ω be a closed k-form defined near $\partial[s]$ for some l-simplex $(s) = (v_{i_0}, \cdots, v_{i_l})$. The key point is to show that ω is exact near $\partial[s]$, i.e. $\omega = d\mu$ for some $(k-1)$-form μ defined near $\partial[s]$. Presuming this is true, let f be a bump function which is supported inside the domain D of μ and equals one near $\partial[s]$. Define

$$\tau \triangleq \begin{cases} d(f\mu), & \text{on } D; \\ 0, & \text{on } D^c. \end{cases}$$

It is apparent that τ is a well-defined closed k-form on M. In addition, one has

$$\tau = df \wedge \mu + f d\mu = 0 \wedge \mu + 1 \cdot \omega = \omega$$

near $\partial[s]$. This proves the extension property $(\mathbf{A})_k$.

Now it remains to prove the exactness of ω near $\partial[s]$. To this end, let $(t) \triangleq (v_{i_1}, \cdots, v_{i_l})$ and choose a star-shaped neighbourhood V of $[s]\backslash(t)$ with respect to the vertex v_{i_0}. According to the Poincaré lemma again, there exists $\mu_1 \in \Omega^{k-1}(V)$ such that $\omega = d\mu_1$ on V. This in particular holds near $\partial[t]$. In the case when $l = k+1$, setting $c \triangleq \partial\langle s\rangle - \langle t\rangle \in C_k(V;\mathbb{R})$ one also has

$$\int_{\langle t\rangle} \omega - \int_{\partial\langle t\rangle} \mu_1 = \int_{\langle t\rangle} \omega + \int_{\partial c} \mu_1 = \int_{\langle t\rangle} \omega + \int_c \omega = \int_{\partial\langle s\rangle} \omega = 0$$

by assumption. It follows from Hypothesis $(\mathbf{B})_k$ that there exists a $(k-1)$-form μ_2 near $[t]$ such that $\mu_2 = \mu_1$ near $\partial[t]$ and $\omega = d\mu_2$ near $[t]$. As a consequence, the forms μ_1 and μ_2 patch to a well-defined form μ near $\partial[s]$ which satisfies $\omega = d\mu$ near $\partial[s]$. $\square$

We are now in a position to prove (a slightly stronger version of) the injectivity lemma.

Lemma 4.18. *Suppose that ω is a closed k-form and $\mathcal{I}_k(\omega) = \partial^*\varphi$ for some $\varphi \in C^{k-1}(K;\mathbb{R})$ $(k \geqslant 1)$. Then there exists a $(k-1)$-form τ such that $d\tau = \omega$ and $\int_{\langle s\rangle} \tau = \varphi$ for any $\langle s\rangle \in C_{k-1}(K;\mathbb{R})$.*

Proof. The main strategy is to construct a sequence $\tau_0, \tau_1, \cdots, \tau_n$ of $(k-1)$-forms inductively with the following properties.

(i) τ_l is defined near $|K^l|$ and $d\tau_l = \omega$ near $|K^l|$.

(ii) $\tau_l = \tau_{l-1}$ near $|K^{l-1}|$.

(iii) $\int_{\langle s \rangle} \tau_{k-1} = \varphi(\langle s \rangle)$ for any $\langle s \rangle \in C_{k-1}(K; \mathbb{R})$.

To construct τ_0, we cover K^0 by disjoint balls and apply the Poincaré lemma to obtain a $(k-1)$-form τ_0' defined near $|K^0|$ such that $d\tau_0' = \omega$ near $|K^0|$. If $k \neq 1$, we just define $\tau_0 \triangleq \tau_0'$. If $k = 1$, we set

$$\tau_0 \triangleq \tau_0' + f,$$

where f is the locally constant function defined near $|K^0|$ by

$$f(v_j) \triangleq \varphi(v_j) - \tau_0'(v_j), \quad v_j \in K^0.$$

It is readily checked that τ_0 satisfies the required properties.

Suppose that $\tau_0, \cdots, \tau_{l-1}$ have already been constructed. Given any l-simplex $(s) \in K$, one has $d\tau_{l-1} = \omega$ near $\partial[s]$. If $l = k$, by applying the assumption of the lemma and Property (iii) for τ_{l-1}, one also has

$$\int_{\langle s \rangle} \omega = \varphi(\partial\langle s \rangle) = \int_{\partial\langle s \rangle} \tau_{l-1}.$$

It follows from Lemma 4.17 (ii) that there exists a $(k-1)$-form $\tau_l(s)$ defined near $[s]$, such that

$$\tau_l(s) = \tau_{l-1} \text{ near } \partial[s], \quad d\tau_l(s) = \omega \text{ near } [s]. \tag{4.29}$$

The property (4.29) allows one to patch the forms $\{\tau_l(s)\}$ to a well-defined $(k-1)$-form τ_l' near $|K^l|$ which satisfies Properties (i) and (ii).

If $l \neq k-1$, we just define $\tau_l \triangleq \tau_l'$. If $l = k-1$, we adjust τ_{k-1}' by setting

$$\tau_{k-1} \triangleq \tau_{k-1}' + \alpha_{k-1}(\psi),$$

where $\psi \in C^{k-1}(K; \mathbb{R})$ is defined by

$$\psi(\langle s \rangle) \triangleq \varphi(\langle s \rangle) - \int_{\langle s \rangle} \tau_{k-1}', \quad \langle s \rangle \in C_{k-1}(K; \mathbb{R}).$$

This precisely ensures that Property (iii) holds for τ_{k-1}. In addition, since $\alpha_{k-1}(\psi)$ vanishes near $|K^{k-2}|$ (see Lemma 4.16 (iii)), one has $\tau_{k-1} = \tau_{k-2}$ near $|K^{k-2}|$. Finally, one also has

$$\begin{aligned} d\tau_{k-1} &= d\tau_{k-1}' + d\alpha_{k-1}(\psi) \\ &= d\tau_{k-1}' + \alpha_k(\partial^*\psi) = d\tau_{k-1}' = \omega \end{aligned}$$

near $|K^{k-1}|$, because $\alpha_k(\partial^*\psi)$ vanishes identically near $|K^{k-1}|$. Therefore, τ_{k-1} satisfies Properties (i), (ii) and (iii). □

4.7.6 Summary of singular and de Rham correspondence

Let $\mathcal{D} : H^*(M;\mathbb{R}) \to H^*_{\mathrm{dR}}(M)$ be the actual isomorphism between singular and de Rham cohomologies (inverse of $\bar{\mathcal{I}}_*$ in Theorem 4.5 composed with the isomorphism between simplicial and singular cohomologies). Here $H^*(M;\mathbb{R})$ refers to the singular cohomology with real coefficients. It can be shown that $\mathcal{D}$ is a ring isomorphism with respect to the (singular) cup and (differential) wedge products respectively. If M is assumed to be oriented, one also has

$$\int_M \mathcal{D}a = \langle a, [M] \rangle$$

for all $a \in H^n(M;\mathbb{R})$, where $[M]$ is the fundamental class of M.

All results we developed so far in both the singular and de Rham settings are consistent under the de Rham theorem. For instance, the de Rham image of the singular Poincaré dual $P_X^{-1}([A])$ of the submanifold A (Definition 3.5) is the differential Poincaré dual $[\omega_A]$ (Definition 4.4). The de Rham image of the absolute singular Thom class u_A^X (Corollary 3.1) is the differential Thom class $\bar{\Phi}$ (Proposition 4.3). More generally, the de Rham image of the singular Thom class (respectively, singular Euler class) of a vector bundle is the differential Thom class (respectively, differential Euler class). For the Thom class, one shall invoke the following version of de Rham isomorphism:

$$H^*(E, E_0;\mathbb{R}) \to H^*_{\mathrm{cv}}(E).$$

There is also a relative version of de Rham isomorphism:

$$H^*(T, \partial T;\mathbb{R}) \to H^*_{\mathrm{dR}}(T, \partial T),$$

where T is the closed tubular neighbourhood in the earlier submanifold context and $H^*_{\mathrm{dR}}(T, \partial T)$ is the *relative de Rham cohomology group* (the cohomology of forms whose pullback to the boundary vanish). This relative de Rham isomorphism sends the singular Thom class u_T (Section 3.5.2) to the previous differential Thom class $\bar{\Phi}$ (viewed as a relative de Rham class in $H^i_{\mathrm{dR}}(T, \partial T)$).

5 The original intrinsic proof of S.S. Chern

In this chapter, we discuss Chern's original intrinsic proof of the CGB theorem in a self-contained way following Chern's original paper [Che44] (cf. [CW93]). Throughout this chapter, let M be a closed, oriented Riemannian manifold of dimension $n = 2m$. Recall from Definition C.2 that the Riemann curvature tensor is given by

$$R(X,Y)Z \triangleq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for $X, Y, Z \in \Gamma(TM)$. In this chapter, ∇ always denotes Levi-Civita connection.

5.1 The Chern-Gauss-Bonnet theorem

In this section, we formulate the CGB theorem and summarise the essential idea behind Chern's intrinsic proof.

5.1.1 The Euler form

The key concept appearing in CGB is the so-called *Euler form* which we shall first define.

Let $e = \{e_1, \cdots, e_n\}$ be a local PONF defined on some open subset $U \subseteq M$. In other words, $\{e_1(x), \cdots, e_n(x)\}$ is a PONB of $T_x M$ at every $x \in U$. To see the existence of such a frame, one can for instance take a star-shaped open set U and parallel transport an ONB of $T_{x^*}M$ at the center x^* of U along rays emitting from x^*. Let $\{\eta^1, \cdots \eta^n\}$ be the dual frame field of $\{e_1, \cdots, e_n\}$, i.e. $\eta^i \in \Omega^1(U)$ with $\eta^i(e_j) = \delta_{ij}$.

Let $A = (A_i^j)_{1 \leqslant i,j \leqslant n}$ be the *local connection matrix* of ∇ defined by

$$\nabla e_i = A_i^j \otimes e_j. \tag{5.1}$$

One can write $A_i^j = \Gamma_{ki}^j \eta^k$, where Γ_{ki}^j are the *Christoffel symbols* of ∇ with respect to $\{e_i\}$, i.e. $\nabla_{e_k} e_i = \Gamma_{ki}^j e_j$ (cf. Section 6.9.5). We will use matrix notation exclusively. For instance, η is viewed as an $n \times 1$ column vector and ω is an $n \times n$ matrix.

The characterising properties of the Levi-Civita connection are summarised as follows (cf. Section 6.10).

(i) The connection ∇ is compatible with the metric if and only if $A_i^j = -A_j^i$, namely, A is an $\mathfrak{so}(n)$-valued 1-form on U.

(ii) The connection ∇ is torsion free if and only if $d\eta = -A \wedge \eta$ (under matrix notation).

The *local curvature matrix* of ∇ with respect to $\{e_i\}$ is the $\mathfrak{so}(n)$-valued 2-form on U defined by

$$F \triangleq dA + A \wedge A.$$

Equivalently, $(F^i_j(e_k,e_l))_{1\leqslant i,j\leqslant n}$ is the matrix of $R(e_k,e_l)$ with respect to $\{e_i\}$:

$$R(e_k,e_l)e_j = F^i_j(e_k,e_l)e_i$$

Note that F is local; if $\tilde{e} = \{\tilde{e}_1,\cdots,\tilde{e}_n\}$ is another PONF which is related to e by $\tilde{e} = e\cdot Q$ for some $Q \in C^\infty(U;\mathrm{SO}(n))$, one has $F = Q\tilde{F}Q^t$ where $\tilde{F}$ is the local curvature form with respect to $\tilde{e}$ and Q^t means the transposition of Q.

Now we are in a position to define the Euler form. Recall that $\dim M = n = 2m$.

Definition 5.1. The *Euler form* is the $2m$-form on M defined by

$$\mathcal{E} \triangleq \frac{1}{2^{2m}\pi^m m!}\sum_{i_1,\cdots,i_{2m}=1}^{2m} \mathrm{sgn}(i_1,\cdots,i_{2m})F^{i_1}_{i_2}\wedge\cdots\wedge F^{i_{2m-1}}_{i_{2m}}. \tag{5.2}$$

The first question we shall address is the intrinsicness of $\mathcal{E}$.

Lemma 5.1. *The Euler form does not depend on the choice of the PONF. In particular, it is intrinsically defined on M.*

Proof. Under a new PONF $\tilde{e} = e\cdot Q$, one has $F = Q\tilde{F}Q^t$. It follows that

$$\begin{aligned}\mathcal{E}_0 &\triangleq \sum_{\sigma\in\mathcal{S}_{2m}} \mathrm{sgn}(\sigma)F^{\sigma(1)}_{\sigma(2)}\wedge\cdots\wedge F^{\sigma(2m-1)}_{\sigma(2m)}\\ &= \sum_{\sigma\in\mathcal{S}_{2m}} \mathrm{sgn}(\sigma)\sum_{i_1,\cdots,i_{2m}} \left(Q^{\sigma(1)}_{i_1}\tilde{F}^{i_1}_{i_2}Q^{\sigma(2)}_{i_2}\right)\wedge\cdots\wedge\left(Q^{\sigma(2m-1)}_{i_{2m-1}}\tilde{F}^{i_{2m-1}}_{i_{2m}}Q^{\sigma(2m)}_{i_{2m}}\right)\\ &= \sum_{i_1,\cdots,i_{2m}} \det(Q_{i_1},\cdots,Q_{i_{2m}})\tilde{F}^{i_1}_{i_2}\wedge\cdots\wedge\tilde{F}^{i_{2m-1}}_{i_{2m}}.\end{aligned}$$

Note that $\det(Q_{i_1},\cdots,Q_{i_{2m}}) = 0$ if $i_k = i_l$ for some $k\neq l$ and

$$\det(Q_{i_1},\cdots,Q_{i_{2m}}) = \mathrm{sgn}(i_1,\cdots,i_{2m})\det Q = \mathrm{sgn}(i_1,\cdots,i_{2m})$$

if $(i_1,\cdots,i_{2m})$ is a permutation of $(1,\cdots,2m)$. Therefore, one obtains that

$$\mathcal{E}_0 = \sum_{i_1,\cdots,i_{2m}} \mathrm{sgn}(i_1,\cdots,i_{2m})\tilde{F}^{i_1}_{i_2}\wedge\cdots\wedge\tilde{F}^{i_{2m-1}}_{i_{2m}}.$$

This shows that the definition of $\mathcal{E}_0$ is independent of the local PONF e. □

5.1.2 Main theorem and idea of Chern's intrinsic proof

The main theorem of S.S. Chern is stated as follows.

Theorem 5.1 (The Chern-Gauss-Bonnet Theorem)**.** *The integral of the Euler form over a closed, oriented, $2m$-dimensional Riemannian manifold M is equal to the Euler characteristic of M:*

$$\int_M \mathcal{E} = \chi(M). \tag{5.3}$$

The essential idea behind Chern's original proof are summarised as follows. The development of the underlying techniques has led to far-reaching implications in global differential geometry.

1. The lift of the Euler form onto the special orthonormal frame bundle $\mathrm{SO}(M)$ is an exact form. In other words, one can write $\pi^*\mathcal{E} = d\Pi$ for some form Π on $\mathrm{SO}(M)$, where $\pi : \mathrm{SO}(M) \to M$ denotes the bundle projection.

2. Due to its specific construction, the form Π descends to a form Π' on the sphere bundle SM. More precisely, let $\pi_1 : \mathrm{SO}(M) \to SM$ and $\pi_2 : SM \to M$ denote the bundle projections. There exists a form Π' such that $\Pi = \pi_1^*\Pi'$. As a result, the lift of $\mathcal{E}$ onto the sphere bundle is exact: $\pi_2^*\mathcal{E} = d\Pi'$. This part is a rather deep insight of Chern.

3. Let V be a smooth vector field on M with isolated zeros $\{p_l : 1 \leqslant l \leqslant r\}$. This defines a section

   $$\bar{V} : M \backslash \bigcup_{l=1}^{r} \{p_l\} \to SM, \ \bar{V}(x) \triangleq \frac{V(x)}{|V(x)|}.$$

   It follows from Step 2 that $\mathcal{E} = d\bar{V}^*\Pi'$ on $M\backslash\{p_1, \cdots, p_r\}$. Note that in general, one cannot expect $\mathcal{E}$ to be exact on M. Here exactness is obtained after puncturing M by the "holes" $\{p_1, \cdots, p_r\}$.

4. Stokes' theorem allows one to transfer the integral $\int_M \mathcal{E}$ to boundary integrals around the isolated zeros p_l:

   $$\int_M \mathcal{E} = -\lim_{\varepsilon \to 0^+} \sum_{l=1}^{r} \int_{\partial B(p_l,\varepsilon)} \bar{V}^*\Pi'.$$

   A careful analysis into the shape of the form Π' shows that the above localised integral at p_l picks up the index of the vector field V at p_l in the limit:

   $$\int_{\partial B(p_l,\varepsilon)} \bar{V}^*(-\Pi') = \mathrm{Ind}_{p_l}(V) + O(\varepsilon) \quad \text{as } \varepsilon \to 0^+.$$

   In view of the Poincaré-Hopf theorem, the CGB formula (5.3) thus follows.

Remark 5.1. One will gain deeper insight into the construction of $\mathcal{E}$ after we discuss the Chern-Weil theory in Chapter 7. Basically, the Euler form is a certain $\mathfrak{so}(2m)$-invariant polynomial of the curvature (the Pfaffian). The Chern-Weil theory shows that such construction is a cohomology class property (Euler forms defined from different geometries yield the same cohomology class). Since the integral of the Euler form depends only on its cohomology class, it is therefore reasonable to expect that one picks up topological information and does not feel the actual geometry of M.

5.2 Orthonormal frame bundle

Chern's proof uses the special orthonormal frame bundle in an essential way. In general, an advantage of working over frame bundles (more generally, over principal bundles) is that there are intrinsic geometric quantities (e.g. connection and curvature forms, horizontal vector fields etc.) which need not admit global projections on M.

In this section, we recall some basic properties of this bundle and examine the Levi-Civita connection from this perspective. Most discussions here follow naturally from the general connection theory developed in Chapter 6 (as a special situation), through which the reader will gain deeper understanding. We will only provide proofs that are not contained in the more general results in that chapter. The reader should also compare the discussion here with the two-dimensional situation (cf. Section 1.3.2 and Section 1.3.3).

Definition 5.2. The *special orthonormal frame bundle* SO(M) over M is the bundle of PONB at every point of M, namely,

$$\mathrm{SO}(M) \triangleq \big\{u = (x; X_1, \cdots, X_n) : x \in M, \{X_1, \cdots, X_n\} \text{ is a PONB of } T_xM\big\}.$$

The bundle SO(M) admits a canonical differential structure which makes it into a manifold of dimension $n(n+1)/2$. The group SO(n) acts transitively on SO(M) from the right along each fiber; under usual matrix notation the action is defined by

$$(u = (x; (X_1, \cdots, X_n)), g) \mapsto R_g(u) \triangleq ug \triangleq (x; (X_1, \cdots, X_n) \cdot g).$$

In other words, the g-action transforms a PONB $(X_1, \cdots, X_n)$ of T_xM into another one given by $\{Y_i = g_i^j X_j : 1 \leqslant i \leqslant n\}$. Any two such bases are related by a unique element in the group. In particular, each fiber $\pi^{-1}(x)$ (π is the bundle projection) is diffeomorphic to SO(n). Let $\mathcal{U}$ be an open cover of M where on each $U \in \mathcal{U}$ is defined a PONF $\{e_i\}$. Then

$$h_U : U \times \mathrm{SO}(n) \to \pi^{-1}U,\ h_U(x, g) \triangleq (x; (e_1(x), \cdots, e_n(x)) \cdot g) \tag{5.4}$$

defines a local trivialisation (as well as a local parametrisation if U is a coordinate chart) of SO(M).

5.2.1 Vertical subspaces and the canonical form

The fiber directions of $\mathrm{SO}(M)$ define the vertical subspaces of $T\mathrm{SO}(M)$ in a canonical way, which do not depend on the geometry of M.

Definition 5.3. Let $u = (x; X) \in \mathrm{SO}(M)$. The *vertical subspace* $\mathcal{V}_u$ of $T_u\mathrm{SO}(M)$ is the space of vectors that are tangential to a curve passing through u and living on the fiber $\pi^{-1}(x)$. Equivalently, $\mathcal{V}_u = \ker(d\pi)_u$.

Since the fiber $\pi^{-1}(x)$ is homeomorphic to $\mathrm{SO}(n)$, it is reasonable to expect that $\mathcal{V}_u \cong \mathfrak{so}(n)$. In fact, such an isomorphism is canonical (see Proposition 6.3).

Proposition 5.1. *Let $u \in \mathrm{SO}(M)$ be given fixed. The map*

$$\mathfrak{so}(n) \ni A \mapsto V_A \triangleq \frac{d}{dt}\Big|_{t=0} ue^{tA} \in \mathcal{V}_u \tag{5.5}$$

is a linear isomorphism.

On the bundle $\mathrm{SO}(M)$, there is a canonical $\mathbb{R}^n$-valued 1-form $\theta = (\theta^1, \cdots, \theta^n)^T$ which is defined as follows. Let $X \in T_u\mathrm{SO}(M)$. We define $\theta(X)_u$ by the unique expansion

$$(d\pi)_u X = \theta^i(X)_u X_i,$$

where $u = (x; X_1, \cdots, X_n)$. In other words, $\{\theta^i(X)_u\}$ are the coefficients of $(d\pi)_u X$ with respect to the basis $\{X_1, \cdots, X_n\}$ of T_xM.

Let $\{e_i\}$ be a local PONF on some open subset $U \subseteq M$ and let $\{\eta^i\}$ be its dual. The canonical form θ admits the following local expression (see Lemma 6.7):

$$\theta^i_{(x,g)} \triangleq (g^{-1})^i_j \pi^* \eta^j_x = g^j_i \pi^* \eta^j_x, \tag{5.6}$$

where (x, g) ($x \in U, g \in \mathrm{SO}(n)$) is the parametrisation of $\pi^{-1}U$ under the local trivialisation $\{e_i\}$ (see (5.4)). In matrix notation, one has $\theta = g^t \cdot \pi^*\eta$. Unlike the forms η^i which are only local, a nice feature of $\mathrm{SO}(M)$ is that the forms θ^i are globally well-defined.

Definition 5.4. The form θ is known as the *canonical form* on $\mathrm{SO}(M)$.

We summarise a few basic properties of the canonical form as follows.

Proposition 5.2. *(i) For each $u \in \mathrm{SO}(M)$, one has*

$$\mathcal{V}_u = \{Y \in T_u\mathrm{SO}(M) : \langle \theta^i, Y\rangle_u = 0 \text{ for every } i\}.$$

(ii) $R_g^*\theta = g^{-1} \cdot \theta$ for all $g \in \mathrm{SO}(n)$.

(iii) The n-form $\theta^1 \wedge \cdots \wedge \theta^n$ descends to the volume form vol_M on M. More precisely, one has $\theta^1 \wedge \cdots \wedge \theta^n = \pi^*\mathrm{vol}_M$.

Proof. The first two parts are proved in Proposition 6.14. To prove Part (iii), we first show that $\theta^1 \wedge \cdots \wedge \theta^n$ descends to M. Indeed, for any $g \in \mathrm{SO}(n)$, one has from Part (ii) that

$$\begin{aligned} R_g^*\big(\theta^1 \wedge \cdots \wedge \theta^n\big) &= R_g^*\theta^1 \wedge \cdots \wedge R_g^*\theta^n = a^1_{j_1} \cdots a^n_{j_n} \theta^{j_1} \wedge \cdots \wedge \theta^{j_n} \\ &= (\det a) \cdot \theta^1 \wedge \cdots \wedge \theta^n = \theta^1 \wedge \cdots \wedge \theta^n, \end{aligned}$$

where $a \triangleq g^{-1}$ and the last equality holds since $a \in \mathrm{SO}(n)$. Together with Part (i), one concludes that the form $\theta^1 \wedge \cdots \wedge \theta^n$ annihilates vertical vectors (i.e. $(\theta^1 \wedge \cdots \wedge \theta^n)(X_1, \cdots, X_n) = 0$ provided that at least one of the X_i's is vertical) and is invariant under $\mathrm{SO}(n)$-actions. As a consequence, it descends to an n-form on M, namely, $\theta^1 \wedge \cdots \wedge \theta^n = \pi^*\tau$ for some $\tau \in \Omega^n(M)$.

It remains to prove that $\tau = \mathrm{vol}_M$. Let $\{e_1, \cdots, e_n\}$ be a local PONF over some open subset $U \subseteq M$. Let $s : U \to \mathrm{SO}(M)$ be the section defined by

$$s(x) \triangleq (x; e_1(x), \cdots, e_n(x)).$$

By the definition of θ, for any $v \in T_xM$ one has

$$(s^*\theta^i)_x(v) = \theta^i((ds)(v))_{s(x)} = \langle (d\pi) \circ (ds)v, e_i \rangle_x = \langle v, e_i \rangle_x.$$

This shows that $\{s^*\theta^1, \cdots, s^*\theta^n\}$ is dual to $\{e_i\}$ on U. It follows that

$$\mathrm{vol}_M = s^*\theta^1 \wedge \cdots \wedge s^*\theta^n = s^*\pi^*\tau = \tau.$$

This proves the final claim. □

5.2.2 Horizontal subspaces and connection form

Essentially, both the vertical subspaces and the canonical form are differential concepts and do not depend on the geometry of M (they are in fact well-defined on the general frame bundle and descend to $\mathrm{SO}(M)$ because of the Riemannian structure). We now introduce another important construction which is based on the Levi-Civita connection on M and is thus geometric.

Definition 5.5. A curve $u_t = (\gamma_t; X_t)$ on $\mathrm{SO}(M)$ is said to be *horizontal* if each component of X_t is parallel (as a vector field) along γ_t. A tangent vector $H \in T_u\mathrm{SO}(M)$ is a *horizontal vector* if it is tangential to a horizontal curve passing through u. The *horizontal subspace* $\mathcal{H}_u$ of $T_u\mathrm{SO}(M)$ is the subspace of horizontal vectors at u.

Note that the horizontal subspace $\mathcal{H}_u$ is determined by the Levi-Civita connection. It is isomorphic to T_xM under the projection $(d\pi)_u$. Indeed, for any curve γ_t passing through x, there is a unique horizontal curve u_t passing through

$u = (x; X) \in \mathrm{SO}(M)$ such that $\pi(u_t) = \gamma_t$; this curve is defined by $u_t = (\gamma_t; X_t)$ where X_t is the parallel transport of X along γ. In addition, if two base curves γ_t and $\tilde{\gamma}_t$ are equivalent at x (i.e. $\gamma_0 = \tilde{\gamma}_0 = x$ and $\gamma_0' = \tilde{\gamma}_0'$), their horizontal lifts u_t and $\tilde{u}_t$ are also equivalent at u. The horizontal vector at u whose projection is $v \in T_xM$ is called the *horizontal lift* of v at u.

Since $\ker(d\pi)_u = \mathcal{V}_u$ and $(d\pi)_u|_{\mathcal{H}_u} : \mathcal{H}_u \to T_xM$ is an isomorphism, it is immediate that $T_u\mathrm{SO}(M) = \mathcal{V}_u \oplus \mathcal{H}_u$. This decomposition gives rise to the connection form ω on $\mathrm{SO}(M)$ which is defined as follows. For

$$X = Y + Z \in \mathcal{V}_u \oplus \mathcal{H}_u,$$

we set $\omega(X)_u$ to be the element of $\mathfrak{so}(n)$ corresponding to the vertical vector Y under the isomorphism (5.5). By varying u, one obtains an $\mathfrak{so}(n)$-valued 1-form $\omega \in \Omega^1(\mathrm{SO}(M))$.

Let $\{e_i\}$ be a local PONF on some open subset $U \subseteq M$. Let A be the local ($\mathfrak{so}(n)$-valued) connection matrix of ∇ with respect to $\{e_i\}$ (i.e. $\nabla e_i = A_i^j \otimes e_j$; see (5.1)). The connection form ω admits the following local representation (see Proposition 6.6):

$$\omega_{(x,g)} \triangleq g^{-1} \cdot dg + g^{-1} \cdot \pi^* A_x \cdot g. \tag{5.7}$$

Here the term $g^{-1} \cdot dg$ is understood as follows. Let Y be a tangent vector at $u = (x, g)$ represented by a curve $u_t = (\gamma_t, g_t)$ ($x_t \in U$, $g_t \in \mathrm{SO}(n)$ with $u_0 = (x, g)$). Then

$$\langle g^{-1} \cdot dg, Y \rangle_u = g^{-1} \cdot g_0' \in \mathfrak{so}(n).$$

In the following proposition, we summarise the essential properties of the connection form ω on $\mathrm{SO}(M)$ in the spirit of connections on principal bundles (cf. Theorem 6.1).

Proposition 5.3. *(i) The kernel of ω at each $u \in \mathrm{SO}(M)$ is precisely the horizontal subspace H_u. This gives rise to a splitting $T_u\mathrm{SO}(M) = \mathcal{V}_u \oplus \mathcal{H}_u$ as a direct sum that varies smoothly in u.*

(ii) For any $u \in \mathrm{SO}(M)$ and $a \in \mathrm{SO}(n)$, one has $\mathcal{H}_{ua} = (dR_a)_u \mathcal{H}_u$.

(iii) For any $V \in \mathcal{V}_u$, the matrix $\langle \omega, V \rangle_u \in \mathfrak{so}(n)$ is the element corresponding to V under the canonical isomorphism (5.5).

(iv) $R_a^* \omega = \mathrm{Ad}(a^{-1})(\omega) = a^{-1} \cdot \omega \cdot a$ for all $a \in \mathrm{SO}(n)$.

Proof. (i) and (iii) just follow from the definition of ω. (ii) is a direct consequence of (i) and (iv). We thus only need to prove (iv). Indeed, by using the local expression

(5.7) one has

$$\begin{aligned} R_a^*\omega &= R_a^*(g^{-1}\cdot dg + g^{-1}\cdot \pi^*A\cdot g) \\ &= g^{-1}(R_a\circ\cdot)\cdot d(g(R_a\circ\cdot)) + g^{-1}(R_a\circ\cdot)\cdot\pi^*A\cdot g(R_a\circ\cdot) \\ &= a^{-1}\cdot g^{-1}\cdot dg\cdot a + a^{-1}\cdot g^{-1}\cdot \pi^*A\cdot g\cdot a \\ &= a^{-1}(g^{-1}\cdot dg + g^{-1}\cdot\pi^*A\cdot g)\cdot a = a^{-1}\cdot\omega\cdot a \end{aligned}$$

for all $a\in \mathrm{SO}(n)$. This gives the desired relation. □

Remark 5.2. Currently, the connection form ω is defined in terms of the Levi-Civita connection (through the induced notion of parallel transport). The general theory of connections on principal bundles takes the opposite viewpoint. A *connection on* $\mathrm{SO}(M)$ is an $\mathfrak{so}(n)$-valued 1-form on $\mathrm{SO}(M)$ which satisfy the four properties of Proposition 5.3 (cf. Definition 6.8 and Proposition 6.5). Suppose that a connection ω is given in this sense. One can then define horizontal curves and vectors (since the horizontal subspaces are well-defined as the kernels of ω), and in particular, the notion of parallel transport. This allows one to make sense of covariant derivative of vector fields on M, therefore giving rise to an associated affine connection on the tangent bundle TM. Such an affine connection must be compatible with the metric because ω is $\mathfrak{so}(n)$-valued. It is the Levi-Civita connection (namely, satisfying the additional torsion free property) if and only if $d\theta+\omega\wedge\theta=0$ (cf. the first structure equation in Theorem 6.6).

5.2.3 Curvature form and the lifted Euler form

We now define the curvature form of ω where ω is the connection form on $\mathrm{SO}(M)$ associated with the Levi-Civita connection.

Definition 5.6. The *curvature form* of ω is the $\mathfrak{so}(n)$-valued 2-form on $\mathrm{SO}(M)$ defined by $\Omega \triangleq d\omega+\omega\wedge\omega$.

One can also write

$$\Omega = d\omega + \frac{1}{2}[\omega\wedge\omega], \tag{5.8}$$

where the *Lie bracket wedge* $[\cdot\wedge\cdot]$ is defined through

$$[(\alpha\otimes A)\wedge(\beta\otimes B)] \triangleq (\alpha\wedge\beta)\otimes[A,B]$$

for $\alpha,\beta\in\Omega(P)$ and $A,B\in\mathfrak{so}(n)$ (cf. (6.29)). It is clear from (5.8) that Ω is $\mathfrak{so}(n)$-valued. Unlike the local curvature matrix $F=dA+A\wedge A$, the curvature form Ω is intrinsically defined over the entire $\mathrm{SO}(M)$. In addition, one has the following basic properties of Ω. A form η on $\mathrm{SO}(M)$ is said to be *horizontal* if $\iota(V)\eta=0$ for all vertical vector fields V, where $\iota(V)$ is the interior product by V (see Definition B.6).

Lemma 5.2. *(i) Locally, one has* $\Omega = g^t \cdot \pi^* F \cdot g$. *In particular,* Ω *is horizontal.*

(ii) $R_a^* \Omega = a^t \cdot \Omega \cdot a$ *for all* $a \in \mathrm{SO}(n)$.

Proof. (i) By using the local expression (6.11) of ω, one finds that

$$\begin{aligned} d\omega &= dg^t \cdot dg + dg^t \cdot \pi^* A \cdot g + g^t \cdot \pi^* dA \cdot g - g^t \cdot \pi^* A \cdot dg, \\ \omega \wedge \omega &= (g^t \cdot dg + g^t \cdot \pi^* A \cdot g) \wedge (g^t \cdot dg + g^t \cdot \pi^* A \cdot g). \end{aligned}$$

The claim follows by adding up the above two identities together with the observation that $dg^t \cdot g + g^t \cdot dg = 0$. The fact that Ω is horizontal is obvious from this.

(ii) This is a direct consequence of Proposition 5.3 (iv). $\square$

The curvature form Ω satisfies the following *Bianchi's identity*, which plays an essential role in Chern's intrinsic proof of the CGB theorem.

Proposition 5.4 (Bianchi's identity)**.** $d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$.

Proof. This is another straight forward computation:

$$\begin{aligned} d\Omega &= d(\omega \wedge \omega) = d\omega \wedge \omega - \omega \wedge d\omega \\ &= (\Omega - \omega \wedge \omega) \wedge \omega - \omega \wedge (\Omega - \omega \wedge \omega) = \Omega \wedge \omega - \omega \wedge \Omega. \end{aligned}$$

$\square$

We now restrict to the even dimensional case ($n = 2m$) and define the lifted Euler form on $\mathrm{SO}(M)$. Recall from Definition 5.1 that the Euler form $\mathcal{E}$ on M is defined by (5.2).

Definition 5.7. The (*lifted*) *Euler form on* $\mathrm{SO}(M)$ is defined by

$$\Xi \triangleq \frac{1}{2^{2m}\pi^m m!} \sum_{i_1,\cdots,i_{2m}=1}^{2m} \mathrm{sgn}(i_1,\cdots,i_{2m}) \Omega^{i_1}_{i_2} \wedge \cdots \wedge \Omega^{i_{2m-1}}_{i_{2m}}.$$

Lemma 5.3. *One has* $\Xi = \pi^* \mathcal{E}$. *In particular,* Ξ *is horizontal and* $\mathrm{SO}(2m)$*-invariant, i.e.* $R_a^* \Xi = \Xi$ *for all* $a \in \mathrm{SO}(2m)$.

Proof. By definition and Lemma 5.2 (i), one has

$$\begin{aligned} \Xi =& \frac{1}{2^{2m}\pi^m m!} \sum_{i_1,\cdots,i_{2m}} \sum_{j_1,\cdots,j_{2m}} \mathrm{sgn}(i_1,\cdots,i_{2m}) \big(g^{j_1}_{i_1} \pi^* F^{j_1}_{j_2} g^{j_2}_{i_2}\big) \wedge \\ & \cdots \wedge \big(g^{j_{2m-1}}_{i_{2m-1}} \pi^* F^{j_{2m-1}}_{j_{2m}} g^{j_{2m}}_{i_{2m}}\big) \\ =& \frac{1}{2^{2m}\pi^m m!} \sum_{j_1,\cdots,j_{2m}} \mathrm{sgn}(j_1,\cdots,j_{2m}) \pi^* F^{j_1}_{j_2} \wedge \cdots \wedge \pi^* F^{j_{2m-1}}_{j_{2m}} = \pi^* \mathcal{E}, \end{aligned}$$

where the second identity follows from the fact that $\det g = 1$. As the pullback of a form on M by π^*, it is immediate that Ξ is horizontal and $\mathrm{SO}(2m)$-invariant. $\square$

Remark 5.3. A differential form η on $\mathrm{SO}(M)$ is the pullback of some form on M (i.e. $\eta = \pi^*\alpha$ for some $\alpha \in \Omega(M)$) if and only if η is horizontal and $\mathrm{SO}(2m)$-invariant. The proof of this fact is left as an exercise.

5.3 Chern's intrinsic proof

In this section, we discuss Chern's intrinsic proof of Theorem 5.1. The fundamental idea is to construct a family of transgression forms on the sphere bundle and use them to establish the exactness of the lifted Euler form.

5.3.1 Transgression forms and their descendence to sphere bundle

We begin by introducing the following key constructions, which are nowadays known as the *Chern transgression forms*.

Definition 5.8. For each $k = 0, 1, \cdots, m-1$, we define

$$\Phi_k \triangleq \sum_{i_1,\cdots,i_{2m-1}=1}^{2m-1} \mathrm{sgn}(i_1, \cdots, i_{2m-1}, 2m)\Omega^{i_1}_{i_2} \wedge \cdots \wedge \Omega^{i_{2k-1}}_{i_{2k}} \wedge \omega^{i_{2k+1}}_{2m} \wedge \cdots \wedge \omega^{i_{2m-1}}_{2m} \tag{5.9}$$

and

$$\Psi_k \triangleq \sum_{i_1,\cdots,i_{2m-1}=1}^{2m-1} \mathrm{sgn}(i_1, \cdots, i_{2m-1}, 2m)\Omega^{i_1}_{i_2} \wedge \cdots \wedge \Omega^{i_{2k-1}}_{i_{2k}} \wedge \Omega^{i_{2k+1}}_{2m} \wedge \omega^{i_{2k+2}}_{2m} \wedge \cdots \wedge \omega^{i_{2m-1}}_{2m}.$$

We also set $\Psi_{-1} \triangleq 0$ as a convention.

Similar to the Euler form, it is routine to check that Φ_k, Ψ_k are intrinsic over $\mathrm{SO}(M)$. Note that $\Phi_k \in \Omega^{2m-1}(\mathrm{SO}(M))$ and $\Psi_k \in \Omega^{2m}(\mathrm{SO}(M))$. A crucial fact is that the forms Φ_k, Ψ_k descend to the sphere bundle. We first recall the basic definition.

Definition 5.9. The *sphere bundle* SM over M is the bundle of unit tangent vectors:

$$SM \triangleq \big\{(x, v) : x \in M,\ v \in T_xM, |v|_{T_xM} = 1\big\}.$$

The bundle projection is denoted as $\pi_2 : SM \to M$. We also define the canonical projection

$$\pi_1 : \mathrm{SO}(M) \to SM,\ \pi_1((x; X_1, \cdots, X_n)) \triangleq (x, X_n).$$

By definition, $\pi = \pi_2 \circ \pi_1$. The bundle $(\mathrm{SO}(M), SM, \pi_1)$ is an $\mathrm{SO}(n-1)$-bundle where one identifies $\mathrm{SO}(n-1)$ with

$$\Big\{a = \begin{pmatrix} g & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} : g \in \mathrm{SO}(n-1)\Big\}. \tag{5.10}$$

The action of a on $(x; X_1, \cdots, X_n) \in \mathrm{SO}(M)$ is defined by transforming the first $n-1$ vectors $(X_1, \cdots, X_{n-1})$ leaving the last vector X_n fixed. We say that a form η on $\mathrm{SO}(M)$ *descends to* SM if there exists a form α on SM such that $\eta = \pi_1^*\alpha$. Similar to Remark 5.3, a form η descends to SM if and only if it is π_1-horizonal (i.e. $\iota_V\eta = 0$ for any π_1-vertical vector field V) and $\mathrm{SO}(n-1)$-invariant.

Lemma 5.4. *The forms Φ_k, Ψ_k descend to the sphere bundle SM.*

Proof. (i) *The forms Φ_k, Ψ_k are* $\mathrm{SO}(2m-1)$*-invariant.* Let a be given by (5.10) with some $g \in \mathrm{SO}(2m-1)$. According to Proposition 5.3 (iv) and Lemma 5.2 (ii),

$$R_a^*\Phi_k = \sum_{i_1,\cdots,i_{2m-1}=1}^{2m-1} \varepsilon(i_1,\cdots,i_{2m-1},2m)(R_a^*\Omega)_{i_2}^{i_1} \wedge \cdots \wedge (R_a^*\Omega)_{i_{2k}}^{i_{2k-1}}$$
$$\wedge (R_a^*\omega)_{2m}^{i_{2k+1}} \wedge \cdots \wedge (R_a^*\omega)_{2m}^{i_{2m-1}}.$$

The $R_a^*\Omega$-terms are of the form

$$(R_a^*\Omega)_\beta^\alpha = \sum_{\delta,\gamma=1}^{2m} a_\alpha^\delta \Omega_\gamma^\delta a_\beta^\gamma, \quad \alpha, \beta = 1, \cdots, 2m-1.$$

Due to the special form of a, the above sum only runs over $\delta, \gamma = 1, \cdots, 2m-1$. The $R_a^*\omega$-terms are of the form

$$(R_a^*\omega)_{2m}^\alpha = \sum_{\delta,\gamma=1}^{2m} a_\alpha^\delta \omega_\gamma^\delta a_{2m}^\gamma, \quad \alpha = 1, \cdots, 2m-1.$$

Note that γ must be equal to $2m$ and δ essentially runs over $1, \cdots, 2m-1$. Based on these observations, one finds that

$$\begin{aligned} R_a^*\Phi_k = {} & \sum_{i_1,\cdots,i_{2m-1}=1}^{2m-1} \sum_{j_1,\cdots,j_{2m-1}=1}^{2m-1} \mathrm{sgn}(i_1,\cdots,i_{2m-1},2m) g_{i_1}^{j_1} \cdots g_{i_{2m-1}}^{j_{2m-1}} \\ & \Omega_{j_2}^{j_1} \wedge \cdots \wedge \Omega_{j_{2k}}^{j_{2k-1}} \wedge \omega_{2m}^{j_{2k+1}} \wedge \cdots \wedge \omega_{2m}^{j_{2m-1}} \\ = {} & \sum_{j_1,\cdots,j_{2m-1}=1}^{2m-1} \mathrm{sgn}(j_1,\cdots,j_{2m-1},2m)\Omega_{j_2}^{j_1} \wedge \cdots \wedge \Omega_{j_{2k}}^{j_{2k-1}} \\ & \wedge \omega_{2m}^{j_{2k+1}} \wedge \cdots \wedge \omega_{2m}^{j_{2m-1}} = \Phi_k. \end{aligned}$$

In a similar way, it can be shown that Ψ_k is also $\mathrm{SO}(2m-1)$-invariant.

(ii) *The forms Φ_k, Ψ_k are π_1-horizontal.* The horizontality of Ω is contained in Lemma 5.2 (i). In addition, recall from (5.7) that locally one has

$$\omega_{2m}^j(x,g) = \sum_{k=1}^{2m} \left(g_j^k dg_{2m}^k + g_j^k g_{2m}^l \pi^* A_l^k\right), \quad j = 1, \cdots, 2m-1. \tag{5.11}$$

Note that $(g_{2m}^k)_{1\leqslant k\leqslant 2m}$ are the coefficients of X_{2m} (the last vector of the frame $u=(x,g)$) with respect to the local PONF $\{e_i\}$. If

$$u_t=(x;X_1(t),\cdots,X_{2m}(t))=(x,g_t)$$

is a π_1-vertical curve, the curve $X_{2m}(t)$ is constant and thus $\dot{g}_{2m}^k(0)=0$ for all k. As a result, $\iota_V dg_{2m}^k=0$ for all π_1-vertical vectors $V\in T_u\mathrm{SO}(M)$. The second part of (5.11) is clearly horizontal. This proves that ω_{2m}^j is also π_1-horizontal. It follows from the definitions of Φ_k,Ψ_k that they are both π_1-horizontal. □

5.3.2 Exactness of the Euler form on sphere bundle

According to Lemma 5.4, one can write

$$\Phi_k=\pi_1^*\Phi_k',\ \ \Psi_k=\pi_1^*\Psi_k' \tag{5.12}$$

for some $\Phi_k'\in\Omega^{2m-1}(SM)$ and $\Psi_k'\in\Omega^{2m}(SM)$. In this subsection, we establish the following key relation.

Lemma 5.5. *The recursive relation*

$$d\Phi_k'=(2m-2k-1)\Psi_k'-2k\Psi_{k-1}' \tag{5.13}$$

holds for all $k=0,1,\cdots,m-1$.

Proof. The argument is quite technical and we divide it into several steps. Recall from the definition of Ω and Bianchi's identity that

$$d\Omega=\Omega\wedge\omega-\omega\wedge\Omega,\ \ d\omega=\Omega-\omega\wedge\omega. \tag{5.14}$$

These are two key relations that will be used in the main calculation. In what follows, we always omit the summation sign over the repeated indices $i_1,\cdots,i_{2m-1}$ (but noting that the range of summation is from 1 to $n-1$).

(i) By using the definition of Φ_k and the relation (5.14), one can write

$$\begin{aligned}d\Phi_k=&\mathrm{sgn}(i_1,\cdots,i_{2m-1},2m)\Big[\sum_{l=1}^{k}\Omega_{i_2}^{i_1}\wedge\cdots\wedge d\Omega_{i_{2l}}^{i_{2l-1}}\wedge\cdots\wedge\Omega_{i_{2k}}^{i_{2k-1}}\wedge\omega_{2m}^{i_{2k+1}}\\&\wedge\cdots\wedge\omega_{2m}^{i_{2m-1}}+\sum_{l=2k+1}^{2m-1}(-1)^{l-1}\Omega_{i_2}^{i_1}\wedge\cdots\wedge\Omega_{i_{2k}}^{i_{2k-1}}\wedge\omega_{2m}^{i_{2k+1}}\\&\wedge\cdots\wedge d\omega_{2m}^{i_l}\wedge\cdots\wedge\omega_{2m}^{i_{2m-1}}\Big]=:A_k+B_k+C_k+D_k,\end{aligned}$$

where

$$A_k \triangleq \sum_{l=1}^{k}\sum_{\alpha=1}^{2m} \mathrm{sgn}(i_1,\cdots,i_{2m-1},2m)\Omega_{i_2}^{i_1}\wedge\cdots\wedge\Omega_{\alpha}^{i_{2l-1}}\wedge\omega_{i_{2l}}^{\alpha}\wedge\cdots\wedge\Omega_{i_{2k}}^{i_{2k-1}}\wedge\omega_{2m}^{i_{2k+1}}\wedge\cdots\wedge\omega_{2m}^{i_{2m-1}},$$

$$B_k \triangleq -\sum_{l=1}^{k}\sum_{\alpha=1}^{2m} \mathrm{sgn}(i_1,\cdots,i_{2m-1},2m)\Omega_{i_2}^{i_1}\wedge\cdots\wedge\omega_{\alpha}^{i_{2l-1}}\wedge\Omega_{i_{2l}}^{\alpha}\wedge\cdots\wedge\Omega_{i_{2k}}^{i_{2k-1}}\wedge\omega_{2m}^{i_{2k+1}}\wedge\cdots\wedge\omega_{2m}^{i_{2m-1}},$$

$$C_k \triangleq \sum_{l=2k+1}^{2m-1}\sum_{\alpha=1}^{2m}(-1)^{l-1}\mathrm{sgn}(i_1,\cdots,i_{2m-1},2m)\Omega_{i_2}^{i_1}\wedge\cdots\wedge\Omega_{i_{2k}}^{i_{2k-1}}\wedge\omega_{2m}^{i_{2k+1}}\wedge\cdots\wedge\omega_{2m}^{\alpha}\wedge\omega_{\alpha}^{i_l}\wedge\cdots\wedge\omega_{2m}^{i_{2m-1}}.$$

$$D_k \triangleq \sum_{l=2k+1}^{2m-1}(-1)^{l-1}\mathrm{sgn}(i_1,\cdots,i_{2m-1},2m)\Omega_{i_2}^{i_1}\wedge\cdots\wedge\Omega_{i_{2k}}^{i_{2k-1}}\wedge\omega_{2m}^{i_{2k+1}}\wedge\cdots\wedge\Omega_{2m}^{i_l}\wedge\cdots\wedge\omega_{2m}^{i_{2m-1}}.$$

(ii) In the expression of D_k, by moving $\Omega_{2m}^{i_l}$ to the place between $\Omega_{i_{2k}}^{i_{2k-1}}$ and $\omega_{2m}^{i_{2k+1}}$ one easily finds that

$$(-1)^{l-1}\mathrm{sgn}(i_1,\cdots,i_{2m-1},2m)\Omega_{i_2}^{i_1}\wedge\cdots\wedge\Omega_{i_{2k}}^{i_{2k-1}}\wedge\omega_{2m}^{i_{2k+1}}\wedge\cdots\wedge\Omega_{2m}^{i_l}\wedge\cdots\wedge\omega_{2m}^{i_{2m-1}} = \Psi_k$$

for every $2k+1\leqslant l\leqslant 2m-1$. As a result, one has

$$D_k = (2m-2k-1)\Psi_k.$$

(iii) Observe that A_k, B_k, C_k all involve summation over $\alpha = 1,\cdots,2m$. We claim that the summation over $\alpha = 1,\cdots,2m-1$ is zero for each of these three terms. Indeed, by rearranging terms one can write

$$A_k + B_k + C_k = \sum_{1\leqslant\alpha<\beta\leqslant 2m-1}\omega_{\alpha}^{\beta}\wedge\eta_{\beta}^{\alpha} + A_k' + B_k',$$

where η_{β}^{α} are suitable $(2m-1)$-forms and the terms A_k', B_k' are defined by freezing $\alpha = 2m$ in the definitions of A_k, B_k respectively. Here the C_k'-term is not needed because C_k contains the factor ω_n^{α} which vanishes when $\alpha = 2m$. It follows that

$$d\Phi_k = (2m-2k-1)\Psi_k + \sum_{1\leqslant\alpha<\beta\leqslant 2m-1}\omega_{\alpha}^{\beta}\wedge\eta_{\beta}^{\alpha} + A_k' + B_k'. \tag{5.15}$$

Our aim is to show that $\eta_{\beta}^{\alpha} = 0$.

To this end, note that the forms $\eta_{\beta}^{\alpha}, A_k', B_k', C_k'$ are all expressed in terms of linear combinations of exterior products among the $\Omega_q^p, \omega_{2m}^r$'s. It was shown in the proof of Lemma 5.4 that $\Omega_p^q, \omega_{2m}^r$ are π_1-horizontal (so are the Φ_k, Ψ_k's). As a consequence, by applying ι_V for any π_1-vertical vector to both sides of (5.15) one finds that

$$\sum_{1\leqslant\alpha<\beta\leqslant 2m-1}(\iota_V\omega_{\alpha}^{\beta})\wedge\eta_{\beta}^{\alpha} = 0. \tag{5.16}$$

Now fix a pair $1 \leqslant \alpha < \beta \leqslant 2m-1$. Let $V_{\alpha,\beta}$ be the vertical vector at $u = (x;(X_1,\cdots,X_{2m}))$ represented by the rotation $u_t \triangleq (x;(X_1(t),\cdots,X_{2m}(t)))$ $(-\varepsilon < t < \varepsilon)$, where

$$X_\alpha(t) \triangleq (\cos t)X_\alpha + (\sin t)X_\beta,\ \ X_\beta(t) \triangleq -(\sin t)X_\alpha + (\cos t)X_\beta$$

and $X_\gamma(t) \triangleq X_\gamma$ for all $\gamma \neq \alpha, \beta$. By the definition (5.7) of ω and the construction of u_t, it is readily checked that

$$\langle \omega_\alpha^\beta, V_{\alpha,\beta}\rangle_u = \langle X_\beta(0), \dot{X}_\alpha(0)\rangle_{T_xM} = \langle X_\beta(0), X_\beta(0)\rangle_{T_xM} = 1$$

and $\langle \omega_\sigma^\tau, V_{\alpha,\beta}\rangle_u = 0$ for all $(\sigma,\tau) \neq (\alpha,\beta)$. It follows from (5.16) with $V = V_{\alpha,\beta}$ that $\eta_\beta^\alpha = 0$ at u. The claim follows since α, β, u are arbitrary.

(iv) From Step (iii), one obtains that

$$d\Phi_k = (2m-2k-1)\Psi_k + A_k' + B_k'.$$

By rearranging the factors and keeping track of sign properly, it is not hard to see that each l-summand in the definition of A_k' or B_k' is equal to $-\Psi_{k-1}$. Therefore, one has

$$d\Phi_k = (2m-2k-1)\Psi_k - 2k\Psi_{k-1}.$$

It follows from (5.12) that

$$\pi_1^*\big(d\Phi_k' - (2m-2k-1)\Psi_k' + 2k\Psi_{k-1}'\big) = 0.$$

On the other hand, by using a local section it is easily seen that π_1^* is injective. Therefore, one concludes that

$$d\Phi_k' - (2m-2k-1)\Psi_k' + 2k\Psi_{k-1}' = 0,$$

which gives the desired relation. □

Let us define the (*lifted*) *Euler form on* SM by $\Xi' \triangleq \pi_2^*\mathcal{E}$. By using Lemma 5.5, one can now establish the exactness of Ξ'.

Corollary 5.1. *One has* $\Xi' = d\Pi'$ *where*

$$\Pi' \triangleq \sum_{k=0}^{m-1} \tilde{\Phi}_k';\ \ \tilde{\Phi}_k' \triangleq \frac{2^{-k}}{(2\pi)^m(2m-2k-1)!!k!}\Phi_k'. \tag{5.17}$$

Proof. By the definition of Ψ_k, it is easily seen that

$$\Xi = \frac{2m}{2^{2m}\pi^m m!}\Psi_{m-1}.$$

This relation descends to SM due to the injectivity of π_1^*, namely, one has

$$\Xi' = \frac{2m}{2^{2m}\pi^m m!}\Psi'_{m-1}.$$

It is now a simple matter of algebra to deduce $\Xi' = d\Pi'$ based on this identity and the recursive relation (5.13). ☐

Remark 5.4. Of course, one also has $\Xi = d\Pi$ on $\mathrm{SO}(M)$, where Π is defined by the same formula (5.17) with Φ'_k replaced by Φ_k. However, exactness of the Euler form *on* SM is the heart of the whole argument.

Remark 5.5. Historically, it was a bit mysterious how Chern came up with the specific construction of the forms Φ_k, Ψ_k in the first place. One possible observation is that Ξ has the form "$\Omega \wedge \cdots \wedge \Omega$" and a reasonable idea is to look for a family of forms of the shape "$\Omega \wedge \cdots \wedge \Omega \wedge \omega \wedge \cdots \wedge \omega$"; the key relations in (5.14) yield certain recursive relations among these forms, which eventually lead to the exactness of Ξ.

5.3.3 Local analysis of transgression forms

Another remarkable quantitative aspect of Chern's proof is that the $(2m-1)$-form Π' restricts to the normalised volume form on each fiber of SM. Let $x \in M$ be given fixed. The orientation on the unit sphere

$$S_x \triangleq \pi_2^{-1}(x) = \big\{v \in T_xM : |v|_{T_xM} = 1\big\}$$

is taken to be the one induced from the oriented Euclidean space T_xM. Recall from Example C.1 that the normalised volume form on S_x is given by

$$\nu = \frac{(m-1)!}{2\pi^m}\sum_{j=1}^{2m}(-1)^{j-1}x^j dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^{2m}. \tag{5.18}$$

Here $(x^i)_{1\leqslant i\leqslant 2m}$ are the extrinsic coordinates on T_xM with respect to any fixed PONB.

Lemma 5.6. *Let* $i_x : S_x \to SM$ *be the inclusion. Then*

$$i_x^*\Pi' = i_x^*\tilde{\Phi}'_0 = -\nu \tag{5.19}$$

and $i_x^*\tilde{\Phi}'_k = 0$ *for all* $k \geqslant 1$.

Proof. Define $\tilde{\Phi}_k$ in the same way as $\tilde{\Phi}'_k$ with Φ'_k replaced by Φ_k (see (5.17)). Locally, $\tilde{\Phi}'_k$ is just the pullback of $\tilde{\Phi}_k$ by any local section s of π_1. Since each $\tilde{\Phi}_k$ $(k \geqslant 1)$ contains the term $\Omega = g^t \cdot \pi^*F \cdot g$, it is obvious that $i_x^*\tilde{\Phi}'_k = 0$. For the same reason, the term $g^t \cdot \pi^*A \cdot g$ appearing in any of the $\omega_{2m}^{\cdot}$-terms in the expression of $\tilde{\Phi}_0$ is annihilated by i_x^* (after restricting to $\tilde{\Phi}_0'$ by s). The only nontrivial contribution thus comes from the term

$$\tilde{\Phi}_{0,\mathrm{main}} \triangleq \frac{1}{(2\pi)^m(2m-1)!!} \sum_{i_1,\cdots,i_{2m-1}=1}^{2m-1} \sum_{j_1,\cdots,j_{2m-1}=1}^{2m} \mathrm{sgn}(i_1,\cdots,i_{2m-1}) \\ g_{i_1}^{j_1}\cdots g_{i_{2m-1}}^{j_{2m-1}} dg_{2m}^{j_1}\wedge\cdots\wedge dg_{2m}^{j_{2m-1}}.$$

This form indeed descends to SM. To see this, one first notes that

$$\sum_{i_1,\cdots,i_{2m-1}=1}^{2m-1} \mathrm{sgn}(i_1,\cdots,i_{2m-1}) g_{i_1}^{j_1}\cdots g_{i_{2m-1}}^{j_{2m-1}} = \det\begin{pmatrix} g_1^{j_1} & \cdots & g_{2m-1}^{j_1} \\ \vdots & & \vdots \\ g_1^{j_{2m-1}} & \cdots & g_{2m-1}^{j_{2m-1}} \end{pmatrix}$$

for each fixed $(j_1,\cdots,j_{2m-1})$. This determinant is nonzero only when $(j_1,\cdots,j_{2m-1})$ are all distinct. For any such distinct $(2m-1)$-tuple, the relation $g^t g = \mathrm{id}$ easily yields that

$$g_{2m}^{j_{2m}} = (-1)^{j_{2m}}\mathrm{sgn}(j_1,\cdots,j_{2m-1}) \det\begin{pmatrix} g_1^{j_1} & \cdots & g_{2m-1}^{j_1} \\ \vdots & & \vdots \\ g_1^{j_{2m-1}} & \cdots & g_{2m-1}^{j_{2m-1}} \end{pmatrix},$$

where $j_{2m} \triangleq (j_1,\cdots,j_{2m-1})^c$. Since

$$\mathrm{sgn}(j_1,\cdots,j_{2m-1}) dg_{2m}^{j_1}\wedge\cdots\wedge dg_{2m}^{j_{2m-1}} = dg_{2m}^1\wedge\cdots\wedge \widehat{dg_{2m}^{j_{2m}}}\wedge\cdots\wedge dg_{2m}^{2m}$$

It follows that

$$\tilde{\Phi}_{0,\mathrm{main}} \triangleq \frac{(2m-1)!}{(2\pi)^m(2m-1)!!} \sum_{j_{2m}=1}^{2m} (-1)^{j_{2m}} g_{2m}^{j_{2m}} dg_{2m}^1\wedge\cdots\wedge\widehat{dg_{2m}^{j_{2m}}}\wedge\cdots\wedge dg_{2m}^{2m}, \quad (5.20)$$

which is indeed a form on SM. For $(x, X_{2m}) \in SM$, note that $(g_{2m}^j)_{1\leqslant j\leqslant 2m}$ are the coefficients of X_{2m} with respect to a local PONF. In view of (5.18), the restriction of (5.20) on the fiber S_x is precisely the negative of the normalised volume form. The relation (5.19) thus follows. □

Chern's proof of Theorem 5.1 actually uses a perturbed version of Lemma 5.6 around isolated zeros of a vector field. The key estimate is contained in Lemma 5.7 below. Let V be a smooth vector field on M with isolated zeros $\{p_1,\cdots,p_r\}$ and we assume that each p_l is nondegenerate (see Proposition 2.4 and Remark 2.4 for a discussion on the existence of such a vector field). For each p_l, we consider a normal chart $(U; x^i)$ around p_l along with a PONF $\{e_i\}$ on U that is parallel along geodesic rays emitting from p_l (see Appendix C.3 for the construction). Let us write

$$V(x) = \lambda^i(x)e_i(x) = V^i(x)\partial_i$$

with some functions λ^i, V^i on U.

We define the normalised vector field

$$\bar{V} : M\backslash\{p_1, \cdots, p_r\} \to SM,\ \bar{V}(x) \triangleq \frac{V(x)}{|V(x)|}.$$

On the normal chart U, one has

$$\bar{V}(x) = \sum_{i=1}^{2m} \frac{\lambda^i(x)}{\big(\sum_j \lambda^j(x)^2\big)^{1/2}} e_i(x).$$

For each small $\varepsilon > 0$, we define

$$\hat{V}_{l,\varepsilon} : \partial B(p_l, \varepsilon) \to S^{2m-1},\ \hat{V}_{l,\varepsilon}(x) \triangleq \frac{(V^1(x), \cdots, V^{2m}(x))}{\big(\sum_j V^j(x)^2\big)^{1/2}}.$$

Here the sphere $\partial B(p_l, \varepsilon)$ is oriented under the outer normal orientation and S^{2m-1} is the standard sphere in $\mathbb{R}^n$ (with the standard orientation). We also denote $\bar{V}_{l,\varepsilon} \triangleq \bar{V}|_{\partial B(p_l,\varepsilon)} : \partial B(p_l, \varepsilon) \to SM$.

Lemma 5.7. *There exist positive constants C, ε_0 depending only on M and V, such that the following estimate*

$$\big|\bar{V}^*_{l,\varepsilon}(-\Pi') - \hat{V}^*_{l,\varepsilon}\nu\big| \leqslant C\varepsilon^{-(2m-2)}$$

holds for all $\varepsilon \in (0, \varepsilon_0)$ and $l = 1, \cdots, r$ uniformly over $\partial B(p_l, \varepsilon)$. Here ν denotes the normalised volume form on S^{2m-1}.

Proof. The computation is essentially the same as the proof of Lemma 5.6. The $1/\varepsilon$-singularity comes from the terms

$$d\Big[\frac{\lambda^i(x)}{\big(\sum_j \lambda^j(x)^2\big)^{1/2}}\Big] \quad (1 \leqslant i \leqslant 2m)$$

inside the expression of $\bar{V}^*_{l,\varepsilon}(-\Pi')$. In the decomposition

$$\bar{V}^*_{l,\varepsilon}(-\Pi') = \sum_{k=0}^{m-1} \bar{V}^*_{l,\varepsilon}\big(-\tilde{\Phi}'_k\big),$$

the terms corresponding to $k \geqslant 1$ are all of order at most $O(\varepsilon^{-(2m-3)})$ (because they contain at most $2m-3$ number of $d\omega^i_{2m}$'s; see (5.9)). The leading term (i.e.

the most singular term) therefore comes from $\tilde\Phi_0'$; in fact, it is from $\tilde\Phi_{0,\mathrm{main}}$ defined by (5.20) since those terms containing $g^t\cdot\pi^*A\cdot g$ have singularity of order at most $\varepsilon^{-(2m-2)}$.

By definition, one can now write

$$\bar V_{l,\varepsilon}^*\big(-\tilde\Phi_{0,\mathrm{main}}\big)=\frac{(2m-1)!}{(2\pi)^m(2m-1)!!}\sum_{j_{2m}=1}^{2m}(-1)^{j_{2m}-1}\bar V^{j_{2m}}(x)d\bar V^1(x)\wedge\cdots\wedge\widehat{d\bar V^{j_{2m}}(x)}\wedge\cdots\wedge d\bar V^{2m}(x),$$

$$\hat V_{l,\varepsilon}^*\nu=\frac{(2m-1)!}{(2\pi)^m(2m-1)!!}\sum_{j_{2m}=1}^{2m}(-1)^{j_{2m}-1}\hat V^{j_{2m}}(x)d\hat V^1(x)\wedge\cdots\wedge\widehat{d\hat V^{j_{2m}}(x)}\wedge\cdots\wedge d\hat V^{2m}(x),$$

where we introduce

$$\bar V^i(x)\triangleq\frac{\lambda^i(x)}{\big(\sum_j\lambda^j(x)^2\big)^{1/2}},\quad \hat V^i(x)\triangleq\frac{V^i(x)}{\big(\sum_j V^j(x)^2\big)^{1/2}}$$

to ease notation. Let (σ_i^j) be the matrix such that $e_i=\sigma_i^j\partial_j$. Then one has

$$\lambda^i(x)\sigma_i^j(x)=V^j(x).$$

It is standard that $\sigma_i^j(x)=\delta_i^j+O(\varepsilon)$ as $\varepsilon\to0^+$ uniformly in $x\in\partial B(p_l,\varepsilon)$ (see (9.10)). It follows that

$$\bar V^i(x)=\hat V^i(x)+O(\varepsilon)$$

for all i uniformly in $x\in\partial B(p_l,\varepsilon)$. Similarly, one also has

$$d\bar V^i-d\hat V^i=O(1);\ \ d\bar V^i,d\hat V^i=O(\varepsilon^{-1}).$$

As a consequence,

$$\bar V_{l,\varepsilon}^*\big(-\tilde\Phi_{0,\mathrm{main}}\big)-\hat V_{l,\varepsilon}^*\nu=O(\varepsilon^{-(2m-2)}).$$

The desired claim thus follows. □

5.3.4 Proof of CGB

We now gather the main ingredients obtained so far to present Chern's proof of Theorem 5.1. The strategy becomes quite clear at this stage.

Proof of Theorem 5.1. First of all, since $\pi_2^*\mathcal{E}=d\Pi'$ and $\pi_2\circ\bar V=\mathrm{id}$, one has

$$\mathcal{E}=\bar V^*d\Pi'=d\bar V^*\Pi'\quad\text{on }M\backslash\{p_1,\cdots,p_r\}.$$

It follows from Stokes' theorem and Lemma 5.7 that

$$\int_{M\backslash\bigcup_{l=1}^r B(p_l,\varepsilon)}\mathcal{E}=-\sum_{l=1}^r\int_{\partial B(p_l,\varepsilon)}\bar V_{l,\varepsilon}^*\Pi'=\sum_{l=1}^r\int_{\partial B(p_l,\varepsilon)}\hat V_{l,\varepsilon}^*\nu+O(\varepsilon)\tag{5.21}$$

for all small ε. Here the boundary orientation of $M \backslash \cup_{l=1}^{r} B(p_l, \varepsilon)$ is the inner-normal orientation of the spheres $\partial B(p_l, \varepsilon)$ as the boundary of $B(p_l, \varepsilon)$.

On the other hand, according to Definition 2.11 of the index and Proposition 2.3, one has

$$\int_{\partial B(p_l,\varepsilon)} \hat{V}_{l,\varepsilon}^{*} \nu = \mathrm{ind}_{p_l}(V) \int_{S^{2m-1}} \nu = \mathrm{ind}_{p_l}(V). \tag{5.22}$$

By substituting (5.22) into (5.21) and taking $\varepsilon \to 0^+$, one concludes that

$$\int_M \mathcal{E} = \lim_{\varepsilon \to 0^+} \int_{M \backslash \bigcup_{l=1}^{r} B(p_l,\varepsilon)} \mathcal{E} = \sum_{l=1}^{r} \mathrm{ind}_{p_l}(V) = \chi(M).$$

The last identity follows from the Poincaré-Hopf theorem.

The proof of Theorem 5.1 is now complete.

□

6 Basic theory of connections

In this chapter, we recall the basic theory of connections from the perspective of principal bundles. This will provide deeper insight into several computations over $\mathrm{SO}(M)$ in the previous chapter as well as some preliminary notions needed for the Chern-Weil theory, which leads to another proof of the CGB theorem in the next chapter. Putting aside CGB, this chapter is of fundamental importance in various geometric contexts (e.g. gauge theory, index theory, low dimensional topology etc.). The contents of this chapter essentially come from [KN63], which is undoubtedly the best reference for this part.

Throughout the rest of this chapter, unless otherwise stated M is an n-manifold and G is a matrix Lie group.

6.1 Principal bundles

We start with the definition of a principal G-bundle.

Definition 6.1. A *principal G-bundle over M* consists of a manifold P, an action of G on P (G is called the *structure group*) and a projection $\pi : P \to M$ which satisfy the following properties.

(i) π is a smooth surjective submersion.

(ii) G acts freely on P from the right and is fiber preserving (i.e. $\pi(ug) = \pi(u)$ for all $u \in P$ and $g \in G$).

(iii) P is locally trivial in the sense that for every $x \in M$, there exists an open neighbourhood $U \subseteq M$ of x and a diffeomorphism

$$\psi : \pi^{-1}U \to U \times G, \ \ \psi(u) = (\pi(u), \varphi(u)),$$

where φ is the G-component of ψ, such that φ is *G-equivariant* in the sense that $\varphi(ug) = \varphi(u)g$ for all $u \in \pi^{-1}U$ and $g \in G$.

We will use the notation $(P, M, \pi; G)$ to denote a principal G-bundle.

Example 6.1. The trivial principal G-bundle is the product space $P = M \times G$, where the group action is defined by multiplication from the right on the G-component.

Example 6.2. Let

$$\mathcal{F}(M) \triangleq \{(x; X_1, \cdots, X_n) : x \in M, \{X_1, \cdots, X_n\} \text{ is a basis of } T_xM\}.$$

The space $\mathcal{F}(M)$ admits a natural differential structure which makes it into an $(n+n^2)$-dimensional manifold. $\mathcal{F}(M)$ is a principal $\mathrm{GL}(n;\mathbb{R})$-bundle over M whose right $\mathrm{GL}(n;\mathbb{R})$-action on $\mathcal{F}(M)$ is defined by the usual matrix multiplication:

$$\big((x;X_1,\cdots,X_n),g\big)\mapsto(x;X_ig_1^i,\cdots X_ig_n^i).$$

The bundle $\mathcal{F}(M)$ is known as the *frame bundle* over M. If M is a Riemannian manifold, the *orthonormal frame bundle* $\mathrm{O}(M)$ is the subbundle consisting of orthonormal bases. This is a principal $\mathrm{O}(n)$-bundle over M (cf. Section 5.2). The bundle $(\mathrm{SO}(M),SM,\pi_1;\mathrm{SO}(n-1))$ we used in Section 5.3 is a principal $\mathrm{SO}(n-1)$-bundle.

It is often useful to describe a principal bundle by means of *local gluing data*. Suppose that $(P,M,\pi;G)$ is a principal G-bundle. By definition, there exists an open cover $\{U_\alpha:\alpha\in\mathcal{A}\}$ of M together with G-equivariant diffeomorphisms $\psi_\alpha:\pi^{-1}U_\alpha\to U_\alpha\times G$ (i.e. $\varphi_\alpha(ug)=\varphi_\alpha(u)g$). Given $\alpha,\beta\in\mathcal{A}$ with $U_\alpha\cap U_\beta\neq\emptyset$, we define

$$g_{\beta\alpha}:U_{\alpha\beta}\triangleq U_\alpha\cap U_\beta\to G,\ \ g_{\beta\alpha}(x)\triangleq\varphi_\beta(u)\varphi_\alpha(u)^{-1}$$

where u is any point on the fiber $\pi^{-1}(x)$. The G-equivariance of each φ_α ensures that $g_{\beta\alpha}$ is well-defined. In other words, the G-components of a point $u\in P$ are related by the transition rule $\varphi_\beta(u)=g_{\beta\alpha}(\pi(u))\varphi_\alpha(u)$. The family of maps $g_{\beta\alpha}$ satisfy the following *cocycle relation*:

- $g_{\gamma\alpha}(x)=g_{\gamma\beta}(x)g_{\beta\alpha}(x)$ for all $x\in U_{\alpha\beta\gamma}\triangleq U_\alpha\cap U_\beta\cap U_\gamma$ provided that this intersection is nonempty.

Note that the above cocycle relation implies that $g_{\beta\alpha}(x)=g_{\alpha\beta}(x)^{-1}$ and $g_{\alpha\alpha}(x)=e$. The family $\{(U_\alpha,\psi_\alpha,g_{\beta\alpha})\}$ is called a *local trivialisation* of P.

Definition 6.2. A *gluing cocycle* with respect to an open cover $\{U_\alpha:\alpha\in\mathcal{A}\}$ is a family of maps $g_{\beta\alpha}:U_{\alpha\beta}\to G$ ($\alpha,\beta\in\mathcal{A}$ with $U_{\alpha\beta}\neq\emptyset$) which satisfy the above cocycle relation.

One has seen that a local trivialisation of a principal G-bundle gives rise to a pair $(\mathcal{U},g_{\bullet\bullet})$ (*local gluing data*) where $\mathcal{U}=\{U_\alpha\}$ is an open cover of M and $g_{\bullet\bullet}=\{g_{\beta\alpha}\}$ is a gluing cocycle. Conversely, given such a pair one can construct a principal G-bundle in the following way. Define the total space

$$P\triangleq\Big(\bigsqcup_\alpha(\{\alpha\}\times U_\alpha\times G)\Big)_\sim$$

where $\sqcup$ means disjoint union and the equivalence relation $\sim$ is defined by

$$(\alpha,x,a)\sim(\beta,y,b)\iff y=x,b=g_{\beta\alpha}(x)a.$$

It is standard to show that P admits a natural differential structure. In addition, with the projection π and G-action defined in the obvious way, $(P,M,\pi;G)$ becomes a principal G-bundle. The family $\{U_\alpha,\psi_\alpha,g_{\beta\alpha}\}$ with $\psi_\alpha([(\alpha,x,a)])\triangleq(x,a)$ is a local trivialisation of this bundle. We refer the reader to [KN63, Proposition 5.2] for the routine details.

Definition 6.3. A *homomorphism* between two principal bundles $(P', M', \pi'; G')$ and $(P, M, \pi; G)$ consists of a smooth map $f_1 : P' \to P$ and a homomorphism $f_2 : G' \to G$ such that

$$f_1(u'g') = f_1(u')f_2(g')$$

for all $u' \in P'$ and $g' \in G'$. It is called an *embedding* if $f_1 : P' \to P$ is an embedding and $f_2 : G' \to G$ is a monomorphism. It is called an *isomorphism* if f_1 is a diffeomorphism and f_2 is an isomorphism.

A homomorphism (f_1, f_2) clearly preserves fibers, hence inducing a smooth map from M' to M. By abuse of notation, the aforementioned maps will all be denoted as f. We are primarily interested in the situation where $M' = M$ and $G' = G$. Suppose that P, P' are principal G-bundles over M. In this case, by a *homomorphism* (*covering the identity* id_M) we specifically mean a smooth map $f : P' \to P$ such that $\pi \circ f = \pi'$ and $f(u'g) = f(u')g$ for all $u' \in P', g \in G$. An *automorphism* is a principal G-bundle isomorphism onto itself.

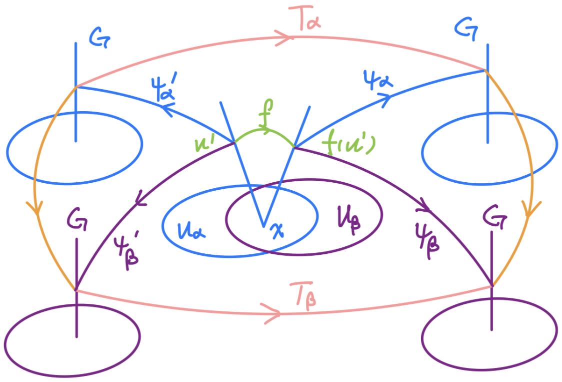
We would like to know under what condition(s) two pairs $(\mathcal{U}, g_{\bullet\bullet})$, $(\mathcal{V}, h_{\bullet\bullet})$ of local gluing data determine isomorphic principal G-bundles over M. To answer this question, we first make a simple observation. A pair $\{W_i, g'_{ji} : i, j \in \mathcal{I}\}$ is said to be a *refinement* of $\{U_\alpha, g_{\beta\alpha} : \alpha, \beta \in \mathcal{A}\}$ if for every $i \in \mathcal{I}$, there exists some $\alpha(i) \in \mathcal{A}$ such that $W_i \subseteq U_{\alpha(i)}$ and $g'_{ji} = g_{\alpha(j)\alpha(i)}|_{W_{ij}}$. It is apparent that $(\mathcal{U}, g_{\bullet\bullet})$ and any of its refinement define isomorphic principal G-bundles. We may therefore assume without loss of generality that $\mathcal{U} = \mathcal{V} = \{U_\alpha : \alpha \in \mathcal{A}\}$. The following result provides an answer to the aforementioned question.

Proposition 6.1. *Let $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ be an open cover of M. Let $g_{\beta\alpha}$ and $h_{\beta\alpha}$ be two gluing cocycles with respect to $\mathcal{U}$. The two pairs $(\mathcal{U}, g_{\bullet\bullet})$ and $(\mathcal{U}, h_{\bullet\bullet})$ define isomorphic principal G-bundles if and only if there exists a smooth map $T_\alpha : U_\alpha \to G$ for each $\alpha \in \mathcal{A}$, such that*

$$h_{\beta\alpha}(x) = T_\beta(x)g_{\beta\alpha}(x)T_\alpha(x)^{-1} \tag{6.1}$$

for all $x \in U_{\alpha\beta}$ provided that $U_{\alpha\beta} \neq \emptyset$.

Proof. Let P', P be the principal G-bundles over M associated with gluing data $(\mathcal{U}, g_{\bullet\bullet})$, $(\mathcal{U}, h_{\bullet\bullet})$ and let $\psi'_\alpha = (\pi', \varphi'_\alpha)$, $\psi_\alpha = (\pi, \varphi_\alpha)$ be the corresponding local trivialisation maps respectively. Suppose that $f : P' \to P$ is an isomorphism. Define $T_\alpha(x) \triangleq \varphi_\alpha(f(u'))\varphi'_\alpha(u')^{-1}$, where u' is any point on the fiber of P' over x. Then $\{T_\alpha\}$ satisfy (6.1).



Conversely, given such maps $\{T_\alpha\}$ we define

$$f(u') \triangleq \psi_\alpha^{-1}(\pi'(u'), T_\alpha(\pi'(u'))\varphi'_\alpha(u')).$$

Using the relation (6.1), one checks that f is a well-defined isomorphism between P' and P. □

Definition 6.4. Two gluing cocycles $\{g_{\beta\alpha}\}$ and $\{h_{\beta\alpha}\}$ are said to be *cohomologous* if there exists a smooth map $T_\alpha : U_\alpha \to G$ for each α such that the relation (6.1) is satisfied.

Another important concept is the reduction of a principal G-bundle.

Definition 6.5. We say that the structure group G of a principal G-bundle P is *reducible* to a closed subgroup G' if there exists an embedding f of some principal G'-bundle $(P', M, \pi'; G')$ into $(P, M, \pi; G)$ such that $f : M \to M$ is the identity map and $f : G' \to G$ is the inclusion. The bundle P' is called a *G'-reduced bundle* of P.

Suppose that P is a principal G-bundle over M with local gluing data $(\mathcal{U}, g_{\bullet\bullet})$. The structure group G of a principal G-bundle $(P, M, \pi; G)$ can be reduced to a closed subgroup G' if and only if there exists a local trivialisation $\{U_\alpha, \psi_\alpha, g_{\beta\alpha}\}$ such that $g_{\beta\alpha}$ takes values in G' (see [KN63, Proposition 5.3]).

Example 6.3. Let M be a Riemannian manifold. The orthonormal frame bundle $\mathrm{O}(M)$ is an $\mathrm{O}(n)$-reduction of the frame bundle $\mathcal{F}(M)$. If M is oriented, one can further reduce $\mathcal{F}(M)$ to the special orthonormal frame bundle $\mathrm{SO}(M)$.

We can also consider the pullback of a principal bundle. Let $(P, M, \pi; G)$ be a principal G-bundle and let $f : N \to M$ be a smooth map. The space

$$f^*P \triangleq \{(n, u) \in N \times P : f(n) = \pi(u)\}$$

admits a principal G-bundle structure over N, where the projection is given by $\pi'(n,u) \triangleq n$ and the right G-action is defined by $(n,u)g \triangleq (n,ug)$. Suppose that $\{U_\alpha, \psi_\alpha = (\pi, \varphi_\alpha), g_{\beta\alpha}\}$ is a local trivialisation of P. Then

$$V_\alpha \triangleq f^{-1}U_\alpha;\ \psi'_\alpha(n,u) \triangleq (n, \varphi_\alpha(u));\ h_{\beta\alpha} \triangleq g_{\beta\alpha} \circ f$$

defines a local trivialisation of f^*P. The principal G-bundle f^*P over N is called the *pullback* of P by f.

6.2 Associated vector bundles

An important reason for considering principal bundles is that *every vector bundle is associated with a principal bundle.* As we will see, geometric properties of a vector bundle (e.g. connection and curvature) are inherited from the principal bundle in a natural way.

6.2.1 Basic construction and examples

Let $(P, M, \pi; G)$ be a principal bundle. Let $\rho : G \to \mathrm{GL}(V)$ be a given representation of G on a $\mathbb{K}$-vector space V ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). We define an equivalence relation $\sim_\rho$ on $P \times V$ by

$$(u, \xi) \sim_\rho (ug^{-1}, \rho(g)\xi), \quad u \in P, \xi \in V, g \in G.$$

Let $P \times_\rho V$ denote the space of equivalence classes. It is easy to show that $P \times_\rho V$ is a $\mathbb{K}$-vector bundle with fiber V; the projection map is induced from P (denoted as $\bar{\pi}$) and the linear structure on each fiber is defined by

$$\lambda[(u,\xi)] + [(u,\eta)] \mapsto [(u, \lambda\xi + \eta)], \quad \xi, \eta \in V, \lambda \in \mathbb{K}.$$

Definition 6.6. $P \times_\rho V$ is called the *associated vector bundle* of (P, ρ).

Fix $u \in P$. The map

$$V \ni \xi \mapsto [(u,\xi)] \in E_{\pi(u)} \tag{6.2}$$

defines a linear isomorphism between V and $E_{\pi(u)}$. By abuse of notation, we will just denote this map as u. Let $\{U_\alpha, \psi_\alpha, g_{\beta\alpha}\}$ be a local trivialisation of P. Then

$$\bar{\psi}_\alpha : \bar{\pi}^{-1}(U_\alpha) \to U_\alpha \times V,\ [(u,\xi)] \mapsto (\pi(u), \rho(\varphi_\alpha(u))\xi)$$

defines a local trivialisation of $P \times_\rho V$ with transition functions $\rho(g_{\beta\alpha}) : U_{\alpha\beta} \to \mathrm{GL}(V)$. Let $\{\varepsilon_1, \cdots, \varepsilon_r\}$ be a given basis of V. Then the family

$$\left\{s_i(x) \triangleq \bar{\psi}_\alpha^{-1}(x, \varepsilon_i) : x \in U_\alpha, 1 \leqslant i \leqslant r\right\} \tag{6.3}$$

defines a local frame field of E (i.e. the family $\{s_1(x), \cdots, s_r(x)\}$ is a basis of E_x at each $x \in U_\alpha$).

Every vector bundle can be realised as an associated vector bundle. Indeed, let E be a $\mathbb{K}$-vector bundle of rank r over M . Let

$$\mathcal{F}(E) = \{(x; e_1, \cdots, e_r) : x \in M, \{e_1, \cdots, e_r\} \text{ is a basis of } E_x\} \tag{6.4}$$

be the frame bundle of E. As in Example 6.2, $\mathcal{F}(E)$ is a principal $\mathrm{GL}(r; \mathbb{K})$-bundle over M whose group action is given by

$$\big((x; e_1, \cdots, e_r), g\big) \mapsto (x; g_1^i e_i, \cdots, g_r^i e_i).$$

The map

$$\big((x; e_1, \cdots, e_r), \xi\big) \mapsto \xi^i e_i$$

induces a vector bundle isomorphism $\mathcal{F}(E) \times_{\mathrm{id}} \mathbb{K}^r \to E$, where id denotes the standard (matrix) representation of $\mathrm{GL}(r; \mathbb{K})$ over $\mathbb{K}^r$. In addition, for fixed $u = (x; e_1, \cdots, e_r)$ the u-map (6.2) is given by $u(\xi) = \xi^i e_i$.

Example 6.4. The tangent bundle TM is isomorphic to the associated vector bundle of $(\mathcal{F}(M), \mathrm{id})$ where $\mathcal{F}(M)$ is the frame bundle defined in Example 6.2. More generally, the tensor bundle

$$T_s^r M \triangleq \underbrace{TM \otimes \cdots \otimes TM}_{r} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{s}$$

is isomorphic to the associated vector bundle of $(\mathcal{F}(M), \mathrm{id}_s^r)$, where id_s^r is the representation of $\mathrm{GL}(n; \mathbb{R})$ on the tensor product space

$$T_s^r \triangleq (\mathbb{R}^n)^{\otimes r} \otimes (\mathbb{R}^n)^{*\otimes s}$$

induced from id in the obvious way.

Example 6.5. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. The *adjoint representation* of G on $\mathfrak{g}$ is defined by

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}),\ \mathrm{Ad}(g)(X) \triangleq gXg^{-1}.$$

The associated vector bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$ is called the *adjoint bundle* of P. It is denoted as $\mathrm{Ad}(P)$.

Example 6.6. Let $\rho : G \to \mathrm{GL}(V)$ be a representation and let $\rho_* \triangleq (d\rho)_e : \mathfrak{g} \to \mathrm{End}(V)$ be the induced representation on the Lie algebra $\mathfrak{g}$. Define $\mathrm{End}_\rho(V) \triangleq \rho_*(\mathfrak{g})$. Elements of $\mathrm{End}_\rho(V)$ are those endomorphisms described by $\mathfrak{g}$-actions (infinitesimal symmetries). For any $g \in G$ and $X \in \mathfrak{g}$, one has

$$\rho_*(\mathrm{Ad}(g)X) = \rho(g)\rho_*(X)\rho(g)^{-1}.$$

In particular, $\mathrm{End}_\rho(V)$ is Ad-invariant. This allows one to define a representation $\mathrm{Ad}_\rho : G \to \mathrm{GL}(\mathrm{End}_\rho(V))$ by

$$\mathrm{Ad}_\rho(g)(T) \triangleq \rho(g)T\rho(g)^{-1}, \quad T \in \mathrm{End}_\rho(V). \tag{6.5}$$

The associated vector bundle $P \times_{\mathrm{Ad}_\rho} \mathrm{End}_\rho(V)$ is called the *bundle of infinitesimal symmetries* of $E \triangleq P \times_\rho V$ and is denoted as $\mathrm{End}_\rho(E)$. Clearly, it is a subbundle of $\mathrm{End}(E)$.

6.2.2 G-structures on a vector bundle

Let E be a vector bundle over M with fiber V. We say that E *admits a G-structure* if there exists a principal G-bundle P over M and a representation $\rho : G \to \mathrm{GL}(V)$ such that E is isomorphic to $P \times_\rho V$ as vector bundles. In terms of local data, E admits a G-structure if and only if there exists local gluing data $(\mathcal{U}, g_{\bullet\bullet})$ and a representation $\rho : G \to \mathrm{GL}(V)$ such that the vector bundle defined by $(\mathcal{U}, \rho(g_{\bullet\bullet}))$ is isomorphic to E.

Example 6.7. Let E be a real vector bundle of rank r and suppose that there is an Euclidean metric on E. Then E admits an $\mathrm{O}(r)$-structure. To see this, consider the orthonormal frame bundle

$$\mathrm{O}(E) = \{(x; e_1, \cdots, e_r) : x \in M, \{e_1, \cdots, e_r\} \text{ is an ONB basis of } E_x\}$$

as a principal $\mathrm{O}(r)$-bundle and the identity representation $\mathrm{id} : \mathrm{O}(r) \hookrightarrow \mathrm{GL}(r; \mathbb{R})$. The map

$$\big((x; e_1, \cdots, e_r), (\lambda^i)_{1 \leqslant i \leqslant r}\big) \mapsto \lambda^i e_i \in E_x$$

induces an isomorphism between $\mathrm{O}(E) \times_{\mathrm{id}} \mathbb{R}^r$ and E, hence yielding an $\mathrm{O}(r)$-structure for E. In particular, the tangent bundle over an n-dimensional Riemannian manifold admits an $\mathrm{O}(n)$-strucutre. Similarly, if E is a complex vector bundle with a Hermitian metric, it admits an $\mathrm{U}(r)$-structure.

Proposition 6.2. *Different Euclidean (respectively, Hermitian) metrics on a real (respectively, complex) vector bundle E of rank r induce isomorphic $\mathrm{O}(r)$-structures (respectively, $\mathrm{U}(r)$-structures).*

Proof. We first recall a simple fact from linear algebra. Let V be a finite dimensional, real vector space and let h_0, h_1 be two Euclidean metrics on V. Then there exists a unique linear isomorphism $F : V \to V$ which satisfies the following three properties:

(i) F is h_0-symmetric: $h_0(F(\xi), \eta) = h_0(\xi, F(\eta))$ for all $\xi, \eta \in V$;

(ii) F is h_0-positive-definite: $h_0(F(\xi), \xi) > 0$ for all $\xi \neq 0$;

(iii) $h_1(\xi, \eta) = h_0(F(\xi), F(\eta))$ for all $\xi, \eta \in V$. □

To prove this, let $\{\varepsilon_1, \cdots, \varepsilon_n\}$ be a fixed ONB of V with respect to h_0. Suppose that such an F exists. Let $A = (a_i^j)$ be its matrix with respect to $\{\varepsilon_i\}$, i.e. $F(\varepsilon_i) = a_i^j \varepsilon_j$. According to (i) and (ii), the matrix A is symmetric and positive definite. Let $B = (b_i^j)$ where $b_i^j \triangleq h_1(\varepsilon_i, \varepsilon_j)$. Property (iii) shows that $B = A^T A = A^2$, in particular, A is a symmetric, positive definite square root of B. But it is well known from linear algebra that such a square root exists uniquely. Therefore, the matrix A is uniquely defined, and so is F accordingly.

In the bundle context, let h_i be an Euclidean metric on E and let P_i be the corresponding orthonormal frame bundle ($i = 0, 1$). At each $x \in M$, one uses the previous fact to find the unique linear isomorphism $F_x : E_x \to E_x$ satisfying the above three properties. It follows that F_x maps an h_1-ONB of E_x to an h_0-ONB of E_x. As a consequence, the family $\{F_x : x \in M\}$ induces a map $P_1 \to P_0$ which is easily seen to be a principal $\mathrm{O}(r)$-bundle isomorphism.

Remark 6.1. Apparently, the isomorphism between the two $\mathrm{O}(r)$-structures constructed in the above proof is canonical.

6.2.3 Construction of global sections

Finally, we recall the construction of global sections from local data. Consider a principal G-bundle $(P, M, \pi; G)$ with a local trivialisation $\{U_\alpha, \psi_\alpha, g_{\beta\alpha}\}$. Let $s_\alpha : U_\alpha \to G$ be a family of smooth maps indexed by α. Suppose that they satisfy the consistency relation $s_\beta(x) = g_{\beta\alpha}(x) s_\alpha(x)$. Then they patch together into a global section of P, more precisely,

$$s(x) \triangleq \psi_\alpha^{-1}(x, s_\alpha(x))$$

is a well-defined global section. The same discussion applies to a vector bundle E. In the context of an associated vector bundle $E = P \times_\rho V$, suppose that $s_\alpha : U_\alpha \to V$ is a family of maps satisfying

$$s_\beta(x) = \rho(g_{\beta\alpha}(x)) s_\alpha(x). \tag{6.6}$$

Then

$$s(x) \triangleq [(\psi_\alpha^{-1}(x, e), s_\alpha(x))]$$

is a well-defined global section of E.

6.3 Connections on a principal bundle

In this section, we study connections on a principal bundle and its associated vector bundles.

6.3.1 Fundamental vector fields

We first recall a basic construction. For each $u \in P$, the *vertical subspace* $\mathcal{V}_u$ is the subspace of T_uP consisting of vectors that are tangential to the fiber at u. The vertical subspace is canonically isomorphic to the Lie algebra $\mathfrak{g}$ (cf. Proposition 5.1 as a special case).

Proposition 6.3. *The map*

$$\mathfrak{g} \ni A \mapsto \frac{d}{dt}\Big|_{t=0} u\exp(tA) \in \mathcal{V}_u \tag{6.7}$$

is a linear isomorphism for every $u \in P$.

Proof. Linearity is obvious. Since the G-action is transitive, every vertical vector V at u is represented by a curve of the form $u_t = ug_t$ ($t \in (-\varepsilon, \varepsilon)$) where $g_t \in G$ and $g_0 = e$. But g_t is equivalent to $\tilde{g}_t \triangleq \exp(tg_0')$ at $t = 0$. Therefore, V is also represented by the curve $\tilde{u}_t \triangleq ue^{tg_0'}$. This shows that the map defined by (6.7) is surjective. By dimension comparison, it is a linear isomorphism. $\square$

Definition 6.7. Given $A \in \mathfrak{g}$, by varying $u \in P$ one obtains from (6.7) a vector field on P. This is called the *fundamental vector field* associated with A and it is denoted as A^*.

The lemma below shows that fundamental vector fields are consistent with the Lie bracket.

Lemma 6.1. $[A^*, B^*] = [A, B]^*$ *for all* $A, B \in \mathfrak{g}$.

Proof. Since the flow of A^* is $R_{e^{tA}}$, one has

$$[A^*, B^*]_u = (\mathcal{L}_{A^*}B^*)_u = \lim_{t\to 0}\frac{1}{t}\big((dR_{e^{-tA}})B^*_{ue^{tA}} - B^*_u\big). \tag{6.8}$$

Note that the $\mathfrak{g}$-element corresponding to the vertical vector $(dR_{e^{-tA}})B^*_{ue^{tA}}$ is

$$e^{tA}Be^{-tA} = \mathrm{Ad}(e^{tA})B.$$

It follows that the $\mathfrak{g}$-element corresponding to the RHS of (6.8) is

$$\lim_{t\to 0}\frac{1}{t}\big(\mathrm{Ad}(e^{tA})B - B\big) = \mathrm{ad}(A)B = [A, B].$$

Therefore, the RHS of (6.8) equals $[A, B]^*_u$ $\square$

6.3.2 The concept of a connection

Now we give the precise definition of a connection on P.

Definition 6.8. A *connection* Γ on P is an assignment of a linear subspace (*horizontal subspace*) $\mathcal{H}_u$ of $T_u(P)$ at every $u \in P$ which satisfies the following properties.

(i) $T_u(P) = \mathcal{V}_u \oplus \mathcal{H}_u$ for all $u \in P$;

(ii) $\mathcal{H}_{ug} = (dR_g)_u\mathcal{H}_u$ for all $u \in P$ and $g \in G$, where R_g denotes the right G-action on P;

(iii) The assignment $u \mapsto \mathcal{H}_u$ is smooth.

Remark 6.2. Property (iii) means that for each $u \in P$, there exists a neighbourhood U of u and a family of smooth vector fields $\{X_1, \cdots, X_n\}$ on U such that $\{X_1(v), \cdots, X_n(v)\}$ is a basis of $\mathcal{H}_v$ for every $v \in U$.

Let Γ be a connection on P. Since $\ker(d\pi)_u = \mathcal{V}_u$, it follows from Property (i) of the connection that $(d\pi)_u|_{\mathcal{H}_u} : \mathcal{H}_u \to T_xM$ $(x \triangleq \pi(u))$ is a linear isomorphism. Given $v \in T_xM$, the horizontal vector $(d\pi)_u^{-1}(v)$ is called the *horizontal lift* of v at u. Let X be a smooth vector field on M. One can define a vector field X^* on P by setting X_u^* to be the horizontal lift of $X_{\pi(u)}$.This is the unique vector field X^* such that $X_u^* \in \mathcal{H}_u$ and $(d\pi)_uX_u^* = X_{\pi(u)}$ for every u. According to Property (ii), one has $(dR_g)_uX_u^* = X_{ug}^*$.

Definition 6.9. The vector field X^* is called the *horizontal lift* of X.

The horizontal lift satisfies the following basic properties. Given a vector field Z on P, we write hZ as the horizontal component of Z.

Proposition 6.4. *(i)* $(X+Y)^* = X^* + Y^*$.

(ii) $(fX)^* = (f \circ \pi)X^*$*where* $f \in C^\infty(M)$.

(iii) $[X,Y]^* = h[X^*, Y^*]$.

Proof. The first two properties are obvious. The third property is a direct consequence of the fact that the Lie bracket is consistent with pushforwards (see (B.2)). ☐

6.3.3 Connection forms and their local representations

A connection Γ can be equivalently described in terms of a $\mathfrak{g}$-valued 1-form on P. Let Γ be a given connection on P. For each $u \in P$ and $X \in T_u(P)$, by writing $X = Y + Z$ where $Y \in \mathcal{V}_u$ and $Z \in \mathcal{H}_u$, we define $\omega(X)_u \triangleq A \in \mathfrak{g}$ where A is the element corresponding to the vertical vector Y under the isomorphism (6.7). By varying u and X, one obtains a $\mathfrak{g}$-valued 1-form ω on P. This 1-form satisfies the following basic properties.

Theorem 6.1. *(i) $\omega(A^*) = A$ for all $A \in \mathfrak{g}$, where A^* is the fundamental vector field associated with A.*

(ii) $R_g^*\omega = \mathrm{Ad}(g^{-1})\omega$ for all $g \in G$. Equivalently,

$$\omega_{ug}\big((dR_g)_u X_u\big) = \mathrm{Ad}(g^{-1})\omega_u(X_u) \tag{6.9}$$

for all $u \in P$, $g \in G$ and $X_u \in T_uP$.

Proof. Property (i) is just definition. For Property (ii), it is obvious if X_u is horizontal (both sides are zero). Now suppose that $X = A^*$ for some $A \in \mathfrak{g}$. The vertical vector $(dR_g)_u A_u^*$ is tangential to the curve

$$(-\varepsilon, \varepsilon) \ni t \mapsto ue^{tA}g = ug \cdot g^{-1}e^{tA}g,$$

which corresponds to the $\mathfrak{g}$-element $\mathrm{Ad}(g^{-1})A$. This gives the relation (6.9) for vertical vectors. ☐

Conversely, suppose that ω is a $\mathfrak{g}$-valued 1-form on P which satisfies (i) and (ii) of Theorem 6.1. Define $\mathcal{H}_u \triangleq \ker \omega_u$ for each $u \in P$. Then the assignment $u \mapsto \mathcal{H}_u$ defines a connection on P. In summary, one obtains the following basic result.

Proposition 6.5. *A connection on P is equivalent to a $\mathfrak{g}$-valued 1-form on P satisfying Theorem 6.1.*

We use $\mathfrak{C}(P)$ to denote the space of $\mathfrak{g}$-valued 1-form on P satisfying Theorem 6.1. Elements in $\mathfrak{C}(P)$ are called *connection forms on P*.

Local representation of a connection form

We now describe a connection in terms of local data, which is sometimes more convenient to work with. Let $\omega \in \mathfrak{C}(P)$ be a given connection form. Let $\{U_\alpha, \psi_\alpha, g_{\beta\alpha}\}$ be a local trivialisation of P. For each α, we define $A_\alpha \triangleq \sigma_\alpha^*\omega$ where $\sigma_\alpha : U_\alpha \to P$ is the local identity section defined by $\sigma_\alpha(x) \triangleq \psi_\alpha^{-1}(x, e)$. Note that A_α is a $\mathfrak{g}$-valued 1-form on U_α.

Lemma 6.2. *The family $\{A_\alpha\}$ satisfies the following consistency relation*

$$A_\beta = g_{\beta\alpha} \cdot A_\alpha \cdot g_{\beta\alpha}^{-1} - dg_{\beta\alpha} \cdot g_{\beta\alpha}^{-1}. \tag{6.10}$$

Proof. We first claim that

$$\sigma_\beta(x) = \sigma_\alpha(x)g_{\alpha\beta}(x), \quad x \in U_{\alpha\beta}.$$

Indeed, let $g \in G$ be such that $\sigma_\beta(x) = \sigma_\alpha(x)g$. By the G-equivariance of φ_β, one has

$$e = \varphi_\beta(\sigma_\beta(x)) = \varphi_\beta(\sigma_\alpha(x))g = g_{\beta\alpha}(x)\varphi_\alpha(\sigma_\alpha(x))g = g_{\beta\alpha}(x)g.$$

Therefore, $g = g_{\alpha\beta}(x)$ which yields the desired claim.

Now let $v \in T_xM$ and let $x(t)$ $(t \in (-\varepsilon, \varepsilon))$ be such that $x(0) = x$, $\dot{x}(0) = v$. By differentiating the relation $\sigma_\beta(x(t)) = \sigma_\alpha(x(t))g_{\alpha\beta}(x(t))$ at $t = 0$, one finds that

$$\begin{aligned}(d\sigma_\beta)_x v =&(dR_{g_{\alpha\beta}})_{\sigma_\alpha(x)}(d\sigma_\alpha)_x v \\ &+ \sigma_\alpha(x)g_{\alpha\beta}(x)\frac{d}{dt}\Big|_{t=0} g_{\alpha\beta}(x)^{-1}g_{\alpha\beta}(x(t)) \\ =&(dR_{g_{\alpha\beta}})_{\sigma_\alpha(x)}(d\sigma_\alpha)_x v - \sigma_\beta(x)(dg_{\beta\alpha}\cdot g_{\beta\alpha}^{-1})_x v.\end{aligned}$$

It follows from Theorem 6.1 that

$$\omega\big((d\sigma_\beta)_x v\big) = \mathrm{Ad}(g_{\beta\alpha}(x))\omega\big((d\sigma_\alpha)_x v\big) - (dg_{\beta\alpha}\cdot g_{\beta\alpha}^{-1})_x v.$$

This gives the desired relation (6.10). □

The family $\{A_\alpha\}$ is called the *local representation* of ω. Conversely, given such a family $\{A_\alpha\}$ one can construct a connection form ω whose local representation is $\{A_\alpha\}$.

Proposition 6.6. *Let $\{U_\alpha, \psi_\alpha, g_{\beta\alpha}\}$ be a local trivialisation of P. Let $\{A_\alpha \in \Omega^1(U_\alpha)\otimes\mathfrak{g}\}$ be a family of local $\mathfrak{g}$-valued 1-forms which satisfy the consistency relation (6.10). Then there exists a unique connection form $\omega \in \mathfrak{C}(P)$, such that $A_\alpha = \sigma_\alpha^*\omega$ on each U_α. More precisely, ω is given by the formula*

$$\omega = \varphi_\alpha^{-1}d\varphi_\alpha + \mathrm{Ad}(\varphi_\alpha^{-1})(\pi^*A_\alpha). \tag{6.11}$$

Proof. For fixed α, let $F \triangleq \psi_\alpha^{-1} : U_\alpha \times G \to \pi^{-1}(U_\alpha)$. We claim that

$$(dF)_{(x,g)}(X,V) = (dR_g)_{\sigma_\alpha(x)}(d\sigma_\alpha)_x(X) + (g^{-1}V)^*_{F(x,g)} \tag{6.12}$$

for all $(x,g) \in U_\alpha \times G$ and $(X,V) \in T_xM \oplus T_gG$. Indeed, one first notes that

$$F(x,g) = \sigma_\alpha(x)g = R_g(\sigma_\alpha(x)).$$

For $X \in T_xM$, one has

$$(dF)_{(x,g)}(X,0) = \frac{d}{dt}\Big|_{t=0} R_g(\sigma_\alpha(x_t)) = (dR_g)_{\sigma_\alpha(x)}(d\sigma_\alpha)_x(X),$$

where x_t is any curve satisfying $x_0 = x$ and $\dot{x}_0 = X$. For $V \in T_gG$, by setting $A \triangleq g^{-1}V \in \mathfrak{g}$ one has

$$(dF)_{(x,g)}(0,V) = \frac{d}{dt}\Big|_{t=0}\sigma_\alpha(x)ge^{tA} = A^*_{F(x,g)} = (g^{-1}V)^*_{F(x,g)}.$$

The relation (6.12) thus follows.

Now suppose that ω is a connection form whose local representation is $\{A_\alpha\}$. By applying ω to both sides of (6.12) and using the properties of connection, one finds that

$$(F^*\omega)_{(x,g)} = \mathrm{Ad}(g^{-1})(\mathrm{pr}_1^* A_\alpha) + \mathrm{pr}_2^*(g^{-1}dg),$$

where pr_i denotes the projection onto the i-th component ($i = 1, 2$). Since $F^{-1} = \psi_\alpha = (\pi, \varphi_\alpha)$, it follows that

$$\omega = \mathrm{Ad}(\varphi_\alpha^{-1})(\pi^* A_\alpha) + \varphi_\alpha^{-1} d\varphi_\alpha.$$

This justifies the formula (6.11) and proves the uniqueness claim. For its existence, by using the consistency relation (6.10) one checks that the formula (6.11) defines a global $\mathfrak{g}$-valued 1-form ω on P. In addition, it is plain that ω satisfies the two properties of Theorem 6.1 and $A_\alpha = \sigma_\alpha^*\omega$. Therefore, ω is a connection form with local representation $\{A_\alpha\}$. ☐

In summary, a connection form $\omega \in \mathfrak{C}(P)$ is also equivalent to a family $\{A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}\}$ of local $\mathfrak{g}$-valued 1-forms satisfying (6.10). The space of such families is denoted as $\mathfrak{C}(\mathcal{U}, g_{\bullet\bullet})$ (clearly, such $\{A_\alpha\}$ is described in terms of the open cover $\mathcal{U} = \{U_\alpha\}$ and the gluing cocycle $g_{\bullet\bullet}$ only). The following result shows how different local representations of a connection are related.

Proposition 6.7. *Let $\{A_\alpha\}$ and $\{B_\alpha\}$ be local representations of a connection form $\theta \in \mathfrak{C}(P)$ with respect to local trivialisations $\{U_\alpha, \psi_\alpha = (\pi, \varphi_\alpha)\}$ and $\{U_\alpha, \psi'_\alpha = (\pi, \varphi'_\alpha)\}$ respectively. Then one has*

$$B_\alpha = -dT_\alpha \cdot T_\alpha^{-1} + T_\alpha \cdot A_\alpha \cdot T_\alpha^{-1} \quad \text{on } U_\alpha,$$

where $T_\alpha : U_\alpha \to G$ is the map defined by $T_\alpha(x) \triangleq \varphi'_\alpha(u)\varphi_\alpha(u)^{-1}$ ($x \in U_\alpha$, $u \in \pi^{-1}(x)$).

Proof. Left as an exercise. ☐

6.4 Horizontal lift of curves

Next, we discuss the important concept of horizontal lift of curves. This allows one to define a notion of parallel transport, which is essential for the construction of covariant differentiation on associated vector bundles later on. Let Γ be a connection on P whose connection form is ω.

Definition 6.10. A piecewise smooth curve $u_t \in P$ is said to be *horizontal* if $\dot{u}_t \in \mathcal{H}_{u_t}$ at every differentiable point t.

Theorem 6.2. *Let $\gamma : [0, 1] \to M$ be a smooth curve and let $u \in \pi^{-1}(\gamma_0)$. There exists a unique horizontal curve $u : [0, 1] \to P$ such that $u_0 = u$ and $\pi(u_t) = \gamma_t$ for all t.*

Proof. Let $(v_t)_{0\leqslant t\leqslant 1}$ be any smooth lift of γ at u, i.e. $v_0 = u$ and $\pi(v_t) = \gamma_t$ for all t. The existence of v_t can be obtained by using the local triviality of P. We therefore look for a "correction path" $g_t \in G$ such that $u_t \triangleq v_t g_t$ is a horizontal lift of γ. Apparently, $\pi(u_t) = \pi(v_t) = \gamma_t$ and one also has $u_0 = u$ if $g_0 = e$. By differentiating u_t, one finds that

$$\dot{u}_t = \dot{v}_t g_t + v_t \dot{g}_t = (dR_{g_t})_{v_t}\dot{v}_t + u_t g_t^{-1}\dot{g}_t.$$

Therefore,

$$\omega(\dot{u}_t) = \mathrm{Ad}(g_t^{-1})\omega(\dot{v}_t) + g_t^{-1}\dot{g}_t.$$

Requiring u_t to be horizontal means that $\omega(\dot{u}_t) = 0$, which is equivalent to saying that

$$\dot{g}_t g_t^{-1} = -\omega(\dot{v}_t). \tag{6.13}$$

The equation (6.13) is a linear ODE for g_t which can be solved uniquely with initial condition $g_0 = e$. This proves the existence of u_t. For its uniqueness, suppose $\tilde{u}_t$ is another horizontal lift of γ at u. Write $\tilde{u}_t = v_t\tilde{g}_t$ for some $\tilde{g}_t \in G$. The same reasoning as above shows that $\tilde{g}_t$ solves the ODE (6.13) with initial condition $\tilde{g}_0 = e$. By uniqueness, one concludes that $\tilde{g}_t = g_t$ and thus $\tilde{u}_t = u_t$. □

Corollary 6.1. *Let $\gamma : [a, b] \to M$ be a piecewise smooth curve and let $u \in \pi^{-1}(\gamma_a)$. There exists a unique horizontal lift of γ starting at u.*

Proof. Apply Theorem 6.2 to each smooth piece of γ_t consecutively. □

Let $\gamma : [a, b] \to M$ be a piecewise smooth curve. We define a map $\tau_\gamma : \pi^{-1}(\gamma_a) \to \pi^{-1}(\gamma_b)$ in the following way. Given $u \in \pi^{-1}(x_a)$, let u_t be the horizontal lift of γ starting at u. Then we set $\tau_\gamma(u) \triangleq u_b$. This map is known as the *parallel transport* along γ. It is easy to show that τ_γ is bijective and does not depend on the parametrisation of γ. If γ is the concatenation of two curves α and β, one has $\tau_\gamma = \tau_\beta \circ \tau_\alpha$. In addition, since horizontal subspaces are invariant under G-actions, it follows that τ_γ is G-equivariant, i.e. $R_g \circ \tau_\gamma = \tau_\gamma \circ R_g$ for all $g \in G$. It is important to note that the parallel transport depends on the entire curve γ.

6.5 Connections on associated vector bundles

Let $E = P \times_\rho V$ be the associated vector bundle of (P, ρ) where $\rho : G \to \mathrm{GL}(V)$ is a given representation. The aim of this section is to construct a suitable notion of "covariant derivative" $\nabla_X s$ of a section $s \in \Gamma(E)$ with respect to a vector field $X \in \Gamma(TM)$.

If the bundle E is trivial (i.e. $E = M \times V$), sections of E are just V-valued functions on M and one can simply define $\nabla_X s$ to be the directional derivative of s along the direction X. More precisely, one defines

$$(\nabla_X s)(x) \triangleq \lim_{t\to 0}\frac{1}{t}\big(s(\gamma_t) - s(x)\big), \quad x \in M, \tag{6.14}$$

where $\gamma : (-\varepsilon, \varepsilon) \to M$ is any smooth curve satisfying $\gamma_0 = x$, $\dot{\gamma}_0 = X_x$. The difference $s(\gamma_t) - s(x)$ makes sense since both $s(\gamma_t)$ and $s(x)$ take values in the same vector space V.

If E is not a trivial bundle, the difference $s(\gamma_t) - s(x)$ does not make sense because $s(\gamma_t) \in E_{\gamma_t}$ and $s(x) \in E_x$ live in different spaces. A natural idea is to "transport" $s(\gamma_t)$ to the fiber E_x along the curve γ so that one can make sense of the limit in (6.14) on the same vector space E_x. This requires a notion of parallel transport, which is naturally induced from a connection on the principal bundle P.

Let Γ be a connection on P. We first define the horizontal subspace $\mathcal{H}_e^E \subseteq T_e E$ (with respect to Γ) at each $e \in E$. Pick a representative (u, ξ) of e and consider the projection map $P \ni v \mapsto [(v, \xi)] \in E$. The *horizontal subspace* $\mathcal{H}_e^E$ is defined to be the image of $\mathcal{H}_u$ under the differential of this projection map at u. It can be shown that $\mathcal{H}_e^E$ does not depend on the choice of representatives of e.

A piecewise smooth curve e_t on E is *horizontal* if $\dot{e}_t \in \mathcal{H}_{e_t}^E$ at every differentiable time t. Let $\gamma : [a, b] \to M$ be a piecewise smooth curve and let $e \in E_{\gamma_a}$. Similar to Theorem 6.2, there exists a unique horizontal curve $e : [a, b] \to E$ such that $e_a = e$ and $\pi(e_t) = \gamma_t$ for all t. This curve is called the *horizontal lift* of γ at e. By varying $e \in \pi^{-1}(\gamma_a)$, one obtains a linear isomorphism $\tau_\gamma^E : E_{\gamma_a} \to E_{\gamma_b}$ $(e \mapsto e_b)$. The map τ_γ^E is called the *parallel transport* along γ.

The horizontal lift e_t and parallel transport can be easily described from the principal bundle perspective. Recall that each $u \in P$ defines a linear isomorphism $u : V \to E_{\pi(u)}$ given by $u(\xi) \triangleq [(u, \xi)]$.

Lemma 6.3. *Let (u, ξ) be a representative of $e \in E_{\gamma_a}$ and let $u_t \in P$ be the horizontal lift of γ at u. Then $e_t = [(u_t, \xi)]$. In particular, the parallel transport of e to E_{γ_t} along γ is given by $u_t u^{-1}(e)$.*

Proof. By definition, the curve $t \mapsto [(u_t, \xi)]$ is horizontal in E. □

With the notion of parallel transport, one can now define the covariant derivative of E-sections in a precise mathematical way. We proceed in the following manner.

Definition 6.11. (i) (*Covariant derivative along a curve*) Let $\gamma : I \to M$ be a smooth curve defined on an interval I and let s be a smooth section over γ (i.e. $s_t \in E_{\gamma_t}$). The *covariant derivative of s along γ* is the section over γ defined by

$$\frac{Ds_t}{dt} \triangleq \lim_{h \to 0} \frac{1}{h} \big(\tau_{t,t+h}^E s_{t+h} - s_t \big) \in E_{\gamma_t},$$

where $\tau_{t,t+h}^E$ denotes the parallel transport from $E_{\gamma_{t+h}}$ to E_{γ_t} along γ.

(ii) (*Covariant derivative along a direction*) Let $v \in T_x M$ and let s be a smooth section of E defined on a neighbourhood of x. The *covariant derivative of s at x*

with respect to the direction v is defined in the following way. Pick a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ satisfying $\gamma_0 = x$ and $\dot{\gamma}_0 = v$. We then set

$$(\nabla_v s)(x) \triangleq \frac{D}{dt}\Big|_{t=0} s(\gamma_t).$$

(iii) (*Covariant derivative operator*) Let $s \in \Gamma(E)$ and $X \in \Gamma(TM)$. By using (ii), one can define the covariant derivative $(\nabla_X s)(x)$ at every $x \in M$. This gives rise to a section $\nabla_X s \in \Gamma(E)$ which is known as the *covariant derivative of* s *along the vector field* X.

The following result shows that the covariant derivative is consistent with the more familiar definition of a connection on a vector bundle from the differential operator perspective (cf. Definition B.17). Its proof is routine and is left as an exercise.

Proposition 6.8. *The covariant derivative operator* $\nabla_X s$ *is well-defined and satisfies the following properties. For any* $X, Y \in \Gamma(TM)$, $s, t \in \Gamma(E)$ *and* $f \in C^\infty(M)$, *one has*

(i) $\nabla_{(fX+Y)} s = f\nabla_X s + \nabla_Y s$*;*

(ii) $\nabla_X(s+t) = \nabla_X s + \nabla_X t$*;*

(iii) $\nabla_X(fs) = (Xf)s + f\nabla_X s$*.*

Here Xf *denotes the directional derivative of* f *with respect to* X*.*

Definition 6.12. A *connection* (or a *covariant derivative*) on a vector bundle $(E, M, \pi; V)$ is a linear operator

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E),\ (X, s) \mapsto \nabla_X s$$

which satisfies the three properties in Proposition 6.8. One can equivalently view

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) = \Omega^1(M, E)$$

by setting $\langle \nabla s, X \rangle \triangleq \nabla_X s$.

Remark 6.3. Every connection on E is the covariant derivative induced from some principal bundle connection. In fact, a connection ∇ on E defines a notion of parallel transport (see Definition B.18). This gives rise to a connection on the frame bundle $\mathcal{F}(E)$ in the following way. Let $u = (x; e_1, \cdots, e_r) \in \mathcal{F}(E)$. Given $v \in T_xM$, draw a curve γ_t with $\gamma_0 = x$ and $\dot{\gamma}_0 = v$. The parallel transport of e_i along γ_t defines a curve $u_t = (\gamma_t; e_1(t), \cdots, e_r(t)) \in \mathcal{F}(E)$ with $u_0 = u$. The set of tangent vectors $\dot{u}_0$ obtained by varying $v \in T_xM$ defines an n-dimensional subspace of $T_u\mathcal{F}(E)$. By declaring this subspace as the horizontal subspace for each

$u \in \mathcal{F}(E)$, one obtains a connection on the principal bundle $\mathcal{F}(E)$. It follows that ∇ is the covariant derivative operator induced from this connection. If E admits an Euclidean metric, there is an $O(r)$-reduction $E \cong O(E) \times_{\mathrm{id}} \mathbb{R}^r$ (see Example 6.7). In this case, metric connections on E (i.e. connections that are compatible with the metric in the sense of (B.21)) are in one-to-one correspondence with connections on the orthonormal frame bundle $O(E)$.

Example 6.8. Recall from Example 6.4 that the tensor bundle $T_s^r M$ is an associated vector bundle of the frame bundle $\mathcal{F}(M)$. A connection Γ on $\mathcal{F}(M)$ therefore induces a connection ∇ on $T_s^r M$. One can view $\nabla : T_s^r M \to T_{s+1}^r M$ by setting

$$(\nabla S)(X_1, \cdots, X_s, X) \triangleq (\nabla_X S)(X_1, \cdots, X_s) \in T_0^r(T_x M), \quad X_i, X \in T_x M.$$

This connection is consistent with the induced connection on tensor bundles defined using (B.23) (cf. (C.1)). A tensor field $S \in \Gamma(T_s^r M)$ is said to be *parallel* if $\nabla S = 0$. This is equivalent to saying that the parallel transport of $S(\gamma_a)$ along any smooth curve $\gamma : [a, b] \to M$ coincides with $S(\gamma_b)$.

The following useful result (and its corollary below) tells one how to compute the covariant derivative from the principal bundle perspective; essentially, it is the directional derivative along horizontal vectors.

Lemma 6.4. *One has $(\nabla_X s)(x) = u((X^* f)(u))$ for all $x \in M$ and $u \in \pi^{-1}(x)$. Here X^* is the horizontal lift of X and f is the V-valued function on P defined by $f(u) \triangleq u^{-1}(s(\pi(u)))$.*

Proof. Fix $u \in P$. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve such that $\gamma_0 = x \triangleq \pi(u)$ and $\dot{\gamma}_0 = X_x$. Let $u : (-\varepsilon, \varepsilon) \to P$ be the horizontal lift of γ at u. Then one has

$$(X^* f)(u) = \lim_{h \to 0} \frac{1}{t}\big(f(u_t) - f(u)\big) = \lim_{h \to 0} \frac{1}{t}\big(u_t^{-1}(s(\gamma_t)) - u^{-1}s(x)\big).$$

It follows that

$$u((X^* f)(u)) = \lim_{h \to 0} \frac{1}{t}\big(uu_t^{-1}(s(\gamma_t)) - s(x)\big). \tag{6.15}$$

According to Lemma 6.3, $uu_t^{-1}(s(\gamma_t))$ is precisely the parallel transport of $s(\gamma_t)$ to E_x along γ. By the definition of covariant derivative, the RHS of (6.15) is equal to $(\nabla_X s)(x)$. □

Corollary 6.2. *Let $s : I \to E$ be a section of E along a smooth curve $\gamma : I \to M$. Let $u : I \to P$ be a horizontal lift of γ and let $\xi_t \triangleq u_t^{-1}(s_t) \in V$. Then one has*

$$\nabla_{\dot{\gamma}_t} s_t = u_t(\dot{\xi}_t)$$

for all $t \in I$.

A section s along γ is said to be *parallel* if $\nabla_{\dot{\gamma}_t} s_t = 0$ for all t. This is equivalent to saying that $u_t^{-1}(s_t)$ is constant in t for any horizontal lift (u_t) of γ.

As an application of Lemma 6.4, one can derive the local representation of ∇. Let $\{U_\alpha, \psi_\alpha, g_{\beta\alpha}\}$ be a local trivialisation of P. Let $\{s_1, \cdots, s_r\}$ be the local frame field of E over U_α induced from a fixed basis $\{\varepsilon_1, \cdots, \varepsilon_r\}$ of V (see (6.3)). More precisely, $s_i(x) = [(\sigma_\alpha(x), \varepsilon_i)]$ where $\sigma_\alpha(x)$ is the identity section on U_α. Let $\{A_\alpha = \sigma_\alpha^* \omega\}$ be the local representation of the connection Γ on P. Recall that $\rho_* \triangleq (d\rho)_e : \mathfrak{g} \to \mathrm{End}(V)$ is the induced representation of ρ.

Proposition 6.9. *Let $\bar{A}_\alpha$ be the matrix of $\rho_*(A_\alpha) \in \Omega^1(U_\alpha) \otimes \mathrm{End}(V)$ with respect to $\{\varepsilon_i\}$, i.e. $\rho_*(A_\alpha)\varepsilon_i = (\bar{A}_\alpha)_i^j \otimes \varepsilon_j$. Then one has*

$$\nabla s_i = (\bar{A}_\alpha)_i^j \otimes s_j.$$

In other words, $\rho_*(A_\alpha)$ is the local connection matrix of ∇ with respect to$\{s_i\}$ on U_α.

Proof. The argument consists of the following basic observations.

(i) Let $s \in \Gamma(E)$. There exists a smooth function $v : U_\alpha \to V$ such that $s(x) = [(\sigma_\alpha(x), v(x))]$ for all $x \in U_\alpha$. Let $f : P \to V$ be defined by $f(u) \triangleq u^{-1}\big(s(\pi(u))\big)$. Then one has

$$v(x) = f(\sigma_\alpha(x)). \tag{6.16}$$

In addition, by applying Lemma 6.4 with $u = \sigma_\alpha(x)$, one finds that

$$(\nabla_X s)(x) = [(\sigma_\alpha(x), (X^* f)(\sigma_\alpha(x)))] \tag{6.17}$$

for any vector field $X \in \Gamma(TU_\alpha)$. Here X^* is the horizontal lift of X.

(ii) We claim that

$$X^*_{\sigma_\alpha(x)} = (d\sigma_\alpha)_x X_x - A_\alpha(X)^*_x, \tag{6.18}$$

where $A_\alpha(X)^*_x$ is the fundamental vector field corresponding to $A_\alpha(X)_x \in \mathfrak{g}$. Indeed, the projection of $(d\sigma_\alpha)_x X_x - A_\alpha(X)^*_x$ down to M is clearly X_x. In addition, by using the relation $A_\alpha = \sigma_\alpha^* \omega$ one finds that

$$\langle \omega, (d\sigma_\alpha)_x X_x - A_\alpha(X)^*_x \rangle = A_\alpha(X)_x - A_\alpha(X)_x = 0.$$

In particular, $(d\sigma_\alpha)_x X_x - A_\alpha(X)^*_x$ is a horizontal vector. Therefore, it is the horizontal lift of X_x at $\sigma_\alpha(x)$.

(iii) Suppose that $F : P \to V$ is a function which satisfies $F(ug) = \rho(g^{-1})F(u)$ for all $u \in P$ and $g \in G$. Let $A \in \mathfrak{g}$ with fundamental vector field A^*. Then one has

$$(A^* F)(u) = \frac{d}{dt}\Big|_{t=0} F(ue^{tA}) = \frac{d}{dt}\Big|_{t=0} \rho(e^{-tA}) F(u) = -\rho_*(A) F(u). \tag{6.19}$$

(iv) By applying the relations (6.16, 6.19) to (6.18) (acting on F), one obtains

$$(X^*f)(\sigma_\alpha(x)) = (dv)_x(X_x) + (\rho_* A_\alpha(X)_x)v(x).$$

It follows from (6.17) that

$$(\nabla_X s)(x) = \big[\big(\sigma_\alpha(x), (dv)_x(X_x) + (\rho_* A_\alpha(X)_x)v(x)\big)\big]. \tag{6.20}$$

(v) Now take $s(x) = s_i(x) = [(\sigma_\alpha(x), \varepsilon_i)]$. In this case, $v(x) \equiv \varepsilon_i$. According to (6.20), one has

$$(\nabla_X s_i)(x) = \big[\big(\sigma_\alpha(x), (\rho_* A_\alpha(X)_x)\varepsilon_i\big)\big] = \big(\bar{A}_\alpha(X)_x\big)_i^j [(\sigma_\alpha(x), \varepsilon_j)] = \big(\bar{A}_\alpha\big)_i^j s_j(x).$$

This completes the proof of the proposition. □

6.6 Exterior covariant derivative

The study of curvature relies on the notion of exterior covariant derivative which we shall first introduce. Let $(P, M, \pi; G)$ be a principal bundle.

Definition 6.13. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. A *type* (ρ, V) k-form on P is a V-valued k-form $\varphi \in \Omega^k(P) \otimes V$ which satisfies the relation $R_g^*\varphi = \rho(g^{-1})\varphi$ for all $g \in G$. We say that φ is *horizontal* if $\varphi(X_1, \cdots, X_k) = 0$ provided at least one of the X_i's is vertical.

Remark 6.4. Definition 6.13 does not rely on a connection on P, although the term "horizontal" is used (just as a convention).

An important fact is that *a horizontal, type* (ρ, V) *k-form* φ *on* P *is equivalent to an* E*-valued k-form* $\bar{\varphi}$ *on* M. Here $E \triangleq P \times_\rho V$ is the associated vector bundle of (P, ρ). Indeed, suppose that φ is a horizontal, type (ρ, V) k-form on P. Given $v_1, \cdots, v_k \in T_xM$, we define

$$\bar{\varphi}(v_1, \cdots, v_k)_x \triangleq u\big(\varphi(X_1, \cdots, X_k)_u\big) \in E_x \tag{6.21}$$

where u is any point in $\pi^{-1}(x)$ and X_i is any lift of v_i at u (i.e. $(d\pi)_u X_i = v_i$). One checks that $\tilde{\varphi}$ is a well-defined E-valued k-form on M. Conversely, suppose that $\bar{\varphi} \in \Omega^k(M, E)$. Given $X_1, \cdots, X_k \in T_uP$, we define

$$\varphi(X_1, \cdots, X_k)_u \triangleq u^{-1}\big(\bar{\varphi}((d\pi)_u X_1, \cdots, (d\pi)_u X_k)_x\big).$$

Then φ is a horizontal, type (ρ, V) k-form on P.

Suppose that Γ is a connection on P whose connection form is ω. Note that ω is a type $(\mathrm{Ad}, \mathfrak{g})$ 1-form but it is not horizontal. It cannot be descended to a $\mathrm{Ad}(P)$-valued form on M.

Lemma 6.5. *Let φ be a type (ρ, V) k-form on P.*

(i) $d\varphi$ is a type (ρ, V) $(k+1)$-form.

(ii) The form φ^h defined by $\varphi^h(X_1, \cdots, X_k) \triangleq \varphi(hX_1, \cdots, hX_k)$ (hX_i denotes the projection of X_i onto the horizontal subspace) is a horizontal, type (ρ, V) k-form.

Proof. The first claim follows from the relation that R_g^* commutes with d. The second claim follows from the invariance of horizontal subspaces under R_g. □

Definition 6.14. The *exterior covariant derivative* of a type (ρ, V) k-form φ on P is the horizontal, type (ρ, V) $(k+1)$-form defined by $D\varphi \triangleq (d\varphi)^h$; more precisely,

$$(D\varphi)(X_1, \cdots, X_{k+1}) \triangleq (d\varphi)(hX_1, \cdots, hX_{k+1}).$$

Definition 6.14 does not require φ to be horizontal. If one further assumes so, its exterior covariant derivative can be equivalently described in terms of the connection ∇ on E (Proposition 6.10). We first give the following definition.

Definition 6.15. The *exterior covariant derivative operator* $d^\nabla : \Omega^*(M, E) \to \Omega^{*+1}(M, E)$ is the unique linear operator which satisfies $d^\nabla(s) = \nabla s$ for $s \in \Gamma(E)$ and the Leibniz rule

$$d^\nabla(\alpha \wedge s) = d\alpha \wedge s + (-1)^k \alpha \wedge d^\nabla s \tag{6.22}$$

for $\alpha \in \Omega^k(M)$ and $s \in \Omega^*(M, E)$ $(k \geqslant 1)$.

Proposition 6.10. *For a horizontal, type (ρ, V) form φ on P, let $\bar{\varphi}$ be the corresponding E-valued form. Then one has $\overline{D\varphi} = d^\nabla \bar{\varphi}$.*

Proof. Let φ be a V-valued function satisfying $\varphi(ug) = \rho(g^{-1})\varphi(u)$. Let ψ be the horizontal, type (ρ, V) 1-form corresponding to $\nabla\varphi \in \Omega^1(M, E)$. We want to show that $D\varphi = \psi$. Indeed, for any $u \in P$ and $X \in T_uP$, one has

$$\begin{aligned}\psi(X)_u &= u^{-1}\big(\nabla_{(d\pi)_u X}\bar{\varphi}\big) = u^{-1}\big(u((hX)\varphi)\big) \quad \text{(by Lemma 6.4)} \\ &= \big((hX)\varphi\big)(u) = (D\varphi)(X)_u.\end{aligned}$$

Therefore, $D\varphi = \psi$. This proves the claim for 0-forms.

For general k-forms, one only needs to observe that the exterior covariant derivative D also satisfies the Leibniz rule on the principal bundle. Namely, for any $\alpha \in \Omega^k(M)$ and any horizontal, type (ρ, V) form η on P, the form $\pi^*\alpha \wedge \eta$ is also horizontal and of type (ρ, V), and one has

$$D\big(\pi^*\alpha \wedge \eta\big) = d\pi^*\alpha \wedge \eta + (-1)^k \pi^*\alpha \wedge D\eta.$$

This part is straight forward. □

In the horizontal case, one can derive an explicit formula for the exterior covariant derivative in terms of the exterior derivative d and the connection form ω. To state this formula, we first introduce some notation.

Definition 6.16. We define a natural pairing $\cdot \wedge \cdot$ between $\Omega(P) \otimes \mathrm{End}(V)$ and $\Omega(P) \otimes V$ by extending the relation

$$(\alpha \otimes S) \wedge (\beta \otimes \xi) \triangleq (\alpha \wedge \beta) \otimes S(\xi), \ \ \alpha, \beta \in \Omega(P), S \in \mathrm{End}(V), \xi \in V$$

in a multilinear manner. In the case when $\Phi \in \Omega^1(P) \otimes \mathrm{End}(V)$ and $\psi \in \Omega^k(P) \otimes V$, it is easily seen that $\Phi \wedge \psi \in \Omega^{k+1}(P) \otimes V$ is given by

$$(\Phi \wedge \psi)(X_0, \cdots, X_k) = \sum_{i=0}^{k} (-1)^i \Phi(X_i)\big(\psi(X_0, \cdots, \widehat{X_i}, \cdots, X_k)\big). \tag{6.23}$$

Proposition 6.11. *Let φ be a horizontal, type (ρ, V) form on P. Then*

$$D\varphi = d\varphi + (\rho_* \omega) \wedge \varphi, \tag{6.24}$$

where the above $\wedge$-pairing is understood in the sense of Definition 6.16. Equivalently, if one views $\varphi \in \Omega(M, E)$ whose local representation with respect to a local trivialisation $\{U_\alpha, \psi_\alpha, g_{\beta\alpha}\}$ is $\varphi_\alpha \in \Omega(U_\alpha) \otimes V$, then the local representation of $d^\nabla \varphi$ is given by

$$(d^\nabla \varphi)_\alpha = d\varphi_\alpha + (\rho_* A_\alpha) \wedge \varphi_\alpha, \tag{6.25}$$

where $\{A_\alpha\}$ is the local representation of ω.

Proof. We only prove (6.24). The local version (6.25) follows immediately from (6.24) as well as Proposition 6.10. Let φ be of degree k. We need to show that

$$(D\varphi)(X_0, \cdots, X_k) = \big(d\varphi + (\rho_* \omega) \wedge \varphi\big)(X_0, \cdots, X_k) \tag{6.26}$$

for all vector fields $X_0, \cdots, X_k \in \Gamma(TP)$.

(i) *The X_i's are all horizontal.* In this case,

$$\big((\rho_* \omega) \wedge \varphi\big)(X_0, \cdots, X_k) = 0$$

due to the presence of the ω-term. The relation (6.26) just reduces to the definition of $D\varphi$.

(ii) *At least one of the X_i's is vertical.* Without loss of generality, let us assume that $X_0 = A^*$ with $A \in \mathfrak{g}$. In this case, the LHS of (6.26) is zero. For the RHS, since φ is horizontal, by using the formula (6.23) one finds that

$$\big((\rho_* \omega) \wedge \varphi\big)(A^*, X_1, \cdots, X_k) = (\rho_* A)\big(\varphi(X_1, \cdots, X_k)\big).$$

On the other hand, according to Cartan's formula for the Lie derivative (see Proposition B.1), one has

$$\mathcal{L}_{A^*}\varphi = \iota(A^*)d\varphi + d\iota(A^*)\varphi = \iota(A^*)d\varphi$$

where $\iota(A^*)$ is the interior product (see Definition B.6). Therefore,

$$\begin{aligned}(d\varphi)(A^*, X_1, \cdots, X_k) &= \big(\iota(A^*)d\varphi\big)(X_1, \cdots, X_k)\\ &= (\mathcal{L}_{A^*}\varphi)(X_1, \cdots, X_k).\end{aligned}$$

It remains to shows that

$$(\mathcal{L}_{A^*}\varphi)(X_1, \cdots, X_k) = -(\rho_* A)\big(\varphi(X_1, \cdots, X_k)\big). \tag{6.27}$$

Indeed, by Definition B.2 of the Lie derivative and the fact that φ is of type (ρ, V), one has

$$(\mathcal{L}_{A^*}\varphi)_u = \lim_{t\to 0}\frac{1}{t}\big(R^*_{e^{tA}}\varphi_{ue^{tA}} - \varphi_u\big) = \lim_{t\to 0}\frac{1}{t}\big(\rho(e^{-tA})\varphi_u - \varphi_u\big) = -(\rho_* A)\varphi_u.$$

This gives the relation (6.27) and thus completes the proof of (6.24). □

Example 6.9. In the case when $\rho = \mathrm{Ad}$ and $E = \mathrm{Ad}(P)$, one has

$$D\varphi = d\varphi + [\omega \wedge \varphi]. \tag{6.28}$$

Here the *Lie bracket wedge* between two $\mathfrak{g}$-valued forms on P is the unique bilinear map

$$[\cdot \wedge \cdot] : (\Omega(P) \otimes \mathfrak{g}) \times (\Omega(P) \otimes \mathfrak{g}) \to \Omega(P) \otimes \mathfrak{g} \tag{6.29}$$

which satisfies

$$[(\alpha \otimes A) \wedge (\beta \otimes B)] \triangleq (\alpha \wedge \beta) \otimes [A, B]$$

for $\alpha, \beta \in \Omega(P)$ and $A, B \in \mathfrak{g}$. This is because $(\mathrm{Ad}_*\omega) \wedge \varphi = [\omega \wedge \varphi]$. From the $\Omega(M, \mathrm{Ad}(P))$ perspective, one has the local representation

$$(d^\nabla \varphi)_\alpha = d\varphi_\alpha + [A_\alpha \wedge \varphi_\alpha] \tag{6.30}$$

in this case.

6.7 Curvature of a connection

Now we discuss a central concept of this chapter: the curvature of a connection. Let Γ be a connection on $(P, M, \pi; G)$ whose connection form is ω. Recall that ω is a type $(\mathrm{Ad}, \mathfrak{g})$ 1-form on P. We define the horizontal, type $(\mathrm{Ad}, \mathfrak{g})$ 2-form $\Omega \triangleq D\omega$. According to the previous discussion, Ω descends to an element of $\Omega^2(M, \mathrm{Ad}(P))$ which is denoted as F_ω.

Definition 6.17. Both Ω and F_ω are called the *curvature form* of Γ (which one is referred to will depend on the context).

6.7.1 Cartan's structure equation and Bianchi's identities

The curvature form Ω satisfies the following (second) *structure equation of Cartan* (cf. Theorem 6.6).

Theorem 6.3. *One has*

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega], \tag{6.31}$$

where $[\cdot \wedge \cdot]$ *is the Lie bracket wedge defined in (6.29). An equivalent way of rewriting (6.31) is that*

$$\Omega(X, Y) = (d\omega)(X, Y) + [\omega(X), \omega(Y)] \tag{6.32}$$

for all $X, Y \in \Gamma(TP)$.

Proof. We verify (6.32) at each $u \in P$. Let $X, Y \in T_uP$.

(i) *Both X, Y are horizontal.* In this case, $\omega(X) = \omega(Y) = 0$ and the relation (6.32) is just the definition of Ω.

(ii) *Both X, Y are vertical.* In this case, one can assume that $X = A_u^*, Y = B_u^*$ with $A, B \in \mathfrak{g}$. By definition, $\Omega(A^*, B^*) = 0$. On the other hand

$$\begin{aligned}
&(d\omega)(A^*, B^*) + [\omega(A^*), \omega(B^*)] \\
&= A^*\omega(B^*) - B^*\omega(A^*) - \omega([A^*, B^*]) + [\omega(A^*), \omega(B^*)] \\
&= -\omega([A^*, B^*]) + [\omega(A^*), \omega(B^*)] \\
&= -\omega([A, B]^*) + [A, B] \\
&= 0,
\end{aligned}$$

where the second last equality follows from Lemma 6.1.

(iii) *X is horizontal and $Y = A_u^*$ ($A \in \mathfrak{g}$) is vertical.* We take a horizontal extension of X (still denoted as X) to a neighbourhood of u. For instance, one can choose an extension of $(d\pi)_u X$ to a neighbourhood U of $\pi(u)$ and take its horizontal lift to $\pi^{-1}(U)$. By definition, one still has $\Omega(X, A^*) = 0$ and also $[\omega(X), \omega(A^*)] = 0$ in this case. It remains to show that

$$(d\omega)(X, A^*) = 0. \tag{6.33}$$

To this end, one first notes that

$$(d\omega)(X, A^*) = X\omega(A^*) - A^*\omega(X) + \omega([A^*, X]) = \omega([A^*, X]).$$

Next, we claim that $[A^*, X]$ is horizontal. In fact, one has

$$[A^*, X]_u = (\mathcal{L}_{A^*}X)_u = \lim_{t\to 0} \frac{1}{t}\big((dR_{e^{tA}})X_{ue^{tA}} - X_u\big).$$

The RHS is clearly horizontal since horizontal subspaces are invariant under G-actions. Therefore, $\omega([A^*, X]) = 0$ and the relation (6.33) follows. □

Remark 6.5. The 1/2-factor appearing in the structure equation (6.31) does not contradict the formula (6.28). In fact, Proposition 6.11 *cannot* be applied to $\varphi = \omega$ because ω is not horizontal. If one examines the proofs of both Proposition 6.11 and the formula (6.31) carefully, the 1/2-factor precisely arises from the nonhorizontality of ω.

Remark 6.6. The structure equation shows that

$$\Omega(X, Y) = -\omega([X, Y]) \tag{6.34}$$

for horizontal vector fields X, Y. In particular, $\Omega(X, Y)_u$ picks up the $\mathfrak{g}$-element corresponding to the vertical component of $[X, Y]_u$. From this perspective, the curvature measures the degree of nonintegrability for the horizontal distribution $\{\mathcal{H}_u\}_{u \in P}$. If the connection is flat (i.e. $\Omega = 0$), the relation (6.34) shows that the horizontal distribution is invariant under Lie bracket. As a consequence, the horizontal distribution is integrable and P is thus foliated by maximal connected horizontal leaves (Frobenius theorem; see [War83, Theorem 1.60]). If M is further assumed to be simply connected, it is isomorphic to the trivial bundle $M \times G$ (see [KN63, Corollary 9.2]).

Remark 6.7. Since G is assumed to be a matrix Lie group, one can also express the structure equation (6.31) in the following form:

$$\Omega = d\omega + \omega \wedge \omega, \tag{6.35}$$

where ω is viewed as a matrix-valued 1-form and the above exterior product is taken with respect to matrix multiplication. This formula is consistent with Definition 5.6 in the orthonormal frame bundle case.

The curvature form satisfies the following *Bianchi's identity*, which is an extension of Proposition 5.4.

Theorem 6.4 (Bianchi's Identity)**.** *One has $D\Omega = 0$, or equivalently, $d^\nabla F_\omega = 0$. Here d^∇ is the exterior covariant derivative on $\Omega(M, \mathrm{Ad}(P))$ induced from Γ.*

Proof. By taking exterior derivative on both sides of (6.35), one obtains that

$$d\Omega = d\omega \wedge \omega - \omega \wedge d\omega.$$

Due to the presence of ω in the above equation, it is apparent that $(d\Omega)(X, Y, Z) = 0$ for any horizontal vector fields X, Y, Z. □

Remark 6.8. By using the formula (6.28) and the relation that $[\omega \wedge \Omega] = \omega \wedge \Omega - \Omega \wedge \omega$, one can rewrite Bianchi's identity as

$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega.$$

This is consistent with Proposition 5.4 in the orthonormal frame bundle case.

Example 6.10. Consider the trivial principal G-bundle $P = M \times G$. The *Maurer-Cartan form* is the $\mathfrak{g}$-valued 1-form on G defined by $\bar{\omega} \triangleq g^{-1}dg$; its evaluation at $X_g \in T_gG$ is the left translation of X_g by g^{-1} as an element in $\mathfrak{g}$. Explicit calculation shows that

$$d\bar{\omega} + \frac{1}{2}[\bar{\omega} \wedge \bar{\omega}] = 0.$$

As a result, the curvature of the connection $\rho_2^*\bar{\omega}$ ($\rho_2 : P \to G$ is the projection onto the second component) is flat.

6.7.2 Local representation of curvature form

Now we consider local representations of Ω. Let $\{U_\alpha, \psi_\alpha, g_{\beta\alpha}\}$ be a local trivialisation of P. Similar to the connection case, we define $F_\alpha \triangleq \sigma_\alpha^*\Omega \in \Omega^2(U_\alpha) \otimes \mathfrak{g}$ where $\sigma_\alpha(x) \triangleq \psi_\alpha^{-1}(x, e)$ is the local identity section. It follows from (6.35) that

$$F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha, \tag{6.36}$$

where we recall that $A_\alpha \triangleq \sigma_\alpha^*\omega$ is the local representation of ω. Using the consistency relation (6.10) for A_α, it is plain to check that

$$F_\beta = g_{\beta\alpha} \cdot F_\alpha \cdot g_{\beta\alpha}^{-1} \quad \text{on } U_{\alpha\beta}. \tag{6.37}$$

In particular, $F_\beta(X, Y) = \mathrm{Ad}(g_{\beta\alpha})(F_\alpha(X, Y))$ on $U_{\alpha\beta}$, which is precisely the consistency relation required for $\{F_\alpha(X, Y)\}$ defining a global section of $\mathrm{Ad}(P)$ (cf. (6.6)). In other words, the family $\{F_\alpha\}$ patches to a global $\mathrm{Ad}(P)$-valued 2-form F on M which is of course just F_ω. Sometimes we also use F_A to denote the curvature form if the connection is given by the local data $A = \{A_\alpha\} \in \mathfrak{C}(\mathcal{U}, g_{\bullet\bullet})$. According to the relation (6.30), Bianchi's identity $d^\nabla F_\omega = 0$ locally reads

$$dF_\alpha + [A_\alpha \wedge F_\alpha] = 0. \tag{6.38}$$

Lemma 6.6. *Let $\{U_\alpha, \psi_\alpha\}$ and $\{U_\alpha, \psi'_\alpha\}$ be two local trivialisations of P. The corresponding local representations of Ω are related by $F'_\alpha = T_\alpha F_\alpha T_\alpha^{-1}$ where $T_\alpha : U_\alpha \to G$ is the map defined by $T_\alpha(x) \triangleq \varphi'_\alpha(p)\varphi_\alpha(p)^{-1}$ ($x \in U_\alpha$, $p \in \pi^{-1}(x)$).*

Proof. This is explicit calculation based on Proposition 6.7. □

6.7.3 Curvature of induced connection on associated vector bundles

Finally, we consider the curvature of the induced connection on an associated vector bundle $E = P \times_\rho V$. Recall from Section 6.5 that the connection Γ on P induces a connection ∇ on E. The form $\rho_*(\Omega)$ is a horizontal, type $(\mathrm{Ad}_\rho, \mathrm{End}_\rho(V))$ 2-form on P, which descends to an element $F_\nabla \in \Omega^2(M, \mathrm{End}_\rho(E))$ (see Example 6.6 for the relevant definitions). The form F_∇ can also be constructed in the following way. Define

$$\bar{F}_\alpha = d\bar{A}_\alpha + \bar{A}_\alpha \wedge \bar{A}_\alpha \in \Omega^2(U_\alpha) \otimes \mathrm{End}_\rho(V),$$

where $\bar{A}_\alpha \triangleq \rho_*(A_\alpha)$ is the local representation of the connection ∇ (see Proposition 6.9). According to the relation (6.37), one has

$$\bar{F}_\beta = \mathrm{Ad}_\rho(g_{\beta\alpha})\big(\bar{F}_\alpha\big).$$

It follows that the family $\{\bar{F}_\alpha\}$ patches to a global element $F_\nabla \in \Omega^2(M, \mathrm{End}_\rho(E))$.

Definition 6.18. The form $F_\nabla \in \Omega^2(M, \mathrm{End}_\rho(E))$ is called the *curvature form* of the connection ∇.

By viewing $\mathrm{End}_\rho(E)$ as a subbundle of $\mathrm{End}(E)$ in a natural way, the 2-form F_∇ is exactly the curvature tensor of ∇ defined by (B.20); indeed, one can show that

$$F_\nabla(X,Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s$$

for all $X, Y \in \Gamma(TM)$ and $s \in \Gamma(E)$ (cf Lemma 6.8). The additional structure here is that the endomorphisms determined by F_∇ are infinitesimal symmetries coming from $\mathfrak{g}$ through the induced representation ρ_* (cf. Example 6.6).

Example 6.11. Let $(E, M, \pi; V)$ be a real Euclidean vector bundle of rank r. Recall from Example 6.7 that $E \cong E' \triangleq \mathrm{O}(E) \times_{\mathrm{id}} \mathbb{R}^r$ where $\mathrm{O}(E)$ is the orthonormal frame bundle of E. A metric connection ∇ on E is therefore equivalent to a connection ω on $\mathrm{O}(E)$ (see Remark 6.3). The curvature tensor F_∇ of ∇ is an element of $\Omega^2(M, \mathrm{End}^-(E))$, where $\mathrm{End}^-(E)$ is the subbundle of $\mathrm{End}(E)$ consisting of endomorphisms that are antisymmetric with respect to the metric. On the other hand, there is a natural isomorphism

$$\Phi : \mathrm{End}^-(E) \to \mathrm{End}_{\mathrm{id}}E', \;\; \Phi(s) \triangleq \big[\big((x; e_1, \cdots, e_r), (T_i^j)_{1\leqslant i,j\leqslant r}\big)\big],$$

where $x = \pi(s)$ and T_i^j is defined by $se_i = T_i^j e_j$. The image of F_∇ under Φ, as an element of $\Omega^2(M, \mathrm{End}_{\mathrm{id}}E')$, is precisely the curvature of the connection on E' induced from ω. Similar discussion applies to the complex Hermitian case.

6.8 Gauge transformations

We take a small detour to briefly discuss how connection and curvature transforms under principal bundle automorphisms. Let $(P, M, \pi; G)$ be a given fixed principal bundle.

Definition 6.19. A *gauge transformation* of P is a principal bundle automorphism $T : P \to P$ such that $\pi \circ T = \pi$. The space of gauge transformations (the *gauge group*) is denoted as $\mathcal{G}(P)$.

Clearly, $\mathcal{G}(P)$ is a group under composition. Let $(\mathcal{U}, g_{\bullet\bullet})$ be given fixed local gluing data of P. Recall from Proposition 6.1 that a gauge transformation $T \in \mathcal{G}(P)$ is equivalent to a family of maps $T_\alpha : U_\alpha \to G$ satisfying $g_{\beta\alpha}(x) = T_\beta(x) g_{\beta\alpha}(x) T_\alpha(x)^{-1}$. We also denote $\mathcal{G}(P)$ as $\mathcal{G}(\mathcal{U}, g_{\bullet\bullet})$ if we prefer to work with their local representations.

Proposition 6.12. *Let $S, T \in \mathcal{G}(P)$ with local representations $\{S_\alpha\}, \{T_\alpha\}$ respectively. Then $\{S_\alpha T_\alpha\}$ (defined by pointwise multiplication) is the local representation of ST.*

Proof. From the proof of Proposition 6.1, one knows that

$$T(u) = \psi_\alpha^{-1}(\pi(u), T_\alpha(x)\varphi_\alpha(u)).$$

Therefore, one has

$$S(T(u)) = \psi_\alpha^{-1}(\pi(u), S_\alpha(x)\varphi_\alpha(T(u))) = \psi_\alpha^{-1}(\pi(u), S_\alpha(x) T_\alpha(x)\varphi_\alpha(u)).$$

This shows that the local representation of ST on U_α is $S_\alpha T_\alpha$. □

Remark 6.9. Let $\{T'_\alpha\}$ be the local representation of T with respect to another gluing data $(\mathcal{U}, h_{\bullet\bullet})$. Then $T'_\alpha = R_\alpha T_\alpha R_\alpha^{-1}$ where $R_\alpha(x) = \varphi'_\alpha(u)\varphi_\alpha(u)^{-1}$ for $u \in \pi^{-1}(x)$.

The gauge group $\mathcal{G}(P)$ acts on the space $\mathfrak{C}(P)$ of connections in the obvious way:

$$T \curvearrowright \omega \triangleq (T^{-1})^*\omega, \quad T \in \mathcal{G}(P), \omega \in \mathfrak{C}(P).$$

This is indeed a group action; one has $(ST) \curvearrowright \omega = S \curvearrowright (T \curvearrowright \omega)$. The following result provides the local formula for this action (cf. Proposition 6.7).

Proposition 6.13. *Let $T \in \mathcal{G}(P)$ and $\omega \in \mathfrak{C}(P)$ be represented by $\{T_\alpha\}$ and $\{A_\alpha\}$ respectively under local trivialisation $\{U_\alpha, \psi_\alpha, g_{\beta\alpha}\}$. The local representation of $T \curvearrowright \omega$ is then given by*

$$B_\alpha = T_\alpha A_\alpha T_\alpha^{-1} - dT_\alpha T_\alpha^{-1}. \tag{6.39}$$

Proof. Let $S \triangleq T^{-1}$ and $\omega' \triangleq T \curvearrowright \omega = S^*\omega$. By definition,

$$B_\alpha = \sigma_\alpha^*\omega' = (S \circ \sigma_\alpha)^*\omega,$$

where $\sigma_\alpha(x) = \psi_\alpha^{-1}(x, e)$ is the local identity section. An important observation is that

$$(S \circ \sigma_\alpha)(x) = S(\psi_\alpha^{-1}(x, e)) = \psi_\alpha^{-1}(x, S_\alpha(x)e) = \psi_\alpha^{-1}(x, eS_\alpha(x)) = \sigma_\alpha(x)S_\alpha(x),$$

where $\{S_\alpha\}$ is the local representation of S. Now let $v \in T_xM$ be given fixed. Setting $u \triangleq \sigma_\alpha(x)$ and $g \triangleq S_\alpha(x)$, one has

$$\langle B_\alpha, v\rangle_x = \langle \omega, (S\circ\sigma_\alpha)_* v\rangle_{ug}. \tag{6.40}$$

Let x_t be a curve representing the tangent vector v. Denoting $X_u \triangleq (\sigma_\alpha)_* v \in T_uP$, it follows that

$$\begin{aligned}(S\circ\sigma_\alpha)_* v &= \frac{d}{dt}\Big|_{t=0}\sigma_\alpha(x_t)S_\alpha(x_t)\\ &= (R_g)_*X_u + ug\frac{d}{dt}\Big|_{t=0}S_\alpha^{-1}(x)S_\alpha(x_t).\end{aligned}$$

By substituting this into (6.40) and using Theorem 6.1 (ii), one finds that

$$\begin{aligned}\langle B_\alpha, v\rangle_x &= \langle R_g^*\omega_{ug}, X_u\rangle + \Big\langle \omega_{ug}, ug\frac{d}{dt}\Big|_{t=0}S_\alpha^{-1}(x)S_\alpha(x_t)\Big\rangle\\ &= g^{-1}\langle\omega_u, X_u\rangle g + \Big\langle \omega_{ug}, ug\frac{d}{dt}\Big|_{t=0}S_\alpha^{-1}(x)S_\alpha(x_t)\Big\rangle\\ &= T_\alpha(x)\langle A_\alpha, v\rangle_x T_\alpha(x)^{-1} + \Big\langle \omega_{ug}, ug\frac{d}{dt}\Big|_{t=0}S_\alpha^{-1}(x)S_\alpha(x_t)\Big\rangle,\end{aligned}$$

To evaluate the second term, one observes that

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}S_\alpha^{-1}(x)S_\alpha(x_t) = \langle S_\alpha^*\bar\omega, v\rangle_x &= \langle S_\alpha dS_\alpha^{-1}, v\rangle_x\\ &= -\langle dT_\alpha T_\alpha^{-1}, v\rangle_x =: A \in \mathfrak{g},\end{aligned}$$

where $\bar\omega$ is the Maurer-Cartan form (see Example 6.10). By using Proposition 6.1 (i), one has

$$\Big\langle \omega_{ug}, ug\frac{d}{dt}\Big|_{t=0}S_\alpha^{-1}(x)S_\alpha(x_t)\Big\rangle = A = -\langle dT_\alpha T_\alpha^{-1}, v\rangle_x.$$

As a consequence,

$$\langle B_\alpha, v\rangle_x = T_\alpha(x)\langle A_\alpha, v\rangle_x T_\alpha(x)^{-1} - \langle dT_\alpha T_\alpha^{-1}, v\rangle_x.$$

The relation (6.39) thus follows. □

The curvature also transforms in an obvious way under the gauge action. More precisely, one has $F_{T\frown\omega} = \hat{T}(F_\omega)$ where $\hat{T} : \mathrm{Ad}(P) \to \mathrm{Ad}(P)$ is the isomorphism induced by $[(u, A)] \mapsto [(Tu, A)]$ ($u \in P$, $A \in \mathfrak{g}$). If $\{T_\alpha\}$ is the local representation of T, one has $(F_{T\frown\omega})_\alpha = T_\alpha \cdot (F_\omega)_\alpha \cdot T_\alpha^{-1}$.

More generally, let $\Phi : P \to Q$ be an isomorphism of principal G-bundles over M (covering the identity id_M). Suppose that $(\mathcal{U}, g_{\bullet\bullet})$ (respectively, $(\mathcal{U}, h_{\bullet\bullet})$) is the local gluing data for P (respectively, for Q). According to Proposition 6.1, Φ is equivalent to a family of maps $T_\alpha : U_\alpha \to G$ satisfying the relation (6.1). Let $\omega \in \mathfrak{C}(P)$ and $\omega' \triangleq (\Phi^{-1})^*\omega$ be the corresponding connection on Q. Their local representations ($\{A_\alpha\}$ and $\{B_\alpha\}$ respectively) are related by

$$B_\alpha = T_\alpha A_\alpha T_\alpha^{-1} - dT_\alpha T_\alpha^{-1}.$$

Their curvatures are related by $F_{\omega'} = \hat{\Phi}(F_\omega)$ where $\hat{\Phi} : \mathrm{Ad}(P) \to \mathrm{Ad}(Q)$ is the vector bundle isomorphism induced by Φ. Locally, one has $(F_{\omega'})_\alpha = T_\alpha \cdot (F_\omega)_\alpha \cdot T_\alpha^{-1}$.

6.9 Affine connections on tangent bundle

In this section, we apply the general theory to a particular situation: the tangent bundle TM.

Let M be an n-dimensional manifold. Recall from Example 6.2 that the frame bundle $\mathcal{F}(M)$ is a principal $\mathrm{GL}(n;\mathbb{R})$-bundle over M. Elements of $\mathcal{F}(M)$ are of the form $u = (x; X_1, \cdots, X_n)$, where $x \in M$ and $\{X_1, \cdots, X_n\}$ is a basis of T_xM. A local frame field $\{e_i\}$ of TM over some open subset $U \subseteq M$ gives rise to a local trivialisation (parametrisation)

$$\psi : \pi^{-1}U : U \times \mathrm{GL}(n;\mathbb{R}),\ \psi((x; X_1, \cdots, X_n)) = (x, g) \tag{6.41}$$

where g is the transformation matrix from $(e_1, \cdots, e_n)$ to $(X_1, \cdots, X_n)$, i.e.

$$(X_1, \cdots, X_n) = (e_1, \cdots, e_n) \cdot g.$$

According to Example 6.4, the tangent bundle TM is isomorphic to the associated vector bundle $\mathcal{F}(M) \times_{\mathrm{id}} \mathbb{R}^n$, where $\mathrm{id} : \mathrm{GL}(n;\mathbb{R}) \curvearrowright \mathbb{R}^n$ is the standard representation defined by matrix multiplication. For $u = (x; X_1, \cdots, X_n) \in \mathcal{F}(M)$, the linear isomorphism $u : \mathbb{R}^n \to T_xM$ is defined by $u(\xi) \triangleq \xi^i X_i$ (cf. (6.2)). In other words, $u(\xi)$ is the tangent vector whose coordinates with respect to the basis u are $(\xi^i)_{1\leqslant i\leqslant n}$.

6.9.1 The canonical form

The frame bundle $\mathcal{F}(M)$ has a special feature: the canonical form (cf. Definition 5.4).

Definition 6.20. The *canonical form* on $\mathcal{F}(M)$ is the $\mathbb{R}^n$-valued 1-form on $\mathcal{F}(M)$ defined by

$$\theta(X)_u \triangleq u^{-1}\big((d\pi)_u X\big), \quad u \in \mathcal{F}(M), X \in T_u\mathcal{F}(M).$$

More explicitly, $\theta^i(X)_u$ is the X_i-coefficient of $(d\pi)_u X$ with respect to the basis $\{X_1, \cdots, X_n\}$ of T_xM defining u.

The form θ admits the following local expression.

Lemma 6.7. *Let $\{e_i\}$ be a local frame field of TM over some open subset $U \subseteq M$ and let $\{\eta^i\}$ be its dual. Then one has*

$$\theta_{(x,g)} = g^{-1} \cdot \pi^*\eta, \tag{6.42}$$

where (x, g) is the local parametrisation of $\mathcal{F}(M)$ given by (6.41).

Proof. Let $X \in T_u\mathcal{F}(M)$ and write $(d\pi)_uX = \lambda^j X_j$, where $u = (x; X_1, \cdots, X_n)$. By the definition of θ, one has

$$\begin{aligned} \langle (g^{-1})^i_j \pi^*\eta^j, X\rangle &= (g^{-1})^i_j \langle \eta^j, (d\pi)_uX\rangle = (g^{-1})^i_j \langle \eta^j, \lambda^k X_k\rangle \\ &= (g^{-1})^i_j \langle \eta^j, \lambda^k g^l_k e_l\rangle = (g^{-1})^i_j g^j_k \lambda^k = \lambda^i = \theta^i(X)_u. \end{aligned}$$

This gives the relation (6.42). □

Proposition 6.14. *The canonical form θ is horizontal and of type $(\mathrm{id}, \mathbb{R}^n)$. It descends to an element $\bar{\theta} \in \Omega^1(M, TM)$ which is given by $\bar{\theta}(v)_x = v$ for all $v \in T_xM$.*

Proof. By definition, one has

$$\langle \theta^i, Y\rangle_u = g^j_i \langle \eta^j, (d\pi)_uY\rangle_x$$

for any $Y \in T_u\mathcal{F}(M)$. Since g is invertible,

$$\langle \theta^i, Y\rangle_u = 0 \ \forall i \iff \langle \eta^j, (d\pi)_uY\rangle_x = 0 \ \forall i \iff (d\pi)_uY = 0 \iff Y \in \mathcal{V}_u.$$

This shows that $\ker \theta_u = \mathcal{V}_u$, and in particular, θ is horizontal.

Next, we need to show that

$$\theta((dR_g)X)_{ug} = g^{-1} \cdot \theta(X)_u \tag{6.43}$$

for all $X \in T_u\mathcal{F}(M)$ and $g \in \mathrm{GL}(n; \mathbb{R})$. Indeed, the LHS of (6.43) is the coordinates of $v \triangleq (d\pi)_uX$ with respect to the basis $(X_1, \cdots, X_n) \cdot g$ where $u = (x; X_1, \cdots, X_n)$. It is therefore related to the vector $\theta(X)$ (coordinates of v with respect to $(X_1, \cdots, X_n)$) by the formula (6.43).

Finally, to derive the expression of $\bar{\theta}$, one simply recalls from (6.21) that

$$\bar{\theta}(v)_x = u(\theta(X)_u), \quad v \in T_xM,$$

where $u \in \pi^{-1}(x)$ and X is any lift of v. It is apparent from definition that $u(\theta(X)_u)$ is just v. □

6.9.2 Affine connections and absolute parallelism

Connections on TM are called *affine connections*. According to Remark 6.3, a connection on TM is equivalent to a connection on $\mathcal{F}(M)$. We will use the terminology affine connections on M / on TM / on $\mathcal{F}(M)$ interchangeably depending on the context.

In what follows, we assume that $\mathcal{F}(M)$ is equipped with an affine connection Γ with connection form ω. Another special feature of $\mathcal{F}(M)$ is that it admits an *absolute parallelism* with respect to Γ, i.e. a global frame field $\{V_j^i, H_k : 1 \leqslant i, j, k \leqslant n\}$ in the sense that $\{V_j^i(u), H_k(u)\}$ is a basis of $T_u\mathcal{F}(M)$ at every $u \in \mathcal{F}(M)$. The vector fields $\{V_j^i\}$ are the fundamental vector fields corresponding to the canonical basis of $\mathfrak{gl}(n;\mathbb{R})$. To construct the H_k's, we need to introduce the following definition.

Definition 6.21. Let $\xi \in \mathbb{R}^n$. For each $u \in \mathcal{F}(M)$, we define $H(\xi)_u \in \mathcal{H}_u$ to be the horizontal lift of $u(\xi) \in T_{\pi(u)}M$. The horizontal vector field $u \mapsto H(\xi)_u$ is called the *standard horizontal vector field corresponding to* ξ.

We summarise a few basic properties of standard horizontal vector fields as follows.

Proposition 6.15. *(i) The map $\xi \mapsto H(\xi)$ is linear. In addition, for each $\xi \neq 0$, the vector field $H(\xi)$ is everywhere nonvanishing.*

(ii) $\theta(H(\xi)) = \xi$ for all $\xi \in \mathbb{R}^n$.

(iii) $(dR_g)H(\xi) = H(g^{-1}\xi)$ for all $\xi \in \mathbb{R}^n$ and $g \in G$.

(iv) $[A^*, H(\xi)] = H(A\xi)$ for all $A \in \mathfrak{gl}(n;\mathbb{R})$ and $\xi \in \mathbb{R}^n$.

Proof. The first three properties are straight forward. In addition, one has

$$\begin{aligned}[A^*, H(\xi)]_u &= (\mathcal{L}_{A^*}H(\xi))_u = \lim_{t\to 0}\frac{1}{t}\big((dR_{e^{-tA}})H(\xi)_{ue^{tA}} - H(\xi)_u\big) \\ &= \lim_{t\to 0}\frac{1}{t}\big(H(e^{tA}\xi)_u - H(\xi)_u\big) \quad \text{(by (iii))} \\ &= H(A\xi).\end{aligned}$$

This proves the last property. □

Now we prove the aforementioned absolute parallelism. Let $E_j^i \in \mathfrak{gl}(n;\mathbb{R})$ be the elementary matrix whose (i,j)-entry is one and all other entries are zero. Note that $\{E_j^i : 1 \leqslant i, j \leqslant n\}$ is a basis of $\mathfrak{gl}(n;\mathbb{R})$. Let $\{\varepsilon_1, \cdots, \varepsilon_n\}$ be the canonical basis of $\mathbb{R}^n$.

Theorem 6.5. *Let $(E^i_j)^*$ be the fundamental vector field corresponding to E^i_j. Then the vector fields $\{(E^i_j)^*, H(\varepsilon_k)\}$ and the 1-forms $\{\omega^i_j, \theta^k\}$ on P are dual to each other. In particular,*

$$\{(E^i_j)^*_u, H(\varepsilon_k)_u : 1 \leqslant i, j, k \leqslant n\}$$

form a basis of $T_u\mathcal{F}(M)$ at every $u \in \mathcal{F}(M)$.

Proof. Note that ω annihilates horizontal vectors and θ annihilates vertical vectors. The result follows immediately from Theorem 6.1 (i) and Proposition 6.15 (ii). $\square$

6.9.3 Structure equations and Bianchi's identities

In Section 6.7, we derived the structure equation and Bianchi's identity for the curvature form $\Omega = D\omega$ in the general principal bundle context. In the frame bundle case, there is an additional structure equation as well as an additional Bianchi's identity which are both related to the so-called torsion form. Let Γ be an affine connection on $\mathcal{F}(M)$.

Definition 6.22. The *torsion form* of Γ is the horizontal, type $(\mathrm{id}, \mathbb{R}^n)$ 2-form defined by $\Theta \triangleq D\theta$ where θ is the canonical form.

Remark 6.10. Recall from (6.34) that the curvature form measures the vertical component of the Lie bracket between two horizontal vector fields. To some extent, the torsion form measures its horizontal component. Let $H(\xi)$ and $H(\eta)$ be the standard horizontal vector fields corresponding to ξ and η respectively. By the definition of Θ, one has

$$\begin{aligned}\Theta(H(\xi), H(\eta)) &= d\theta(H(\xi), H(\eta)) \\ &= H(\xi)\theta(H(\eta)) - H(\eta)(\theta H(\xi)) - \theta([H(\xi), H(\eta)]) \\ &= H(\xi)\eta - H(\eta)\xi - \theta(h[H(\xi), H(\eta)]) \\ &= -\theta(h[H(\xi), H(\eta)]). \end{aligned} \tag{6.44}$$

The vector $\theta(h[H(\xi), H(\eta)])_u$ picks up the coordinates of $(d\pi)_u(h[H(\xi), H(\eta)]_u)$ with respect to the basis u of $T_{\pi(u)}M$.

We now state the full structure equations and Bianchi's identities for an affine connection.

Theorem 6.6. *(i) (The first structure equation) $d\theta = -\omega \wedge \theta + \Theta$.*

(ii) (*The second structure equation*) $d\omega = -\omega \wedge \omega + \Omega$.

Proof. The second structure equation was proven in Theorem 6.3 and we only consider the first one. Fixing $u \in \mathcal{F}(M)$, we test the equation on a pair of vectors $X, Y \in T_u\mathcal{F}(M)$.

(i) *Both X, Y are vertical.* We assume that $X = A_u^*$, $Y = B_u^*$ for some $A, B \in \mathfrak{g}$. In this case, the RHS is zero. For the LHS, one has from Lemma 6.1 that

$$\begin{aligned} d\theta(A^*, B^*) &= A^*\theta(B^*) - B^*\theta(A^*) - \theta([A^*, B^*]) \\ &= -\theta([A, B]^*) = 0. \end{aligned}$$

(ii) *Both X, Y are horizontal.* In this case, the claim just becomes the definition of Θ.

(iii) *X is vertical and Y is horizontal.* We assume that $X = A_u^*$ and $Y = H(\xi)_u$ for some $A \in \mathfrak{g}$, $\xi \in \mathbb{R}^n$. In this case, one has from Proposition 6.15 that

$$\begin{aligned} d\theta(A^*, H(\xi)) &= A^*\theta(H(\xi)) - H(\xi)\theta(A^*) - \theta([A^*, H(\xi)]) \\ &= 0 - \theta(H(A\xi)) = -A\xi. \end{aligned}$$

On the other hand,

$$(-\omega \wedge \theta + \Theta)(A^*, H(\xi)) = -\omega(A^*)\theta(H(\xi)) = -A\xi.$$

The desired claim thus follows. ☐

Theorem 6.7. *(i) (The first Bianchi's identity) $D\Theta = \Omega \wedge \theta$.*

(ii) (The second Bianchi's identity) $D\Omega = 0$.

Proof. The second Bianchi's identity was proven in Theorem 6.4 and we only prove (i). According to the structure equations, one has

$$\begin{aligned} d\Theta &= d\omega \wedge \theta - \omega \wedge d\theta \\ &= (\Omega - \omega \wedge \omega) \wedge \theta - \omega \wedge (-\omega \wedge \theta + \Theta) \\ &= \Omega \wedge \theta - \omega \wedge \Theta. \end{aligned}$$

It follows that

$$d\Theta(X, Y, Z) = (\Omega \wedge \theta)(X, Y, Z)$$

for any horizontal vector fields X, Y, Z. This implies $D\Theta = \Omega \wedge \theta$. ☐

6.9.4 Descendence to tangent bundle

In this subsection, we consider the equivalent formulation of the torsion and curvature forms on the tangent bundle. Let ∇ denote the connection (covariant derivative) on TM induced from the affine connection (cf. Definition 6.11). Firstly, Θ descends to a TM-valued 2-form on M which will be denoted as T (the torsion tensor). In addition, Ω descends to an $\mathrm{Ad}(\mathcal{F}(M))$-valued 2-form on M. There is an isomorphism $\mathrm{Ad}(\mathcal{F}(M)) \to \mathrm{End}(TM)$ defined by

$$[(u, A)] \mapsto [\ni v \mapsto u(A \cdot u^{-1}v)] \in \mathrm{End}(T_x M) \quad (u \in \pi^{-1}(x)).$$

This allows one to view Ω as an element in $\Omega^2(M, \mathrm{End}(TM))$ which will be denoted as R (the curvature tensor). The tensors T and R have the following more familiar expressions in Riemannian geometry.

Lemma 6.8. *One has*

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{6.45}$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{6.46}$$

for all $X, Y, Z \in \Gamma(TM)$.

Proof. We only prove (6.46) and leave the other one as an exercise. Fix $u \in \mathcal{F}(M)$. Let X^*, Y^*, Z^* be the horizontal lifts of X, Y, Z respectively. Let $f \triangleq \theta(Z^*)$ be the $\mathbb{R}^n$-valued function on P corresponding to Z. According to Lemma 6.4,

$$\begin{aligned}&(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z)_x \\ &= u\big(X_u^*(Y^* f) - Y_u^*(X^* f) - h[X^*, Y^*]_u f\big) \\ &= u\big(v[X^*, Y^*]_u f\big),\end{aligned}$$

where $v[X^*, Y^*]$ denotes the vertical component of $[X^*, Y^*]$. Let $A \in \mathfrak{g}$ be the element such that $A_u^* = v[X^*, Y^*]_u$. Since f satisfies $f(ug) = g^{-1} f(u)$, the same reasoning leading to (6.27) shows that

$$v[X^*, Y^*]_u f = -Af(u).$$

Therefore,

$$(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z)_x = -u\big(Af(u)\big). \tag{6.47}$$

On the other hand, one has from (6.34) that

$$\Omega(X^*, Y^*)_u = -\omega([X^*, Y^*])_u = -A.$$

It follows that

$$(R(X,Y)Z)_x = u\big(\Omega(X^*, Y^*)_u f(u)\big) = -u(Af(u)). \tag{6.48}$$

Combining (6.47) and (6.48), the relation (6.46) thus follows. ☐

The following lemma is an almost immediate application of Lemma 6.4. It is useful for deriving Bianchi's identities for T and R.

Lemma 6.9. *Fix* $u \in \pi^{-1}(x)$ *and let* $X, Y, Z, W \in T_x M$. *Let* X^*, Y^*, Z^* *be the standard horizontal vector fields corresponding to the vectors* $u^{-1}X$, $u^{-1}Y$, $u^{-1}Z$ *respectively. Then the following two relations hold at* u:

$$(\nabla_X T)(Y,Z) = u\big(X^*(\Theta(Y^*, Z^*))\big), \tag{6.49}$$

$$(\nabla_X R)(Y,Z)W = u\big(X^*(\Omega(Y^*, Z^*)) \cdot u^{-1} W\big).$$

Proof. We only prove (6.49) as the other one follows from similar reasoning. By viewing $T \in \Gamma(T_2^1 M)$, let

$$f : \mathcal{F}(M) \to \mathbb{R}^n \otimes (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*, \ f_u \triangleq u^{-1}(T_{\pi(u)})$$

be the function corresponding to T. Let $\eta \triangleq u^{-1}Y$ and $\zeta \triangleq u^{-1}Z$. According to Lemma 6.4, one has

$$(\nabla_X T)(Y, Z) = u\big((X_u^* f)(\eta, \zeta)\big) \quad \text{at } u. \tag{6.50}$$

On the other hand, let $u : (-\varepsilon, \varepsilon) \to \mathcal{F}(M)$ be a curve satisfying $u_0 = u$ and $\dot{u}_0 = X_u^*$. By the definition of f,

$$X_u^*(\Theta(Y^*, Z^*)) = \frac{d}{dt}\Big|_{t=0} \Theta(Y^*, Z^*)_{u_t} = \frac{d}{dt}\Big|_{t=0} f_{u_t}\big(u_t^{-1}(d\pi)Y_{u_t}^*, u_t^{-1}(d\pi)Z_{u_t}^*\big).$$

Since Y^*, Z^* are standard horizontal vector fields, one has

$$u_t^{-1}(d\pi)Y_{u_t}^* = \eta, \ u_t^{-1}(d\pi)Z_{u_t}^* = \zeta$$

for all t. As a result,

$$X_u^*(\Theta(Y^*, Z^*)) = \frac{d}{dt}\Big|_{t=0} f_{u_t}(\eta, \zeta) = (X_u^* f)(\eta, \zeta).$$

It follows that (6.50) that

$$u\big(X_u^*(\Theta(Y^*, Z^*)_u)\big) = (\nabla_X T)(Y, Z) \quad \text{at } u,$$

which yields the relation (6.49). □

We now give the equivalent formulation of Theorem 6.7 in terms of the tensors T and R. It describes the basic symmetries of the curvature tensor R. We use $\mathfrak{C}(\cdots)$ to denote a cyclic symmetrisation, i.e.

$$\mathfrak{C}(\Phi(X, Y, Z)) \triangleq \Phi(X, Y, Z) + \Phi(Y, Z, X) + \Phi(Z, X, Y).$$

Theorem 6.8. *(i) (The first Bianchi's identity)* $\mathfrak{C}(R(X, Y)Z) = \mathfrak{C}((\nabla_X T)(Y, Z) + T(T(X, Y), Z))$.

(ii) (The second Bianchi's identity) $\mathfrak{C}((\nabla_X R)(Y, Z) + R(T(X, Y), Z)) = 0$.

In particular, if the connection is torsion free (i.e. $T \equiv 0$*), one has*

$$\mathfrak{C}(R(X, Y)Z) = 0, \ \mathfrak{C}((\nabla_X R)(Y, Z)) = 0$$

for all $X, Y, Z \in \Gamma(TM)$.

Proof. The result is just the descendence of Theorem 6.7 to the tangent bundle. We only prove the first identity. Recall from Theorem 6.7 (i) that $D\Theta = \Omega \wedge \theta$. Fix $u \in \mathcal{F}(M)$ and let $X, Y, Z \in T_{\pi(u)}M$. Let X^*, Y^*, Z^* be the standard horizontal vector fields corresponding to the vectors $u^{-1}X$, $u^{-1}Y$, $u^{-1}Z$ respectively. On the one hand,

$$\begin{aligned} D\Theta(X_u^*, Y_u^*, Z_u^*) &= \mathfrak{C}\big(X_u^*(\Theta(Y^*, Z^*)) - \Theta([X^*, Y^*], Z^*)_u\big) \\ &= u^{-1}\Big[\mathfrak{C}\big((\nabla_X T)(Y, Z) - T((d\pi)_u[X^*, Y^*]_u, Z)\big)\Big] \\ &= u^{-1}\Big[\mathfrak{C}\big((\nabla_X T)(Y, Z) + T(T(X, Y), Z)\big)\Big]. \end{aligned} \tag{6.51}$$

The second equality follows from (6.49). The third equality follows from the relation

$$(d\pi)_u[X^*, Y^*]_u = -T(X, Y),$$

which is just a restatement of (6.44). On the other hand,

$$\begin{aligned} (\Omega \wedge \theta)(X_u^*, Y_u^*, Z_u^*) &= \mathfrak{C}\big(\Omega(X^*, Y^*)_u \theta(Z^*)_u\big) \\ &= u^{-1}\Big[\mathfrak{C}\big(R(X, Y)Z\big)_u\Big]. \end{aligned} \tag{6.52}$$

Equating (6.51) and (6.52) gives the first Bianchi's identity. □

An important geometric concept associated with an affine connection is the notion of geodesics.

Definition 6.23. A smooth curve $\gamma : I \to M$ is called a *geodesic* with respect to the connection ∇ if $\dot{\gamma}_t$ is parallel along γ, i.e. $\nabla_{\dot{\gamma}_t}\dot{\gamma}_t = 0$ for all $t \in I$.

The following result provides a neat characterisation of geodesics from the principal bundle perspective.

Proposition 6.16. *A smooth curve in M is a geodesic if and only if it is the projection of an integral curve of a standard horizontal vector field in P.*

Proof. Let γ_t be a smooth curve in M and let u_t be a horizontal lift of γ_t. Define $\xi_t \triangleq u_t^{-1}(\dot{\gamma}_t) \in \mathbb{R}^n$. According to Corollary 6.2, one has

$$\nabla_{\dot{\gamma}_t}\dot{\gamma}_t = u_t(\dot{\xi}_t) \tag{6.53}$$

for all t.

Now suppose that γ_t is a geodesic. One knows from (6.53) that $\dot{\xi}_t = 0$ and thus ξ_t is constant (say ξ). In other words, $\dot{u}_t$ is the horizontal lift of $u_t(\xi)$ for all t. By definition, this shows that u_t is the integral curve of the standard horizontal vector field $H(\xi)$. Conversely, if u_t is such an integral curve, ξ_t must be constant and one concludes from (6.53) that $\nabla_{\dot{\gamma}_t}\dot{\gamma}_t = 0$. Therefore, γ_t is a geodesic. □

Proposition 6.16 immediately gives the existence and uniqueness of geodesics.

Corollary 6.3. *Let $x \in M$ and $v \in T_xM$. There exists a unique geodesic γ defined on a maximal interval I containing the origin, such that $\gamma_0 = x$ and $\dot{\gamma}_0 = v$.*

Proof. Fix $u \in \pi^{-1}(x)$ and set $\xi \triangleq u^{-1}(v)$. According to Proposition 6.16, the claim is equivalent to the existence and uniqueness of the integral curve $u : I \to \mathcal{F}(M)$ of $H(\xi)$ with $u_0 = u$. The latter follows from the standard existence and uniqueness theorem for first order ODEs (the integral curve is defined by the ODE $\dot{u}_t = H(\xi)_{u_t}$ with $u_0 = u$). It is easy to see that the geodesic is independent of the choice of u (of course, the vector ξ will change accordingly). □

An affine connection is said to be *complete* if all geodesics are defined on $(-\infty, \infty)$. According to Corollary 6.3, this is equivalent to saying that $H(\xi)$ is complete on $\mathcal{F}(M)$ (i.e. integral curves are all well-defined on $(-\infty, \infty)$) for all $\xi \in \mathbb{R}^n$.

6.9.5 The Christoffel symbols

In this subsection, we compute the torsion and curvature tensors under a local coordinate chart. The key ingredient is the expression of the connection form ω in terms of local coordinates.

Let $(U; x^i)$ be a local chart on M. This provides a local parametrisation of $\mathcal{F}(M)$ by defining

$$\pi^{-1}(U) \ni u \mapsto (x^i, X^j_k) \in U \times \mathrm{GL}(n; \mathbb{R}),$$

where (x^i) are the coordinates of $\pi(u)$ and (X^j_k) is the matrix of the basis $u = (x; X_1, \cdots, X_n)$ with respect to $\{\partial_j\}$, i.e. $X_k = X^j_k \partial_j$. Recall that $A_U \triangleq \sigma_U^* \omega \in \Omega^1(U) \otimes \mathfrak{gl}(n; \mathbb{R})$ is the local connection matrix of ω on U, where σ_U is the identity section on U. Let $\{E^i_j\}$ be the canonical basis of $\mathfrak{gl}(n; \mathbb{R})$ consisting of the elementary matrices, one can now write

$$A_U = \Gamma^i_{jk} dx^j \otimes E^k_i$$

for some $\Gamma^i_{jk} \in C^\infty(U)$.

Definition 6.24. The functions Γ^i_{jk} are called the *Christoffel symbols* of the connection ω with respect to the local chart $(U; x^i)$.

The following result gives the local expression of ω in terms of the Christoffel symbols.

Lemma 6.10. *One has*

$$\omega^i_j = Y^i_k dX^k_j + Y^i_k X^l_j (\pi^* \Gamma^k_{ml}) dx^m,$$

where the matrix (Y^i_j) is the inverse of (X^i_j).

Proof. This is a direct corollary of the formula (6.11). □

Corollary 6.4. *The horizontal lift of* ∂_m *is given by*

$$X_m^* = \partial_m - (\pi^*\Gamma_{ml}^k)\frac{\partial}{\partial X_j^k}X_j^l.$$

Proof. Since $(d\pi)X_m^* = \partial_m$, one can write

$$X_m^* = \partial_m + b_j^k \frac{\partial}{\partial X_j^k}$$

with some $b_j^k \in C^\infty(\pi^{-1}(U))$. By applying Lemma 6.10 to the relation $\omega(X_m^*) = 0$, one finds that $b_j^k = -(\pi^*\Gamma_{ml}^k)X_j^l$. □

We now derive the local formula for the connection ∇ on TM, which yields a more familiar characterisation of the Christoffel symbols.

Proposition 6.17. $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\partial_k$.

Proof. According to Lemma 6.4,

$$\nabla_{\partial_i}\partial_j = u(X_i^* f_j),$$

where $f_j(u) \triangleq u^{-1}(\partial_j)$ is the $\mathbb{R}^n$-valued function on $\pi^{-1}(U)$ corresponding to ∂_j and X_i^* is the horizontal lift of ∂_i. It is straight forward to see that $f_j(u) = (Y_j^p)_{1\leqslant p\leqslant n}$, where (Y_j^i) is the inverse of (X_j^i). In addition, one has from Corollary 6.4 that

$$X_i^* f^p = \big(\partial_i - (\pi^*\Gamma_{il}^k)X_j^l \frac{\partial}{\partial X_j^k}\big)Y_j^p = \Gamma_{ij}^k Y_k^p.$$

Since $(Y_k^p)_{1\leqslant p\leqslant n}$ are the coordinates of ∂_k with respect to u, it follows that

$$u(X_i^* f_j) = \Gamma_{ij}^k\partial_k.$$

This yields the claimed identity. □

As an immediate application, we derive the local formulae for the tensors T, R as well as the geodesic equation.

Proposition 6.18. *(i) One has*

$$T(\partial_j, \partial_k) = T_{jk}^i\partial_i,\ \ R(\partial_k, \partial_l)\partial_j = R_{jkl}^i\partial_i,$$

where

$$T_{jk}^i = \Gamma_{jk}^i - \Gamma_{kj}^i,$$

$$R^i_{jkl} = \frac{\partial \Gamma^i_{lj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^l} + \Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm}.$$

(ii) A curve $\gamma_t = (\gamma^i_t)_{1\leqslant i\leqslant n}$ *(*$t \in I$*) in* U *is a geodesic if and only if*

$$\frac{d^2\gamma^i_t}{dt^2} + \Gamma^i_{jk}(\gamma_t)\dot{\gamma}^j_t\dot{\gamma}^k_t = 0$$

for all $t \in I$ *and* $1 \leqslant i \leqslant n$.

Proof. This follows immediately from Proposition 6.17 and Lemma 6.8. □

6.10 Riemannian connections

We now further restrict ourselves to the Riemannian context. In what follows, let (M, g) be an n-dimensional Riemannian manifold.

Definition 6.25. An affine connection ∇ on TM is said to be *a metric connection* (or simply *metric*) if

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

for all $X, Y, Z \in \Gamma(TM)$.

The Riemannian structure allows one to introduce the *orthonormal frame bundle*

$$\mathrm{O}(M) \triangleq \{(x; X_1, \cdots, X_n) : \{X_1, \cdots, X_n\} \text{ is an ONB of } T_xM\}.$$

Recall that $\mathrm{O}(M)$ is a principal $\mathrm{O}(n)$-bundle over M. In addition, the inclusion $f : \mathrm{O}(M) \hookrightarrow \mathcal{F}(M)$ defines a principal bundle embedding.

A connection Γ' on $\mathrm{O}(M)$ induces a connection Γ on $\mathcal{F}(M)$ in the following way. Let $u \in \mathcal{F}(M)$. Choose an ONB u' of $T_{\pi(u)}M$ and let $g \in \mathrm{GL}(n;\mathbb{R})$ be the matrix such that $u = u'g$. We define the horizontal subspace $\mathcal{H}_u \subseteq T_u\mathcal{F}(M)$ by

$$\mathcal{H}_u \triangleq (dR_g)_{u'}(df)_{u'}(\mathcal{H}'_{u'}),$$

where $\mathcal{H}'_{u'} \subseteq T_{u'}\mathrm{O}(M)$ is the horizontal subspace defined by the connection Γ'. One can check that $\mathcal{H}_u$ is independent of the choice of u'. The assignment $u \mapsto \mathcal{H}_u$ defines a connection Γ on $\mathcal{F}(M)$ in the sense of Definition 6.8.

Proposition 6.19. *Let* Γ *be an affine connection on* M *with induced connection* ∇ *on* TM*. The following statements are equivalent.*

(i) ∇ *is metric.*

(ii) The metric tensor g *is parallel, i.e.* $\nabla g = 0$ *(recall from Example 6.8 for the definition).*

(iii) Parallel transports in TM *are isometries.*

(iv) The connection Γ *is induced from a connection* Γ' *on* $\mathrm{O}(M)$*.*

Proof. (i) $\iff$ (ii). This follows from the Leibniz rule for the induced connection on tensor fields (cf. Example 6.8):

$$(\nabla_X g)(Y,Z) = X\langle Y,Z\rangle - \langle \nabla_X Y, Z\rangle - \langle Y, \nabla_X Z\rangle$$

for all $X,Y,Z \in \Gamma(TM)$.

(ii) $\iff$ (iii). Let $\gamma : I \to M$ be a smooth curve and let Y_t, Z_t be parallel vector fields along γ. By the definition of ∇g, one has

$$(\nabla_{\dot{\gamma}_t} g)(Y_t, Z_t) = \lim_{h\to 0} \frac{1}{h}\big(\langle Y_{t+h}, Z_{t+h}\rangle - \langle Y_t, Z_t\rangle\big). \tag{6.54}$$

Suppose that g is parallel. One obtains from (6.54) that the function $\varphi(t) \triangleq \langle Y_t, Z_t\rangle$ has zero derivative. Therefore, $\langle Y_t, Z_t\rangle$ is constant in t. This shows that the parallel transport along γ is an isometry. Conversely, suppose that parallel transports are isometries. It follows that the LHS of (6.54) is identically zero for all curves γ_t and all parallel vector fields Y_t, Z_t along γ_t. This clearly implies $\nabla g = 0$.

(iii) $\iff$ (iv). Let $\gamma : [0,1] \to M$ be a smooth curve and fix $Y, Z \in T_{\gamma_0}M$. Let $u' \in \pi^{-1}(\gamma_0) \cap \mathrm{O}(M)$ and let $(u_t)_{0\leqslant t\leqslant 1}$ be the horizontal lift of γ in $\mathcal{F}(M)$ starting at u'. Recall from Lemma 6.3 that the parallel transport of Y (respectively, Z) along γ is defined by $Y_t \triangleq u_t\xi$ (respectively, $Z_t \triangleq u_t\eta$), where $\xi \triangleq (u')^{-1}Y$ (respectively, $\eta \triangleq (u')^{-1}Z$). Since $u' \in \mathrm{O}(M)$, it is clear that $\langle Y, Z\rangle = \langle \xi, \eta\rangle_{\mathbb{R}^n}$.

Suppose that Γ is induced from a connection Γ' on $\mathrm{O}(M)$. Then one has $u_t \in \mathrm{O}(M)$ for all t. As a result,

$$\langle Y_t, Z_t\rangle = \langle u_t\xi, u_t\eta\rangle = \langle \xi, \eta\rangle_{\mathbb{R}^n} = \langle Y, Z\rangle.$$

This shows that the parallel transport along γ is an isometry. Conversely, suppose that parallel transports are isometries. We define a connection on $\mathrm{O}(M)$ by reversing the above consideration. More specifically, let $u' \in \mathrm{O}(M)$. Given any smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma_0 = \pi(u')$, let u_t be the horizontal lift of γ in $\mathcal{F}(M)$ starting at u'. By assumption,

$$\langle u_t\xi, u_t\eta\rangle = \langle u'\xi, u'\eta\rangle = \langle \xi, \eta\rangle_{\mathbb{R}^n}$$

for all t and all $\xi, \eta \in \mathbb{R}^n$. This implies that u_t must be an ONB of $T_{\gamma_t}M$, i.e. $u_t \in \mathrm{O}(M)$. We now define $\mathcal{H}'_{u'}$ to be the collection of vectors $\dot{u}_0 \in T_{u'}\mathrm{O}(M)$ which are obtained in the above manner (by varying γ in an arbitrary way). This defines an assignment $\mathrm{O}(M) \ni u' \mapsto \mathcal{H}'_{u'}$ which gives rise to a connection Γ' on $\mathrm{O}(M)$. It is plain to check that Γ is induced from Γ'. □

Remark 6.11. Proposition 6.19 naturally extends to general vector bundles. Let $E = P \times_\rho \mathbb{R}^n$ be an associated vector bundle of a principal G-bundle P and a representation $\rho : G \to \mathrm{GL}(\mathbb{R}^n)$. Suppose that E admits an Euclidean metric. Then the structure group G can be reduced to the subgroup

$$H \triangleq \{g \in G : \rho(g) \in \mathrm{O}(n)\}.$$

The principal H-bundle

$$Q \triangleq \left\{u \in P : \langle u(\xi), u(\eta)\rangle_{E_{\pi(u)}} = \langle \xi, \eta\rangle_{\mathbb{R}^n} \text{ for all } \xi, \eta \in \mathbb{R}^n\right\}$$

is an H-reduced bundle of P (cf. Definition 6.5). The G-structure of E is also reducible to H, i.e. $E \cong Q \times_\rho \mathbb{R}^n$. Now let Γ be a connection on P. Parallel to the proof of Proposition 6.19, Γ is *metric* iff parallel transports in E determined by Γ are isometries and iff Γ is induced from a connection on Q (in the same manner as in the tangent bundle case). Similar discussion holds in the Hermitian setting with $\mathbb{R}^n$ replaced by $\mathbb{C}^n$ and $\mathrm{O}(n)$ replaced by $\mathrm{U}(n)$.

Let Γ be a connection on $\mathrm{O}(M)$. The connection form of Γ is now an $\mathfrak{so}(n)$-valued 1-form ω on $\mathrm{O}(M)$. All previous results on $\mathcal{F}(M)$ (canonical form, torsion and curvature forms, structure equations, Bianchi's identities) descend to $\mathrm{O}(M)$ in the obvious way by replacing the structure group $\mathrm{GL}(n;\mathbb{R})$ with $\mathrm{O}(n)$ and the Lie algebra $\mathfrak{gl}(n;\mathbb{R})$ with $\mathfrak{so}(n)$. In particular, the structure equations

$$d\theta = -\omega \wedge \theta + \Theta,\ d\omega = -\omega \wedge \omega + \Omega$$

hold in the same way as before.

In Riemannian geometry, it is particularly advantageous to work with a "special" connection which is on the one hand metric (i.e. coming from a connection on $\mathrm{O}(M)$) and on the other hand with vanishing torsion (so that the first structure equation simplifies to $d\theta = -\omega \wedge \theta$).

Definition 6.26. An affine connection on a Riemannian manifold M is a *Riemannian connection* if it is both metric and torsion free.

We conclude this chapter by establishing the following *fundamental theorem of Riemannian geometry.*

Theorem 6.9. *There exists a unique Riemannian connection on any Riemannian manifold M.*

Proof. Uniqueness. Let φ, ω be two connections on $\mathrm{O}(M)$ which are both torsion free. Since $\varphi(A^*) = \omega(A^*) = A$ for all $A \in \mathfrak{so}(n)$, the form $\varphi - \omega$ annihilates vertical vectors. As a result, one can write

$$\varphi^i_j - \omega^i_j = a^i_{jk}\theta^k$$

with some $a^i_{jk} \in C^\infty(\mathrm{O}(M))$. Since $\varphi - \omega$ takes values in $\mathfrak{so}(n)$, one has

$$a^i_{jk} = -a^j_{ik} \tag{6.55}$$

for all i, j, k. On the other hand, since both φ and ω are torsion free, the first structure equation yields that

$$0 = (\varphi^i_j - \omega^i_j) \wedge \theta^j = a^i_{jk}\theta^k \wedge \theta^j,$$

which further implies that

$$a^i_{jk} = a^i_{kj}. \tag{6.56}$$

It is an elementary algebra exercise to show that the relations (6.55) and (6.56) force $a^i_{jk} = 0$ for all i, j, k. Therefore, $\varphi = \omega$.

Existence. Let φ be any connection form on $\mathrm{O}(M)$ (the existence of a connection on a general principal bundle can be proved e.g. by using a partition of unity argument in a way similar to the existence of a Riemannian metric on any manifold). Let Θ be the torsion form of φ. Since Θ is horizontal, one can write

$$\Theta^i = \frac{1}{2}T^i_{jk}\theta^j \wedge \theta^k$$

where the functions $T^i_{jk} \in C^\infty(\mathrm{O}(M))$ satisfy $T^i_{jk} = -T^i_{kj}$. Consider the 1-forms defined by

$$\tau^i_j \triangleq \frac{1}{2}(T^k_{ij} + T^j_{ik} - T^i_{jk})\theta^k, \quad 1 \leqslant i, j \leqslant n.$$

Note that $\tau \triangleq (\tau^i_j)_{1\leqslant i,j\leqslant n}$ is $\mathfrak{so}(n)$-valued and satisfies $\Theta = -\tau \wedge \theta$. It is plain to check that the $\mathfrak{so}(n)$-valued 1-form

$$\omega \triangleq \varphi + \tau$$

is a connection form on $\mathrm{O}(M)$ which satisfies $d\theta = -\omega \wedge \theta$. According to the first structure equation, one concludes that ω is torsion free. Therefore, ω defines a Riemannian connection. $\square$

Remark 6.12. Theorem 6.9 can also be proved from the (more standard) covariant derivative perspective. By using the formula (6.45) for the torsion tensor, the metric-compatibility and torsion free properties force $\nabla_X Y$ to satisfy

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle =& X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle \end{aligned} \tag{6.57}$$

for all $X, Y, Z \in \Gamma(TM)$. This formula uniquely determines the covariant derivative operator $(X, Y) \mapsto \nabla_X Y$, which proves existence and uniqueness at the same time. It is also immediate from the formula (6.57) and Proposition 6.17 that the Christoffel symbols of ∇ under a local chart $(U; x^i)$ are given by

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \Big(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \Big).$$

Here $g_{ij} \triangleq g(\partial_i, \partial_j)$ and (g^{ij}) is the inverse matrix of (g_{ij}). The approach we took before emphasises the principal bundle perspective.

Definition 6.27. The unique Riemannian connection given by Theorem 6.9 is known as the *Levi-Civita connection*.

Remark 6.13. If M is oriented, all the previous discussions further descend to the bundle $\mathrm{SO}(M)$ with structure group $\mathrm{SO}(n)$, which was the principal bundle set-up in Chapter 5.

7 The Chern-Weil theory

In this chapter, we discuss some basics of the Chern-Weil theory which is a fundamental technique in global differential geometry. This will provide deeper insight into the CGB formula (in particular, the construction of the Euler form $\mathcal{E}$). The main goal of this chapter is to provide an alternative proof of the CGB theorem from the perspective of the Chern-Weil theory.

The Chern-Weil construction provides a general way of constructing cohomology classes of a manifold M that are associated with principal bundles over M. To some extent, these cohomology classes measure the topological twisting (nontriviality) of the bundle. To be more specific, let G be a Lie group with Lie algebra $\mathfrak{g}$. The Chern-Weil construction requires the input of an adjoint invariant polynomial ϕ on $\mathfrak{g}$ and a principal G-bundle $(P, M, \pi; G)$. Its output is a de Rham cohomology class $[\phi(P)] \in H^*_{\mathrm{dR}}(M)$ over the base manifold M. The actual construction of the differential form $\phi(P)$ requires an additional piece of *geometric* data: a connection on P (more precisely, the curvature form of a connection). The remarkable point is that, although the form $\phi(P)$ does depend on the choice of the connection, its cohomology class does *not* and thus encodes topological information about the principal bundle P.

In the context of the CGB theorem, the construction of the form $\phi(P)$ (with suitable choices of ϕ and P) naturally yields the Euler form $\mathcal{E}$ which depends on the Riemannian curvature. Its cohomology class, however, coincides with the Thom class of the tangent bundle, whose integral yields the Euler characteristic. This is essentially the CGB formula.

Convention. In this chapter, we always work with the de Rham cohomology (over real coefficients unless otherwise stated). We will therefore only write $H^*(M)$ without the subscript "dR".

7.1 Adjoint invariant polynomials

We first discuss the notion of adjoint invariant polynomials. Let V be a real, finite dimensional vector space. A *symmetric, k-linear functional* over V is a map

$$\phi : \underbrace{V \times \cdots \times V}_{k} \to \mathbb{R}$$

which is linear in each variable and symmetric in all variables. The space of all such functionals is denoted as $S^k(V^*)$. Note that ϕ is uniquely determined by the functional

$$\tilde{\phi}(v) \triangleq \phi(v, \cdots, v)$$

through polarisation:

$$\phi(v_1,\cdots,v_k)=\frac{1}{k!}\frac{\partial^k}{\partial t_1\cdots\partial t_k}\tilde{\phi}(t_1v_1+\cdots+t_kv_k).$$

If $\mathcal{A}$ is a real algebra with unit, any $\phi\in S^k(V^*)$ determines a unique k-linear map

$$\phi_{\mathcal{A}}:(\mathcal{A}\otimes V)\times\cdots\times(\mathcal{A}\otimes V)\to\mathcal{A}$$

which satisfies

$$\phi_{\mathcal{A}}(a_1\otimes v_1,\cdots,a_k\otimes v_k)\triangleq\phi(v_1,\cdots,v_k)a_1\cdots a_k,\quad a_i\in\mathcal{A},v_j\in V. \tag{7.1}$$

Note that $\phi_{\mathcal{A}}$ is also symmetric if $\mathcal{A}$ is commutative.

The main interesting situation is when V is a Lie algebra. Let G be a given matrix Lie group with Lie algebra $\mathfrak{g}$. A symmetric, k-linear functional $\phi\in S^k(\mathfrak{g}^*)$ is said to be Ad-*invariant* if

$$\phi(gX_1g^{-1},\cdots,gX_kg^{-1})=\phi(X_1,\cdots,X_k)$$

holds for all $g\in G$ and $X_j\in\mathfrak{g}$. The space of such functionals is denoted as $I^k(G)$. We introduce the spaces

$$I(G)\triangleq\bigoplus_{k=0}^{\infty}I^k(G),\ I[G]\triangleq\prod_{k=0}^{\infty}I^k(G).$$

Elements in $I(G)$ (respectively, in $I[G]$) are called Ad-*invariant polynomials* (respectively, Ad-*invariant formal power series*).

Example 7.1. Let $G=\mathrm{GL}(n;\mathbb{C})$ with $\mathfrak{g}=\mathfrak{gl}(n;\mathbb{C})$. The function $\mathrm{Tr}\exp:\mathfrak{g}\to\mathbb{C}$ can be viewed as an element in $I[G]$. In fact, its k-th level component is the functional

$$\tilde{\phi}_k(X)\triangleq\frac{1}{k!}\mathrm{tr}[X^k];$$

more precisely, the corresponding symmetric, k-linear functional is given by

$$\phi_k(X_1,\cdots,X_k)=\frac{1}{(k!)^2}\sum_{\sigma\in\mathcal{S}_k}\mathrm{tr}\big[X_{\sigma(1)}\cdots X_{\sigma(k)}\big].$$

It is clear that ϕ_k is Ad-invariant.

Example 7.2. Let $G=\mathbb{T}^r=\mathrm{U}(1)\times\cdots\times\mathrm{U}(1)$ be the r-torus. Its Lie algebra is $\mathfrak{g}=\mathfrak{u}(1)\times\cdots\times\mathfrak{u}(1)$. Under the parametrisation

$$(\theta_1,\cdots,\theta_r)\in\mathbb{R}^r\mapsto(i\theta_1,\cdots,i\theta_r)\in\mathfrak{g},$$

any functional on $\mathfrak{g}$ can be viewed as a function of the real variables $(\theta_1,\cdots,\theta_r)$. Since G is abelian, the Ad-invariance property becomes trivial. As a result, $I(G)\cong\mathbb{R}[\theta_1,\cdots,\theta_r]$ (the $\mathbb{R}$-algebra of polynomials in r commutative variables).

Example 7.3. Let $G = \mathrm{SO}(2m)$ with $\mathfrak{g} = \mathfrak{so}(2m)$. Any $X \in \mathfrak{g}$ is $\mathrm{SO}(2m)$-conjugate to a block diagonal form

$$\lambda_1 J \oplus \cdots \oplus \lambda_m J$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\lambda_1, \cdots, \lambda_m \in \mathbb{R}$. The polynomial

$$\mathfrak{g} \ni X \mapsto \lambda_1 \cdots \lambda_m \in \mathbb{R}$$

defines an element in $I^m(G)$. As we will see later on, this polynomial will play an essential role in the Chern-Weil proof of the CGB theorem (the Euler form will be constructed from this polynomial).

7.2 The Chern-Weil construction

We now explain how one can use Ad-invariant polynomials to construct cohomology classes associated with principal bundles. In what follows, we fix a matrix Lie group G with Lie algebra $\mathfrak{g}$.

Let P be a principal G-bundle over a manifold M. Let $\omega \in \mathfrak{C}(P)$ be a connection on P with curvature $F_\omega \in \Omega^2(M, \mathrm{Ad}P)$. Under a local trivialisation $(\mathcal{U}, g_{\bullet\bullet})$, the curvature F_ω admits the following local representation (see (6.36)):

$$F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha = dA_\alpha + \frac{1}{2}[A_\alpha \wedge A_\alpha] \in \Omega^2(U_\alpha) \otimes \mathfrak{g},$$

where $\{A_\alpha\}$ is the local representation of ω. Let $\phi \in I^k(G)$ be a given fixed Ad-invariant polynomial. By abuse of notation, we define

$$\phi(F_\alpha) \triangleq \phi(F_\alpha, \cdots, F_\alpha) \in \Omega^{2k}(U_\alpha)$$

according to (7.1) with $\mathcal{A} \triangleq \Omega^*(U_\alpha)$ and $V = \mathfrak{g}$. Recall from (6.37) that

$$F_\beta = g_{\beta\alpha} \cdot F_\alpha \cdot g_{\beta\alpha}^{-1} \quad \text{on } U_{\alpha\beta} = U_\alpha \cap U_\beta.$$

Due to the Ad-invariance of ϕ, one has $\phi(F_\beta) = \phi(F_\alpha)$ on the intersection $U_{\alpha\beta}$. As a consequence, the local forms $\{\phi(F_\alpha)\}$ patch to a globally well-defined form $\phi(F_\omega) \in \Omega^{2k}(M)$. Since the local expressions of the curvature form under different trivialisations are also Ad-related (see Lemma 6.6), it follows that $\phi(F_\omega)$ does not depend on the particular choice of local trivialisation. The following result lies at the heart of the Chern-Weil theory.

Theorem 7.1 (The Chern-Weil Theorem)**.** *(i) The form $\phi(F_\omega)$ is closed.*

(ii) Suppose that $\omega^0, \omega^1 \in \mathfrak{C}(P)$. Then $\phi(F_{\omega^1}) - \phi(F_{\omega^0})$ is an exact form. In particular, the cohomology class $[\phi(F_\omega)] \in H^{2k}(M)$ is independent of the connection $\omega \in \mathfrak{C}(P)$.

We first state a simple observation that is particularly useful for proving the Chern-Weil theorem.

Lemma 7.1. *(i) For any* $X, X_1, \cdots, X_k \in \mathfrak{g}$, *one has*

$$\phi([X, X_1], X_2, \cdots, X_k) + \cdots + \phi(X_1, X_2, \cdots, [X, X_k]) = 0.$$

(ii) Let $F_1, \cdots, F_{k-1} \in \Omega^{even}(U) \otimes \mathfrak{g}$ *and* $A, F_k \in \Omega^*(U) \otimes \mathfrak{g}$. *Then*

$$\sum_{i=1}^{k} \phi(F_1, \cdots, F_{i-1}, [A \wedge F_i], F_{i+1}, \cdots, F_k) = 0.$$

Proof. (i) This follows from the relation

$$\frac{d}{dt}\Big|_{t=0} \phi\big(e^{tX} X_1 e^{-tX}, \cdots, e^{tX} X_k e^{-tX}\big) = 0,$$

which is a direct consequence of the Ad-invariance of ϕ.

(ii) Due to multilinearity, let us assume that

$$F_i = \alpha_i \otimes X_i, \ \ A = \alpha \otimes X.$$

One has

$$\begin{aligned}&\phi(F_1, \cdots, F_{i-1}, [A \wedge F_i], F_{i+1}, \cdots, F_k) \\ &= (-1)^{d(d_1 + \cdots + d_{i-1})} \alpha \wedge \alpha_1 \wedge \cdots \wedge \alpha_k \phi(X_1, \cdots, [X, X_i], \cdots, X_k),\end{aligned}$$

where $d \triangleq \deg\alpha$ and $d_i \triangleq \deg\alpha_i$. Since $F_1, \cdots, F_{k-1}$ are all even, the sign on the RHS of the above identity is positive. The result thus follows from Part (i) after summing over i. □

Proof of the Chern-Weil theorem. (i) According to Bianchi's identity (6.38), one has

$$\begin{aligned} d\phi(F_\alpha) &= \phi(dF_\alpha, F_\alpha, \cdots, F_\alpha) + \cdots + \phi(F_\alpha, \cdots, F_\alpha, dF_\alpha) \\ &= -\big[\phi([A_\alpha \wedge F_\alpha], F_\alpha, \cdots, F_\alpha) + \cdots + \phi(F_\alpha, \cdots, F_\alpha, [A_\alpha \wedge F_\alpha])\big]. \end{aligned}$$

Since F_α is even, the above expression is zero as a consequence of Lemma 7.1 (ii).

(ii) We consider the connection A^t defined by

$$A_\alpha^t \triangleq A_\alpha^0 + tC_\alpha,$$

where $C_\alpha \triangleq A_\alpha^1 - A_\alpha^0$. Note that the local forms$\{C_\alpha\}$ patch to a global element in $\Omega^1(\mathrm{Ad}(P))$. By explicit calculation, one has

$$F_\alpha^t = F_\alpha^0 + t(dC_\alpha + [A_\alpha^0 \wedge C_\alpha]) + \frac{t^2}{2}[C_\alpha \wedge C_\alpha] \tag{7.2}$$

$$\dot{F}^t_\alpha = dC_\alpha + [A^0_\alpha \wedge C_\alpha] + [tC_\alpha \wedge C_\alpha] = dC_\alpha + [A^t_\alpha \wedge C_\alpha].$$

It follows from the symmetry of ϕ that

$$\begin{aligned}\phi(F^1_\alpha) - \phi(F^0_\alpha) &= k\int_0^1 \phi(\dot{F}^t_\alpha, F^t_\alpha, \cdots, F^t_\alpha)dt \\ &= k\int_0^1 \phi(dC_\alpha + [A^t_\alpha \wedge C_\alpha], F^t_\alpha, \cdots, F^t_\alpha)dt.\end{aligned}$$

Our aim is to show that the LHS is exact.

Indeed, we claim that

$$\phi(dC_\alpha + [A^t_\alpha \wedge C_\alpha], F^t_\alpha, \cdots, F^t_\alpha) = d\phi(C_\alpha, F^t_\alpha, \cdots, F^t_\alpha). \tag{7.3}$$

To see this, one computes by using Bianchi's identity that

$$\begin{aligned}&d\phi(C_\alpha, F^t_\alpha, \cdots, F^t_\alpha) = d\phi(F^t_\alpha, \cdots, F^t_\alpha, C_\alpha) \\ &= \big[\phi(-[A^t_\alpha \wedge F^t_\alpha], F^t_\alpha, \cdots, F^t_\alpha, C_\alpha) + \cdots + \phi(F^t_\alpha, \cdots, F^t_\alpha, -[A^t_\alpha \wedge F^t_\alpha], C_\alpha) \\ &\quad + \phi(F^t_\alpha, \cdots, F^t_\alpha, -[A^t_\alpha \wedge C_\alpha])\big] + \phi(F^t_\alpha, \cdots, F^t_\alpha, dC_\alpha + [A^t_\alpha \wedge C_\alpha]).\end{aligned} \tag{7.4}$$

Observe that the summation inside the bracket $[\cdots]$ on the RHS of (7.4) is zero, which is again a consequence of Lemma 7.1 (ii). This yields the relation (7.3). It follows that

$$\phi(F^1_\alpha) - \phi(F^0_\alpha) = d\Big[k\int_0^1 \phi(C_\alpha, F^t_\alpha, \cdots, F^t_\alpha)dt\Big].$$

The local $(2k-1)$-forms $\{\tau_\alpha\}$ defined by

$$\tau_\alpha \triangleq k\int_0^1 \phi(C_\alpha, F^t_\alpha, \cdots, F^t_\alpha)dt \tag{7.5}$$

patch to a global $(2k-1)$-form on M since

$$F^t_\beta = g_{\beta\alpha}F^t_\alpha g^{-1}_{\beta\alpha},\ C_\beta = g_{\beta\alpha}C_\alpha g^{-1}_{\beta\alpha}.$$

Therefore, $\phi(F^0_\alpha)$ and $\phi(F^1_\alpha)$ define the same cohomology class.

□

Remark 7.1. The $(2k-1)$-form τ defined by (7.5) is called the (ϕ-)*transgression* from ω^0 to ω^1 and it is denoted as $T_\phi(\omega^0, \omega^1)$. The formula

$$\phi(F_{\omega^1}) - \phi(F_{\omega^0}) = dT_\phi(\omega^0, \omega^1) \tag{7.6}$$

is known as the *transgression formula*.

The cohomology class $[\phi(F_\omega)] \in H^{2k}(M)$, which is independent of the choice of the connection $\omega \in \mathfrak{C}(P)$, is denoted as $\phi(P)$. The following result shows that $\phi(P)$ is also invariant under principal bundle isomorphisms.

Proposition 7.1. *Suppose that* $\Phi : P \to Q$ *is an isomorphism of principal* G*-bundles over* M *(covering the identity* id_M*). Then* $\phi(P) = \phi(Q)$*.*

Proof. Suppose that P, Q admit local trivialisations $(\mathcal{U}, g_{\bullet\bullet}), (\mathcal{U}, h_{\bullet\bullet})$ respectively. The isomorphism Φ is locally represented by a family of maps $T_\alpha : U_\alpha \to G$ (see Proposition 6.1). Let $\omega \in \mathfrak{C}(P)$ with local representation $\{A_\alpha\}$. Then

$$B_\alpha \triangleq T_\alpha A_\alpha T_\alpha^{-1} - dT_\alpha \cdot T_\alpha^{-1}$$

defines a connection $\omega' \in \mathfrak{C}(Q)$. Their curvature forms are related by $(F_{\omega'})_\alpha = T_\alpha \cdot (F_\omega)_\alpha \cdot T_\alpha^{-1}$ (see Lemma 6.6). It follows from the Ad-invariance of ϕ that $\phi((F_{\omega'})_\alpha) = \phi((F_\omega)_\alpha)$. $\square$

To some extent, the nontriviality of the cohomology class $\phi(P)$ reflects the global nontriviality of the bundle P. This is suggested by the proposition below.

Proposition 7.2. *Let* $P = M \times G$ *be a trivial principal* G*-bundle. Then* $\phi(P) = 0$ *for all* $\phi \in I^k(G)$ *with* $k \geqslant 1$*.*

Proof. Consider the connection induced by the Maurer-Cartan form (see Example 6.10). Its curvature is flat, thus yielding a trivial cohomology class $\phi(P)$ for any $\phi \in I^k(G)$. $\square$

Let us summarise the Chern-Weil construction we discussed so far. Let G be a given matrix Lie group and let P be a principal G-bundle over M. Then there is the so-called *Chern-Weil correspondence*

$$\mathrm{CW}_P : I(G) \to H^*(M),\ \phi \mapsto \phi(P).$$

It is easily seen that CW_P is an algebra homomorphism. In addition, the map CW_P depends only on the isomorphism class of P and is in particular gauge invariant. The following proposition also shows the naturality of the Chern-Weil construction under pullbacks.

Proposition 7.3. *Let* $f : N \to M$ *be a smooth map and let* P *be a principal* G*-bundle over* M*. Then*

$$\phi(f^*P) = f^*\phi(P) \in H^*(N)$$

for any $\phi \in I(G)$*.*

Proof. Let $\{A_\alpha\}$ be a connection on P with curvature $\{F_\alpha\}$. Then $\{f^*A_\alpha\}$ is a connection on f^*P with curvature $\{f^*F_\alpha\}$. The result follows from the fact that $f^*\phi(F_\alpha) = \phi(f^*F_\alpha)$ (f^* is consistent with the wedge product). $\square$

7.3 Characteristic classes

In this section, we investigate a few concrete examples of the Chern-Weil construction which yield some important characteristic classes of vector bundles.

7.3.1 Chern classes

Consider $G = \mathrm{U}(r)$ ($r \times r$ unitary matrices) and $\mathfrak{g} = \mathfrak{u}(r)$ ($r \times r$ skew-Hermitian matrices). Recall from linear algebra that any $X \in \mathfrak{u}(r)$ is $\mathrm{U}(r)$-conjugate to a diagonal matrix of the form

$$\Lambda = \begin{pmatrix} i\lambda_1 & & \\ & \ddots & \\ & & i\lambda_r \end{pmatrix} \tag{7.7}$$

for some real numbers $\lambda_1, \cdots, \lambda_r$. In addition, permutations of the λ_k's can also be achieved by $\mathrm{U}(r)$-conjugations. Therefore, Ad-*invariant polynomials on* $\mathfrak{u}(r)$ *are precisely the symmetric polynomials of* $(\lambda_1, \cdots, \lambda_r)$. As a convention, we will use the variables

$$x_k \triangleq -\frac{1}{2\pi}\lambda_k, \quad k = 1, \cdots, r. \tag{7.8}$$

For $1 \leqslant k \leqslant r$, let $c_k : \mathfrak{u}(r) \to \mathbb{R}$ be the *k-th elementary symmetric polynomial*:

$$c_k(X) \triangleq \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant r} x_{i_1} \cdots x_{i_k},$$

where the x_k's are defined by (7.8) with the λ_k's coming from the diagonal form (7.7) of X. It is standard that the space of symmetric polynomials of $(x_1, \cdots, x_r)$ has a basis given by the elementary symmetric polynomials $c_1, \cdots, c_r$ (see [Wey39]). It follows that $I(G) \cong \mathbb{R}[c_1, \cdots, c_r]$.

The c_k's can be defined in a more unified manner using the following expansion:

$$\begin{aligned} c_t(X) &\triangleq \det\big(\mathrm{id} - \frac{t}{2\pi i}X\big) = \prod_{k=1}^{r}\big(1 - \frac{t\lambda_k}{2\pi}\big) \\ &= \prod_{k=1}^{r}(1 + tx_k) = \sum_{k=0}^{r} c_k(X)t^k. \end{aligned} \tag{7.9}$$

The polynomial $c_t(X) \in I(\mathrm{U}(r))[t]$ is called the *universal rank-r Chern polynomial*. The polynomials $\{c_1(X), \cdots, c_r(X)\}$ are called the *universal rank-r Chern classes*.

Now suppose that E is a complex vector bundle of rank r over a manifold M. Let E be equipped with a Hermitian metric so that it admits a $\mathrm{U}(r)$-structure (cf. Example 6.7). Let ∇ be a connection which is compatible with the given metric.

This is equivalent to a connection on the principal U(r)-bundle of unitary frames. Let F_∇ be the curvature of this connection. For each $t \in \mathbb{R}$, define the cohomology class

$$c_t(E) \triangleq [c_t(F_\nabla)] = \big[\det\big(\mathrm{id} - \frac{t}{2\pi i}F_\nabla\big)\big] \in H^*(M).$$

In view of (7.9), one can write

$$c_t(E) = \sum_{k=0}^{r} c_k(E)t^k$$

where $c_k(E) \triangleq [c_k(F_\nabla)]$ $(k = 1, \cdots, r)$. According to the Chern-Weil theorem, the cohomology classes $c_t(E), c_k(E)$ are independent of the connection ∇. They are also independent of the metric, since different Hermitian metrics yield isomorphic U(r)-structures (see Proposition 6.2 and Proposition 7.1). As a consequence, these cohomology classes contain topological information about the vector bundle E.

Definition 7.1. The polynomial $c_t(E) \in H^*(M)[t]$ is called the *Chern polynomial* of E. The cohomology class $c_k(E) \in H^{2k}(M)$ is called the *k-th Chern class* of E. As a convention, we also set $c_0(E) \triangleq \mathbf{1}$ (the unit of $H^*(M)$).

Proposition 7.4. *Suppose that $E = E_1 \oplus E_2$ where E_1, E_2 are complex vector bundles over M. Then*

$$c_t(E) = c_t(E_1)c_t(E_2),$$

where the multiplication is the cohomology wedge product. In particular, one has

$$c_k(E) = \sum_{j=0}^{k} c_j(E_1)c_{k-j}(E_2)$$

for every k.

Proof. Fix a Hermitian metric as well as a compatible connection ∇^i on E_i $(i = 1, 2)$. Consider the direct sum metric and the compatible connection $\nabla \triangleq \nabla^1 \oplus \nabla^2$ on E. It is apparent that $F_\nabla = F_{\nabla^1} \oplus F_{\nabla^2}$; in local matrix form, one has

$$(F_\nabla)_\alpha = \begin{pmatrix} (F_{\nabla^1})_\alpha & 0 \\ 0 & (F_{\nabla^2})_\alpha \end{pmatrix}.$$

As a consequence,

$$\begin{aligned} c_t(E) &= \big[\det\big(\mathrm{id} - \frac{t}{2\pi i}F_\nabla\big)\big] \\ &= \big[\det\big(\mathrm{id} - \frac{t}{2\pi i}F_{\nabla^1}\big)\det\big(\mathrm{id} - \frac{t}{2\pi i}F_{\nabla^2}\big)\big] = c_t(E_1)c_t(E_2). \end{aligned}$$

□

Remark 7.2. Although $\mathfrak{u}(r)$ consist of complex matrices, the Chern classes are real cohomology classes by definition.

7.3.2 Pontryagin classes

Consider $G = \mathrm{O}(r)$ ($r \times r$ real orthogonal matrices) and $\mathfrak{g} = \mathfrak{so}(r)$ ($r \times r$ real anti-symmetric matrices). Every $X \in \mathfrak{so}(r)$ is $\mathrm{O}(r)$-conjugate to the following block-diagonal form:

$$X \overset{\mathrm{O}(r)}{\sim} \begin{cases} \mathrm{diag}(\lambda_1 J, \cdots, \lambda_m J), & \text{if } r = 2m; \\ \mathrm{diag}(\lambda_1 J, \cdots, \lambda_m J, 0), & \text{if } r = 2m+1 \end{cases} \tag{7.10}$$

with certain real numbers $\lambda_1, \cdots, \lambda_m$, where

$$J \triangleq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Block permutations of the $\lambda_k J$'s and sign flips of any of the λ_k's can also be achieved by $\mathrm{O}(r)$-conjugations. As a consequence, Ad-*invariant polynomials are precisely the symmetric polynomials of* $(\lambda_1^2, \cdots, \lambda_m^2)$. Using the variables x_k introduced in (7.8), we define

$$p_k(X) \triangleq \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant m} (x_{i_1} \cdots x_{i_k})^2, \quad X \in \mathfrak{so}(r). \tag{7.11}$$

It follows that $I(\mathrm{O}(r)) \cong \mathbb{R}[p_1, \cdots, p_m]$. Similar to (7.9), the p_k's can be obtained as coefficients of the polynomial

$$p_t(X) \triangleq \det\big(\mathrm{id} - \frac{t}{2\pi} X\big) = \prod_{k=1}^{m} \big(1 + t x_k^2\big) = \sum_{k=0}^{m} p_k(X) t^{2k} \in I(\mathrm{O}(r))[t].$$

The polynomial $p_t(X) \in I(\mathrm{O}(r))[t]$ is called the *universal rank-r Pontryagin polynomial*. The polynomials $\{p_1(X), \cdots, p_m(X)\}$ are called the *universal rank-r Pontryagin classes*.

Now suppose that E is a real vector bundle of rank r over a manifold M. Similar to the complex case, let E be equipped with an Euclidean metric and let ∇ be a compatible connection with curvature F_∇. For each $t \in \mathbb{R}$, define the cohomology class

$$p_t(E) \triangleq [p_t(F_\nabla)] = \big[\det\big(\mathrm{id} - \frac{t}{2\pi} F_\nabla\big)\big] \in H^*(M).$$

One can also write

$$p_t(E) = \sum_{k=0}^{r} p_k(E) t^{2k}$$

where $p_k(E) \triangleq [p_k(F_\nabla)] \in H^{4k}(M)$ $(k = 1, \cdots, m)$. The cohomology classes $p_t(E), p_k(E)$ are independent of the connection ∇ and the metric.

Definition 7.2. The polynomial $p_t(E) \in H^*(M)[t]$ is called the *Pontryagin polynomial* of E. The cohomology class $p_k(E) \in H^{4k}(M)$ is called the *k-th Pontryagin class* of E. As a convention, we also set $p_0(E) \triangleq \mathbf{1}$.

Similar to Proposition 7.4, one has the following decomposition of Pontryagin classes with respect to a direct sum.

Proposition 7.5. *Suppose that $E = E_1 \oplus E_2$ where E_1, E_2 are real vector bundles over M. Then*

$$p_t(E) = p_t(E_1)p_t(E_2),$$

In particular, one has

$$p_k(E) = \sum_{j=0}^{k} p_j(E_1)p_{k-j}(E_2)$$

for every k.

7.3.3 The Euler class

Now we consider the case when $G = \mathrm{SO}(r)$ and $\mathfrak{g} = \mathfrak{so}(r)$. Again, every $X \in \mathfrak{so}(r)$ is $\mathrm{SO}(r)$-conjugate to the block-diagonal form (7.10). If $r = 2m+1$, block permutations of the $\lambda_k J$'s and sign flips of any of the λ_k's can be achieved by $\mathrm{SO}(r)$-conjugations. In this case, Ad-invariant polynomials are again just symmetric polynomials of the λ_k^2's; namely, one has

$$I(\mathrm{SO}(2m+1)) \cong I(\mathrm{O}(2m+1)) \cong \mathbb{R}[p_1, \cdots, p_m],$$

where p_k is the polynomial defined by (7.11).

The situation when $r = 2m$ is more interesting. In this case, $\mathrm{SO}(2m)$-conjugations can permute the $\lambda_k J$ blocks in any arbitrary way, however, they can only flip the signs of an *even* number of λ_k's. This observation gives rise to a new invariant polynomial which is defined as follows (cf. Example 7.3).

Definition 7.3. The *Pfaffian* on $\mathfrak{so}(2m)$ is the polynomial defined by

$$\mathrm{Pf}(X) \triangleq \lambda_1 \cdots \lambda_m,$$

where $\lambda_1, \cdots, \lambda_m$ are the real numbers appearing in the block-diagonal form (7.10) of X.

Remark 7.3. Definition 7.3 differs from the standard convention of Pfaffian by a sign factor $(-1)^m$.

Note that $\mathrm{Pf}(X)$ is not invariant in the $r = 2m+1$ case; an $\mathrm{SO}(2m+1)$-conjugation can possibly flip the sign of a single λ_k, hence changing the sign of $\mathrm{Pf}(X)$. The following alternative characterisation of the Pfaffian is also useful.

Lemma 7.2. *One has*

$$\mathrm{Pf}(X) = \frac{(-1)^m}{2^m m!} \sum_{\sigma \in \mathcal{S}_{2m}} \mathrm{sgn}(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2m-1)\sigma(2m)} \tag{7.12}$$

for all $X = (a_{ij})_{1 \leqslant i,j \leqslant 2m} \in \mathfrak{so}(2m)$.

Proof. Denote the RHS of (7.12) by $F(X)$. It is easy to show that $F(X)$ is invariant under $\mathrm{SO}(2m)$-conjugations. As a result, one may assume WLOG that $X = \mathrm{diag}(\lambda_1 J, \cdots, \lambda_m J)$. In this case, a summand in $F(X)$ is nonzero exactly when σ is a permutation of the m diagonal blocks and each $a_{\sigma(2k-1)\sigma(2k)}$ is one of the two off-diagonal elements in the corresponding block. More precisely,

$$a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2m-1)\sigma(2m)} \neq 0$$

if and only if there exists $\hat{\sigma} \in \mathcal{S}_m$ such that

$$\big(\sigma(2k-1), \sigma(2k)\big) = \big(2\hat{\sigma}(k) - 1, 2\hat{\sigma}(k)\big) \text{ or } \big(2\hat{\sigma}(k), 2\hat{\sigma}(k) - 1\big) \tag{7.13}$$

for all $k = 1, \cdots, m$. Note that the block permutation $\hat{\sigma}$ has no effect on the sign of σ. For each k, Case I of (7.13) yields $a_{\sigma(2k-1)\sigma(2k)} = -\lambda_{\hat{\sigma}(k)}$ with a "+1" contribution to the sign of σ, while Case II of (7.13) yields $a_{\sigma(2k-1)\sigma(2k)} = \lambda_{\hat{\sigma}(k)}$ with a "−1" contribution to the sign of σ. Therefore, one has

$$\begin{aligned} &\varepsilon(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2m-1)\sigma(2m)} \\ &= (-1)^{\#\{k : k \in \text{Case I}\}} \times (-1)^{\#\{k : k \in \text{Case II}\}} \times \lambda_1 \cdots \lambda_m \\ &= (-1)^m \lambda_1 \cdots \lambda_m. \end{aligned}$$

It follows that

$$F(X) = \frac{(-1)^m}{2^m m!} \times m! 2^m \times (-1)^m \lambda_1 \cdots \lambda_m = \mathrm{Pf}(X).$$

This completes the proof of the lemma. □

Conventionally, we will use the variable x_k defined by (7.8) and introduce the $\mathrm{SO}(2m)$-invariant polynomial

$$\Delta(X) \triangleq x_1 \cdots x_m = \big(-\frac{1}{2\pi}\big)^m \mathrm{Pf}(X)$$

accordingly. The polynomial $\Delta(X)$ is called the *universal rank-$2m$ Euler class.*

It is clear that the p_k's are $\mathrm{SO}(2m)$-invariant polynomials. The following result shows that $I(\mathrm{SO}(2m))$ *is precisely the algebra generated by the* p_k*'s and* Δ *modulo the obvious relation that* $\Delta^2 = p_m$.

Proposition 7.6. *The algebra homomorphism* $\Psi : \mathbb{R}[P_1, \cdots, P_m, D] \to I(\mathrm{SO}(2m))$ *induced by*

$$P_k \mapsto p_k, \ \ D \mapsto \Delta$$

is surjective whose kernel is the ideal generated by $D^2 - P_m$. *In particular, one has*

$$\mathbb{R}[P_1, \cdots, P_m, D]/(D^2 - P_m) \cong I(\mathrm{SO}(2m)).$$

Proof. *Surjectivity of* Ψ. Let $\phi(x_1, \cdots, x_m) \in I(\mathrm{SO}(2m))$. Note that

$$\phi(-x_1, x_2, \cdots, x_m) = \phi(x_1, \cdots, -x_k, \cdots, x_m) \tag{7.14}$$

for each $k \geqslant 2$. Define

$$\begin{aligned} \phi_+(x_1, \cdots, x_m) &\triangleq \phi(x_1, x_2, \cdots, x_m) + \phi(-x_1, x_2, \cdots, x_m), \\ \phi_-(x_1, \cdots, x_m) &\triangleq \phi(x_1, x_2, \cdots, x_m) - \phi(-x_1, x_2, \cdots, x_m). \end{aligned}$$

Since ϕ_+ is $\mathrm{O}(2m)$-invariant, one has $\phi_+ = f(p_1, \cdots, p_m)$ for some polynomial f. To analyse the shape of ϕ_-, let $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ be a monomial appearing in ϕ. If α_k is even for some k, this monomial must be annihilated in the expression of ϕ_- due to the relation (7.14). As a result, only those monomials where all the α_k's are odd can survive in the expression of ϕ_-. Therefore, one can write $\phi_- = \Delta \cdot \psi$ where ψ is an $\mathrm{O}(2m)$-invariant polynomial. In other words, $\phi_0 = \Delta \cdot g(p_1, \cdots, p_m)$ for some polynomial g. It follows that

$$\phi = \frac{1}{2}\big(f(p_1, \cdots, p_m) + \Delta \cdot g(p_1, \cdots, p_m)\big) \in \mathrm{Im}(\Psi).$$

This shows that Ψ is surjective.

Kernel of Ψ. First of all, it is obvious that $(D^2 - P_m) \subseteq \ker\Psi$. Conversely, let $F \in \ker \Psi$. By viewing $F \in \mathbb{R}[P_1, \cdots, P_m][D]$, one can write $F = Q \cdot (D^2 - P_m) + R$ where

$$R = f(P_1, \cdots, P_m)D + g(P_1, \cdots, P_m)$$

for some polynomials f, g. Note that $R \in \ker \Psi$. In other words, one has

$$f(p_1, \cdots, p_m)x_1 \cdots x_m + g(p_1, \cdots, p_m) = 0 \tag{7.15}$$

for all $X \in \mathfrak{so}(2m)$. Since (7.15) is true as a function of $(x_1, \cdots, x_m) \in \mathbb{R}^m$, it follows that both f and g must be zero. Therefore, $R = 0$ and thus $F \in (D^2 - P_m)$. $\square$

Now suppose that E is a real, oriented vector bundle of rank $2m$ over a manifold M. Let E be equipped with an Euclidean metric and let ∇ be a compatible connection with curvature F_∇.

Definition 7.4. The differential form

$$e(\nabla) \triangleq \Delta(F_\nabla) = \frac{(-1)^m}{(2\pi)^m}\mathrm{Pf}(F_\nabla) \in \Omega^{2m}(M)$$

is called the *Euler form* of ∇. The cohomology class $[e(\nabla)]$ is called the *geometric Euler class* of E. It is denoted as $e_{\mathrm{geo}}(E)$.

Clearly, the Euler form $e(\nabla)$ depends on the connection ∇. However, the geometric Euler class $e_{\mathrm{geo}}(E)$ is independent of the connection ∇ and the metric. As suggested in the proposition below, the geometric Euler class provides the obstruction of possessing nonvanishing sections by E.

Proposition 7.7. *Suppose that E admits a nonvanishing section. Then $e_{\mathrm{geo}}(E) = 0$.*

Proof. A nonvanishing section s of E defines an obvious line bundle L (as a subbundle of E):

$$L \triangleq \big\{u \in E : u \in \mathrm{Span}\{s(\pi(u))\}\big\}.$$

Let E be equipped with an Euclidean metric. Then one obtains the global splitting $E = L \oplus L^\perp$ where $L^\perp$ is the (fiberwise) orthogonal complement of L with respect to the metric. Now let us pick metric connections ∇^L, $\nabla^{L^\perp}$ on L, $L^\perp$ respectively and define $\nabla \triangleq \nabla^L \oplus \nabla^{L^\perp}$ on E. Since ∇^L is always flat, the local shape of the curvature F_∇ (as a $\mathfrak{so}(2m)$-valud 2-form) is of the form

$$F_\nabla = \begin{pmatrix} 0 & 0 \\ 0 & F_{\nabla^\perp} \end{pmatrix} \overset{\mathrm{SO}(2m)}{\sim} \mathrm{diag}(0, 0, \lambda_1 J, \cdots, \lambda_{m-1} J).$$

In particular, $\mathrm{Pf}(F_\nabla) = 0$. □

Example 7.4. Let M be a closed, oriented Riemannian manifold of dimension $2m$. The form $e(\nabla^M) \in \Omega^{2m}(M)$, where ∇^M is the Levi-Civita connection on $E = TM$, is precisely the Euler form $\mathcal{E}$ defined by (5.2) in Chapter 5. This follows immediately from Lemma 7.2. In the 2D case, let $\{e_1, e_2\}$ be a local PONF. The Gaussian curvature is given by

$$K = \langle R(e_2, e_1)e_1, e_2\rangle,$$

where R is the Riemann curvature tensor (see Appendix C.1). By locally writing R as $\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$, one easily finds that $\omega = -K \times \mathrm{vol}_M$. As a result,

$$e(\nabla^M) = -\frac{1}{2\pi}\mathrm{Pf}(R) = -\frac{1}{2\pi}\omega = \frac{K}{2\pi}\mathrm{vol}_M.$$

This is consistent with the Gauss-Bonnet theorem for surfaces (cf. Theorem 1.1 and Corollary 7.1).

7.4 Chern-Gauss-Bonnet revisited: the Chern-Weil perspective

In this section, we give another proof of CGB from the perspective of the Chern-Weil theory. If one re-examines Chern's original proof, a key insight is that the Euler form $\mathcal{E}$, when lifted to the sphere bundle, becomes an exact form. The proof we will present here is not essentially different from the Chern's original proof, in the sense that this insight remains a key ingredient. Based on the Chern-Weil theory, this point can now be understood in a more fundamental way.

Let E be a real, oriented vector bundle of rank $2m$ over a closed manifold M. Recall that the (differential) topological Euler class is defined by $e_{\text{top}}(E) \triangleq s_0^*\Phi$, where $s_0 : M \to E$ is the zero section and Φ is the (differential) Thom class (see Definition 4.3 and Definition 4.6). The class $e_{\text{top}}(E)$ is constructed through topological data (one always identifies singular and de Rham cohomologies through the de Rham theorem). On the other hand, the geometric Euler class $e_{\text{geo}}(E)$ (see Definition 7.4) is constructed by means of geometric data (a metric connection). The CGB theorem, stated in the context of vector bundles, asserts that these two cohomology classes are identical.

Theorem 7.2 (CGB on Vector Bundles). $e_{\text{top}}(E) = e_{\text{geo}}(E)$.

We first show that the CGB theorem is a direct corollary of Theorem 7.2.

Corollary 7.1. *Let M be a closed, oriented Riemannian manifold with dimension $2m$. Then one has*

$$\chi(M) = \int_M e_{\text{geo}}(\nabla^M),$$

where ∇^M denotes the Levi-Civita connection on the tangent bundle.

Proof. According to Proposition 3.7 and the de Rham theorem (cf. Section 4.7.6), the integral of $e_{\text{top}}(TM)$ gives the Euler characteristic. On the other hand, Theorem 7.2 shows that $e_{\text{geo}}(TM) = e_{\text{top}}(TM)$. The result thus follows. ☐

Remark 7.4. If the rank of E is odd, the topological Euler class is trivial. In fact, let $a : E \to E$ be the antipodal map defined by $a(u) \triangleq -u$. It is clear that $a^*\Phi = -\Phi$ because $\text{rank}(E)$ is odd (Φ is the Thom class). After applying pullback by the zero section, one finds that $e_{\text{top}}(E) = -e_{\text{top}}(E)$ which implies that $e_{\text{top}}(E) = 0$.

In the rest of this section, we develop the proof of Theorem 7.2. The main argument is essentially a higher rank extension of Section 4.6.

7.4.1 Global angular form and construction of the Thom class

Let ∇ be a connection which is compatible with some Euclidean metric on E. To prove Theorem 7.2, one essentially needs to construct a closed form $\phi \in \Omega^{2m}_{\mathrm{cv}}(E)$ which satisfies the following two properties.

(i) $\pi_*\phi = 1 \in \Omega^0(M)$. Here π_* is the integration along fiber defined by (4.13). This will imply that ϕ is a representative of the Thom class Φ of E.

(ii) $s_0^*\phi = e(\nabla)$. Together with (i), this clearly implies that

$$[e(\nabla)] = s_0^*[\phi] = s_0^*\Phi,$$

which yields the conclusion of the theorem.

Similar to the spirit of Chern's original proof, let us consider the *sphere bundle*

$$S(E) \triangleq \{u \in E : |u| = 1\},$$

where $|\cdot|$ denotes the given metric on E. Let $\bar{\nabla}$ be the induced connection on the pullback bundle π_0^*E over $S(E)$, where $\pi_0 : S(E) \to M$ is the projection map.

$$\begin{tikzcd}
& E \arrow[dl] \arrow[d, "\pi"] \\
S(E) \arrow[r, "\pi_0"] & M.
\end{tikzcd}$$

According to Proposition 7.3, one has

$$e(\bar{\nabla}) = \pi_0^* e(\nabla) = \big(-\frac{1}{2\pi}\big)^m \mathrm{Pf}(F_{\bar{\nabla}}).$$

The key lemma for constructing the aforementioned Thom class representative ϕ is stated as follows.

Lemma 7.3. *There exists a $(2m-1)$-form Ψ on $S(E)$ such that $e(\bar{\nabla}) = d\Psi$ and*

$$\int_{S(E_x)} i_x^*\Psi = -1 \tag{7.16}$$

for all $x \in M$. Here $S(E_x)$ denotes the unit sphere on the fiber E_x. The orientation of $S(E_x)$ is chosen according to the convention that

$$\mathrm{or}(\langle e\rangle) \oplus \mathrm{or}(T_eS(E_x)) = \mathrm{or}(E_x)$$

for all $e \in S(E_x)$. The map $i_x : S(E_x) \to S(E)$ is the canonical inclusion.

The form Ψ given by the above lemma is called the *global angular form*. Its general existence (without the integral property (7.16)) is an easy consequence of the Chern-Weil theory. To see this, the main observation is that the vector bundle π_0^*E admits a canonical nonvanishing section defined by the identity map $u \mapsto u$. As a consequence of Proposition 7.7, the Euler class $[e(\bar{\nabla})]$ is trivial. In other words, one has $e(\bar{\nabla}) = d\Psi$ for some $(2m-1)$-form on $S(E)$.

The nontrivial part of the lemma is that $-\Psi$ *restricts to the normalised volume form on each sphere* $S(E_x)$ (i.e. the integral property (7.16) holds). The last quantitative property is essential for constructing a representative of the Thom class. We shall use the transgression formula (7.6) to write down Ψ explicitly and deduce the integral property (7.16) from that.

We first complete the proof of the CGB theorem by presuming the correctness of Lemma 7.3. After that, we will come back to prove this key lemma.

Proof of Theorem 7.2. Let $E_0 \triangleq \{u \in E : u \neq 0\}$. One can identify E_0 with $(0,\infty) \times S(E)$ by $u \mapsto (|u|, u/|u|)$. The two projections under this identification are denoted as ρ_i $(i = 1, 2)$ respectively. Let Ψ be the global angular form given by Lemma 7.3. Let $\eta : [0,\infty) \to [0,\infty)$ be a smooth function such that $\eta = 1$ on $[0, 1/2]$ and $\eta = 0$ on $[1,\infty)$. Using the above objects, one can construct a form ϕ on E by

$$\phi \triangleq \rho_1^*(\eta'(r)dr) \wedge \rho_2^*(\Psi) + (\eta \circ \rho_1) \cdot \pi^* e(\nabla). \tag{7.17}$$

It is clear that ϕ is well-defined on E with compact support along fibers.

(i) ϕ *is closed*. Indeed, one has

$$d\phi = -\rho_1^*(\eta'(r)dr) \wedge \rho_2^*(d\Psi) + \rho_1^*(\eta'(r)dr) \wedge \pi^* e(\nabla).$$

Since $\eta'(r)$ vanishes near the zero section, the above $\pi^* e(\nabla)$ can be replaced by $\rho_2^* \pi_0^*(e(\nabla))$. The claim thus follows from the fact that $d\Psi = e(\bar{\nabla}) = \pi_0^* e(\nabla)$.

(ii) $s_0^*\phi = e(\nabla)$. This follows immediately from the definition (7.17) of ϕ and the properties of η.

(iii) $\pi_*\phi = 1$. By the definition of the integration map π_*, one has

$$(\pi_*\phi)_x = \left(\int_0^\infty \eta'(r)dr\right) \times \left(\int_{S(E_x)} i_x^*\Psi\right) = (-1) \times (-1) = 1.$$

The proof of the CGB theorem is therefore complete.

□

7.4.2 Construction of global angular form

We now prove Lemma 7.3. The crucial point is the global splitting $\pi_0^*E = L \oplus L^\perp$, where L is the canonical line bundle over $S(E)$ induced by the identity section

$u \mapsto u$ and $L^\perp$ is its orthogonal complement with respect to the given metric. Recall that $\bar{\nabla}$ is the connection on π_0^*E induced by ∇.

Exactness of $e(\bar{\nabla})$ and transgression form

We define another connection $\hat{\nabla}$ on π_0^*E in the following way.

(i) $\hat{\nabla}$ is trivial on L: $\hat{\nabla}(fe) = df \otimes e$ where e denotes the identity section of L.

(ii) For any section s of $L^\perp$, $\hat{\nabla}s$ is defined to be the orthogonal projection of $\bar{\nabla}s$ onto the $L^\perp$-component.

Similar to the proof of Proposition 7.7, it is easily seen that $\mathrm{Pf}(F_{\hat{\nabla}}) = 0$ (because the L-component is flat). It follows from the transgression formula (7.6) that $e(\bar{\nabla}) = d\Psi$, where

$$\Psi \triangleq \frac{m(-1)^m}{(2\pi)^m} \int_0^1 \mathrm{Pf}(\bar{\nabla} - \hat{\nabla}, F^t, \cdots, F^t)dt$$

is the transgression form and F^t is the curvature of $\hat{\nabla} + t(\bar{\nabla} - \hat{\nabla})$.

Induced connections at x

Let $x \in M$ be given fixed. Our aim is to prove the integral property (7.16), where

$$i_x^*\Psi = \frac{m(-1)^m}{(2\pi)^m} \int_0^1 \mathrm{Pf}\big(i_x^*(\bar{\nabla} - \hat{\nabla}), i_x^*F^t, \cdots, i_x^*F^t\big)dt.$$

To this end, let $S(E_x)$ be the unit sphere on E_x and consider the pullback vector bundle π_x^*E over $S(E_x)$, where $\pi_x \triangleq \pi_0 \circ i_x$. Note that this bundle is trivial; one has $\pi_x^*E \cong S(E_x) \times E_x$. In particular, sections of π_x^*E are just E_x-valued functions on the sphere $S(E_x)$. Let us write $\bar{\nabla}^x \triangleq i_x^*\bar{\nabla}$ and $\hat{\nabla}^x \triangleq i_x^*\hat{\nabla}$ as the induced connections on $S(E_x)$ respectively.

Lemma 7.4. *The connection $\bar{\nabla}^x$ is trivial. In other words, $\bar{\nabla}^x V = dV$ for any smooth section $V : S(E_x) \to E_x$.*

Proof. Let A_α be the local connection form of ∇. Then $\pi_x^*A_\alpha$ is the local connection form of $\bar{\nabla}^x$. Since the map π_x is degenerate, the pullback of any differential form (with positive degree) by π_x^* is trivial. In particular, one has $\pi_x^*A_\alpha = 0$. □

Lemma 7.5. *Consider the canonical splitting $\pi_x^*E = N_x \oplus TS(E_x)$, where N_x is the normal bundle and $TS(E_x)$ is the tangent bundle over $S(E_x)$. Then $\hat{\nabla}^x|_{TS(E_x)}$ is the Levi-Civita connection with respect to the standard spherical metric on $S(E_x)$ induced from the Euclidean space E_x. As a consequence, $\hat{\nabla}^x$ is the direct sum of the trivial connection on N_x and the Levi-Civita connection on $TS(E_x)$.*

Proof. By definition, $\hat{\nabla}^x$ is precisely the orthogonal projection of $\bar{\nabla}^x$ onto the tangential component $TS(E_x)$. The claim thus follows from the consistency property of Levi-Civita connection with respect to isometric embeddings (see Proposition C.1). □

Explicit computation of $i_x^*\Psi$

Now we proceed to evaluate $i_x^*\Psi$. Let $u \in S(E_x)$ be given fixed. Let $\{s_1, \cdots, s_{2m-1}\}$ be a local PONF of $TS(E_x)$ near u, which is chosen such that the corresponding Christoffel symbols of the Levi-Civita connection vanish at u (see Appendix C.3). Let $\{\theta^1, \cdots, \theta^{2m-1}\}$ be the dual coframe of $\{s_1, \cdots, s_{2m-1}\}$. We also take s_0 to be the unit normal vector field near u so that $s_0(u) = u$ and $s = \{s_0, s_1, \cdots, s_{2m-1}\}$ is a local PONF of π_x^*E. Let A^0 (respectively, A^1) be the local connection matrix of $\hat{\nabla}^x$ (respectively, $\bar{\nabla}^x$) with respect to the frame field s near u. Their local curvature matrices are denoted as F^0, F^1 respectively.

Lemma 7.6. *At the point u, one has $A^0(u) = 0$ and*

$$dA^0(u) = F^0(u) = \begin{pmatrix} 0 & 0 \\ 0 & \left(\theta^i \wedge \theta^j\right)_{i,j=1}^{2m-1} \end{pmatrix}. \tag{7.18}$$

Proof. The relation $A^0(u) = 0$ follows from the triviality of $\hat{\nabla}^x|_{N_x}$ and the fact that the local connection matrix of $\hat{\nabla}^x|_{TS(E_x)}$ with respect to $\{s_1, \cdots, s_{2m-1}\}$ vanishes at u. In addition, the curvature F^0 is the direct sum of the curvature of the trivial connection on N_x (which is flat) and the curvature F^0_{LC} of the Levi-Civita connection on $TS(E_x)$. Under the PONF $\{s_1, \cdots, s_{2m-1}\}$, the matrix F_k^l of the curvature F^0_{LC} is defined by

$$F^0_{\mathrm{LC}}(\cdot, \cdot)s_j = \sum_{i=1}^{2m-1} F_j^i(\cdot, \cdot)s_i, \quad j = 1, \cdots, 2m-1.$$

Since $S(E_x)$ has constant sectional curvature $K \equiv 1$ (see Example C.1), it is easily seen from the definition of the sectional curvature that

$$\langle F^0_T(s_k(u), s_l(u))s_j(u), s_i(u)\rangle = \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}.$$

In particular,

$$F_j^i(s_k(u), s_l(u)) = \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}.$$

This shows that $F_j^i = \theta^i \wedge \theta^j$ at u and the curvature expression (7.18) thus follows. Finally, one also has

$$F^0(u) = dA^0(u) + A^0(u) \wedge A^0(u) = dA^0(u),$$

since $A^0(u) = 0$. □

Lemma 7.7. *At the point u, one has*

$$A^1(u) = \begin{pmatrix} 0 & -\theta^1 & \cdots & -\theta^{2m-1} \\ \theta^1 & & & \\ \vdots & & 0 & \\ \theta^{2m-1} & & & \end{pmatrix}, \quad dA^1(u) = \begin{pmatrix} 0 & 0 \\ 0 & (\theta^i \wedge \theta^j)_{i,j=1}^{2m-1} \end{pmatrix}.$$

Proof. Recall from Lemma 7.4 that $\bar{\nabla}^x$ is the trivial connection. By the construction of s_i, one easily finds that $\bar{\nabla}^x_{s_j} s_0 = s_j$ for each $j = 1, \cdots, 2m-1$. This is equivalent to saying that

$$\bar{\nabla}^x s_0 = \sum_{j=0}^{2m-1} \theta^j \otimes s_j,$$

which gives the zeroth column of $A^1(u)$. In addition, for $i \geqslant 1$ one can write

$$\bar{\nabla}^x_{s_j} s_i = \langle \bar{\nabla}^x_{s_j} s_i, s_0 \rangle s_0 + (\bar{\nabla}^x_{s_j} s_i)^T, \quad j = 1, \cdots, 2m-1,$$

where $(\cdot)^T$ denotes the orthogonal projection onto the tangent bundle $TS(E_x)$. The tangential term $(\bar{\nabla}^x_{s_j} s_i)^T$ vanishes at u since it becomes the Levi-Civita connection. The normal term is computed as

$$\langle \bar{\nabla}^x_{s_j} s_i, s_0 \rangle = -\langle s_i, \bar{\nabla}^x_{S_j} s_0 \rangle = -\langle s_i, \theta^j \rangle = -\delta_{ij}.$$

It follows that $\bar{\nabla}^x s_i = -\theta^i \otimes s_0$, which gives the i-th column of $A^1(u)$. The expression of $dA^1(u)$ follows from the fact that $F^1 \equiv 0$ (because $\bar{\nabla}^x$ is trivial). □

With the aid of the previous two lemmas, one can now compute $i_x^* \Psi$ easily.

Lemma 7.8. *The form $i_x^* \Psi$ is the negative of the normalised volume form on $S(E_x)$. In particular, the integral property (7.16) holds for every $x \in M$.*

Proof. We continue to use the previous notation. Let C (respectively, G^t) denote the matrix of $\bar{\nabla}^x - \hat{\nabla}^x$ (respectively, $i_x^* F^t$) with respect to the local PONF s at u. By using the formula (7.2) and Lemmas 7.6, 7.7, one computes that

$$C \triangleq \begin{pmatrix} 0 & -\theta^1 & \cdots & -\theta^{2m-1} \\ \theta^1 & & & \\ \vdots & & 0 & \\ \theta^{2m-1} & & & \end{pmatrix}, \; G^t \triangleq \begin{pmatrix} 0 & 0 \\ 0 & (1-t^2)(\theta^i \wedge \theta^j)_{i,j=1}^{2m-1} \end{pmatrix}.$$

According to Lemma 7.2, one has

$$\begin{aligned} &\mathrm{Pf}(i_x(\bar{\nabla}^x - \hat{\nabla}^x), i_x^* F^t, \cdots, i_x^* F^t)(u) \\ &= \frac{(-1)^m}{2^m m!} \sum_{\sigma \in \mathcal{S}_{2m}} \mathrm{sgn}(\sigma) C_{\sigma(0)\sigma(1)} F^t_{\sigma(2)\sigma(3)} \cdots F^t_{\sigma(2m-2)\sigma(2m-1)}. \end{aligned} \tag{7.19}$$

Note that the term $C_{\sigma(0)\sigma(1)}$ is nonzero if and only if $\sigma(0) = 0, \sigma(1) \neq 0$ or $\sigma(1) = 0, \sigma(0) \neq 0$. The first case in the sum of (7.19) gives that

$$\begin{aligned} &\sum_{\sigma \in \mathcal{S}_{2m}:\sigma(0)=0} \operatorname{sgn}(\sigma)(-\theta^{\sigma(1)}) \wedge \theta^{\sigma(2)} \wedge \cdots \wedge \theta^{\sigma(2m-1)}(1-t^2)^{m-1} \\ &= -\sum_{\tau \in \mathcal{S}_{2m-1}} \operatorname{sgn}(\tau)\theta^{\tau(1)} \wedge \cdots \wedge \theta^{\tau(2m-1)} = -(2m-1)!\mathrm{vol}_{S(E_x)}. \end{aligned}$$

The second case gives the same expression by similar calculation. As a consequence,

$$i_x^*\Psi = \frac{m(-1)^m}{(2\pi)^m}\frac{(-1)^{m+1}2(2m-1)!}{2^m m!}\int_0^1 (1-t^2)^{m-1}dt \times \mathrm{vol}_{S(E_x)}. \tag{7.20}$$

Using the elementary relation

$$\int_0^1 (1-t^2)^{m-1}dt = \frac{2 \cdot 4 \cdot \cdots \cdot (2m-2)}{1 \cdot 3 \cdot \cdots \cdot (2m-1)},$$

one finds that the entire coefficient in front of $\mathrm{vol}_{S(E_x)}$ in (7.20) is equal to $-\mathrm{vol}(S(E_x))^{-1}$. Therefore, one concludes that $i_x^*\Psi$ is the negative of the normalised volume form on $S(E_x)$. This completes the proof of the lemma. □

8 The Hodge theory

In this chapter, we discuss a rather beautiful result in global differential geometry: *the Hodge theorem*. Let M be a closed, oriented manifold. The de Rham theorem (see Theorem 4.5) asserts that the de Rham cohomology of M, which is defined in terms of the differential structure, is isomorphic to the singular cohomology with real coefficients and is thus a topological invariant. Now suppose that M is equipped with a Riemannian metric. One can then define the so-called Hodge Laplacian Δ on forms and consider harmonic forms on M (those forms ω satisfying $\Delta\omega = 0$). The Hodge theorem asserts that *the space of harmonic k-forms is isomorphic to the k-th de Rham cohomology group* and hence encoding topological information. This is a profound theorem, since the notion of harmonic forms relies on the underlying geometry (the Riemannian metric), while (surprisingly) the dimension of the space of harmonic forms turns out to be a topological invariant. This result will also be used in the later heat equation proof of the CGB theorem.

8.1 The Hodge star operator

We need some linear algebra preparation (Hodge duality) before moving to the definition of the Hodge Laplacian. Let W be a real, oriented Euclidean space of dimension n. This means that W is equipped with an inner product $\langle\cdot,\cdot\rangle$ and a preferred equivalence class of bases (orientation) is fixed. Let ΛW denote the exterior algebra over W.

Definition 8.1. The *Hodge star operator* $* : \Lambda W \to \Lambda W$ is defined in the following way. Let $\{\theta^1, \cdots, \theta^n\}$ be a PONB of W. We define $* : \Lambda^k W \to \Lambda^{n-k} W$ to be the unique linear operator such that

$$*(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}) \triangleq \mathrm{sgn}(I, J)\theta^{j_1} \wedge \cdots \wedge \theta^{j_{n-k}}, \tag{8.1}$$

for all $I \triangleq (i_1 < \cdots < i_k)$ and $J = (j_1 < \cdots < j_{n-k}) \triangleq I^c$.

Lemma 8.1. *The Hodge star operator is well-defined, i.e. it is independent of the choice of PONB. In addition, one has* $*^2|_{\Lambda^k W} = (-1)^{k(n-k)}\mathrm{id}$.

Proof. Left as an exercise. □

The relation (8.1) remains valid even if the indices $i_1, \cdots, i_k \in I$ and $\{j_1, \cdots, j_{n-k}\} = I^c$ are unordered (but still distinct). Let $\mathrm{vol} = \theta^1 \wedge \cdots \wedge \theta^n$ denote the volume element. By definition,

$$(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}) \wedge *(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}) = \mathrm{vol}.$$

It is also true that $*1 = \mathrm{vol}$ and $*\mathrm{vol} = 1$.

By using the Hodge star operator, one can define a canonical inner product structure on ΛW.

Definition 8.2. The inner product $\langle \cdot, \cdot \rangle$ on Λ is defined in the following way. For k-vectors $\alpha, \beta \in \Lambda^k W$, we set

$$\langle \alpha, \beta \rangle \triangleq *(\alpha \wedge *\beta).$$

We also assume that $\Lambda^k W$ and $\Lambda^l W$ are orthogonal if $k \neq l$.

It is obvious that $\langle \cdot, \cdot \rangle|_W$ is the original inner product. It also admits the following alternative expression.

Lemma 8.2. *One has*

$$\langle \alpha, \beta \rangle = \det \big((\langle v_i, w_j \rangle_{1 \leqslant i,j \leqslant k}) \big) \tag{8.2}$$

for any $\alpha = v_1 \wedge \cdots \wedge v_k$ *and* $\beta = w_1 \wedge \cdots \wedge w_k$ $(v_i, w_j \in W)$.

Proof. It suffices to consider the case when $\alpha = \theta^{i_1} \wedge \cdots \wedge \theta^{i_k}$ and $\beta = \theta^{j_1} \wedge \cdots \wedge \theta^{j_k}$, where $\{\theta^1, \cdots, \theta^n\}$ is a positive ONB of W and $i_1 < \cdots < i_k$, $j_1 < \cdots < j_k$. To this end, one simply observes that both sides of (8.2) are zero if $(i_1, \cdots, i_k) \neq (j_1, \cdots, j_k)$ and are equal to one if $(i_1, \cdots, i_k) = (j_1, \cdots, j_k)$. ☐

Lemma 8.3. *One has* $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \mathrm{vol}$ *for all* k*-vectors* $\alpha, \beta \in \Lambda^k W$.

Proof. By Lemma 8.1, one has $** = \mathrm{id}$ on n-vectors, and in particular,

$$\alpha \wedge *\beta = * * (\alpha \wedge *\beta) = *\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle * 1 = \langle \alpha, \beta \rangle \mathrm{vol}.$$

This gives the desired relation. ☐

8.2 Divergence operator and the Hodge Laplacian

We now return to the geometric setting. Let M be a closed, oriented, n-dimensional Riemannian manifold.

Definition 8.3. The (*Riemannian*) *Hodge star operator* $* : \Lambda^k T^*M \to \Lambda^{n-k} T^*M$ $(0 \leqslant k \leqslant n)$ is defined by applying the Hodge star operator to $W = T_x^*M$ at every $x \in M$, where the metric on T_x^*M is induced by its canonical identification with $T_x M$ under the Riemannian metric.

The fiberwise inner product naturally induces an inner product on the space of differential forms via integration.

Definition 8.4. We define an inner product on $\Omega(M)$ by setting

$$\langle \alpha, \beta \rangle_{L^2} \triangleq \int_M \langle \alpha, \beta \rangle_x dx, \quad \alpha, \beta \in \Omega(M).$$

Here $\langle \alpha, \beta \rangle_x$ denotes the fiberwise inner product on ΛT_x^*M (see Definition 8.2) and dx denotes the volume form on M. We define $L^2(\Lambda T^*M)$ to be the completion of $\Omega(M)$ with respect to $\langle \alpha, \beta \rangle_{L^2}$.

By Lemma 8.3, one also has the following alternative expression:

$$\langle \alpha, \beta \rangle_{L^2} = \int_M \alpha \wedge *\beta, \quad \alpha, \beta \in \Omega^k(M). \tag{8.3}$$

Using the L^2-inner product and duality, one can define the divergence operator on forms.

Definition 8.5. The *divergence operator* $\delta : \Omega(M) \to \Omega(M)$ is the formal L^2-adjoint of d. More precisely, given $\alpha \in \Omega(M)$ we define $\delta\alpha$ to be the unique differential form such that

$$\langle \delta\alpha, \beta \rangle_{L^2} = \langle \alpha, d\beta \rangle_{L^2}$$

for all $\beta \in \Omega(M)$.

Since $d^2 = 0$, it is clear from the definition that $\delta^2 = 0$. In addition, δ maps $(k+1)$-forms to k-forms. Sometimes we use δ^{k+1} to emphasise its action on $(k+1)$-forms (and similarly for d^k on k-forms). The following lemma allows one to express δ^{k+1} in terms of d and the Hodge star.

Lemma 8.4. *For each $1 \leqslant k \leqslant n$, one has $\delta^k = (-1)^{nk+n+1} * d *$.*

Proof. For any $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k-1}(M)$, one has

$$\begin{aligned} \langle d\beta, \alpha \rangle_{L^2} &= \int_M d\beta \wedge *\alpha \quad (\text{by } (8.3)) \\ &= \int_M d(\beta \wedge *\alpha) - (-1)^{k-1} \int_M \beta \wedge d * \alpha \\ &= (-1)^k \int_M \beta \wedge d * \alpha \quad (\text{by Stokes' theorem}) \\ &= (-1)^{nk+n+1} \int_M \beta \wedge *(*d * \alpha) \quad (\text{by } (8.1)) \\ &= \langle \beta, (-1)^{nk+n+1} * d * \alpha \rangle_{L^2}. \end{aligned}$$

The result thus follows. □

Recall that the divergence of a vector field X is defined by (C.9), which is uniquely characterised by Green's formula (see (C.11)):

$$\int_M f \mathrm{div} X dx = -\int_M \langle \nabla^M f, X \rangle dx$$

for all $f \in C^\infty(M)$. In particular, one sees from the duality $TM \cong T^*M$ and Lemma 8.4 (with $k = 0$) that

$$\mathrm{div} X = -\delta\alpha_X = *d*\alpha_X, \tag{8.4}$$

where $\alpha_X \in \Omega^1(M)$ is the metric dual of X.

Example 8.1. Let X be a vector field on $\mathbb{R}^3$ (equipped with standard metric and orientation). As we just saw, $\mathrm{div}X = *d*\alpha_X$. In a similar way, one can also show that $\mathrm{curl}X = *d\alpha_X$, where we identity a vector field with a one form as usual.

Now we introduce the central concept of this chapter: the Hodge Laplacian.

Definition 8.6. The *Hodge Laplacian* is the linear operator $\Delta : \Omega(M) \to \Omega(M)$ defined by $\Delta \triangleq d\delta + \delta d$. One also writes $\Delta^k \triangleq d^{k-1}\delta^k + \delta^{k+1}d^k : \Omega^k(M) \to \Omega^k(M)$ to keep track of the degree.

Note that Δ^0 is just the negative of the Laplace-Beltrami operator on functions; this follows directly from the relation (8.4). It is immediate from definition that Δ is self-adjoint:

$$\begin{aligned}\langle \Delta\alpha, \beta\rangle_{L^2} &= \langle d\delta\alpha, \beta\rangle_{L^2} + \langle \delta d\alpha, \beta\rangle_{L^2}\\ &= \langle \delta\alpha, \delta\beta\rangle_{L^2} + \langle d\alpha, d\beta\rangle_{L^2}\\ &= \langle \alpha, d\delta\beta\rangle_{L^2} + \langle \alpha, \delta d\beta\rangle_{L^2} = \langle \alpha, \Delta\beta\rangle_{L^2}.\end{aligned}$$

It is also nonnegative definite:

$$\langle \Delta\alpha, \alpha\rangle_{L^2} = \langle d\delta\alpha, \alpha\rangle_{L^2} + \langle \delta d\alpha, \alpha\rangle_{L^2} = \langle \delta\alpha, \delta\alpha\rangle_{L^2} + \langle d\alpha, d\alpha\rangle_{L^2} \geqslant 0.$$

Before delving into deeper properties, we make one simple observation to conclude this subsection.

Definition 8.7. A form α is *harmonic* if $\Delta\alpha = 0$.

Lemma 8.5. *A form $\alpha \in \Omega(M)$ is harmonic if and only if it is closed (i.e. $d\alpha = 0$) and co-closed (i.e. $\delta\alpha = 0$).*

Remark 8.1. The definitions of Hodge star, divergence and Hodge Laplacian can be extended to noncompact manifolds in the obvious way (even for Lorentzian manifolds though the Hodge theorem will break down in a fundamental way). We restrict ourselves to compact Riemannian manifolds for the development of the (classical) Hodge theorem.

Example 8.2. Using differential forms and the aforementioned operators, Maxwell's equations in classical electromagnetism can be formulated in an elegant intrinsic form:

$$dF = 0,\ \delta F = -J.$$

Here

$$F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

is the electromagnetic tensor (which is a 2-form) and

$$J = \begin{pmatrix} \text{charge density } \rho \\ \text{current density } \mathbf{J} \end{pmatrix}$$

is the 4-current over Minkowski spacetime. The divergence operator here (as well as the underlying Hodge star) is defined with respect to the flat Lorentzian metric. Maxwell's equation in vacuum (i.e. $J = 0$) just becomes the search of a (Lorentzian) harmonic 2-form as a result of Lemma 8.5. Since Minkowski spacetime is topologically $\mathbb{R}^4$, one knows that $dF = 0 \iff F = dA$ for some 1-form potential A. Under spacetime coordinates (t, x, y, z), the equation $\delta F = 0$ therefore becomes the standard wave equation

$$\frac{\partial^2 A}{\partial t^2} - \Big(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}\Big) = 0.$$

8.3 The L^2-Hodge theorem

We first establish the L^2-version of the Hodge theorem, whose proof is a neat application of heat kernel technique. In this section, we assume that M is a closed, oriented, n-dimensional Riemannian manifold. We also fix a degree k between 0 and n.

Theorem 8.1 (The L^2-Hodge Theorem)**.** *There exists a sequence $\{\lambda_j, \omega_j : j \geqslant 1\}$ such that the following properties hold true.*

(i) $\{\omega_j : j \geqslant 1\}$ is an ONB of $L^2(\Lambda^k T^* M)$ and ω_j is smooth (i.e. $\omega_j \in \Omega^k(M)$).

(ii) λ_j is an eigenvalue of Δ with eigenform ω_j, i.e. $\Delta\omega_j = \lambda_j\omega_j$.

(iii) The eigenvalues are arranged in nondecreasing order:

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \uparrow \infty$$

and each λ_j has finite multiplicity.

We first summarise the main tool for proving Theorem 8.1 in the lemma below. This lemma will be fully proved in Chapter 9 when we study heat kernels on vector bundles in depth.

Lemma 8.6. *(i) There exists a map*

$$(0,\infty)\times M^2 \ni (t,x,y) \mapsto p(t,x,y) \in \Lambda^k T_x^*M \otimes \Lambda^k T_y^*M$$

such that the following properties hold true.

(a) The map $(t,x,y)\mapsto p(t,x,y)$ is smooth.

(b) (Heat equation) $p(t,x,y)$ satisfies the following PDE:

$$(\partial_t + \Delta_x)p(t,x,y) = 0 \quad \forall (t,x,y)\in(0,\infty)\times M^2.$$

(c) (Initial condition) For any $\omega\in\Omega^k(M)$, one has

$$\lim_{t\to 0^+}\int_M \langle p(t,x,y),\omega(y)\rangle_y dy = \omega(x) \tag{8.5}$$

uniformly for $x\in M$. Here $\langle\cdot,\cdot\rangle_y$ denotes the fiberwise inner product on $\Lambda^k T_y^*M$ with respect to the y-variable. If $\omega\in L^2(\Lambda^k T^*M)$, the above convergence holds in the L^2-sense.

(d) (Symmetry) One has

$$\langle p(t,x,y),\xi\rangle_2 = \langle p(t,y,x),\xi\rangle_1 \in \Lambda^k T_x^*M$$

for all $\xi\in\Lambda^k T_y^*M$. Here $\langle\cdot,\cdot\rangle_i$ means taking inner product with respect to the i-th spatial variable of p $(i=1,2)$.

(ii) For each $t\geqslant 0$, define the linear operator

$$e^{-t\Delta}: L^2(\Lambda^k T^*M)\to L^2(\Lambda^k T^*M),$$
$$(e^{-t\Delta}\omega)(x) \triangleq \int_M \langle p(t,x,y),\omega(y)\rangle_y dy,$$

where we set $e^{-t\Delta}\triangleq \mathrm{id}$ if $t=0$. Then $\{e^{-t\Delta}: t\geqslant 0\}$ is a semigroup, i.e. $e^{-(s+t)\Delta}=e^{-s\Delta}\circ e^{-t\Delta}$. In addition, $e^{-t\Delta}$ is a compact, nonnegative definite, self-adjoint operator for every $t>0$.

Proof. According to the Weitzenböck formula (see Proposition 8.1), the Hodge Laplacian Δ is a generalised Laplacian on the vector bundle of k-forms in the sense of Definition 9.1. This ensures the existence and uniqueness of a smooth kernel $p(t,x,y)$ for Δ (see Theorem 9.1). The symmetry of $p(t,x,y)$ follows from the self-adjointness of Δ. The statement of (ii) is contained in Corollary 9.3, where compactness comes from the fact that $e^{-t\Delta}$ is a Hilbert-Schmidt operator (see Lemma 9.7). □

Definition 8.8. The map $(t,x,y)\mapsto p(t,x,y)$ (respectively, $\{e^{-t\Delta}:t\geqslant 0\}$) is called the *heat kernel* (respectively, the *heat semigroup*) of the Hodge Laplacian on k-forms.

Remark 8.2. In general, $p(t,x,y)$ should belong to $\Lambda^k T_x^*M\otimes(\Lambda^k T_y^*M)^*$. Here we identified $(\Lambda^k T_y^*M)^*$ with $\Lambda^k T_y^*M$ using the Riemannian metric.

Proof of Theorem 8.1. Fix $t_0>0$. Since $e^{-t_0\Delta}$ is a compact and self-adjoint, it follows from the spectral theorem that the Hilbert space $L^2(\Lambda^k T^*M)$ admits an orthogonal decomposition

$$L^2(\Lambda^k T^*M)=\bigoplus_{n=1}^{\infty}E_n.$$

Here $\{E_n:n\geqslant 1\}$ are all the eigenspaces of $e^{-t_0\Delta}$. All eigenvalues of $e^{-t_0\Delta}$ are nonnegative (because $e^{-t_0\Delta}$ is nonnegative definite) and has finite multiplicity (except for the zero eigenvalue if it exists). In addition, the only possible accumulation point of the eigenvalues is zero. We now break down the proof of the theorem into the following main steps.

(i) $e^{-t_0\Delta}$ *does not have zero eigenvalue. In particular, all eigenvalues of* $e^{-t_0\Delta}$ *are strictly positive and thus* $\dim E_n<\infty$ *for all* n. Suppose on the contrary that $e^{-t_0\Delta}\omega=0$ for some nonzero k-form ω. Then

$$\langle e^{-t_0\Delta}\omega,\omega\rangle_{L^2}=\langle e^{-t_0\Delta/2}\omega,e^{-t_0\Delta/2}\omega\rangle_{L^2}=0\implies e^{-t_0\Delta/2}\omega=0.$$

Arguing inductively, one obtains that $e^{-t_0\Delta/2^n}\omega=0$ for all n. According to (8.5), one has $\omega=0$ which leads to a contradiction.

(ii) E_n *is invariant under* $e^{-t\Delta}$ *for all* t. Let $\gamma>0$ be the eigenvalue of $e^{-t_0\Delta}$ on E_n. For any $\omega\in E_n$, one has

$$e^{-t_0\Delta}(e^{-t\Delta}\omega)=e^{-t\Delta}(e^{-t_0\Delta}\omega)=\gamma e^{-t\Delta}\omega.$$

This implies that $e^{-t\Delta}\omega\in E_n$.

(iii) Since the family $\mathcal{P}\triangleq\{e^{-t\Delta}:t\geqslant 0\}$ is commutative, Part (ii) shows that $\mathcal{P}$ is simultaneously diagonalisable on E_n. It follows that there exists an ONB $\{\omega_1,\omega_2,\cdots\}$ of $L^2(\Lambda^k T^*M)$, such that

$$E_1=\mathrm{Span}\{\omega_1,\cdots,\omega_{k_1}\},\ E_2=\mathrm{Span}\{\omega_{k_1+1},\cdots,\omega_{k_1+k_2}\},\ \cdots.$$

Each ω_i is a common eigenform of $e^{-t\Delta}$ for all t.

(iv) For each ω_i, let $\gamma_i(t)>0$ be its eigenvalue for $e^{-t\Delta}$, i.e.

$$e^{-t\Delta}\omega_i=\gamma_i(t)\omega_i\ \text{ or }\ \int_M\langle p(t,x,y),\omega_i(y)\rangle_y dy=\gamma_i(t)\omega_i(x).$$

Since the heat kernel $p(t,x,y)$ is smooth, it follows that $\gamma_i(t)$ is smooth in t and $\omega_i(x)$ is smooth in x. By using the heat equation, one has

$$\gamma_i'(t)\omega_i + \gamma_i(t)\Delta\omega_i = 0 \ \text{ or } \ \Delta\omega_i = -\frac{\gamma_i'(t)}{\gamma_i(t)}\omega_i.$$

This in particular shows that $\gamma_i'(t)/\gamma_i(t)$ is independent of t. Call this number $-\lambda_i$ so that $\Delta\omega_i = \lambda_i\omega_i$. Using the initial condition $\gamma_i(0) = 1$, one finds that $\gamma_i(t) = e^{-\lambda_i t}$. Note that $\lambda_i \geqslant 0$ since Δ is nonnegative definite.

(v) Since $\gamma_i(t_0) = e^{-\lambda_i t_0}$ are all equal for ω_i's belonging to the same E_n, those λ_i's are repeated eigenvalues of Δ with multiplicity $\dim E_n$. The result of the theorem follows by rearranging the λ_i's in nondecreasing order, which is possible because ∞ is their only possible accumulation point.

□

Remark 8.3. The $k=0$ case corresponds to functions on M. In this case, $\Delta f = 0$ if and only if f is constant on each connected component of M. Therefore, the dimension of the space of harmonic 0-forms coincides with the number of connected components of M (i.e. $\dim H^0_{\mathrm{dR}}(M)$). The Hodge theorem for de Rham cohomology is an extension of this fact to higher degrees (see Theorem 8.3).

Remark 8.4. One can also formulate the L^2-Hodge theorem (with no essential changes) on ΛT^*M (forms with mixed degrees). This is sometimes more convenient for applications.

8.4 C^∞-regularity and the Hodge decomposition theorem

The L^2-Hodge theorem, as a standard consequence of compactness and the spectral theorem, is the easier part of the story. The harder part concerns with C^∞-regularity. To discuss the main theorems here, we first introduce some basic definitions. As before, we assume that M is a closed, oriented, n-dimensional Riemannian manifold. We introduce the operator $D \triangleq d + \delta$. Note that D is a "square root" of Δ in the sense that $D^2 = \Delta$; this follows from the fact that $d^2 = \delta^2 = 0$. In addition, D is also self-adjoint: $\langle D\alpha, \beta\rangle_{L^2} = \langle \alpha, D\beta\rangle_{L^2}$ for all $\alpha, \beta \in \Omega(M)$.

Definition 8.9. Let $\alpha, \beta \in L^2(\Lambda T^*M)$. We say that $D\alpha = \beta$ *in the weak sense* (or just *weakly*) if

$$\langle \alpha, D\varphi\rangle_{L^2} = \langle \beta, \varphi\rangle_{L^2}$$

for all $\varphi \in \Omega(M)$. Similarly, we say that $\Delta\alpha = \beta$ *in the weak sense* if

$$\langle \alpha, \Delta\varphi\rangle_{L^2} = \langle \beta, \varphi\rangle_{L^2}$$

for all $\varphi \in \Omega(M)$.

Definition 8.10. The *space of harmonic forms* is defined by

$$\mathcal{H} \triangleq \ker \Delta \triangleq \{\omega \in L^2(\Lambda T^*M) : \Delta\omega = 0 \text{ weakly}\}.$$

We also introduce the space

$$\mathcal{H}_\infty^\perp \triangleq \{\alpha \in \Omega(M) : \langle \alpha, \omega\rangle_{L^2} = 0 \ \forall \omega \in \mathcal{H}\}$$

of smooth forms that are orthogonal to $\mathcal{H}$.

The two main theorems of Hodge are stated as follows.

Theorem 8.2 (The Hodge Decomposition Theorem)**.** *(i) (Regularity of harmonic forms) The space $\mathcal{H}$ is finite dimensional and all elements of $\mathcal{H}$ are smooth.*

(ii) (Green's operator) The restriction $\Delta|_{\mathcal{H}_\infty^\perp} : \mathcal{H}_\infty^\perp \to \mathcal{H}_\infty^\perp$ defines a linear isomorphism. Define the (C^∞-)Green's operator $G : \Omega(M) \to \Omega(M)$ by setting

$$G|_{\mathcal{H}} \triangleq 0, \ G|_{\mathcal{H}_\infty^\perp} \triangleq \Delta|_{\mathcal{H}_\infty^\perp}^{-1}. \tag{8.6}$$

Then one has

$$\mathrm{id} = \pi_{\mathcal{H}} + \Delta \circ G \tag{8.7}$$

on $\Omega(M)$, where $\pi_{\mathcal{H}}$ denotes the orthogonal projection onto $\mathcal{H}$.

(iii) (Hodge decomposition) The space $\Omega(M)$ admits the following direct sum decomposition:

$$\Omega(M) = \mathcal{H} \oplus \mathrm{Im}(d) \oplus \mathrm{Im}(\delta), \tag{8.8}$$

where $\mathrm{Im}(d) \triangleq \{d\alpha : \alpha \in \Omega(M)\}$ and similarly for $\mathrm{Im}(\delta)$. In addition, the three components in (8.8) are orthogonal in L^2.

Theorem 8.3 (The Hodge Theorem for de Rham Cohomology)**.** *Let $0 \leqslant k \leqslant n$. Every de Rham cohomology class in $H^k(M)$ admits a unique harmonic representative, which minimises the L^2-norm over all representatives of the same class. In particular, $H^k(M)$ is isomorphic to the space of harmonic k-forms.*

8.4.1 Basic analytic ingredients

The proofs of Theorem 8.2 and Theorem 8.3 rely on two key analytic lemmas (*Gårding's inequality* and *the regularity of D*). In this section, we only state these lemmas and then proceed to prove the main theorems. After that, we come back to prove these analytic lemmas in Section 8.5. For each $s \in \mathbb{N}$, let $(H_s(\Lambda T^*M), \langle\cdot,\cdot\rangle_s)$ denote the order-s Sobolev space on forms (see Definition C.8). In the sequel, we will just write H_s to ease notation. Note that H_s is a Hilbert space and $(H_0, \|\cdot\|_0) = (L^2(\Lambda T^*M), \|\cdot\|_{L^2})$.

Gårding's inequality allows one to control first order derivatives *in all directions* in terms of the operator D which only involves a mixture of specific directions.

Lemma 8.7 (Gårding's inequality)**.** *For each $s \in \mathbb{N}$, there exists $C_s > 0$ such that*

$$\|\omega\|_{s+1} \leqslant C_s\big(\|\omega\|_s + \|D\omega\|_s\big)$$

for all $\omega \in H_{s+1}(\Lambda T^*M)$.

The regularity lemma asserts that D increases regularity by one degree. Intuitively, this is a natural consequence of Gårding's inequality (D controls first order regularity). We say $D\alpha \in H_s$ if $D\alpha = \beta$ weakly for some $\beta \in H_s$.

Lemma 8.8 (Regularity of D)**.** *Suppose that $\omega \in L^2$ and $D\omega \in H_s$. Then $\omega \in H_{s+1}$. In particular, $D\omega \in \Omega(M) \implies \omega \in \Omega(M)$.*

Remark 8.5. What lies at the heart of the proofs of these two lemmas is the so-called *Weitzenböck formula* (see Proposition 8.1 and Proposition C.2). It asserts that the Hodge Laplacian Δ differs from the Bochner Laplacian $\nabla^*\nabla$ by a zeroth order operator $\mathcal{R}$ (the curvature endomorphism), i.e. $\Delta = \nabla^*\nabla + \mathcal{R}$. Using this relation, one sees that

$$\|D\omega\|_0^2 = \langle \Delta\omega, \omega\rangle = \langle \nabla^*\nabla\omega + \mathcal{R}\omega, \omega\rangle = \|\nabla\omega\|_0^2 + \langle \mathcal{R}\omega, \omega\rangle \geqslant c\|\omega\|_1^2 - C\|\omega\|_0^2,$$

which basically yields Gårding's inequality (at least for $s = 0$). In addition, Δ is an elliptic operator, which intuitively explains the regularity lemma from the perspective of classical elliptic theory (D is the "square root" of an elliptic operator and the latter increases regularity by two).

8.4.2 Green's operator and the Hodge decomposition

We now proceed to prove Theorem 8.2. The use of Green's operator plays an essential role in the argument. We first consider Claims (i) and (ii) of the theorem.

Proof of Theorem 8.2 (i). Define $\ker D \triangleq \{\omega \in L^2(\Lambda T^*M) : D\omega = 0 \text{ weakly}\}$. Elements in $\ker D$ are smooth as a direct consequence of the regularity lemma. For elements in $\mathcal{H}$, one has

$$\Delta\omega = 0 \text{ weakly} \implies D(D\omega) = 0 \text{ weakly} \implies D\omega \in \Omega(M) \implies \omega \in \Omega(M).$$

It is obvious that $\ker D \subseteq \mathcal{H}$. Conversely, one has

$$\omega \in \mathcal{H} \implies \Delta\omega = 0 \implies d\omega = \delta\omega = 0 \implies \omega \in \ker D.$$

Therefore, $\mathcal{H} = \ker D = \{\omega \in \Omega(M) : \Delta\omega = 0\}$. For the finite dimensionality of $\mathcal{H}$, suppose on the contrary that $\dim \mathcal{H} = \infty$. One can then choose an orthonormal $\{x_n\} \subseteq \mathcal{H}$, i.e. $Dx_n = 0$ and $\langle x_m, x_n\rangle_{L^2} = \delta_{mn}$. It follows from Gårding's inequality that $\{x_n\}$ is bounded in H_1. According to Rellich's compactness theorem (see Theorem C.3), $\{x_n\}$ contains a convergent subsequence in L^2. But this is impossible since $\|x_m - x_n\|_0 = \sqrt{2}$ for all $m \neq n$.

□

Remark 8.6. $\mathcal{H}$ is just the zero-eigenspace of Δ (equivalently, the 1-eigenspace of $e^{-t\Delta}$) appearing in the L^2-Hodge theorem.

Proof of Theorem 8.2 (ii). It is obvious from self-adjointness that $\Delta(\mathcal{H}_\infty^\perp) \subseteq \mathcal{H}_\infty^\perp$. In addition, $\Delta|_{\mathcal{H}_\infty^\perp}$ is injective;

$$\Delta\alpha = 0 \ \& \ \alpha \in \mathcal{H}_\infty^\perp \implies \alpha \in \mathcal{H} \cap \mathcal{H}_\infty^\perp \implies \alpha = 0.$$

It remains to show that $\Delta|_{\mathcal{H}_\infty^\perp}$ is surjective. The argument is quite standard from the perspective of weak solution theory of elliptic PDEs.

Consider the bilinear form

$$[\psi, \varphi] \triangleq \langle \psi, \Delta\varphi \rangle_{L^2} = \langle D\psi, D\varphi \rangle_{L^2} \quad , \psi, \varphi \in \mathcal{H}_\infty^\perp.$$

It is apparent that $[\cdot, \cdot]$ is symmetric and positive definite. In addition, one has

$$[\varphi, \varphi] \leqslant C\|\varphi\|_1^2 \quad \forall \varphi \in \mathcal{H}_\infty^\perp \tag{8.9}$$

with some positive constant C.

The key observation is that

$$[\varphi, \varphi] \geqslant c\|\varphi\|_1^2 \quad \forall \varphi \in \mathcal{H}_\infty^\perp \tag{8.10}$$

for another positive constant c. Suppose on the contrary that there exists a sequence $\{\varphi_n\} \subseteq \mathcal{H}_\infty^\perp$ with $\|\varphi_n\|_1 = 1$ but

$$[\varphi_n, \varphi_n] = \|D\varphi_n\|_0^2 \to 0.$$

According to Rellich's compactness theorem, $\varphi_{n_k} \to$ some φ in L^2 along some subsequence n_k. We claim that $D\varphi = 0$ weakly (and hence $\varphi \in \mathcal{H}$). Indeed, for any $\psi \in \Omega(M)$ one has

$$\langle \varphi, D\psi \rangle_{L^2} = \lim_{k\to\infty} \langle \varphi_{n_k}, D\psi \rangle_{L^2} = \lim_{k\to\infty} \langle D\varphi_{n_k}, \psi \rangle_{L^2} = 0.$$

On the other hand, $\varphi_{n_k} \perp \mathcal{H} \implies \varphi \perp \mathcal{H}$. Therefore, $\varphi \in \mathcal{H} \cap \mathcal{H}^\perp = \{0\}$, and in particular, $\|\varphi_{n_k}\|_0 \to 0$. This contradicts Gårding's inequality which reads

$$\|\varphi_{n_k}\|_1 \lesssim \|D\varphi_{n_k}\|_0 + \|\varphi_{n_k}\|_0.$$

The LHS is equal to one while the RHS converges to zero. The relations (8.9, 8.10) show that $[\cdot, \cdot]$ and $\langle \cdot, \cdot \rangle_1$ are equivalent inner products on $\mathcal{H}_\infty^\perp$. As a result, $[\cdot, \cdot]$ admits a unique extension to $H_1^\perp \triangleq \{\omega \in H_1 : \omega \perp \mathcal{H} \text{ in } L^2\}$ (which is also the completion of $\mathcal{H}_\infty^\perp$ under $\| \cdot \|_1$).

Now suppose that $\beta \in \mathcal{H}_\infty^\perp$ and let us define

$$l(\varphi) \triangleq \langle \beta, \varphi \rangle_{L^2}, \quad \varphi \in H_1^\perp.$$

Since

$$|l(\varphi)| \leqslant \|\beta\|_{L^2} \|\varphi\|_{L^2} \leqslant \|\beta\|_{L^2} \|\varphi\|_1,$$

one knows that $l(\cdot)$ is a continuous linear functional with respect to the inner product $[\cdot, \cdot]$. It follows from the Riesz representation theorem that there exists a unique $\alpha \in H_1^\perp$, such that

$$l(\varphi) = [\alpha, \varphi] \ (\iff \langle \alpha, \Delta\varphi \rangle_{L^2} = \langle \beta, \varphi \rangle) \tag{8.11}$$

for all $\varphi \in H_1^\perp$. The relation (8.11) also holds trivially for $\varphi \in \mathcal{H}$. As a result, $\Delta\alpha = \beta$ weakly. Since β is smooth, one concludes from the regularity lemma that α is also smooth. This proves that $\Delta|_{\mathcal{H}_\infty^\perp} : \mathcal{H}_\infty^\perp \to \mathcal{H}_\infty^\perp$ is a linear isomorphism.

Finally, let us define *Green's operator* $G : \Omega(M) \to \Omega(M)$ by (8.6). For any $\omega \in \Omega(M)$, one has $\omega - \pi_{\mathcal{H}}\omega \in \mathcal{H}_\infty^\perp$ and therefore, it is equal to $\Delta\alpha$ where

$$\alpha = \Delta|_{\mathcal{H}_\infty^\perp}^{-1}(\omega - \pi_{\mathcal{H}}\omega) = G(\omega - \pi_{\mathcal{H}}\omega) = G(\omega).$$

It follows that $\omega = \pi_{\mathcal{H}}\omega + \Delta G(\omega)$. This proves the decomposition (8.7).

□

By extending the above argument further, one can construct Green's operator for D which is sometimes quite useful. For each $s \in \mathbb{N}$, we define $H_s^\perp \triangleq \{\omega \in H_s : \omega \perp \mathcal{H}\}$ as usual.

Lemma 8.9. *For each $s \geqslant 1$, the operator $D|_{H_s^\perp} : H_s^\perp \to H_{s-1}^\perp$ has a well-defined, bounded inverse.*

Proof. It is easily seen that $D(H_s^\perp) \subseteq H_{s-1}^\perp$ and $D|_{H_s^\perp}$ is injective. Let $R_s \triangleq D(H_s^\perp)$. The same argument leading to (8.10) shows that $D|_{H_s^\perp}^{-1} : R_s \to H_s^\perp$ is a bounded linear operator. It remains to prove that R_s is dense in $H_{s-1}^\perp$. But this follows from the facts that $H_{s-1}^\perp$ is the $\|\cdot\|_{s-1}$-closure of $\mathcal{H}_\infty^\perp$ and any element $\beta \in \mathcal{H}_\infty^\perp$ can be expressed as

$$\beta = \Delta\alpha = D(D\alpha) \in R_s$$

for some $\alpha \in \mathcal{H}_\infty^\perp$ as a consequence of Theorem 8.2 (ii). □

Remark 8.7. The composition $H_{s-1}^\perp \xrightarrow{D^{-1}} H_s^\perp \hookrightarrow H_{s-1}^\perp$ is a compact operator, because the latter map is compact due to Rellich's compactness theorem.

We now prove the Hodge decomposition (8.8). This is a simple application of Green's operator.

Proof of Theorem 8.2 (iii). The orthogonality of $\mathcal{H}, \mathrm{Im}d, \mathrm{Im}\delta$ follows easily from duality; for instance, one has

$$\langle \alpha, \delta\beta \rangle = \langle d\alpha, \beta \rangle = 0 \quad \forall \alpha \in \mathcal{H}, \delta\beta \in \mathrm{Im}\delta.$$

The decomposition (8.8) follows from (8.7), as one can write

$$\omega = \pi_{\mathcal{H}}\omega + \Delta \circ G(\omega) = \pi_{\mathcal{H}}\omega + d(\delta G(\omega)) + \delta(dG(\omega))$$

for any $\omega \in \Omega(M)$.

□

8.4.3 Proof of the Hodge theorem for de Rham cohomology

In this subsection, we present two proofs of Theorem 8.3: the standard functional-analytic proof and the more enlightening heat equation proof.

The functional-analytic proof. Let $[\omega] \in H^k(M)$. By Theorem 8.2 (ii), one can write

$$\omega = \pi_{\mathcal{H}}(\omega) + \Delta \circ G(\omega) = \pi_{\mathcal{H}}(\omega) + d(\delta G(\omega)) + \delta(dG(\omega)).$$

Note that G commutes with d (because Δ does and $G = \Delta^{-1}$ on $\mathcal{H}_\infty^\perp$) and therefore, $dG(\omega) = G(d\omega) = 0$. It follows that

$$\omega = \pi_{\mathcal{H}}(\omega) + d(\delta G(\omega)).$$

In particular, the harmonic form $\pi_{\mathcal{H}}(\omega)$ is a representative of $[\omega]$.

Suppose that both h_1, h_2 are representatives of $[\omega]$. Then $h_1 - h_2 = d\beta$ for some $\beta \in \Omega^{k-1}(M)$. It follows that

$$\|h_1 - h_2\|_{L^2}^2 = \langle h_1 - h_2, d\beta \rangle_{L^2} = \langle \delta(h_1 - h_2), \beta \rangle_{L^2} = 0.$$

As a result, one has $h_1 = h_2$. This shows the uniqueness of harmonic representatives of $[\omega]$.

For the last claim, note that any representative of $[\omega]$ must have the form $h + d\beta$ where h is the unique harmonic representative and $\beta \in \Omega^{k-1}(M)$. One then obtains that

$$\|h + d\beta\|_{L^2}^2 = \|h\|_{L^2}^2 + \|d\beta\|_{L^2}^2 + 2\langle \delta h, \beta \rangle_{L^2} = \|h\|_{L^2}^2 + \|d\beta\|_{L^2}^2 \geqslant \|h\|_{L^2}^2.$$

This shows that h minimises the L^2-norm in the class $[\omega]$.

□

The heat equation proof. We again assume that ω is a given closed k-form. It is enough to prove the existence of a harmonic representative of $[\omega]$; uniqueness and the L^2-minimising property follow from the same argument as before.

For each $t > 0$, one has

$$\frac{d}{dt}e^{-t\Delta}\omega = -e^{-t\Delta}\Delta\omega = d(-e^{-t\Delta}\delta\omega), \tag{8.12}$$

where the last equality comes from the facts that Δ commutes with d, δ and ω is a closed form. By integrating (8.12) from 0 to t, one finds that

$$e^{-t\Delta}\omega - \omega = d\beta(t);\ \ \beta(t) \triangleq -\int_0^t e^{-s\Delta}\delta\omega ds \in \Omega^{k-1}(M). \tag{8.13}$$

On the other hand, according to the L^2-Hodge theorem,

$$\lim_{t\to\infty} e^{-t\Delta}\omega = \pi_{\mathcal{H}}(\omega) \quad \text{in } L^2. \tag{8.14}$$

In fact, the $e^{-t\Delta}$-action on those eigenform directions with strictly positive eigenvalues will all vanish in the large t limit due to the exponential decay. The only surviving term is the projection on the zero-eigenspace (i.e. space of harmonic k-forms). Now the relations (8.13, 8.14) imply that

$$d\beta(t) \to \pi_{\mathcal{H}}(\omega) - \omega \quad \text{in } L^2 \tag{8.15}$$

as $t \to \infty$.

We claim that $\beta(t)$ converges to some $\beta \in \Omega^{k-1}(M)$ under $\|\cdot\|_1$. Indeed, since $\beta(t) \in \text{Im}\delta \subseteq \mathcal{H}_1^{\perp}$ and $D\beta(t) = d\beta(t)$, one knows from Lemma 8.9 that

$$\|\beta(t) - \beta(s)\|_1 \leqslant C\|d\beta(t) - d\beta(s)\|_0 \quad \forall s, t$$

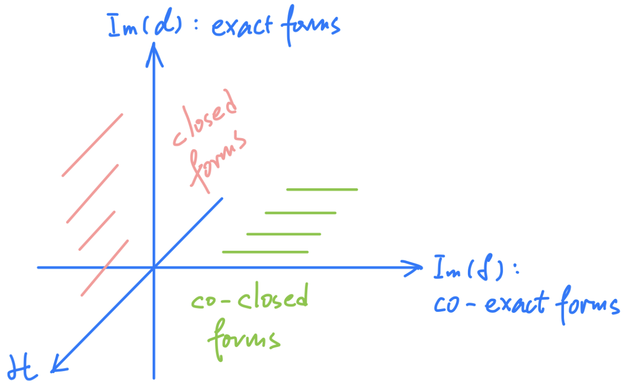
with some universal constant C. It follows from (8.15) that $\{\beta(t)\}$ is a Cauchy sequence in H_1 as $t \to \infty$, hence converging to some β in H_1. In particular, $d\beta(t) \to d\beta$ in L^2, and as a result, $d\beta = \pi_{\mathcal{H}}\omega - \omega$ (in the weak sense). But one also has $\delta\beta = 0$ (because $\delta\beta(t) = 0$). Therefore, $D\beta = \pi_{\mathcal{H}}\omega - \omega$ (in the weak sense). Now the regularity of D (Lemma 8.8) shows that β is smooth. Consequently, the relation

$$D\beta = d\beta = \pi_{\mathcal{H}}\omega - \omega$$

holds in the classical sense. This proves that both $\pi_{\mathcal{H}}\omega$ and ω represent the same cohomology class.

$\square$

Remark 8.8. By using Lemma 8.4 and Theorem 8.3, one knows that every co-closed form α admits a unique decomposition $\alpha = \omega + \delta\beta$ where $\omega \in \mathcal{H}$. The following picture is a nice illustration of the Hodge decomposition (8.8).



8.5 Proof of basic analytic lemmas

In this section, we return to the proofs of the two key analytic lemmas. As we mentioned, the core ingredient is the Weitzenböck formula which we shall first discuss.

8.5.1 The Weitzenböck formula

In this subsection, we assume that M is an oriented, n-dimensional Riemannian manifold. Let ∇ be the Levi-Civita connection. Let $\bar{\nabla} : \Omega(M) \to \Gamma(T^*M \otimes \Lambda T^*M)$ be the induced connection on the exterior algebra bundle ΛT^*M (see Appendix C.1), whose associated connection Laplacian is denoted as $\Delta^{\Lambda T^*M}$ (see Definition C.6). On the other hand, let $\Delta_{\mathrm{B}} \triangleq \bar{\nabla}^*\bar{\nabla} : \Omega(M) \to \Omega(M)$ be the Bochner Laplacian, where $\bar{\nabla}^* : \Gamma(T^*M \otimes \Lambda T^*M) \to \Omega(M)$ denotes the formal L^2-adjoint of $\bar{\nabla}$ (see Definition C.7). Since the connection $\bar{\nabla}$ is compatible with the Euclidean metric on ΛT^*M given by Definition 8.2, it follows from Proposition C.2 that $\Delta_{\mathrm{B}} = \Delta^{\Lambda T^*M}$.

The *Weitzenböck formula* asserts that the operator

$$\mathcal{R} \triangleq \Delta - \Delta^{\Lambda T^*M} = \Delta - \Delta_{\mathrm{B}}$$

is tensorial (i.e. $\mathcal{R} \in \Gamma(\mathrm{End}(\Lambda T^*M))$). In fact, it provides an explicit expression of $\mathcal{R}$ in terms of the Riemann curvature tensor (see Definition 8.12 and Proposition 8.1).

Clifford multipliers on exterior algebra

We need some multilinear algebra tools in order to define the aforementioned curvature endomorphism $\mathcal{R}$. Let V be a finite dimensional, real Euclidean space and we always identity $V^* \cong V$ through the given metric.

Definition 8.11. Let $v \in V$ be given fixed. We define the following linear operators on the exterior algebra ΛV^*.

(i) The operator $a(v)$ is the exterior product by v^* (v^* is the dual of v), i.e.

$$a(v)\omega \triangleq v^* \wedge \omega, \quad \omega \in \Lambda V^*.$$

(ii) The operator $\iota(v)$ is the interior product by v, i.e. the unique linear operator $\iota : \Lambda V^* \to \Lambda V^*$ satisfying $\iota(v)\omega = \omega(v)$ for all $\omega \in V^*$ and

$$\iota(v)(\omega \wedge \eta) = (\iota(v)\omega) \wedge \eta + (-1)^k \omega \wedge \iota(v)\eta. \tag{8.16}$$

for all $\omega \in \Lambda^k V^*$ and $\eta \in \Lambda V^*$.

(iii) We define $c(v), \hat{c}(v) \in \mathrm{End}(\Lambda V^*)$ by

$$c(v) \triangleq a(v) - \iota(v), \ \hat{c}(v) \triangleq a(v) + \iota(v). \tag{8.17}$$

These two operators are called *Clifford multipliers.*

By definition, $a(v)$ increases degree by one while $\iota(v)$ reduces degree by one. Note that $\iota(v)$ does not depend on the metric but $a(v)$ does (because the latter uses the metric dual v^*). It is elementary to check that $a(v)^* = \iota(v)$ ($a(v)^*$ is the adjoint of $a(v)$ under the metric given by Definition 8.2 with $W = V^*$). Let $\{\varepsilon_1, \cdots, \varepsilon_n\}$ be an ONB basis of V and let $\{\theta^1, \cdots, \theta^n\}$ be its dual. It is also clear from definition that

$$\iota(\varepsilon_i)\theta^{j_1} \wedge \cdots \wedge \theta^{j_k} = \begin{cases} 0, & \text{if } i \notin \{j_1, \cdots, j_k\}; \\ (-1)^{q-1}\theta^{j_1} \wedge \cdots \wedge \widehat{\theta^{j_q}} \wedge \cdots \wedge \theta^{j_k}, & \text{if } i = j_q \in \{j_1, \cdots, j_k\} \end{cases}$$

for any $j_1 < \cdots < j_k$.

Notation. Under an ONB, we simply write

$$a_i \triangleq a(\varepsilon_i), \ \iota_i \triangleq \iota(\varepsilon_i), \ c_i \triangleq c(\varepsilon_i), \ \hat{c}_i \triangleq \hat{c}(\varepsilon_i). \tag{8.18}$$

Given multi-index $I = (i_1, \cdots, i_k)$, we set $a_I \triangleq a(\varepsilon_{i_1}) \cdots a(\varepsilon_{i_k})$ (as a convention, $a_\emptyset \triangleq \mathrm{id}$) and similarly for ι_I, c_I, $\hat{c}_I$. We also write $\theta^I \triangleq \theta^{i_1} \wedge \cdots \wedge \theta^{i_k}$.

The following commutation relations are particularly useful. Given two operators A, B, we write $\{A, B\} \triangleq AB + BA$.

Lemma 8.10. *One has*

$$\{a_i, a_j\} = \{\iota_i, \iota_j\} = 0, \ \{\iota_i, a_j\} = \delta_{ij}. \tag{8.19}$$

In addition, one has

$$\{c(v), c(w)\} = -2\langle v, w\rangle, \ \{\hat{c}(v), \hat{c}(w)\} = 2\langle v, w\rangle, \ \{c(v), \hat{c}(w)\} = 0 \tag{8.20}$$

for any $v, w \in V$.

Proof. Left as an exercise. □

Lemma 8.11. *Let $\mathcal{A}$ denote the set of multi-indices $I = (i_1 < \cdots < i_k)$ for all $k = 0, \cdots, n$. Then both families $\{a_I \iota_J : I, J \in \mathcal{A}\}$ and $\{c_I \hat{c}_J : I, J \in \mathcal{A}\}$ are bases of* $\mathrm{End}(\Lambda V^*)$.

Proof. It suffices to prove the claim for $\{a_I \iota_J : I, J \in \mathcal{A}\}$, since $c_i, \hat{c}_i$ can be expressed linearly in terms of a_i, ι_i and vice versa. To this end, let us define

$$n_i \triangleq a_i \iota_i, \ p_i \triangleq \mathrm{id} - a_i \iota_i.$$

It is readily checked that

$$n_i \theta^K = \begin{cases} \theta^K, & \text{if } i \in K; \\ 0, & \text{if } i \notin K, \end{cases} \quad p_i \theta^K = \begin{cases} 0, & \text{if } i \in K; \\ \theta^K, & \text{if } i \notin K \end{cases}$$

for any i and multiindex K. Define

$$q_J \triangleq \prod_{j \in J} n_j \prod_{j \notin J} p_j.$$

It follows that

$$q_J \theta^K = \begin{cases} \theta^J, & \text{if } J = K; \\ 0, & \text{if } J \neq K \end{cases}$$

for any pair of multiindices (J, K). As a consequence, the operator $E_{IJ} \triangleq a_I \iota_J q_J$ sends θ^J to $\pm\theta^I$ (the actual sign does not matter) and sends θ^K to zero for any $K \neq J$. This shows that $\{E_{IJ} : I, J \in \mathcal{A}\}$ is a basis of $\mathrm{End}(\Lambda V^*)$. On the other hand, one knows from the relations (8.19) that

$$\mathrm{Span}\{E_{IJ} : I, J \in \mathcal{A}\} \subseteq \mathrm{Span}\{a_I \iota_J : I, J \in \mathcal{A}\}.$$

By dimension comparison, one concludes that $\{a_I \iota_J\}$ is also a basis of $\mathrm{End}(\Lambda V^*)$. □

We conclude this part with a lemma that will be useful for studying the curvature endomorphism. Given $A \in \mathrm{End}(V)$, let $A^* \in \mathrm{End}(V^*)$ denote its adjoint and we further extend it to a linear operator on ΛV^* (still denoted as A^*) via the Leibniz rule:

$$A^*(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}) \triangleq \sum_{p=1}^{k} \theta^{i_1} \wedge \cdots \wedge A^*\theta^{i_p} \wedge \cdots \wedge \theta^{i_k}, \quad i_1 < \cdots < i_k. \tag{8.21}$$

Lemma 8.12. *Let $A \in \mathrm{End}(V)$ with matrix (A_i^j), i.e. $A(\varepsilon_i) = A_i^j \varepsilon_j$. Then one has $A^* = \sum_{i,j=1}^n A_i^j a_i \iota_j$.*

Proof. Note that

$$a_q \iota_r(\theta^I) = \begin{cases} 0, & \text{if } r \notin I; \\ \theta^{i_1} \wedge \cdots \wedge \theta^q \wedge \cdots \wedge \theta^{i_k}, & \text{if } r = i_p \in I. \end{cases}$$

Therefore, one has

$$\begin{aligned} \sum_{q=1}^{n} A_q^r a_q \iota_r(\theta^I) &= \sum_{q=1}^{n} \sum_{p=1}^{k} A_q^{i_p} \theta^{i_1} \wedge \cdots \wedge \theta^q \wedge \cdots \wedge \theta^{i_k} \\ &= \sum_{p=1}^{k} \theta^{i_1} \wedge \cdots \wedge A^*\theta^{i_p} \wedge \cdots \wedge \theta^{i_k} = A^*(\theta^I). \end{aligned}$$

where the second identity follows from the relation that $A^*\theta^i = A_j^i \theta^j$. □

The curvature endomorphism

We now return to the geometric setting. Recall that the Riemann curvature tensor $R \in \Omega^2(M, \mathrm{End}(TM))$ is defined by (C.2). Under a local ONF $\{e_1, \cdots, e_n\}$, the curvature coefficients are defined by (C.7) and they satisfy the basic symmetry relations (C.8).

We now introduce a few curvature-type operators acting on the exterior algebra bundle, which are all canonically induced from R. Firstly, we define $\bar{R} \in \Omega^2(M, \mathrm{End}(\Lambda T^*M))$ to be the curvature tensor of $\bar{\nabla}$, namely,

$$\bar{R}(X,Y)\alpha \triangleq \bar{\nabla}_X \bar{\nabla}_Y \alpha - \bar{\nabla}_Y \bar{\nabla}_X \alpha - \bar{\nabla}_{[X,Y]}\alpha, \quad X, Y \in \Gamma(TM), \alpha \in \Omega(M).$$

Next, let $R(X,Y)^* \in \Gamma(\mathrm{End}(\Lambda T^*M))$ be the fiberwise endomorphism defined by (8.21) induced from $R(X,Y) \in \Gamma(\mathrm{End}(TM))$.

Lemma 8.13. *$R(X,Y)^* = -\bar{R}(X,Y)$ for all $X, Y \in \Gamma(TM)$.*

We now introduce the curvature endomorphism which will be the operator appearing in the Weitzenböck formula. Let $\{e_1, \cdots, e_n\}$ be a local ONF of TM. We continue to use the notation (8.18), which is now understood fiberwisely.

Definition 8.12. The *curvature endomorphism* $\mathcal{R} \in \Gamma(\mathrm{End}(\Lambda T^*M))$ is defined by $\mathcal{R} \triangleq -R_{ijkl}a_i\iota_j a_k\iota_l$.

We point out that the definition of $\mathcal{R}$ does not depend on the choice of the local ONF and it is therefore an intrinsic tensor field on M. This fact is immediate from the Weitzenböck formula which will be established in the next part. Here we present an alternative representation of $\mathcal{R}$ in terms of the Clifford multipliers.

Lemma 8.14. *One has*

$$\mathcal{R} = \frac{1}{2}\sum_{i,j=1}^{n} c(e_i)c(e_j)\bar{R}(e_i, e_j).$$

Proof. Note that $R(e_i, e_j)e_k = R_{lkij}e_l$. According to Lemma 8.12 and Lemma 8.13, one has

$$\bar{R}(e_i, e_j) = -R(e_i, e_j)^* = R_{ijkl}a_k\iota_l. \tag{8.22}$$

It follows that

$$\begin{aligned}\sum_{i,j=1}^{n} c(e_i)c(e_j)\bar{R}(e_i, e_j) &= \sum_{i,j=1}^{n}(a_i - \iota_i)(a_j - \iota_j)R_{ijkl}a_k\iota_l \\ &= \sum_{i,j=1}^{n}\big(R_{ijkl}a_ia_ja_k\iota_l + R_{ijkl}\iota_i\iota_j a_k\iota_l \\ &\qquad - R_{ijkl}\iota_i a_j a_k\iota_l - R_{ijkl}a_i\iota_j a_k\iota_l\big).\end{aligned} \tag{8.23}$$

The first sum is zero; according to the relations (8.19, C.8) one has

$$\begin{aligned}3R_{ijkl}a_ia_ja_k\iota_l &= R_{ijkl}a_ia_ja_k\iota_l + R_{ijkl}a_ja_ka_i\iota_l + R_{ijkl}a_ka_ia_j\iota_l \\ &= (R_{ijkl} + R_{kijl} + R_{jkil})a_ia_ja_k\iota_l = 0.\end{aligned}$$

The second sum in (8.23) is also zero by the same argument (first interchange a_k and ι_l). The last two sums are equal to

$$-\sum_{i,j=1}^{n}\big(R_{jikl}a_j\iota_i a_k\iota_l + R_{ijkl}a_i\iota_j a_k\iota_l\big) = 2\mathcal{R}.$$

The desired relation thus follows. $\square$

The Weitzenböck and Lichnerowicz formulae

Our main goal in this part is to prove the following result.

Proposition 8.1 (The Weitzenböck formula)**.** *Let $\mathcal{R}$ be the curvature endomorphism introduced by Definition 8.12. Then one has*

$$\Delta = \Delta^{\Lambda T^*M} + \mathcal{R}.$$

In particular, the operator $\mathcal{R} \in \Gamma(\mathrm{End}(\Lambda T^*M))$ is intrinsic on M.

The proof of Proposition 8.1 relies on the following two lemmas. The first lemma gives a representation of $d+\delta$ in terms of the connection and the Clifford action, which is the key to connecting the Hodge Laplacian with the connection Laplacian.

Lemma 8.15. *Under any local PONF $\{e_1, \cdots, e_n\}$, one has*

$$d = \sum_{i=1}^{n} a(e_i) \circ \bar{\nabla}_{e_i}, \ \ \delta = -\sum_{i=1}^{n} \iota(e_i) \circ \bar{\nabla}_{e_i}. \tag{8.24}$$

In particular, $d + \delta = c(e_i) \circ \bar{\nabla}_{e_i}$.

Proof. The computation is lengthy and tedious. We divide the argument into several steps. Let $\{\eta^1, \cdots, \eta^n\}$ be the dual coframe field. Recall that the Christoffel symbols with respect to $\{e_i\}$ are defined by $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$.

(i) We claim that

$$\bar{\nabla}_{e_i} \eta^I = -\sum_{s=1}^{r} \Gamma^{i_s}_{ij} a(e_j) \iota(e_{i_s}) \eta^I, \tag{8.25}$$

for any $I = (i_1 < \cdots < i_r)$. To prove this, first note that $\bar{\nabla}_{e_i} \eta^l = -\Gamma^l_{ij} \eta^j$. By the definition of $\bar{\nabla}$, one has

$$\begin{aligned}
\bar{\nabla}_{e_i} \eta^I &= \sum_{s=1}^{r} \eta^{i_1} \wedge \cdots \wedge \bar{\nabla}_{e_i} \eta^{i_s} \wedge \cdots \wedge \eta^{i_r} \\
&= -\sum_{s=1}^{r} \Gamma^{i_s}_{ij} \eta^{i_1} \wedge \cdots \wedge \eta^j \wedge \cdots \wedge \eta^{i_r} \\
&= -\sum_{s=1}^{r} \Gamma^{i_s}_{ij} a(e_j) \iota(e_{i_s}) \eta^I.
\end{aligned}$$

(ii) We claim that

$$d\eta^l = -\Gamma^l_{ij} \eta^i \wedge \eta^j, \ \ d\eta^I = -\sum_{s=1}^{r} \Gamma^{i_s}_{ij} a(e_i) a(e_j) \iota(e_{i_s}) \eta^I \tag{8.26}$$

Indeed, by using the relation (B.9) and the torsion freeness of ∇, one finds that

$$d\eta^l(e_i, e_j) = -\eta^l([e_i, e_j]) = -\eta^l\big(\nabla_{e_i} e_j - \nabla_{e_j} e_i\big) = \Gamma^l_{ji} - \Gamma^l_{ij}.$$

This gives the first relation of (8.26). In addition, one finds by using (8.26) that

$$\begin{aligned} d\eta^I &= \sum_{s=1}^{r} (-1)^{s-1} \eta^{i_1} \wedge \cdots \wedge d\eta^{i_s} \wedge \cdots \wedge \eta^{i_r} \\ &= -\sum_{s=1}^{r} (-1)^{s-1} \Gamma^{i_s}_{ij} \eta^{i_1} \wedge \cdots \wedge \eta^i \wedge \eta^j \wedge \cdots \wedge \eta^{i_r} \\ &= -\sum_{s=1}^{r} \Gamma^{i_s}_{ij} a(e_i) a(e_j) \iota(e_{i_s}) \eta^I. \end{aligned}$$

This gives the second relation of (8.26).

(iii) Consider a form $\alpha = \alpha_I \eta^I$. By using the relation (8.25), one has

$$\begin{aligned} d\alpha &= d\alpha_I \wedge \eta^I + \alpha_I d\eta^I \\ &= (\bar{\nabla} e_i \alpha_I \eta^i) \wedge \eta^I - \alpha_I \sum_{i_s \in I} \Gamma^{i_s}_{ij} a(e_i) a(e_j) \iota(e_{i_s}) \eta^I \\ &= a(e_i)\big(\bar{\nabla}_{e_i} \alpha - \alpha_I \bar{\nabla}_{e_i} \eta^I\big) - \alpha_I \sum_{i_s \in I} \Gamma^{i_s}_{ij} a(e_i) a(e_j) \iota(e_{i_s}) \eta^I \\ &= a(e_i) \bar{\nabla}_{e_i} \alpha + \alpha_I \Gamma^{i_s}_{ij} a(e_i) a(e_j) \iota(e_{i_s}) \eta^I - \alpha_I \sum_{i_s \in I} \Gamma^{i_s}_{ij} a(e_i) a(e_j) \iota(e_{i_s}) \eta^I \\ &= a(e_i) \bar{\nabla}_{e_i} \alpha. \end{aligned}$$

The first relation of (8.24) thus follows.

(iv) To prove the second relation of (8.24), one first notes that the result is invariant under the change of ONFs. Therefore, it is enough to prove the identity at a fixed $x \in M$ under the normal frame field $\{e_i\}$ around x. The main simplification here is that all Christoffel symbols vanish at x (see Appendix C.3).

Consider a k-form $\alpha = f\eta^I$ where $I = (i_1 < \cdots < i_k)$ and let $J \triangleq I^c = (j_1 < \cdots < j_{n-k})$. By using Lemma 8.4 and the definition of the Hodge star, one can write

$$\begin{aligned} \delta(f\eta^I) &= (-1)^{nk+n+1} (-1)^{\varepsilon(I,J)} * d(f\eta^J) = (-1)^{nk+n+1+\varepsilon(I,J)} * (df \wedge \eta^J) \quad \text{(at } x) \\ &= (\nabla_{e_i} f)(-1)^{nk+n+1+\varepsilon(I,J)} * (\eta^i \wedge \eta^J). \end{aligned}$$

The second equality follows from (8.26) as well as the fact that the Christoffel symbols vanish at x. On the other hand, one has

$$-\iota(e_i) \bar{\nabla}_{e_i} (f\eta^I) = (\nabla_{e_i} f)(-\iota(e_i)\eta^I).$$

Therefore, it suffices to show that

$$(-1)^{nk+n+\varepsilon(I,J)} * (\eta^i \wedge \eta^J) = \iota(e_i)\eta^I \tag{8.27}$$

for every fixed i.

The relation (8.27) is obvious if $i \notin I$; both sides are zero in this case. Now suppose that $i = i_s \in I$. By the definition of the Hodge star, one has

$$*(\eta^{i_s} \wedge \eta^J) = (-1)^{\sigma+\mathrm{sgn}(I,J)}\eta^{i_1} \wedge \cdots \wedge \hat{\eta}^{i_s} \wedge \cdots \wedge \eta^{i_k},$$

where σ is the number of transpositions needed to turn $(i_s, j_1, \cdots, j_{n-k}, i_1, \cdots, \hat{i}_s, \cdots, i_k)$ into (I, J). Simple counting shows that $\sigma = (n-k)(k-1) + s - 1$. Therefore,

$$\begin{aligned}(-1)^{nk+n+\varepsilon(I,J)} * (\eta^{i_s} \wedge \eta^J) &= (-1)^{nk+n+(n-k)(k-1)+s-1}\eta^{i_1} \wedge \cdots \wedge \hat{\eta}^{i_s} \wedge \cdots \wedge \eta^{i_k} \\ &= (-1)^{s-1}\eta^{i_1} \wedge \cdots \wedge \hat{\eta}^{i_s} \wedge \cdots \wedge \eta^{i_k} = \iota(e_{i_s})\eta^I.\end{aligned}$$

The relation (8.27) thus follows. □

The second lemma is concerned with the commutation between the connection and Clifford multipliers. Given two operators A, B, we denote $[A, B] \triangleq AB - BA$.

Lemma 8.16. *Let $X, Y \in \Gamma(TM)$. Then one has*

$$[\bar{\nabla}_X, c(Y)] \triangleq c(\bar{\nabla}_X Y), \ \ [\bar{\nabla}_X, \hat{c}(Y)] \triangleq \hat{c}(\bar{\nabla}_X Y). \tag{8.28}$$

Proof. (i) Firstly, we claim that

$$[\bar{\nabla}_X, a(Y)] = a(\nabla_X Y).$$

Indeed, let $\zeta \in \Omega^1(M)$ be the dual of Y. For any $\omega \in \Omega(M)$, one has

$$\bar{\nabla}_X(\zeta \wedge \omega) = (\bar{\nabla}_X\zeta) \wedge \omega + \zeta \wedge \bar{\nabla}_X\omega$$

which implies that

$$[\bar{\nabla}_X, a(Y)](\omega) = (\bar{\nabla}_X\zeta) \wedge \omega = a(\nabla_X Y)(\omega).$$

The last identity follows from the simple observation that $\bar{\nabla}_X\zeta$ is the dual of $\nabla_X Y$.

(ii) Next, we claim that

$$[\bar{\nabla}_X, \iota(Y)] = \iota(\nabla_X Y).$$

Indeed, one finds from explicit calculation that $[\bar{\nabla}_X, \iota(Y)](\omega)$ for any $\omega \in \Omega^1(M)$ and the operator $[\bar{\nabla}_X, \iota(Y)]$ satisfies the relation (8.16). The claim follows since these two properties uniquely characterise the interior product.

(iii) The relation (8.28) follows trivially from (i) and (ii). □

We are now in a position to prove the Weitzenböck formula.

Proof of Proposition 8.1. According to Lemma 8.15 and Lemma 8.16, the Hodge Laplacian is expressed as

$$\Delta = (d+\delta)^2 = c(e_i)\bar{\nabla}_{e_i}c(e_j)\bar{\nabla}_{e_j} = c(e_i)\big(c(e_j)\bar{\nabla}_{e_i} + c(\nabla_{e_i}e_j)\big)\bar{\nabla}_{e_j}.$$

By writing $\nabla_{e_i}e_j = \langle \nabla_{e_i}e_j, e_k\rangle e_k$,

$$\begin{aligned}\text{RHS} &= c(e_i)c(e_j)\bar{\nabla}_{e_i}\bar{\nabla}_{e_j} + c(e_i)c(e_k)\bar{\nabla}_{\langle \nabla_{e_i}e_j, e_k\rangle e_j}\\ &= c(e_i)c(e_j)\bar{\nabla}_{e_i}\bar{\nabla}_{e_j} - c(e_i)c(e_k)\bar{\nabla}_{\nabla_{e_i}e_k}.\end{aligned}$$

By using the relation (8.20) and the expression (C.16) for the connection Laplacian, one then finds that

$$\text{RHS} = \Delta^{\Lambda T^*M} + \sum_{i\neq j} c(e_i)c(e_j)\bar{\nabla}_{e_i}\bar{\nabla}_{e_j} - \sum_{i\neq j} c(e_i)c(e_j)\bar{\nabla}_{\nabla_{e_i}e_j}.$$

We claim that

$$\sum_{i\neq j} c(e_i)c(e_j)\bar{\nabla}_{e_i}\bar{\nabla}_{e_j} = \frac{1}{2}\sum_{i,j} c(e_i)c(e_j)[\bar{\nabla}_{e_i}, \bar{\nabla}_{e_j}], \tag{8.29}$$

$$\sum_{i\neq j} c(e_i)c(e_j)\bar{\nabla}_{\nabla_{e_i}e_j} = \frac{1}{2}\sum_{i,j} c(e_i)c(e_j)\bar{\nabla}_{[e_i,e_j]}. \tag{8.30}$$

Suppose these relations are true for now. By using the expression (C.16) for $\Delta^{\Lambda T^*M}$, one finds that

$$\begin{aligned}\Delta &= \Delta^{\Lambda T^*M} + \frac{1}{2}\sum_{i,j} c(e_i)c(e_j)\big([\bar{\nabla}_{e_i}, \bar{\nabla}_{e_j}] - \bar{\nabla}_{[e_i,e_j]}\big)\\ &= \Delta^{\Lambda T^*M} + \frac{1}{2}\sum_{i,j} c(e_i)c(e_j)\bar{R}(e_i,e_j) = \Delta^{\Lambda T^*M} + \mathcal{R},\end{aligned}$$

where the last identity follows from Lemma 8.14. This produces the Weitzenböck formula.

Now it remains to prove (8.29) and (8.30). We only consider the latter and leave the former as an exercise. One has

$$\begin{aligned}
&\sum_{i\neq j} c(e_i)c(e_j)\bar{\nabla}_{\nabla_{e_i}e_j}\\
&=\sum_{i<j}\left(c(e_i)c(e_j)\bar{\nabla}_{\nabla_{e_i}e_j}+c(e_j)c(e_i)\bar{\nabla}_{\nabla_{e_j}e_i}\right)\\
&=\sum_{i<j}\left(c(e_i)c(e_j)\bar{\nabla}_{\nabla_{e_i}e_j}-c(e_i)c(e_j)\bar{\nabla}_{\nabla_{e_j}e_i}\right)\quad\text{(by (8.20))}\\
&=\sum_{i<j} c(e_i)c(e_j)\bar{\nabla}_{[e_i,e_j]}\quad\text{(by torsion freeness of }\nabla\text{)}\\
&=\frac{1}{2}\sum_{i,j} c(e_i)c(e_j)\bar{\nabla}_{[e_i,e_j]}.
\end{aligned}$$

This proves the desired relation (8.30).

□

To conclude this part, we provide another representation of the curvature endomorphism $\mathcal{R}$ (*the Lichnerowicz formula*). This will be useful in the later heat equation proof of the CGB theorem.

Proposition 8.2 (The Lichnerowicz formula)**.** *One has*

$$\Delta=\Delta^{\Lambda T^*M}+\frac{1}{8}R_{ijkl}c(e_i)c(e_j)\hat{c}(e_k)\hat{c}(e_l)+\frac{\mathrm{Scal}}{4},$$

where Scal $= R_{ijij}$ *denotes the scalar curvature (see Appendix C.1).*

Proof. According to Lemma 8.14 as well as the relation (8.22), one has

$$\begin{aligned}
\mathcal{R}&=\frac{1}{2}R_{ijkl}c(e_i)c(e_j)a(e_k)\iota(e_l)\\
&=\frac{1}{2}R_{ijkl}c(e_i)c(e_j)\frac{c(e_k)+\hat{c}(e_k)}{2}\,\frac{\hat{c}(e_l)-c(e_l)}{2}\\
&=\frac{1}{8}R_{ijkl}c(e_i)c(e_j)\hat{c}(e_k)\hat{c}(e_l)-\frac{1}{8}R_{ijkl}c(e_i)c(e_j)c(e_k)c(e_l).
\end{aligned}$$

The last identity follows since the c-$\hat{c}$ cross terms vanish by symmetry consideration.

To complete the proof, it suffices to show that

$$R_{ijkl}c(e_i)c(e_j)c(e_k)c(e_l)=-2\mathrm{Scal}.\tag{8.31}$$

To ease notation, we use $c_i, \hat{c}_j$ to denote $c(e_i), \hat{c}(e_j)$. Then one has

$$\begin{aligned}
R_{ijkl}c_ic_jc_kc_l &= -R_{ijkl}c_jc_ic_kc_l = -R_{ijkl}c_j(-2\delta_{ik} - c_kc_i)c_l \\
&= 2R_{ijil}c_jc_l + R_{ijkl}c_jc_kc_ic_l \\
&= 2R_{ijil}c_jc_l - (R_{jkil} + R_{kijl})c_jc_kc_ic_l \\
&= 2R_{ijil}c_jc_l - R_{jkil}c_jc_kc_ic_l - R_{kijl}(-2\delta_{jk} - c_kc_j)c_ic_l \\
&= 4R_{ijil}c_jc_l - R_{jkil}c_jc_kc_ic_l + R_{kijl}c_kc_jc_ic_l \\
&= 4R_{ijil}c_jc_l - R_{jkil}c_jc_kc_ic_l + R_{kijl}c_k(-2\delta_{ij} - c_ic_j)c_l \\
&= 6R_{ijil}c_jc_l - R_{jkil}c_jc_kc_ic_l - R_{kijl}c_kc_ic_jc_l.
\end{aligned}$$

It follows that

$$\begin{aligned}
3R_{ijkl}c_ic_jc_kc_l &= 6R_{ijil}c_jc_l = 3(R_{ijil} + R_{ilij})c_jc_l \\
&= 3R_{ijil}(c_jc_l + c_lc_j) = 3R_{ijil} \times (-2\delta_{jl}) \\
&= -6R_{ijij} = -6\mathrm{Scal}.
\end{aligned}$$

This gives the desired relation (8.31). □

8.5.2 Gårding's inequality

In this subsection, we prove Lemma 8.7 by induction on s.

(i) *The $s = 0$ case.*

According to the Weitzenböck formula and Proposition C.2, one has

$$\|D\omega\|_0^2 = \|\bar{\nabla}\omega\|_0^2 + \langle \mathcal{R}\omega, \omega\rangle_{L^2}$$

for all $\omega \in H_1$. By the compactness of M, there exists $C_1 > 0$ such that

$$\langle \mathcal{R}\omega, \omega\rangle_{L^2} \geqslant -C_1\|\omega\|_0^2. \tag{8.32}$$

On the other hand, by using a partition of unity one can express the integral

$$\|\bar{\nabla}\omega\|_0^2 = \int_M \langle \bar{\nabla}\omega, \bar{\nabla}\omega\rangle_x dx$$

as a sum of local integrals, each having the form

$$I_U(\omega) \triangleq \int_U g^{ij}\langle \bar{\nabla}_{\partial_i}\omega, \bar{\nabla}_{\partial_j}\omega\rangle_x dx$$

where U is a local chart and (g^{ij}) is the inverse metric tensor. Note that $\bar{\nabla}_i = \partial_i - \Gamma_i$ where $\Gamma_i \in \Gamma(\mathrm{End}(\Lambda T^*U))$ is a zeroth order operator. Since (g^{ij}) is positive definite, one has

$$\int_U g^{ij}\langle \partial_i\omega, \partial_j\omega\rangle_x dx \geqslant C_{1,U}\|\omega\|_1^2 - C_{2,U}\|\omega\|_0^2$$

for some constants $C_{1,U}, C_{2,U}$ depending only on the local chart U. Together with the boundedness of g^{ij} and Γ_i, it follows that

$$\begin{aligned} I_U(\omega) =& \int_U g^{ij}\langle(\partial_i - \Gamma_i)\omega, (\partial_j - \Gamma_j)\omega\rangle_x dx \\ \geqslant& C_{1,U}\|\omega\|_1^2 - C_{3,U}\|\omega\|_0^2 - C_{4,U}\|\omega\|_1\|\omega\|_0 \\ \geqslant& C_{5,U}\|\omega\|_1^2 - C_{6,U}\|\omega\|_0^2. \end{aligned}$$

By compactness again, there are finitely many such U's and the above estimate patches globally to yield

$$\|\bar{\nabla}\omega\|_0^2 \geqslant C_2\|\omega\|_1^2 - C_3\|\omega\|_0^2 \tag{8.33}$$

with suitable constants $C_2, C_3 > 0$ depending on M. Combining (8.32) and (8.33), one concludes that

$$\|D\omega\|_0^2 \geqslant C_2\|\omega\|_1^2 - (C_1 + C_3)\|\omega\|_0^2,$$

which is Gårding's inequality in the $s = 0$ case.

(ii) *The induction step.*

This step is indeed quite straightforward. To ease notation, we write $\lesssim$ for inequalities up to multiplicative constants that are independent of ω. Suppose that Gårding's inequality is valid for $s - 1$ and let $\omega \in H_{s+1}$. By the same partition of unity argument, one can assume that ω is supported in a local chart. By the definition of the $\|\cdot\|_{s+1}$-norm and the induction hypothesis, one has

$$\begin{aligned} \|\omega\|_{s+1} \lesssim& \sum_i \|\partial_i\omega\|_s + \|\omega\|_0 \\ \lesssim& \sum_i \big(\|\partial_i\omega\|_{s-1} + \|D\partial_i\omega\|_{s-1}\big) + \|\omega\|_0. \end{aligned} \tag{8.34}$$

Now the main observation is that $[D, \partial_i] \triangleq D\partial_i - \partial_i D$ is a first order differential operator. In particular,

$$\|D\partial_i\omega\|_{s-1} \leqslant \|\partial_i D\omega\|_{s-1} + \|[D, \partial_i]\omega\|_{s-1} \lesssim \|D\omega\|_s + \|\omega\|_s. \tag{8.35}$$

By substituting (8.35) back into (8.34), one obtains that

$$\|\omega\|_{s+1} \lesssim \|\omega\|_s + \|D\omega\|_s.$$

This completes the induction step.

The proof of Gårding's inequality is now complete.

Remark 8.9. Gårding's inequality allows one to control all first order derivatives (the operator $\bar{\nabla}$) in terms of a specific mixture of directional derivatives (the operator D), which is a surprising fact at first glance. In the current argument, the key connecting D and $\bar{\nabla}$ to make this work is precisely the Weitzenböck formula. However, the essence behind the inequality is ellipticity; it holds more generally that $\|\omega\|_1 \lesssim \|\omega\|_0 + \langle L\omega, \omega\rangle_{L^2}$ for any second order elliptic operator L (and in particular, Δ is such an operator).

8.5.3 Regularity of $d + \delta$

In this subsection, we prove the regularity lemma (Lemma 8.8) for the operator $D = d + \delta$. As we mentioned before, this is almost a direct consequence of Gårding's inequality (at least at a formal level). The argument below develops the unpleasant technicalities and can be skipped if the reader is convinced by this point.

We consider a slightly weaker claim:

$$\omega \in H_s, D\omega \in H_s \implies \omega \in H_{s+1}. \tag{8.36}$$

The proof of this claim is divided into the following steps.

(i) *Localisation.*

We first claim that in order to prove (8.36), it is enough to consider the case when ω is supported in a local chart. The main observation is the following lemma.

Lemma 8.17. *Suppose that $\omega \in H_s$ and $D\omega \in H_s$. Then for any $f \in C^\infty(M)$, one has $D(f\omega) \in H_s$.*

Proof. For any smooth form ω and $f \in C^\infty(M)$, one finds by using Lemma 8.15 that

$$D(f\omega) = (e_i f)(a(e_i)\omega - \iota(e_i)\omega) + fD(\omega).$$

Note that the first term is independent of the PONF $\{e_i\}$ and is hence global. If $\omega \in H_s$ and $D\omega = \eta \in H_s$ weakly, one has

$$D(f\omega) = (e_i f)(a(e_i)\omega - \iota(e_i)\omega) + f\eta \in H_s \text{ weakly.}$$

□

Suppose that (8.36) has already been proven for forms supported in local coordinate charts. Now let $\omega \in H_s$ and $D\omega \in H_s$. Using a partition of unity $\{\varphi_i\}_{i=1}^r$ on M, one can write

$$\omega = \sum_{i=1}^r \varphi_i \omega \implies D\omega = \sum_{i=1}^r D(\varphi_i\omega).$$

Since $\omega, D\omega \in H_s$, Lemma 8.17 shows that $D(\varphi_i\omega) \in H_s$, and since $\varphi_i\omega$ is supported in a chart, one further knows that $\varphi_i\omega \in H_{s+1}$. It follows that $\omega = \sum_{i=1}^{r} \varphi_i\omega \in H_{s+1}$.

(ii) *Reduction to the $s = 0$ case.*

Next, we claim that it is sufficient to prove the following assertion:

$$\omega \in L^2(\Lambda T^*V),\ D\omega \in L^2(\Lambda T^*V) \implies \omega \in H_1(\Lambda T^*V), \tag{8.37}$$

where V is any local chart on M with compact closure. Indeed, suppose that (8.37) is true and let

$$\omega \in H_s,\ \mathrm{supp}\omega \subseteq V,\ D\omega \in H_s.$$

Given any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_1+\cdots+\alpha_n \leqslant s$, explicit computation shows that $\partial^\alpha D - D\partial^\alpha$ is a differential operator of order at most s (here $\partial^\alpha \triangleq \partial_{x^1}^{\alpha_1} \cdots \partial_{x^n}^{\alpha_n}$ is the usual partial differentiation in the chart V). Now observe that

$$\omega \in H_s \implies (\partial^\alpha D - D\partial^\alpha)\omega \in L^2,\ D\omega \in H_s \implies \partial^\alpha D\omega \in L^2.$$

Therefore, $D\partial^\alpha\omega \in L^2(\Lambda T^*V)$ and one concludes from (8.37) that $\partial^\alpha\omega \in H_1(\Lambda T^*M)$. Since α is arbitrary, it follows that $\omega \in H_{s+1}(\Lambda T^*M)$.

(iii) *It remains to prove the claim (8.37).*

For technical reason, we will prove the following small variant of the claim which is clearly sufficient. Let V be a local chart and let $U \Subset V$.

Revised Claim (8.37). Suppose that $\omega \in L^2(\Lambda T^*V)$, $\mathrm{supp}\omega \subseteq U$ and $D\omega \in L^2(\Lambda T^*V)$. Then $\omega \in H_1(\Lambda T^*V)$.

Under the local chart V, one can perform analysis under the usual Euclidean notation (e.g. convolution, partial differentiation etc.). For each $\varepsilon > 0$, let χ_ε be the standard mollifier and let $\omega * \chi_\varepsilon$ denote the convolution between ω and χ_ε. Note that $\omega * \chi_\varepsilon \to \omega$ in L^2 as $\varepsilon \to 0^+$. To prove $\omega \in H_1(\Lambda T^*V)$, it is enough to show that $\{\omega * \chi_\varepsilon\}$ is Cauchy under H_1-norm. According to Gårding's inequality, one only needs to show that $\{D(\omega * \chi_\varepsilon)\}$ is Cauchy in L^2.

To this end, let $\omega_{\varepsilon,\delta} \triangleq \omega * \chi_\varepsilon - \omega * \chi_\delta$ and write

$$\begin{aligned} D\omega_{\varepsilon,\delta} =& D(\omega * \chi_\varepsilon) - (D\omega) * \chi_\varepsilon + (D\omega) * \chi_\varepsilon - (D\omega) * \chi_\delta \\ &+ (D\omega) * \chi_\delta - D(\omega * \chi_\delta). \end{aligned}$$

The middle term $(D\omega) * \chi_\varepsilon - (D\omega) * \chi_\delta$ vanishes as $\varepsilon, \delta \to 0^+$ because $D\omega \in L^2$. To complete the proof, it remains to show the following result.

Lemma 8.18. *One has*

$$\lim_{\varepsilon \to 0^+} \|D(\omega * \chi_\varepsilon) - (D\omega) * \chi_\varepsilon\|_{L^2} = 0. \tag{8.38}$$

Proof. This result is true for any first order differential operator D (ellipticity does not play any role). Let us write $D = a^i\partial_{x^i} + b$ where $a^i, b \in C_b^\infty(\bar{V})$. The identity (8.38) clearly holds for the b-component and we therefore only consider the case when $D = a^i\partial_{x^i}$.

(a) For each $\omega \in \Omega_c^*(\bar{U})$ (smooth forms supported on $\bar{U}$), we define

$$T_\varepsilon(\omega) \triangleq D(\omega * \chi_\varepsilon) - (D\omega) * \chi_\varepsilon.$$

We claim that there exist positive constants C and ε_0, such that

$$\|T_\varepsilon\omega\|_{L^2(\Lambda T^*V)} \leqslant C\|\omega\|_{L^2(\Lambda T^*U)} \tag{8.39}$$

for all $\omega \in \Omega_c^*(\bar{U})$ and $\varepsilon \in (0, \varepsilon_0)$. Indeed, simple integration by parts yields

$$\begin{aligned}(T_\varepsilon\omega)(x) =&a_i(x)\int_V \partial_{x^i}\omega(x-\varepsilon y)\chi(y)dy - \int_V a_i(x-\varepsilon y)\partial_{x^i}\omega(x-\varepsilon y)\chi(y)dy \\ =&\int_V \omega(x-\varepsilon y)\cdot\frac{a_i(x)-a_i(x-\varepsilon y)}{\varepsilon}\cdot\partial_{y^i}\chi(y)dy \\ &+\int_V \omega(x-\varepsilon y)\chi(y)\partial_{x^i}a_i(x-\varepsilon y)dy,\end{aligned}$$

from which the claim (8.39) easily follows.

Since $\Omega_c^*(\bar{U})$ is dense in $L^2(\Lambda T^*U)$, one obtains a bounded linear operator $\bar{T}_\varepsilon : L^2(\Lambda T^*U) \to L^2(\Lambda T^*V)$ which is the unique extension of T_ε.

(b) Suppose that $\omega, D\omega \in L^2(\Lambda T^*V)$ and $\text{supp}\omega \subseteq U$. Then one has

$$\bar{T}_\varepsilon\omega = D(\omega * \chi_\varepsilon) - (D\omega) * \chi_\varepsilon. \tag{8.40}$$

To see this, let $\omega_k \in \Omega_c^*(\bar{U})$ be a sequence converging to ω in L^2. Part (a) shows that $T_\varepsilon\omega_k \to \bar{T}_\varepsilon\omega$ in L^2. For any test form $\varphi \in \Omega_c^*(V)$, one has

$$\begin{aligned}\langle\bar{T}_\varepsilon\omega,\varphi\rangle_{L^2} &= \lim_{k\to\infty}\langle T_\varepsilon\omega_k,\varphi\rangle_{L^2} = \lim_{k\to\infty}\langle D(\omega_k * \chi_\varepsilon) - (D\omega_k) * \chi_\varepsilon, \varphi\rangle_{L^2} \\ &= \lim_{k\to\infty}\big(\langle\omega_k * \chi_\varepsilon, D^*\varphi\rangle_{L^2} - \langle\omega_k, D^*(\varphi * \chi_\varepsilon^-)\rangle_{L^2}\big),\end{aligned}$$

where D^* denotes the formal adjoint of D and $\chi_\varepsilon^-(y) \triangleq \chi_\varepsilon(-y)$. Since $D\omega \in L^2(\Lambda T^*V)$, the RHS is equal to

$$\begin{aligned}\langle\omega * \chi_\varepsilon, D^*\varphi\rangle_{L^2} - \langle\omega, D^*(\varphi * \chi_\varepsilon^-)\rangle_{L^2} &= \langle D(\omega * \chi_\varepsilon),\varphi\rangle_{L^2} - \langle D\omega, \varphi * \chi_\varepsilon^-\rangle_{L^2} \\ &= \langle D(\omega * \chi_\varepsilon) - (D\omega) * \chi_\varepsilon, \varphi\rangle_{L^2}.\end{aligned}$$

This gives the relation (8.40).

(c) For smooth forms $\omega \in \Omega^*(\bar{U})$, it is obvious that $T_\varepsilon\omega \to 0$ in L^2 as $\varepsilon \to 0^+$.

(d) We now finish the proof of the lemma. Suppose that $\omega, D\omega \in L^2(\Lambda T^*V)$ and supp$\omega \subseteq U$. Given $\eta > 0$, let $\beta \in \Omega^*(\bar{U})$ be such that $\|\omega - \beta\|_{L^2} < \eta$. According to (8.39), one has

$$\|D(\omega * \chi_\varepsilon) - (D\omega) * \chi_\varepsilon\|_{L^2} \leqslant \|\bar{T}_\varepsilon \omega - T_\varepsilon \beta\|_{L^2} + \|T_\varepsilon \beta\|_{L^2} \leqslant C\eta + \|T_\varepsilon \beta\|_{L^2}$$

for all $\varepsilon \in (0, \varepsilon_0)$. It follows from Step (c) that

$$\overline{\lim_{\varepsilon \to 0^+}} \|D(\omega * \chi_\varepsilon) - (D\omega) * \chi_\varepsilon\|_{L^2} \leqslant C\eta.$$

The result follows since η is arbitrary. □

Now the proof of the revised Claim (8.37) (and thus of (8.36)) is complete.

To prove the regularity Lemma 8.8, we assume that $\omega \in L^2$ and $D\omega = \eta \in H_s$ (weakly). The claim (8.36) shows that $\omega \in H_1$, and using the claim again one finds $\omega \in H_2$. By iterating this argument, one obtains that $\omega \in H_{s+1}$. Note that if $D\omega \in \Omega(M)$, one has $\omega \in H_s$ for all s, and it follows from the Sobolev embedding theorem (see Theorem C.2) that ω is smooth in the classical sense, i.e. $\omega \in \Omega(M)$. The proof of Lemma 8.8 is therefore complete.

The following result is an immediate consequence of Lemma 8.8. It asserts that the Hodge Laplacian increases regularity by two.

Corollary 8.1 (Weyl's lemma). *Suppose that $\omega \in H_1$ and $\Delta\omega = \eta \in H_s$ weakly. Then $\omega \in H_{s+2}$. In particular, eigenforms of Δ are smooth.*

Proof. Let $\beta \triangleq D\omega$ which makes sense since $\omega \in H_1$. For any test form $\varphi \in \Omega(M)$, one has by duality that

$$\langle \beta, D\varphi \rangle_{L^2} = \langle D\omega, D\varphi \rangle_{L^2} = \langle \omega, \Delta\varphi \rangle_{L^2} = \langle \eta, \varphi \rangle_{L^2}.$$

As a result, $D\beta = \eta$ weakly. It follows from Lemma 8.8 that $\beta \in H_{s+1}$ and therefore $\omega \in H_{s+2}$. □

Remark 8.10. The result of Weyl's lemma holds without assuming $\omega \in H_1$; the proof in this more general case requires additional tools from distribution theory.

Remark 8.11. The current proof of Weyl's lemma relies on the regularity lemma for D which is in turn based on the geometric Weitzenböck formula. There are analytic proofs which rely only on ellipticity and work for general elliptic operators (see e.g. [GT77]).

8.6 An application: Bochner's theorem on first Betti number

We conclude this chapter with a nice topological application of the Hodge theorem. Let M be a connected, closed, oriented Riemannian manifold of dimension n. Recall that the *first Betti number* $b_1(M)$ of M is the dimension of the first de Rham cohomology group, which is a topological invariant. Also recall that the Ricci tensor Ric is defined by (C.6).

Theorem 8.4 (Bochner's Theorem)**.** *Suppose that* $\mathrm{Ric} \geqslant 0$*. Then* $b_1(M) \leqslant n$*. If one assumes additionally that* $\mathrm{Ric}(x) > 0$ *at some* $x \in M$*, then* $b_1(M) = 0$*.*

Proof. The Ricci tensor induces an endomorphism $\mathrm{Ric} : \Omega^1(M) \to \Omega^1(M)$ through the identification $TM \cong T^*M$, which further induces an endomorphism (still denoted as) $\mathrm{Ric} \in \Gamma(\mathrm{End}(\Lambda T^*M))$ on forms via the Leibniz rule (see (8.21)).

Firstly, we claim that $\mathrm{Ric} = \mathrm{Ric}_{ij} a(e_i)\iota(e_j)$ with respect to a local PONF $\{e_i\}$. In fact, one easily checks that

$$\mathrm{Ric}(\theta^i) = \mathrm{Ric}_{ji}\theta^j = \mathrm{Ric}_{kl} a(e_k)\iota(e_l)(\theta^i)$$

and additionally, the operator $\mathrm{Ric}_{ij} a(e_i)\iota(e_j)$ satisfies the Leibniz rule (8.21) on forms.

Next, we claim that

$$\Delta(\omega) = \bar{\nabla}^*\bar{\nabla}(\omega) + \mathrm{Ric}(\omega) \tag{8.41}$$

for all $\omega \in \Omega^1(M)$. Indeed, according to the Weitzenböck formula, one has

$$\begin{aligned}\mathcal{R}\omega &= -R_{ijkl} a(e_i)\iota(e_j)a(e_k)\iota(e_l)\omega \\ &= -R_{ijkl} a(e_i)\big(\delta_{jk} - a(e_k)\iota(e_j)\big)\iota(e_l)\omega \\ &= \mathrm{Ric}(\omega) + R_{ijkl} a(e_i)a(e_k)\iota(e_j)\iota(e_l)\omega.\end{aligned}$$

The second term vanishes if ω is a 1-form, hence yielding the relation (8.41).

Suppose that $\mathrm{Ric} \geqslant 0$ on M. In this case, we claim that harmonic 1-forms must be parallel. To see this, let $\omega \in \mathcal{H} \cap \Omega^1(M)$. According to the formula (8.41), one has

$$0 = \langle \Delta\omega, \omega\rangle_{L^2} = \langle \bar{\nabla}\omega, \bar{\nabla}\omega\rangle_{L^2} + \langle \mathrm{Ric}(\omega), \omega\rangle_{L^2} \geqslant 0.$$

As a result, $\bar{\nabla}\omega = 0$. Now assume on the contrary that $b_1(M) > n$. By Theorem 8.3 and the previous claim, one would find more than n linearly independent parallel 1-forms. By using parallel transport, this would lead to more than n linearly independent elements in T_x^*M (for any fixed $x \in M$) which is absurd. Therefore, $b_1(M) \leqslant n$.

Finally, suppose further that $\mathrm{Ric}(x) > 0$ at some $x \in M$. Let ω be a harmonic 1-form. In the relation (8.41), by taking inner product with $\omega(x)$ on T_x^*M one finds that $\omega(x) = 0$. Since $\bar{\nabla}\omega = 0$ and M is connected, the value of ω at any location

$y \in M$ can be obtained through parallel transport of $\omega(x)$ along any smooth curve from x to y. As a result, $\omega(y) = 0$ for all $y \in M$. In particular, there are no nonzero harmonic 1-forms on M and hence $b_1(M) = 0$. $\square$

9 Heat kernels on vector bundles

In this chapter, we prove the existence of a unique smooth heat kernel for any generalised Laplace operator on a vector bundle over a closed, oriented Riemannian manifold. This fact was used in the earlier proof of the L^2-Hodge theorem and will also be needed (in a more essential way) in the later heat equation proof of the CGB theorem. The existence theorem here is not abstract functional-analytic nonsense. In fact, the technique we shall develop yields rather precise quantitative information and asymptotic properties of the heat kernel, which are crucial for CGB.

Throughout the rest of this chapter, we always assume that (E, M, π) is a $\mathbb{K}$-vector bundle over a closed, oriented Riemannian manifold of dimension n ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). We begin by introducing the definition of a generalised Laplacian and the associated heat kernel. Recall Definition C.6 for the Laplacian associated with a connection on E.

Definition 9.1. A *generalised Laplacian* on E is a differential operator of the form $H = \Delta + F : \Gamma(E) \to \Gamma(E)$ where Δ is the Laplacian associated with some connection on E and $F \in \Gamma(\mathrm{End}(E))$ is a zeroth order operator.

Let $E \boxtimes E^*$ be the bundle over $M \times M$ whose fiber at (x, y) is $E_x \boxtimes E_y^*$, where the notation $\boxtimes$ means the usual tensor product.

Definition 9.2. Let H be a generalised Laplacian on E. A *heat kernel* for H is a map $p : (0, \infty) \to \Gamma(M \times M, E \boxtimes E^*)$ which satisfies the following properties.

(i) ($\mathcal{C}^1$ in t) The map

$$(t, x, y) \mapsto p_t(x, y),\ (t, x, y) \mapsto \partial_t p_t(x, y)$$

are continuous in (t, x, y).

(ii) ($\mathcal{C}^2$ in x) For each fixed (t, y), the map $x \mapsto p_t(x, y)$ has continuous derivative up to second order with respect to any coordinate chart of M and any local trivialisation of E.

(iii) (Heat equation) $(\partial_t + H_x)p_t(x, y) = 0$ for all $t > 0$ and $x, y \in M$. Here H_x means applying H to the x-variable of p.

(iv) (Initial condition) For any $s \in \Gamma(E)$, one has

$$\lim_{t \to 0^+} \sup_{x \in M} \|(P_t s)(x) - s(x)\|_{E_x} = 0 \tag{9.1}$$

under any metric on E. Here the operator $P_t : \Gamma(E) \to \Gamma(E)$ is defined by

$$(P_t s)(x) \triangleq \int_M p_t(x, y) s(y) dy$$

for any $s \in \Gamma(E)$.

Remark 9.1. Due to the compactness of M, the property (iv) does not depend on the particular choice of metric on E.

The main goal of this chapter is to prove the following fundamental result. In fact, much more quantitative properties will be established along the way.

Theorem 9.1. *Let H be a generalised Laplacian on E. There exists a unique heat kernel $p_t(x,y)$ for H. In addition, $p_t(x,y)$ is smooth in all variables.*

Example 9.1. The main example of our interest is $E = \Lambda T^*M$ (the exterior algebra bundle) and $H = \Delta$ (the Hodge Laplacian). From the Weitzenböck formula, one knows that H is a generalised Laplacian and therefore admits a unique smooth heat kernel.

9.1 Existence of heat kernel

In this section, we prove the existence of a heat kernel by using the so-called *parametrix method*. The main idea is to first write down an approximating solution and then use it to iterate the heat equation to obtain a true solution. It will be clear from the method that the heat kernel obtained in this way is smooth. An important by-product of this method is a precise small-time asymptotic expansion of the heat kernel, whose coefficients are described quite explicitly and encode useful geometric information (such as curvature properties). Such an expansion plays an essential role in the heat equation proof of the CGB theorem.

9.1.1 The parametrix method: a toy example

We use a toy model to explain the parametrix method in an elementary way. Let V be a finite dimensional vector space and let $H \in \mathcal{L}(V)$ be a bounded linear operator on V. Suppose that one wants to solve the ODE

$$\frac{dP_t}{dt} + HP_t = 0,\ 0 \leqslant t \leqslant T \tag{9.2}$$

with initial condition $P_0 = \mathrm{id}$.

The parametrix method begins with an approximating solution (the *parametrix*) $K_t \in \mathcal{L}(V)$, in the sense that

$$\frac{dK_t}{dt} + HK_t =: R_t,\ K_0 = \mathrm{id}, \tag{9.3}$$

where the "remainder" R_t is assumed to satisfy $|R_t| \leqslant C_1 t^{\alpha}$ for all $t \in [0,T]$ with some $\alpha \geqslant 0$. Given any continuous $\varphi : [0,T] \to \mathcal{L}(V)$, we define the convolution $Q \triangleq K * \varphi$ by

$$Q_t \triangleq (K * \varphi)_t \triangleq \int_0^t K_{t-s}\varphi_s ds.$$

It is easily seen by using (9.3) that

$$\big(\frac{d}{dt}+H\big)Q=\varphi+R*\varphi. \tag{9.4}$$

The relation (9.4) motivates the construction of

$$P=K+\hat{R}, \tag{9.5}$$

with $\hat{R}$ being defined by

$$\hat{R}\triangleq K*\tilde{R},\ \tilde{R}\triangleq\sum_{k=1}^{\infty}(-1)^k R^{*k}. \tag{9.6}$$

Here R^{*k} means the k-fold convolution of R, i.e.

$$R_t^{*1}\triangleq R_t;\ R_t^{*k}\triangleq\int_{0<t_1<\cdots<t_{k-1}<t}R_{t-t_{k-1}}\cdots R_{t_2-t_1}R_{t_1}dt_1\cdots dt_{k-1},\ k\geqslant 2.$$

At a formal level, one obtains from (9.4) that

$$\begin{aligned}\big(\frac{d}{dt}+H\big)P&=\big(\frac{d}{dt}+H\big)(K+\hat{R})=R+\tilde{R}+R*\tilde{R}\\&=R+\sum_{k=1}^{\infty}(-1)^kR^{*k}+\sum_{k=1}^{\infty}(-1)^kR^{*(k+1)}=0.\end{aligned} \tag{9.7}$$

Therefore, P solves the heat equation (9.2).

The above argument is rigorous. Indeed, one can write

$$P_t=\sum_{k=0}^{\infty}(-1)^kQ_t^{(k)}, \tag{9.8}$$

where

$$Q_t^{(0)}\triangleq K_t;\ Q_t^{(k)}\triangleq\int_0^t K_{t-s}R_s^{*k}ds,\ k\geqslant 1,$$

Assuming that $|K_t|\leqslant C_0$ for all $t\in[0,T]$, one concludes by induction that

$$|Q_t^{(k)}|\leqslant\frac{C_0C_1^kt^{k(1+\alpha)}}{k!}\quad\forall k\geqslant 1,t\in[0,T].$$

In particular, the series in (9.8) is uniformly convergent. Therefore, the computation performed in (9.7) is justified. In addition, K_t is a small-time approximation of P_t in the sense that

$$|P_t-K_t|=\Big|\sum_{k=1}^{\infty}(-1)^kQ_t^{(k)}\Big|\leqslant\sum_{k=1}^{\infty}\frac{C_0C_1^kt^{k(1+\alpha)}}{k!}\leqslant C_2t^{1+\alpha}$$

for all $t\in[0,T]$.

9.1.2 Local formulae under normal chart

Before moving to the construction of the heat kernel, we first discuss a few local computations under a normal coordinate chart. The formulae we derive here are not only useful for the current purpose but also of independent interest in various geometric contexts.

The construction and basic properties of a normal chart are recalled in Appendix C.3. Here we just provide a brief summary. Let $y \in M$ be a fixed based point and let $U = B(y, \eta)$ be a geodesic ball centered at y. We assume that η is small enough so that

$$\exp : V \triangleq \{v \in T_yM : |v| < \eta\} \to U$$

is a diffeomorphism. Let $\{\varepsilon_i : 1 \leqslant i \leqslant n\}$ be a given fixed PONB of T_yM and we identify V with the η-ball in $\mathbb{R}^n$ through

$$\mathbf{x} = (x^1, \cdots, x^n) \leftrightarrow x^i \varepsilon_i.$$

The normal coordinates on U are defined by

$$V \ni \mathbf{x} = (x^1, \cdots, x^n) \mapsto \exp(x^i \varepsilon_i) \in U.$$

We use x or $\mathbf{x}$ interchangeably to denote a point in U. Note that $\mathbf{x} = \mathbf{0}$ corresponds to the base point y.

We consider two local frame fields of the tangent bundle over U: the natural coordinate field $\{\partial_i\}$ with respect to the above normal coordinates and the PONF $\{e_i\}$ which are parallel along geodesic rays from y. These two frame fields agree at y and they are related by

$$e_i = \sigma_i^j \partial_j, \ \partial_j = \tau_j^i e_i \tag{9.9}$$

with some matrix-valued functions $\sigma, \tau \in C^\infty(U)$ where $\tau = \sigma^{-1}$. Since $e_i = \partial_i = \varepsilon_i$ at y, one has

$$\sigma_i^j(\mathbf{x}) = \delta_i^j + o(1), \ \tau_i^j(\mathbf{x}) = \delta_i^j + o(1) \quad \text{as } |\mathbf{x}| \to 0. \tag{9.10}$$

An important property of $\{e_i\}$ is that $\nabla_{e_i} e_j = 0$ at y (vanishing Christoffel symbols at y).

The radial vector field

The *radial vector field* $R \triangleq x^i \partial_i$ on U plays a central role in the local computation we shall perform. We summarise its basic properties in the following two lemmas.

Lemma 9.1. *(i) Let $f \in C^\infty(U)$ be such that $f(\mathbf{0}) = 0$ and $Rf = f$ in U. Then $f(\mathbf{x}) = \partial_i f(\mathbf{0}) x^i$.*

(ii) Let $f \in C^\infty(U)$ be such that $f(\mathbf{0}) = 0$ and $Rf = x^i$ in U. Then $f(\mathbf{x}) = x^i$.

Proof. (i) Fix $\delta \in (0, \eta)$. It is enough to prove the claim on the ball $B(y, \delta)$. By assumption, one can write $f = \partial_i f(\mathbf{0}) x^i + g$, where g is a smooth function satisfying $|g(\mathbf{x})| \leqslant C|\mathbf{x}|^2$ for all $|\mathbf{x}| \leqslant \delta$ with some constant C depending on δ. Let $|\mathbf{x}| \leqslant \delta$ be given fixed and define $\varphi(t) \triangleq g(t\mathbf{x})$ $(0 \leqslant t \leqslant 1)$. Since $Rf = f$, one has $Rg = g$. It follows that

$$\varphi'(t) = \frac{1}{t}(Rg)(t\mathbf{x}) = \frac{1}{t} g(t\mathbf{x}) \implies g(\mathbf{x}) = \int_0^1 \frac{1}{t} g(t\mathbf{x}) dt.$$

Therefore,

$$|g(\mathbf{x})| \leqslant \int_0^1 \frac{1}{t} \cdot Ct^2 |\mathbf{x}|^2 dt = \frac{1}{2} C|\mathbf{x}|^2.$$

Note that this is true for all $|\mathbf{x}| \leqslant \delta$. By iteration, one finds that $|g(\mathbf{x})| \leqslant 2^{-m} C|\mathbf{x}|^2$ for all m. As a result, $g(\mathbf{x}) = 0$.

(ii) As before, given fixed $\mathbf{x}$ we define $\varphi(t) \triangleq f(t\mathbf{x})$. Then

$$f(\mathbf{x}) = \int_0^1 \frac{1}{t}(Rf)(t\mathbf{x}) dt = \int_0^1 \frac{1}{t} \cdot t x^i dt = x^i.$$

The result thus follows. □

Lemma 9.2. *The following relations hold true:*

$$\text{(i) } \nabla_R R = R; \quad \text{(ii) } R = x^i e_i; \quad \text{(iii) } \langle R, \partial_i \rangle = x^i.$$

In particular, $|R|^2 = |\mathbf{x}|^2$.

Proof. (i) By definition,

$$\nabla_R R = x^i \nabla_{\partial_i}(x^j \partial_j) = x^i \partial_i + x^i x^j \Gamma_{ij}^k \partial_k = R + x^i x^j \Gamma_{ij}^k \partial_k.$$

where $\Gamma_{ij}^k \triangleq \langle dx^k, \nabla_{\partial_i} \partial_j \rangle$ are the Christoffel symbols of ∇ with respect to $\{\partial_i\}$. Since the ray $t \mapsto t\mathbf{x}$ is a geodesic (for each fixed $\mathbf{x}$), the geodesic equation (C.14) yields that $x^i x^j \Gamma_{ij}^k \partial_k = 0$. Therefore, $\nabla_R R = R$.

(ii) Note that

$$R\langle R, e_i \rangle = \langle \nabla_R R, e_i \rangle + \langle R, \nabla_R e_i \rangle = \langle R, e_i \rangle,$$

where the second identity follows from Part (i) and the fact that $\nabla_R e_i = 0$. It follows from Lemma 9.1 (i) that $\langle R, e_i \rangle$ is a homogeneous linear function of $\mathbf{x}$. On the other hand, according to (9.10) one has

$$\langle R, e_i \rangle = \langle x^j \partial_j, e_i \rangle = x^j (\delta^{ij} + o(1)) = x^i + o(|\mathbf{x}|).$$

As a result, it must hold that $\langle R, e_i \rangle = x^i$ and thus $R = x^i e_i$. In particular, $|R|^2 = |\mathbf{x}|^2$.

(iii) First of all, one has

$$\begin{aligned} R\langle R, \partial_i\rangle &= \langle R, \partial_i\rangle + \langle R, \nabla_R \partial_i\rangle \\ &= \langle R, \partial_i\rangle + \langle R, \nabla_{\partial_i} R\rangle + \langle R, [R, \partial_i]\rangle \\ &= \langle R, \partial_i\rangle + \langle R, \nabla_{\partial_i} R\rangle + \langle R, -\partial_i\rangle = \langle R, \nabla_{\partial_i} R\rangle. \end{aligned}$$

In addition, one knows from Part (ii) that

$$\langle R, \nabla_{\partial_i} R\rangle = \frac{1}{2}\partial_i \langle R, R\rangle = \frac{1}{2}\partial_i |\mathbf{x}|^2 = x^i.$$

Therefore, $R\langle R, \partial_i\rangle = x^i$. It follows from Lemma 9.1 (ii) that $\langle R, \partial_i\rangle = x^i$. $\square$

Remark 9.2. A more explicit way of expressing Lemma 9.2 (iii) is that $x^i g_{ij} = x^j$, or equivalently, $x^i = x^j g^{jk}$. Here $(g_{ij} \triangleq \langle \partial_i, \partial_j\rangle)$ is the metric tensor and (g^{ij}) is its inverse.

Local expansion of connection matrix

Let (E, M, π) be a real vector bundle over M of rank r and let ∇ be a connection on E. Recall from (B.20) that the curvature tensor of ∇ is defined by

$$F(X,Y)s \triangleq \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

for $X, Y \in \Gamma(TM)$ and $s \in \Gamma(E)$. We choose a normal chart U as before and let $\{s_1, \cdots, s_r\}$ be a local frame field of E defined by parallel transporting a given basis $\{s_1(y), \cdots, s_r(y)\}$ of E_y along geodesic rays. This gives a local trivialisation of E on U.

Let ω be the local connection matrix of ∇ with respect to $\{s_\alpha\}$, namely, $\nabla s_\alpha = \omega_\alpha^\beta \otimes s_\beta$ for each α. It follows that

$$\nabla(\lambda^\alpha s_\alpha) = (d\lambda^\alpha + \lambda^\beta \omega_\beta^\alpha) \otimes s_\alpha$$

for any smooth section $\lambda^\alpha s_\alpha$ on U. Under matrix notation, one can simply write $\nabla = d + \omega$. In addition, by viewing $F \in \Omega^2(M, \mathrm{End}(E))$, one can locally express the curvature matrix as

$$F = d\omega + \omega \wedge \omega$$

with respect to $\{s_\alpha\}$.

In this part, we establish the following local expansion of the connection matrix ω in terms of F. We write $\omega = \omega_i dx^i$, where ω_i is an $r \times r$ matrix-valued C^∞-function on U.

Proposition 9.1. *One has*

$$\omega_i(\mathbf{x}) = -\frac{1}{2}F(\partial_i, \partial_j)|_y x^j + o(|\mathbf{x}|) \tag{9.11}$$

as $|\mathbf{x}| \to 0$.

Proof. The key observation is that

$$\mathcal{L}_R\omega = \iota(R)F, \tag{9.12}$$

where $\mathcal{L}_R$ is the Lie derivative and $\iota(R)$ is the interior product by R. Let us presume for now the correctness of (9.12). On the one hand, one has

$$\mathcal{L}_R\omega = \mathcal{L}_R(\omega_i dx^i) = (x^j\partial_j\omega_i + \omega_i)dx^i.$$

On the other hand, using $F = \sum_{i<j} F(\partial_i, \partial_j)dx^i \wedge dx^j$ one sees that

$$\iota(R)F = x^j F(\partial_j, \partial_i)dx^i.$$

Therefore,

$$x^j\partial_j\omega_i + \omega_i = x^j F(\partial_j, \partial_i) \tag{9.13}$$

for every i. By writing $\omega_i = \mu_{ij}x^j + o(|\mathbf{x}|)$, it follows from (9.13) that $\mu_{ij} = -\frac{1}{2}F(\partial_i, \partial_j)|_y$. This gives the desired expansion (9.11).

We now prove (9.12). Since s_α is parallel along R, one has

$$0 = \nabla_R s_\alpha = \iota(R)(\nabla s_\alpha) = \big(\iota(R)\omega_\alpha^\beta\big)s_\beta = 0 \implies \iota(R)\omega = 0. \tag{9.14}$$

It follows from Cartan's formula (see Proposition B.1) that

$$\mathcal{L}_R\omega = \big(\iota(R)d + d\iota(R)\big)\omega = \iota(R)(d\omega) = \iota(R)\big(d\omega + \omega\wedge\omega\big) = \iota(R)F.$$

This yields the relation (9.12) and completes the proof of the proposition. □

Remark 9.3. In a similar way, partial derivatives of ω at y up to order k are determined by partial derivatives of F at y up to order $k-1$.

Local expansion of the metric tensor

We now restrict to the tangent bundle and consider the Levi-Civita connection ∇. In this part, we establish the following local expansion of the metric tensor in terms of curvature coefficients. It shows that any Riemannian metric can be approximated by the Euclidean one up to second order. We again work on the normal chart U and consider the PONF $\{e_i\}$ as before.

Proposition 9.2. *One has*

$$g_{ij}(\mathbf{x}) = \delta_{ij} - \frac{1}{3}\sum_{k,l} R_{ikjl}(y)x^k x^l + o(|\mathbf{x}|^2) \quad \text{as } |\mathbf{x}| \to 0,$$

where $R_{ikjl} \triangleq \langle R(e_j, e_l)e_k, e_i\rangle$ *are the curvature coefficients of the Riemann curvature tensor.*

The proof of Proposition 9.2 uses the canonical form $\bar{\theta}$, which is the TM-valued 1-form on M defined by $\bar{\theta}(X)_x \triangleq X_x$ for all $X \in \Gamma(TM)$ (cf. Definition 6.20 and Proposition 6.14 for the definition). Under any coordinate chart, one has $\bar{\theta} = dx^i \otimes \partial_i$. Recall from Section 6.9.4 that $d^\nabla \bar{\theta} = T$ where d^∇ is the exterior covariant derivative induced by ∇ (see Definition 6.15) and T is the torsion form.

Let ω be its local connection matrix with respect to $\{e_i\}$. Using normal coordinates, one can write

$$\bar{\theta} = dx^i \otimes \partial_i = \bar{\theta}^i \otimes e_i \tag{9.15}$$

with $\bar{\theta}^i \triangleq \tau^i_j dx^j$, where τ^i_j is defined by the relation (9.9). Define the column vector $\bar{\boldsymbol{\theta}} \triangleq (\bar{\theta}^i)_{1\leqslant i\leqslant n}$.

Corollary 9.1. *Under matrix notation, one has* $d\bar{\boldsymbol{\theta}} + \omega \wedge \bar{\boldsymbol{\theta}} = 0$.

Proof. Using $\bar{\theta} = \bar{\theta}^i \otimes e_i$, it is plain to see that $d^\nabla \bar{\theta} = (d\bar{\theta}^i + \omega^i_j \wedge \bar{\theta}^j) \otimes e_i$. On the other hand, since ∇ is torsion free, one knows that $d^\nabla \theta = 0$. Therefore, $d\bar{\theta}^i + \omega^i_j \wedge \bar{\theta}^j = 0$. □

We are now in a position to prove Proposition 9.2.

Proof of Proposition 9.2. By using the relation (9.9), one can write

$$g_{ij} = \langle \partial_i, \partial_j\rangle = \langle \tau^k_i e_k, \tau^l_j e_l\rangle = \tau^k_i \tau^k_j. \tag{9.16}$$

The main idea is to work out the expansion of τ^k_i and then apply the above relation. The computation is lengthy and we divide the argument into several steps. Recall that $R = x^i \partial_i$ is the radial vector field on U.

(i) First of all, note that $\mathcal{L}_R(dx^j) = dx^j$. Therefore, one has

$$\mathcal{L}_R \bar{\theta}^i = \mathcal{L}_R(\tau^i_j dx^j) = R(\tau^i_j)dx^j + \tau^i_j \mathcal{L}_R(dx^j) = \big(x^k \partial_k \tau^i_j + \tau^i_j\big)dx^j.$$

One computes $\mathcal{L}_R^2 \bar{\theta}^i$ in a similar way and finds that

$$(\mathcal{L}_R^2 - \mathcal{L}_R)\theta^i = \big(2x^k \partial_k \tau^i_j + x^k x^l \partial^2_{kl} \tau^i_j\big)dx^j. \tag{9.17}$$

(ii) By using Cartan's formula (Proposition B.1) and Corollary 9.1, one can also compute $\mathcal{L}_R \bar{\boldsymbol{\theta}}$ as

$$\mathcal{L}_R \bar{\boldsymbol{\theta}} = \iota(R)d\bar{\boldsymbol{\theta}} + d\iota(R)\bar{\boldsymbol{\theta}} = \iota(R)(-\omega \wedge \bar{\boldsymbol{\theta}}) + d\iota(R)\bar{\boldsymbol{\theta}} = \omega \wedge \iota(R)\bar{\boldsymbol{\theta}} + d\iota(R)\bar{\boldsymbol{\theta}},$$

where the last identity follows from the fact that $\iota(R)\omega = 0$ (cf. (9.14)). According to Lemma 9.2 (ii) and the definition of $\bar{\theta}$,

$$x^i e_i = R = \iota(R)\bar{\theta} = (\iota(R)\bar{\theta}^i)e_i \implies \iota(R)\bar{\theta}^i = x^i.$$

As a result,

$$\mathcal{L}_R\bar{\boldsymbol{\theta}} = \omega \cdot \mathbf{x} + d\mathbf{x} \tag{9.18}$$

where $\mathbf{x} = (x^i)_{1\leqslant i\leqslant n}$ is viewed as a column vector. It follows that

$$\mathcal{L}_R^2\bar{\boldsymbol{\theta}} = \mathcal{L}_R(\omega \cdot \mathbf{x} + d\mathbf{x}) = (\mathcal{L}_R\omega)\cdot\mathbf{x} + \omega\cdot\mathbf{x} + d\mathbf{x}. \tag{9.19}$$

One computes

$$\mathcal{L}_R\omega = \iota(R)d\omega + d\iota(R)\omega = \iota(R)d\omega = \iota(R)F, \tag{9.20}$$

where $F = d\omega + \omega\wedge\omega$ is the local curvature matrix with respect to $\{e_i\}$. By substituting (9.20) into (9.19) and using (9.18), one obtains that

$$\mathcal{L}_R^2\bar{\boldsymbol{\theta}} - \mathcal{L}_R\bar{\boldsymbol{\theta}} = (\iota(R)F)\cdot\mathbf{x}. \tag{9.21}$$

By comparing (9.17) and (9.21), one thus arrives at the following relation:

$$\big(2x^k\partial_k\tau^i_j + x^kx^l\partial^2_{kl}\tau^i_j\big)dx^j = (\iota(R)F)^i_k x^k. \tag{9.22}$$

(iii) Now we evaluate the RHS of (9.22) in terms of curvature coefficients. Firstly, using $R = x^l\partial_l$ it is easily seen that $\iota(R)F = x^lF(\partial_l,\partial_j)dx^j$. In particular, one has

$$(\iota(R)F)^i_k x^k = x^lF(\partial_l,\partial_j)^i_k x^k dx^j. \tag{9.23}$$

Next, recall that $F(\partial_l,\partial_j)$ is defined by the relation

$$R(\partial_l,\partial_j)e_k = F(\partial_l,\partial_j)^p_k e_p.$$

As a result, one has

$$\begin{aligned} x^kF(\partial_l,\partial_j)^i_k &= \langle x^kR(\partial_l,\partial_j)e_k, e_i\rangle \\ &= \langle x^kR(\partial_l,\partial_j)\partial_k, e_i\rangle \quad (x^ke_k = x^k\partial_k = R) \\ &= x^k\sigma^q_i R_{qklj} \quad (\text{by } (9.9)). \end{aligned} \tag{9.24}$$

By substituting (9.24) into (9.23) and using (9.22), one obtains that

$$2x^k\partial_k\tau^i_j + x^kx^l\partial^2_{kl}\tau^i_j = x^kx^l\sigma^q_i R_{qklj}.$$

Since $(\theta^i_j) = \sigma^{-1}$, it follows that

$$2x^k\tau^i_q\partial_k\tau^i_j + x^kx^l\tau^i_q\partial^2_{kl}\tau^i_j = x^kx^lR_{qklj}. \tag{9.25}$$

(iv) Consider the expansion

$$\tau_j^i = \delta_j^i + a_{j,k}^i x^k + b_{j,kl}^i x^k x^l + \cdots , \ R_{qklj} = R_{qklj}(y) + \cdots . \tag{9.26}$$

By substituting (9.26) into (9.25), one finds that

$$a_{j,k}^i = 0, \ b_{j,kl}^i = -\frac{1}{6} R_{ikjl}(y).$$

Therefore,

$$\tau_j^i = \delta_j^i - \frac{1}{6} R_{ikjl}(y) x^k x^l + o(|\mathbf{x}|^2).$$

After substituting the above expansion back into the relation (9.16), one concludes that

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl}(y) x^k x^l + o(|\mathbf{x}|^2),$$

which gives the desired metric expansion.

□

Corollary 9.2. *One has*

$$(\Delta^M g_{ij})(y) = \frac{2}{3} \mathrm{Ric}_{ij}(y),$$

where Δ^M is the Laplace-Beltrami operator.

Proof. This is a direct consequence of Proposition 9.2, noting that $\Delta^M = \sum_i \partial_i^2$ at y under the normal chart. □

9.1.3 The formal solution

We now address the problem of constructing a heat kernel $p_t(x, y)$ for a generalised Laplacian $H = \Delta + F$ on the vector bundle E, where Δ is the Laplacian associated with a connection ∇. In what follows, we always view $p_t(x, y) \in E_x \boxtimes E_y^* \cong \mathrm{End}(E_y, E_x)$.

The starting point is to understand the shape of $p_t(x, y)$ when x is near y. Proposition 9.2 tells one that the metric tensor is locally approximated by the Euclidean one. It is therefore reasonable to "model" the heat kernel on the Euclidean formula and look for an equation for the "corrector". To make this idea precise, let $y \in M$ be given fixed. Let $U = B(y, \eta)$ be a normal chart around y and let $\{e_i\}$ be a PONF on U as before (η is a fixed small number). Consider the Euclidean heat kernel

$$q_t(x, y) \triangleq \frac{1}{(4\pi t)^{n/2}} e^{-|\mathbf{x}|^2/4t}, \quad t > 0, x \in U, \tag{9.27}$$

where $\mathbf{x} = (x^i)_{1\leqslant i\leqslant n}$ are the normal coordinates of $x \in U$. With y fixed, as a function of (t,x) we postulate that $p_t(x,y) = s_t(x,y)q_t(x,y)$ with some $s_t(x,y) \in \mathrm{End}(E_y, E_x)$. The following lemma gives the correct equation of $s_t(x,y)$ in order for $p_t(x,y)$ to satisfy the heat equation.

Proposition 9.3. *For any smooth section $(t,x) \mapsto s_t(x,y)$, one has*

$$(\partial_t + H_x)(s_t q_t) = \Big[\Big(\partial_t + H + \frac{\nabla_R}{t} + \frac{R\log j}{2t}\Big)s_t\Big]q_t, \tag{9.28}$$

where R is the radial vector field and $j(\mathbf{x}) \triangleq \sqrt{\det g(\mathbf{x})}$ on U. As a consequence, the section $(t,x) \mapsto s_t(x,y)q_t(x,y)$ satisfies the heat equation (i.e. $(\partial_t + H_x)(s_t q_t) = 0$ for all $t > 0$ and $x \in U$) if and only if

$$\Big(\partial_t + H_x + \frac{1}{t}\nabla_R + \frac{R\log j}{2t}\Big)s_t = 0. \tag{9.29}$$

To prove Proposition 9.3, we need the following lemma.

Lemma 9.3. *One has*

$$(\partial_t + \Delta^M)q_t = \frac{R\log j}{2t}q_t, \tag{9.30}$$

where Δ^M is the Laplace-Beltrami operator.

Proof. By the definition of $q_t(x,y)$, one easily finds that

$$\partial_t q_t = \Big(-\frac{n}{2t} + \frac{|\mathbf{x}|^2}{4t^2}\Big)q_t. \tag{9.31}$$

Next, we use the formula $\Delta^M = -\mathrm{tr}[\nabla^M d]$ (see (C.10)) to compute $\Delta^M q_t$ (∇^M denotes the Levi-Civita connection). Explicit calculation shows that

$$\nabla^M d(e^{-|\mathbf{x}|^2/4t}) = \frac{1}{16t^2}e^{-|\mathbf{x}|^2/4t}d|\mathbf{x}|^2 \otimes d|\mathbf{x}|^2 - \frac{1}{4t}e^{-|\mathbf{x}|^2/4t}\nabla^M d|\mathbf{x}|^2.$$

Therefore, one has

$$\Delta^M e^{-|\mathbf{x}|^2/4t} = -\frac{1}{16t^2}e^{-|\mathbf{x}|^2/4t}\mathrm{tr}\big(d|\mathbf{x}|^2 \otimes d|\mathbf{x}|^2\big) - \frac{1}{4t}e^{-|\mathbf{x}|^2/4t}\Delta^M(|\mathbf{x}|^2). \tag{9.32}$$

To proceed further, we first claim that

$$\mathrm{tr}\big[d|\mathbf{x}|^2 \otimes d|\mathbf{x}|^2\big] = 4|\mathbf{x}|^2. \tag{9.33}$$

Indeed, one has

$$\mathrm{tr}\big[d|\mathbf{x}|^2 \otimes d|\mathbf{x}|^2\big] = 4x^i x^j \mathrm{tr}\big[dx^i \otimes dx^j\big]$$

and

$$\mathrm{tr}\big[dx^i \otimes dx^j\big] = \sum_k \langle dx^i, e_k\rangle\langle dx^j, e_k\rangle = \langle dx^i, dx^j\rangle = g^{ij}.$$

It follows from Lemma 9.2 (cf. Remark 9.2) that

$$\mathrm{tr}\big[d|\mathbf{x}|^2 \otimes d|\mathbf{x}|^2\big] = 4x^ix^jg^{ij} = 4x^ix^i = 4|\mathbf{x}|^2.$$

Next, we claim that

$$\Delta^M(|\mathbf{x}|^2) = -2(n + R\log j). \tag{9.34}$$

Indeed, let $\phi \in C_c^\infty(U)$ be any test function. By Green's formula (C.12), one has

$$\int_U \phi\Delta^M(|\mathbf{x}|^2)dx = \int_U \langle\nabla^M\phi, \nabla^M(|\mathbf{x}|^2)\rangle dx.$$

In addition,

$$\begin{aligned}\langle\nabla^M\phi, \nabla^M(|\mathbf{x}|^2)\rangle &= \langle d\phi, d|\mathbf{x}|^2\rangle = \langle\partial_i\phi dx^i, 2x^jdx^j\rangle \\ &= 2x^jg^{ij}\partial_i\phi = 2x^i\partial_i\phi.\end{aligned}$$

Therefore,

$$\int_U \phi\Delta^M(|\mathbf{x}|^2)dx = 2\int_U x^i\partial_i\phi dx = 2\int_U x^i\partial_i\phi\cdot j(\mathbf{x})d\mathbf{x}$$

where we used the fact that $dx = j(\mathbf{x})d\mathbf{x}$ (dx is the volume form and $d\mathbf{x} \triangleq dx^1 \wedge \cdots \wedge dx^n$). Using Euclidean integration by parts, one finds that

$$\begin{aligned}\mathrm{RHS} &= -2\int_U \phi\partial_i\big(x^ij(\mathbf{x})\big)d\mathbf{x} = -2\int_U \phi\cdot\big(nj(\mathbf{x}) + (Rj)(\mathbf{x})\big)d\mathbf{x} \\ &= -2\int_U \phi\cdot\Big(n + \frac{(Rj)(\mathbf{x})}{j(\mathbf{x})}\Big)dx = -2\int_U \phi\cdot\big(n + R\log j\big)dx.\end{aligned}$$

As a result,

$$\int_U \phi\Delta^M(|\mathbf{x}|^2)dx = -2\int_U \phi\cdot\big(n + R\log j\big)dx$$

for all $\phi \in C_c^\infty(U)$. The relation (9.34) thus follows.

By substituting (9.33, 9.34) into (9.32), one obtains that

$$\Delta^M q_t = \frac{1}{(4\pi t)^{n/2}}\Delta^M\big(e^{-|\mathbf{x}|^2/4t}\big) = -\frac{1}{(4\pi t)^{n/2}}\Big(\frac{|\mathbf{x}|^2}{4t} - \frac{1}{2t}\big(n + R\log j\big)\Big)e^{-|\mathbf{x}|^2/4t}.$$

The desired equation (9.30) follows by combining this with (9.31). □

Proof of Proposition 9.3. By using Lemma 9.3, one first computes that

$$\begin{aligned}(\partial_t + H)(s_t q_t) &= \big((\partial_t + H)s_t\big)q_t + s_t\big((\partial_t + \Delta^M)q_t\big) - 2\langle \nabla s_t, \nabla^M q_t\rangle \\ &= \Big[\Big(\partial_t + H + \frac{R\log j}{2t}\Big)s_t\Big]q_t - 2\langle \nabla s_t, \nabla^M q_t\rangle. \end{aligned} \tag{9.35}$$

Next, noting that $\nabla^M q_t = -\frac{R}{2t}q_t$ one has

$$\langle \nabla s_t, \nabla^M q_t\rangle = -\frac{q_t}{2t}\nabla_R s_t. \tag{9.36}$$

The equation (9.28) follows by substituting (9.36) into (9.35).

□

The next idea is to look for a formal solution to the equation (9.29) in terms of t-series. To be more specific, one formally writes

$$s_t(x,y) = \sum_{i=0}^{\infty} t^i U^{(i)}(x,y),$$

where the endomorphisms $U^{(i)}(x,y) \in \mathrm{End}(E_y, E_x)$ are to be determined. In order for $(t,x) \mapsto s_t(x,y)$ to satisfy the equation (9.29), it is routine to show that the $U^{(i)}$'s must satisfy the following system of equations:

$$\begin{cases} \Big(\nabla_R + \dfrac{Rj}{2j}\Big)U^{(0)}(\cdot,y) = 0; \\ \Big(\nabla_R + i + \dfrac{Rj}{2j}\Big)U^{(i)}(\cdot,y) = -H_x U^{(i-1)}(\cdot,y), \quad i \geqslant 1. \end{cases} \tag{9.37}$$

In addition, by the initial condition (9.1) of the heat kernel, it is natural to require $U^{(0)}(y,y) = \mathrm{id}_{E_y}$.

Proposition 9.4. *The system (9.37) with initial condition $U^{(0)}(y,y) = \mathrm{id}_{E_y}$ admits a unique solution which is given by*

$$U^{(0)}(x,y) = \frac{1}{\sqrt{j(\mathbf{x})}}\tau(\mathbf{x},y) \tag{9.38}$$

and

$$U^{(i)}(x,y) = -\frac{1}{\sqrt{j(\mathbf{x})}}\int_0^1 v^{i-1}\sqrt{j(v\mathbf{x})}\,\tau(\mathbf{x},v\mathbf{x})(H_x U^{(i-1)})(v\mathbf{x},y)dv \tag{9.39}$$

for every $i \geqslant 1$. Here we identify $x \in U$ with its normal coordinates $\mathbf{x}$. The map $\tau(\mathbf{x},y) : E_y \to E_x$ (respectively, $\tau(\mathbf{x},v\mathbf{x}) : E_{v\mathbf{x}} \to E_{\mathbf{x}}$) denotes the parallel transport along the ray from y to x (respectively, the ray from $v\mathbf{x}$ to $\mathbf{x}$).

Proof. Let $\rho(\mathbf{x}) = |\mathbf{x}|$ be the radial part of $\mathbf{x} \in U$. Then $R = \rho\partial_\rho$ as a first order differential operator on functions. One checks explicitly that

$$\nabla_R(\rho^i\sqrt{j}U^{(i)}(\cdot, y)) = \rho^i\sqrt{j}\Big(\nabla_R + i + \frac{Rj}{2j}\Big)U^{(i)}(\cdot, y).$$

As a consequence, the system (9.37) is equivalent to that

$$\begin{cases}\nabla_R(\sqrt{j}U^{(0)}(\cdot, y)) = 0, \\ \nabla_R(\rho^i\sqrt{j}U^{(i)}(\cdot, y)) = -\rho^i\sqrt{j}H_xU^{(i-1)}(\cdot, y), \quad i \geqslant 1.\end{cases} \tag{9.40}$$

Under the initial condition $U^{(0)}(y, y) = \mathrm{id}$, the first equation is uniquely solved as (9.38).

To solve the second equation, let $\{s_\alpha\}$ be a local frame field of $E|_U$ that is parallel along rays emitting from y. Fix $i \geqslant 1$ and $\xi \in E_y$. We define

$$V(x) \triangleq \rho^i(\mathbf{x})\sqrt{j(\mathbf{x})}U^{(i)}(\mathbf{x}, y)\xi, \; W(x) \triangleq -\rho^i(\mathbf{x})\sqrt{j(\mathbf{x})}H_xU^{(i-1)}(\mathbf{x}, y)\xi. \tag{9.41}$$

By writing

$$V(x) = V^\alpha(\mathbf{x})s_\alpha(\mathbf{x}), \; W(x) = W^\alpha(\mathbf{x})s_\alpha(\mathbf{x})$$

and noting that $\nabla_R s_\alpha = 0$, the second equation of (9.40) becomes

$$\rho\partial_\rho V^\alpha = W^\alpha.$$

Since $V(\mathbf{0}) = 0$, it follows that

$$V^\alpha(\mathbf{x}) = \int_0^\rho \frac{1}{t}W^\alpha\Big(\frac{t\mathbf{x}}{\rho}\Big)dt.$$

Therefore,

$$\begin{aligned} V(x) &= \Big[\int_0^\rho \frac{1}{t}W^\alpha\Big(\frac{t\mathbf{x}}{\rho}\Big)dt\Big]s_\alpha(\mathbf{x}) \\ &= \Big[\int_0^\rho \frac{1}{t}W^\alpha\Big(\frac{t\mathbf{x}}{\rho}\Big)dt\Big]\tau\Big(\mathbf{x}, \frac{t\mathbf{x}}{\rho}\Big)s_\alpha\Big(\frac{t\mathbf{x}}{\rho}\Big) \\ &= \int_0^\rho \frac{1}{t}\tau\Big(\mathbf{x}, \frac{t\mathbf{x}}{\rho}\Big)W\Big(\frac{t\mathbf{x}}{\rho}\Big)dt \\ &= -\int_0^\rho t^{i-1}\sqrt{j\Big(\frac{t\mathbf{x}}{\rho}\Big)}\tau\Big(\mathbf{x}, \frac{t\mathbf{x}}{\rho}\Big)(H_xU^{(i-1)})\Big(\frac{t\mathbf{x}}{\rho}, y\Big)\xi dt \\ &= -\rho^i\int_0^1 v^{i-1}\sqrt{j(v\mathbf{x})}\tau(\mathbf{x}, v\mathbf{x})(H_xU^{(i-1)})(v\mathbf{x}, y)\xi dv, \end{aligned}$$

where the last equality is obtained by a change of variables $t = \rho v$. Recalling the definition (9.41) of $V(x)$, this is exactly the formula (9.39). □

9.1.4 Construction of parametrix and existence of heat kernel

We now mimic the strategy developed in the toy example of Section 9.1.1 to construct the heat kernel of H. The essential point is to write down a parametrix (the counterpart of the operator K_t).

Let $\eta = 2\varepsilon$ be smaller than the global injectivity radius of M (which is strictly positive since M is compact; see Appendix C.3). The geodesic ball $U_y = B(y, \eta)$ can be used as a normal chart for all $y \in M$. Let $\psi : [0, \infty) \to [0, \infty)$ be a smooth function such that $\psi(r) = 1$ for $r \leqslant (\varepsilon/2)^2$ and $\psi(r) = 0$ for $r \geqslant \varepsilon^2$. Given a fixed number $N \in \mathbb{N}$ to be specified later on, we define

$$k_t^N(x,y) \triangleq \psi(d(x,y)^2) \cdot q_t(x,y) \cdot \sum_{i=0}^{N} t^i U^{(i)}(x,y), \tag{9.42}$$

where $q_t(x,y)$ is the Euclidean heat kernel defined by (9.27) and $\{U^{(i)}\}_{i=0}^{\infty}$ is the solution to the system (9.37) whose expression is explicitly given by Proposition 9.4. Note that $k_t^N(x,y)$ is globally C^∞ in all variables due to the cut-off function ψ. This $k_t^N(x,y)$ will serve as our parametrix for the actual heat kernel, provided N is chosen suitably large.

Lemma 9.4. *Let $K_t^N : \Gamma(E) \to \Gamma(E)$ be the integral operator defined by*

$$(K_t^N s)(x) \triangleq \int_M k_t^N(x,y)s(y)dy, \quad s \in \Gamma(E).$$

Given any metric on E, the operator K_t^N is a bounded linear operator and one has

$$\lim_{t\to 0^+} \sup_{x\in M} \left|(K_t^N s)(x) - s(x)\right|_{E_x} = 0. \tag{9.43}$$

Proof. Boundedness of K_t^N follows from the uniform boundedness of $U^{(i)}(x,y)$ over $\{x, y \in M : d(x,y) \leqslant \varepsilon\}$ and the fact that

$$\sup_{x\in M} \left| \int_{\{y:d(x,y)\leqslant\varepsilon\}} \psi(d(x,y)^2) \cdot \frac{1}{(4\pi t)^{n/2}} e^{-\frac{d(x,y)^2}{4t}} dy \right| < \infty.$$

For the initial condition, given $x \in M$, by considering the normal chart

$$U_x \triangleq \{y \in M : d(x,y) \leqslant \varepsilon\} \cong \{\mathbf{z} \in \mathbb{R}^n : |\mathbf{z}| \leqslant \varepsilon\},$$

one can write

$$(K_t^N s)(x) = \int_{\{\mathbf{z}:|\mathbf{z}|\leqslant\varepsilon\}} \psi(|\mathbf{z}|^2) \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\mathbf{z}|^2}{4t}} \sum_{i=0}^{N} t^i U^{(i)}(x,\mathbf{z}) s(\mathbf{z}) \sqrt{\det g_x(\mathbf{z})} d\mathbf{z},$$

where g_x is the local metric tensor on U_x. By applying a change of variables $\mathbf{w} = \mathbf{z}/\sqrt{t}$ and using Proposition 9.4, it is easily seen that

$$(K_t^N s)(x) = \int_{\left\{\mathbf{w}:|\mathbf{w}|\leqslant\frac{\varepsilon}{2\sqrt{t}}\right\}} \frac{1}{(4\pi)^{n/2}} e^{-|\mathbf{w}|^2/4} \tau(x, \sqrt{t}\mathbf{w}) s(\sqrt{t}\mathbf{w}) d\mathbf{w} + O(t)$$

where the $O(t)$ term is uniform in x. On the other hand, one can also write

$$s(x) = \int_{\left\{\mathbf{w}:|\mathbf{w}|\leqslant\frac{\varepsilon}{2\sqrt{t}}\right\}} \frac{1}{(4\pi)^{n/2}} e^{-|\mathbf{w}|^2/4} s(x) d\mathbf{w} + O(e^{-C/t}).$$

As a consequence,

$$(K_t s)(x) - s(x) = \int_{\left\{\mathbf{w}:|\mathbf{w}|\leqslant\frac{\varepsilon}{2\sqrt{t}}\right\}} \frac{1}{(4\pi)^{n/2}} e^{-|\mathbf{w}|^2/4} \big(\tau(x, \sqrt{t}\mathbf{w}) s(\sqrt{t}\mathbf{w}) - s(x)\big) + O(t)$$

which converges to zero as $t \to 0^+$ uniformly in x. □

From now on, we will fix N and omit the superscript N for simplicity. Define

$$r_t(x, y) \triangleq (\partial_t + H_x) k_t(x, y).$$

We set $r_t^1(x, y) \triangleq r_t(x, y)$ and for $k \geqslant 2$,

$$r_t^k(x, y) \triangleq \int_{0<t_1<\cdots<t_{k-1}<t} r_{t-t_{k-1}}(x, z_{k-1}) \cdots r_{t_2-t_1}(z_2, z_1) r_{t_1}(z_1, y) d\mathbf{t} d\mathbf{z}.$$

In other words, r^k is the k-fold space-time convolution of r. Mimicking (9.6), we define

$$\tilde{r}_t(x, y) \triangleq \sum_{k=1}^{\infty} (-1)^k r_t^k(x, y)$$

and

$$\hat{r}_t(x, y) \triangleq \int_0^t \int_M k_{t-s}(x, z) \tilde{r}_s(z, y) dz ds.$$

The following two lemmas justify the constructions of $\tilde{r}, \hat{r}$ and establish their regularity properties.

Lemma 9.5. *Suppose that $N \geqslant n/2$. Then the following statements hold true.*

(i) The function $r \in C^{q,l_1,l_2}([0,\infty) \times M \times M)$ provided that $2q + l_1 + l_2 \leqslant N - n/2$. In addition, one has

$$|D^{q,l_1,l_2} r_t(x, y)| \leqslant C_T t^{N-n/2-2q-l_1-l_2} \tag{9.44}$$

for all $t \in [0, T]$ and $x, y \in M$, where $C_T > 0$ is a constant depending on T.

(ii) The function $\tilde{r}$ is well-defined and $\tilde{r} \in C^{q,l_1,l_2}([0,\infty) \times M \times M)$ provided that $2q + l_1 + l_2 \leqslant N - n/2$.

Remark 9.4. Including the $t = 0$ endpoint here means that the derivatives exist continuously up to $t = 0$ (no singularity at $t = 0$).

Proof. (i) Explicit calculation based on equation (9.28) and the system (9.37) satisfied by the $U^{(i)}$'s shows that

$$\begin{aligned} r_t(x,y) = & \underbrace{\psi(d(x,y)^2)q_t(x,y)\cdot t^N H_x U^{(N)}(x,y)}_{(*)} \\ & \underbrace{-2\nabla_x^M \psi(d(x,y)^2)\cdot\nabla_x\Phi_t(x,y) + \Delta^M\psi(d(x,y)^2)\cdot\Phi_t(x,y)}_{(\#)}. \end{aligned}$$

where we set

$$\Phi_t(x,y) \triangleq q_t(x,y)\cdot\sum_{i=0}^N t^i U^{(i)}(x,y).$$

If $d(x,y) \leqslant \varepsilon/2$, one has $(\#) = 0$. In addition, the regularity of $(*)$ is determined by the factor

$$e^{-\frac{d(x,y)^2}{4t}} t^{N-n/2}$$

contained in it. Clearly, every differentiation in time picks up a singularity t^{-2} while every differentiation in space picks up a singularity t^{-1} (as the worst scenario). As a result, the worst terms (in terms of t-regularity) in the derivative $D^{q,l_1,l_2} r_t(x,y)$ are those containing the factor

$$t^{N-n/2-2q-l_1-l_2}.$$

If $2q + l_1 + l_2 \leqslant N - n/2$, this does not produce a t-singularity and therefore, one has $r \in C^{q,l_1,l_2}$. In this regime, one apparently has the estimate (9.44).

If $d(x,y) > \varepsilon/2$, the factor $e^{-d(x,y)^2/4t}$ smoothens all possible t-singularities (of any algebraic order) appearing in the space-time differentiation. One therefore has $r \in C^\infty$ in this regime. In addition, $|r_t(x,y)| \lesssim e^{-C/t}$ for the same reason.

Combining the above discussions, the conclusion of Part (i) thus follows.

(ii) One knows from Part (i) that $|r_t(x,y)| \leqslant C_T t^{N-n/2}$ for all $t \in [0,T]$ and $x, y \in M$. It follows from the definition of r^k that

$$|r_t^k(x,y)| \leqslant \frac{C_T^k \mathrm{vol}(M)^k}{k!} t^{k(N-n/2)}.$$

As a result, the series defining $\tilde{r}$ is uniformly convergent on $[0,T]\times M\times M$. Similarly, $\tilde{r}$ has the same regularity as r and also satisfies (9.44). □

Lemma 9.6. *Suppose that* $N \geqslant n/2$. *The function* $\hat{r} \in C^{q,l_1,l_2}((0,\infty)\times M\times M)$ *provided that* $2q+l_1+l_2 \leqslant N-n/2$. *In addition, for each* $T > 0$ *there exists* $C_T > 0$ *such that*

$$|\hat{r}_t(x,y)| \leqslant C_T t^{N-n/2+1} \tag{9.45}$$

for all $t \in [0,T]$ *and* $x, y \in M$.

Proof. Since

$$\hat{r}_t(x,y) \triangleq \int_0^t \int_M k_s(x,z)\tilde{r}_{t-s}(z,y)dzds,$$

the regularity in t and y follows directly from Lemma 9.5. We now consider the regularity in x. Here one cannot directly apply the smoothness of k_s in x because differentiating k_s will introduce additional s-singularities to invalidate the ds-integral.

By using the definition (9.42) of $k_s(x,z)$, one has

$$\hat{r}_t(x,y) = \int_0^t \int_M q_s(x,z)h(x,z;t,s,y)dzds, \tag{9.46}$$

where

$$h(x,z;t,s,y) \triangleq \psi(d(x,z)^2)\sum_{i=0}^{N} s^i U^{(i)}(x,z)\tilde{r}_{t-s}(z,y).$$

Observe that h is smooth in x and no s-singularities will be introduced after differentiation in x. If one differentiates $q_s(x,z)$ in x, since it is a function of $d(x,z)$ the x-derivative can be transferred to a z-derivative (possibly with the introduction of lower order terms). As a result, one can use integration by parts to transfer the z-derivative to the function h. The z-regularity of h is controlled by the regularity of $\tilde{r}$ (the ψ- and $U^{(i)}$-terms are smooth), which is contained in Lemma 9.5. This shows that the x-regularity of $\hat{r}$ is at most $N - n/2$.

The estimate (9.45) follows from (9.44) as well as Lemma 9.4. □

Now one can use the formula (9.5) to construct a heat kernel for H, hence establishing its existence.

Theorem 9.2. *Suppose that $N \geqslant n/2 + 2$. Then*

$$p_t(x,y) \triangleq k_t(x,y) + \hat{r}_t(x,y)$$

is a heat kernel for H.

Proof. According to Lemma 9.6 and the smoothness of $k_t(x,y)$, one knows that $p_t(x,y)$ is C^1 in t and C^2 in x. Due to Lemma 9.5 and Lemma 9.6, one can perform exactly the same calculation as in (9.7) to conclude that $p_t(x,y)$ satisfies the heat equation. The initial condition (9.1) follows from the one for $k_t(x,y)$ (see (9.43)) and the fact that $\hat{r}_t(x,y) \to 0$ as $t \to 0^+$ uniformly in x, y (see (9.45)). □

Remark 9.5. At this stage, the construction of $p_t(x,y)$ depends on N. After we prove uniqueness, it will be clear that $p_t(x,y)$ is independent of N and is hence also smooth.

9.2 Uniqueness of heat kernel

We now address the question of uniqueness. As before, let $H = \Delta^E + F$ be a given generalised Laplacian on E, where Δ^E is the Laplacian associated with a connection ∇^E on E and $F \in \Gamma(\mathrm{End}(E))$.

Theorem 9.3. *There is at most one heat kernel for H.*

Proof. The main insight is that *existence of heat kernel for H^* (the formal adjoint of H) implies the uniqueness of heat kernel for H.*

To see this, one first recalls that H^* is characterised by the relation

$$\int_M \langle Hs, \xi \rangle_x dx = \int_M \langle s, H^*\xi \rangle_x dx \quad \forall s \in \Gamma(E), \xi \in \Gamma(E^*),$$

where $\langle \cdot, \cdot \rangle_x$ denotes the pairing between E_x and E_x^*. It is standard that $H^* = \Delta^{E^*} + F^*$, where Δ^{E^*} is the Laplacian associated with the induced connection ∇^{E^*} from ∇^E on E^* and F^* is the fiberwise dual of F. One already knows from Theorem 9.2 that a heat kernel for H^* exists. Call it $p_t^*(x,y)$ and define

$$P_t^* : \Gamma(E^*) \to \Gamma(E^*), \ (P_t^*\xi)(x) \triangleq \int_M p_t^*(x,y)\xi(y)dy \tag{9.47}$$

for every $t > 0$.

Next, let $q_t(x,y)$ be any heat kernel for H and we define $Q_t : \Gamma(E) \to \Gamma(E)$ in the same way as in (9.47). Given $t \geqslant 0$, $s \in \Gamma(E)$ and $\xi \in \Gamma(E^*)$, consider the function

$$f(r) \triangleq \int_M \langle (Q_r s)(x), (P_{t-r}^*\xi)(x) \rangle_x dx, \quad 0 \leqslant r \leqslant t.$$

By using the heat equations for Q and P^*, it is readily checked that $f'(r) = 0$ for every r. As a result, $f(r)$ is constant, and in particular,

$$\langle Q_t s, \xi \rangle_{L^2} = f(t) = f(0) = \langle s, P_t^*\xi \rangle_{L^2}. \tag{9.48}$$

Now if $q, \tilde{q}$ are both heat kernels of H, the relation (9.48) would imply that

$$\int_M \Big\langle \int_M q_t(x,y)s(y)dy, \xi(x) \Big\rangle_x dx = \int_M \Big\langle \int_M \tilde{q}_t(x,y)s(y)dy, \xi(x) \Big\rangle_x dx$$

for all $s \in \Gamma(E)$ and $\xi \in \Gamma(E^*)$. A density argument then shows that $q = \tilde{q}$. This proves the uniqueness of heat kernel for H. □

One has essentially obtained Theorem 9.1 on the existence, uniqueness and smoothness of the heat kernel.

Proof of Theorem 9.1. Existence and uniqueness has been proved as before. For smoothness, one just picks an arbitrarily large N and apply Theorem 9.2 to see that the heat kernel constructed from that theorem has C^{q,l_1,l_2}-regularity provided that $2q+l_1+l_2 \leqslant N-n/2$. But all these heat kernels coincide due to uniqueness. Since N is arbitrary, one obtains its differentiability for all orders.

□

By using the heat kernel, one can construct the associated heat semigroup which describes solutions to the Cauchy problem for H with smooth initial conditions.

Corollary 9.3. *(i) Define $P_t : \Gamma(E) \to \Gamma(E)$ by*

$$(P_t s)(x) \triangleq \int_M p_t(x,y)s(y)dy, \quad s \in \Gamma(E).$$

Then for every $s \in \Gamma(E)$, the section $u_t(x) \triangleq (P_t s)(x)$ is the unique solution to the Cauchy problem

$$(\partial_t + H)u_t(x) = 0, \ t > 0, x \in M$$

with initial condition $u_0 = s$ in the space of $C^{1,2}$-sections. In addition, $\{P_t : t \geqslant 0\}$ is a semigroup, i.e. $P_{t+s} = P_t \circ P_s$.

Proof. The fact that $P_t s$ solves the Cauchy problem directly follows from properties of the heat kernel. Suppose that u, v are both solutions with the same initial condition. To prove $w \triangleq u - v = 0$, it suffices to show that

$$\int_M \langle w_t(x), \xi(x)\rangle_x dx = 0 \tag{9.49}$$

for all $\xi \in \Gamma(E^*)$. Let us consider

$$f(r) \triangleq \int_M \langle w_r(x), (P_{t-r}^* \xi)(x)\rangle_x dx, \quad 0 \leqslant r \leqslant t.$$

Similar to the proof of Theorem 9.3, one finds that $f'(r) = 0$. Since $f(0) = 0$ (because $w_0 = 0$), one concludes that $f(t) = 0$ which is exactly (9.49). Therefore, the solution to the Cauchy problem is unique. The semigroup property is a direct consequence of uniqueness; for fixed $s \geqslant 0$ and $u \in \Gamma(E)$, both $t \mapsto P_{t+s}u$ and $t \mapsto P_t \circ P_s u$ solve the Cauchy problem with the same initial condition $P_s u$ and hence they are equal. □

Definition 9.3. $\{P_t : t \geqslant 0\}$ is called the *heat semigroup* associated with H. It is denoted as $P_t = e^{-tH}$.

Remark 9.6. The semigroups for H and H^* are dual to each other. For their heat kernels, one has

$$p_t(x,y) = p_t^*(y,x)^*,$$

where we identify E_x with E_x^{**} in the canonical way.

Along the path of proving Theorem 9.1, one has also obtained the following small-time asymptotic expansion of the heat kernel. This expansion will be particularly useful in the later heat equation proof of the CGB theorem.

Corollary 9.4. *For any fixed $N \geqslant n/2$, one has*

$$p_t(x,y) - \frac{1}{(4\pi t)^{n/2}} e^{-\frac{d(x,y)^2}{4t}} \sum_{i=0}^{N} t^i U^{(i)}(x,y) = O(t^{N-n/2+1})$$

as $t \to 0^+$, where the $U^{(i)}$'s are given by (9.38, 9.39) and the above $O(t^{N-n/2+1})$ term is uniform over all x, y such that $d(x,y) \leqslant \eta/4$ (η is the global injectivity radius of M).

9.3 Trace of heat semigroup

We now further assume that the bundle E is equipped with a metric (i.e. a fiberwise inner product that depends smoothly on the base point). Let $L^2(E)$ be the Hilbert space consisting of those sections s that are square integrable over M, i.e. $\int_M |s(x)|^2_{E_x} dx < \infty$. We first make the following observation.

Lemma 9.7. *For each $t > 0$, the operator $P_t : L^2(E) \to L^2(E)$ is a trace-class operator.*

Proof. Since $p_t(x,y)$ is smooth in (x,y), it is in particular uniformly bounded on $M \times M$ and hence square integrable. Let $\{s_n\}$ be an ONB of $L^2(E)$. Then one has

$$\begin{aligned} \|P_t\|^2_{\mathrm{HS}} &= \sum_{n=1}^{\infty} \int_M |(P_t s_n)(x)|^2_{E_x} dx = \sum_{n=1}^{\infty} \int_M \Big| \int_M p_t(x,y) s_n(y) dy \Big|^2_{E_x} dx \\ &= \int_M \|p_t(x,\cdot)\|_{L^2(dy)} dx = \|p_t\|_{L^2(dxdy)} < \infty. \end{aligned}$$

This shows that P_t is a Hilbert-Schmidt operator. Note that $P_t = P_{t/2} \circ P_{t/2}$. As a composition of Hilbert-Schmidt operators, it is therefore of trace-class. □

In what follows, we further assume that H is self-adjoint. Since P_t is of trace-class and self adjoint, one can apply the spectral theorem in exactly the same way as in the proof of the L^2-Hodge theorem (Theorem 8.1) to obtain a spectral decomposition $\{\lambda_j, \phi_j : j \geqslant 1\}$ for H. Here each λ_j is an eigenvalue of H with finite multiplicity and ϕ_j is a smooth λ_j-eigensection of H (smoothness of ϕ_j is a consequence of smoothness of the heat kernel). The eigenvalues are arranged in nondecreasing order:

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \uparrow \infty.$$

The eigensections $\{\phi_j\}$ form an ONB of $L^2(E)$.

The following result computes the trace of P_t in terms of either the spectrum of H or the fiberwise trace of the heat kernel. This result will be useful in the heat equation proof of CGB.

Proposition 9.5. *One has*

$$\mathrm{Tr}[P_t] = \sum_{j=1}^{\infty} e^{-\lambda_j t} = \int_M \mathrm{tr}[p_t(x,x)]dx.$$

Proof. The first identity follows directly from the fact that P_t is of trace-class with eigenvalues $\{e^{-\lambda_j t} : j \geqslant 1\}$. To prove the second identity, one first notes from the semigroup property and self-adjointness that

$$\begin{aligned} \mathrm{tr}[p_t(x,x)] &= \int_M \mathrm{tr}\big[p_{t/2}(x,y)p_{t/2}(y,x)\big]dy \\ &= \int_M \mathrm{tr}\big[p_{t/2}(x,y)p_{t/2}(x,y)^*\big]dy \\ &= \int_M |p_{t/2}(x,y)|^2_{\mathrm{HS}}dy. \end{aligned} \tag{9.50}$$

The next key observation is that

$$\int_M \langle p_s(x,y)v, \phi_j(x)\rangle_{E_x}dx = e^{-\lambda_j s}\langle v, \phi_j(y)\rangle_{E_y}, \quad \forall s \geqslant 0, \tag{9.51}$$

for any given $v \in E_y$ and $j \geqslant 1$. Presuming the correctness of (9.51), for any $v \in E_y$ one has

$$\begin{aligned} \int_M |p_{t/2}(x,y)v|^2_{E_x}dx &= \sum_{j=1}^{\infty}\Big|\int_M \langle p_{t/2}(x,y)v, \phi_j(x)\rangle_{E_x}dx\Big|^2 \\ &= \sum_{j=1}^{\infty} e^{-\lambda_j t}|\langle v, \phi_j(y)\rangle_{E_y}|^2. \end{aligned}$$

By summing over all v's from an ONB of E_y, one obtains that

$$\int_M |p_{t/2}(x,y)|^2_{\mathrm{HS}}dx = \sum_{j=1}^{\infty} e^{-\lambda_j t}|\phi_j(y)|^2_{E_y}.$$

Since $\|\phi_j\|_{L^2} = 1$, after integrating over $y \in M$ it follows that

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} = \int_{M\times M} |p_{t/2}(x,y)|_{\mathrm{HS}}^2 dx dy = \int_M \mathrm{tr}[p_t(x,x)]dx,$$

which is exactly what we want to prove.

Now it remains to prove the relation (9.51). To this end, we define

$$f_j(s;y,v) \triangleq \int_M \langle p_s(x,y)v, \phi_j(x)\rangle_{E_x} dx.$$

Note that

$$f_j(s;y,v) = \int_M \langle v, p_s(x,y)^* \phi_j(x)\rangle_{E_y} dx = \int_M \langle v, p_s(y,x)\phi_j(x)\rangle_{E_y} dx.$$

By using the initial condition for the heat kernel, one finds that

$$f_j(0;y,v) = \langle v, \phi_j(y)\rangle_{E_y}. \tag{9.52}$$

In addition, one has

$$\begin{aligned} \partial_s f_j(s;y,v) &= -\int_M \langle H_x p_s(x,y)v, \phi_j(x)\rangle_{E_x} dx \\ &= -\int_M \langle p_s(x,y)v, H_x\phi_j(x)\rangle_{E_x} dx \\ &= -\lambda_j \int_M \langle p_s(x,y)v, \phi_j(x)\rangle_{E_x} dx = -\lambda_j f_j(s;y,v). \end{aligned} \tag{9.53}$$

By solving the ODE (9.53) with initial condition (9.52), one obtains that

$$f_j(s;y,v) = e^{-\lambda_j s}\langle v, \phi_j(y)\rangle_{E_y}.$$

This gives the desired equation (9.51). □

10 A heat equation approach

In this chapter, we present a heat equation proof of the CGB theorem which was essentially due to McKean-Singer [MS67] and Patodi [Pat71]. Let M be a closed, oriented, n-dimensional Riemannian manifold.

10.1 The McKean-Singer theorem

Recall from the Weitzenböck formula that the Hodge Laplacian Δ is a generalised Laplacian on $E = \Lambda T^*M$ and therefore admits a smooth heat kernel $p_t(x, y)$. In particular, the trace formula given by Proposition 9.5 applies.

The starting point of the argument is the following beautiful observation of McKean-Singer [MS67]. If one considers the "supertrace" of the heat semigroup instead (a notion of trace that respects the even / odd grading on forms), the terms coming from nonzero eigenvalues of Δ cancel out nicely, leaving only those terms corresponding to the zero eigenvalue. As a consequence, what one actually sees from the supertrace is the kernel of Δ (space of harmonic forms in each degree). The Euler characteristic appears naturally according to the Hodge theorem and the de Rham theorem.

To make this precise, we first introduce the definition of supertrace (cf. Definition 11.3 in the general setting of super vector spaces). Let Δ_{even} (respectively, Δ_{odd}) denote the restriction of Δ to the space $\Omega^{\text{even}}(M)$ of even forms (respectively, the space $\Omega^{\text{odd}}(M)$ of odd forms). Since Δ respects degree, it is apparent that the heat kernel $p_t(x, y)$ also respects degree. The restriction of $p_t(x, y)$ to even (respectively, odd) elements is the heat kernel of Δ_{even} (respectively, Δ_{odd}).

Definition 10.1. The *supertrace* of the heat semigroup $e^{-t\Delta}$ is defined by

$$\mathrm{Str}\big[e^{-t\Delta}\big] \triangleq \mathrm{Tr}\big[e^{-t\Delta_{\text{even}}}\big] - \mathrm{Tr}\big[e^{-t\Delta_{\text{odd}}}\big].$$

For each $t > 0$ and $y \in M$, the *supertrace* of $p_t(y, y)$ is defined by

$$\mathrm{str}[p_t(y, y)] \triangleq \mathrm{tr}\big[p_t(y, y)|_{\Lambda^{\text{even}} T_y^* M}\big] - \mathrm{tr}\big[p_t(y, y)|_{\Lambda^{\text{odd}} T_y^* M}\big]. \tag{10.1}$$

The following lemma is a key observation. Let E_λ^{even} (respectively, E_λ^{odd}) be the λ-eigenspace of Δ_{even} (respectively, Δ_{odd}).

Lemma 10.1. *For any $\lambda \neq 0$, the operator $D = d + \delta : E_\lambda^{\text{even}} \to E_\lambda^{\text{odd}}$ is a linear isomorphism.*

Proof. Suppose $\omega \in E_\lambda^{\text{even}}$. Then

$$\Delta D\omega = D\Delta\omega = \lambda D\omega \implies D\omega \in E_\lambda^{\text{odd}}.$$

This shows that D is well-defined. Suppose that $\omega \in E_\lambda^{\mathrm{even}}$ and $D\omega = 0$. Then

$$\lambda\omega = \Delta\omega = D^2\omega = 0 \implies \omega = 0$$

because $\lambda \neq 0$. This shows that D is injective. Finally, let $\eta \in E_\lambda^{\mathrm{odd}}$ and define $\omega \triangleq D\eta/\lambda$. Then $\omega \in E_\lambda^{\mathrm{even}}$ and one has

$$D\omega = \frac{\Delta\eta}{\lambda} = \eta.$$

This shows that D is surjective. $\square$

Theorem 10.1 (McKean-Singer)**.** *One has*

$$\chi(M) = \mathrm{Str}\big[e^{-t\Delta}\big] \tag{10.2}$$

for all $t > 0$.

Proof. By the definition of supertrace, Proposition 9.5 and Lemma 10.1, one has

$$\begin{aligned}\mathrm{Str}\big[e^{-t\Delta}\big] &= \mathrm{Tr}\big[e^{-t\Delta_{\mathrm{even}}}\big] - \mathrm{Tr}\big[e^{-t\Delta_{\mathrm{odd}}}\big] = \sum_{\lambda\in\mathrm{Spec}(\Delta_{\mathrm{even}})} e^{-\lambda t} - \sum_{\lambda\in\mathrm{Spec}(\Delta_{\mathrm{odd}})} e^{-\lambda t} \\ &= \dim(\ker\Delta_{\mathrm{even}}) - \dim(\ker\Delta_{\mathrm{odd}}) = \sum_{k=0}^{n}(-1)^k\dim(\mathcal{H}^k) = \chi(M).\end{aligned}$$

Here $\mathcal{H}^k \triangleq \ker\Delta^k$ denotes the space of harmonic k-form. The last identity follows from the Hodge theorem and the de Rham theorem. $\square$

10.2 Patodi's local index formula

In view of Theorem 10.1, our next task is to compute $\mathrm{Str}\big[e^{-t\Delta}\big]$ in terms of the heat kernel. First of all, according to Proposition 9.5 and Definition 10.1, one has

$$\chi(M) = \mathrm{Str}\big[e^{-t\Delta}\big] = \int_M \mathrm{str}[p_t(y,y)]dy.$$

It follows from the small-time expansion of the heat kernel (see Corollary 9.4) that

$$\chi(M) = \frac{1}{(4\pi)^{n/2}}\sum_{i=0}^{[n/2]} t^{i-n/2}\int_M \mathrm{str}\big[U^{(i)}(y,y)\big]dy + o(1), \tag{10.3}$$

where the $U^{(i)}$'s are given by (9.38, 9.39) and the $o(1)$ term vanishes as $t \to 0^+$. Since the relation (10.3) is valid for all t, one immediate obtains the following formula for the Euler characteristic.

Proposition 10.1. *One has*

$$\int_M \mathrm{str}\big[U^{(i)}(y,y)\big]\,dy = 0$$

for all $i < n/2$. In addition, $\chi(M) = 0$ if n is odd and

$$\chi(M) = \frac{1}{(4\pi)^m}\int_M \mathrm{str}\big[U^{(m)}(y,y)\big]\,dy \tag{10.4}$$

if $n = 2m$ is even.

To prove the CGB theorem, it remains to compute $\mathrm{str}[U^{(m)}(y,y)]$ and see how it is related to the Pfaffian of the curvature. This was achieved by Patodi [Pat71] and the main result stated as follows is known as *Patodi's local index formula* for the operator $D = d + \delta$.

Theorem 10.2. *Suppose that* $\dim M = 2m$. *Then one has*

$$\frac{1}{(4\pi)^m}\mathrm{str}\big[U^{(m)}(y,y)\big]\,dy = \frac{(-1)^m}{(2\pi)^m}\mathrm{Pf}(R), \tag{10.5}$$

where $\mathrm{Pf}(R)$ *is the Pfaffian of the Riemannian curvature R (cf. Definition 5.1 and Example 7.4). As a consequence of (10.4) and (7.12), the CGB formula (5.3) holds.*

In the following subsections, we develop the proof of Patodi's local index formula (10.5) following the approach of Yu [Yu87] (cf. [ZF22] for an excellent exposition).

10.2.1 An algebraic supertrace formula

We need a few algebraic preliminaries. Let V be a real, n-dimensional Euclidean space. Let V^* be its dual and let ΛV^* be the exterior algebra over V^*. We define the automorphism $\tau : \Lambda V^* \to \Lambda V^*$ by

$$\tau \triangleq \begin{cases} \mathrm{id}, & \text{on } \Lambda^{\mathrm{even}}V^*; \\ -\mathrm{id}, & \text{on } \Lambda^{\mathrm{odd}}V^*. \end{cases}$$

Definition 10.2. Let $A \in \mathrm{End}(\Lambda V^*)$. The *supertrace* of A is defined by $\mathrm{str}[A] \triangleq \mathrm{tr}[\tau A]$.

Remark 10.1. It is easily checked that the definition here is consistent with (10.1) (cf. (11.1) below). A more general discussion is given in Section 11.1 below in the context of super vector spaces. Here we only introduce the definitions that are relevant to the current proof of CGB.

The following elementary fact plays an important role in the proof of Theorem 10.2 (and also in the superconnection approach to be developed in the next chapter). Recall that $c(v), \hat{c}(v)$ $(v \in V)$ are the Clifford multipliers defined by (8.17). Let $\{\varepsilon_1, \cdots, \varepsilon_n\}$ be a PONB of V.

Lemma 10.2. *For any words $I = (i_1 < \cdots < i_k)$ and $J = (j_1 < \cdots < j_l)$, one has*

$$\mathrm{tr}[c(\varepsilon_I)\hat{c}(\varepsilon_J)] = \begin{cases} 2^n, & \textit{if } I = J = \emptyset; \\ 0, & \textit{otherwise,} \end{cases}$$

where $c(\varepsilon_I) \triangleq c(\varepsilon_{i_1}) \cdots c(\varepsilon_{i_k})$ and we define $\hat{c}(\varepsilon_J)$ similarly.

Proof. (i) <u>*Case I: $|I| + |J|$ is even and > 0.*</u> In this case, at least one of I, J is nonempty, say $i \in I$ for some I. By the relations (8.20), one has

$$\begin{aligned} \mathrm{tr}\big[c(\varepsilon_I)\hat{c}(\varepsilon_J)\big] &= \mathrm{tr}\big[c(\varepsilon_i)c(\varepsilon_I)\hat{c}(\varepsilon_J)c(\varepsilon_i)^{-1}\big] \\ &= -\mathrm{tr}\big[c(\varepsilon_i)c(\varepsilon_I)\hat{c}(\varepsilon_J)c(\varepsilon_i)\big] \\ &= -(-1)^{|J|}\mathrm{tr}\big[c(\varepsilon_i)c(\varepsilon_I)c(\varepsilon_i)\hat{c}(\varepsilon_J)\big] \\ &= -(-1)^{|J|}(-1)^{|I|-1}\mathrm{tr}\big[c(\varepsilon_i)c(\varepsilon_i)c(\varepsilon_I)\hat{c}(\varepsilon_J)\big] \\ &= -\mathrm{tr}\big[c(\varepsilon_I)\hat{c}(\varepsilon_J)\big]. \end{aligned}$$

Therefore, $\mathrm{tr}[c(\varepsilon_I)\hat{c}(\varepsilon_J)] = 0$. If $j \in J$, one considers $\hat{c}(\varepsilon_j)c(\varepsilon_I)\hat{c}(\varepsilon_J)\hat{c}(\varepsilon_j)^{-1}$ instead.
(ii) <u>*Case II: $|I| + |J|$ is odd.*</u> In this case, at least one of I^c, J^c is nonempty. Assuming some $i \notin I$, a similar calculation also yields the relation that

$$\mathrm{tr}\big[c(\varepsilon_I)\hat{c}(\varepsilon_J)\big] = -\mathrm{tr}\big[c(\varepsilon_I)\hat{c}(\varepsilon_J)\big] \implies \mathrm{tr}\big[c(\varepsilon_I)\hat{c}(\varepsilon_J)\big] = 0.$$

The proof of the lemma is thus complete. □

The following corollary is the main relation that will be used in the current argument.

Corollary 10.1. *For any words $I = (i_1 < \cdots < i_k)$ and $J = (j_1 < \cdots < j_l)$, one has*

$$\mathrm{str}\big[c(\varepsilon_I)\hat{c}(\varepsilon_J)\big] = \begin{cases} (-1)^{n(n+1)/2}2^n, & \textit{if } I = J = (1, \cdots, n); \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We first claim that

$$\tau = \hat{c}(\varepsilon_1)c(\varepsilon_1) \cdots \hat{c}(\varepsilon_n)c(\varepsilon_n). \tag{10.6}$$

To see this, consider a basis element $\theta^I \triangleq \theta^{i_1} \wedge \cdots \wedge \theta^{i_k}$ of ΛV^* where $\{\theta^i\}$ is the dual basis of $\{\varepsilon_i\}$ and $I = (i_1 < \cdots < i_k)$. For any $1 \leqslant j \leqslant n$, one computes by definition that

$$\hat{c}(\varepsilon_j)c(\varepsilon_j)\theta^I = \begin{cases} \iota(\varepsilon_j)a(\varepsilon_j)\theta^I = \theta^I, & \text{if } j \notin I; \\ -\iota(\varepsilon_j)a(\varepsilon_j)\theta^I = -\theta^I, & \text{if } j \in I. \end{cases}$$

Therefore,

$$\hat{c}(\varepsilon_1)c(\varepsilon_1)\cdots\hat{c}(\varepsilon_n)c(\varepsilon_n)\theta^I = (-1)^{|I|}\theta^I,$$

which is exactly the definition of τ.

By using the expression (10.6) as well as the Clifford relations (8.20), one obtains that

$$\begin{aligned} \operatorname{str}\big[c(\varepsilon_I)\hat{c}(\varepsilon_J)\big] &= \operatorname{tr}\big[\hat{c}(\varepsilon_1)c(\varepsilon_1)\cdots\hat{c}(\varepsilon_n)c(\varepsilon_n)c(\varepsilon_I)\hat{c}(\varepsilon_J)\big] \\ &= (-1)^{n(n+1)/2}\operatorname{tr}\big[c(\varepsilon_1)\cdots c(\varepsilon_n)\hat{c}(\varepsilon_1)\cdots\hat{c}(\varepsilon_n)c(\varepsilon_I)\hat{c}(\varepsilon_J)\big]. \end{aligned}$$

According to Lemma 10.2, the RHS is nonzero only when $I = J = (1, \cdots, n)$, in which case a further application of (8.20) yields that the RHS is equal to $(-1)^{n(n+1)/2}2^n$. □

10.2.2 Structure of $U^{(i)}(x, y)$

Throughout the rest, we assume that $\dim M = 2m$. The main idea of proving Theorem 10.2 is to express the endomorphisms $U^{(i)}(x, y)$ in terms of Clifford multipliers, so that one can apply Corollary 10.1 to compute their supertraces.

In what follows, let $y \in M$ be given fixed. Consider a normal chart U around y and a parallel PONF $\{e_i\}$ on U as usual. We will continue to use the notation

$$c_i \triangleq c(e_i),\ \hat{c}_j \triangleq \hat{c}(e_j),\ c_I \triangleq c(e_{i_1})\cdots c(e_{i_k}),\ \hat{c}_J \triangleq \hat{c}(e_{j_1})\cdots\hat{c}(e_{j_l})$$

as in (8.18). We first introduce two basic definitions.

Definition 10.3. We define the endomorphism

$$\mathfrak{R} \triangleq -\frac{1}{8}R_{ijkl}c_ic_j\hat{c}_k\hat{c}_l \in \Gamma(\operatorname{End}(\Lambda T^*M)),$$

where $R_{ijkl} \triangleq \langle R(e_k, e_l)e_j, e_i\rangle$ are the curvature coefficients with respect to $\{e_i\}$.

Remark 10.2. $\mathfrak{R}$ is exactly the endomorphism appearing in the Lichnerowicz formula (see Proposition 8.2) and is therefore intrinsically defined on M.

Definition 10.4. We define $\mathcal{S}$ to be the subalgebra of $\Gamma(\operatorname{End}(\Lambda T^*U))$ generated by elements of the form $\Phi = \varphi c_I\hat{c}_J$, where $\varphi \in C^\infty(U)$, $I = (i_1 < \cdots < i_k)$ and $J = (j_1 < \cdots < j_l)$ with $0 \leqslant k, l \leqslant 2m$.

Remark 10.3. The operator Φ is understood fiberwise as $x \mapsto \Phi_x = \varphi(x) \cdot c(e_I(x))\hat{c}(e_J(x)) \in \mathrm{End}(\Lambda T_x^* M)$.

The main structural result of $U^{(i)}(x,y)$ is stated as follows. As in Section 8.5.1, let $\bar{\nabla}$ be the connection on ΛT^*M induced from the Levi-Civita connection and let $\Delta^{\Lambda T^*M}$ be the associated Laplacian.

Proposition 10.2. *For each $i \geqslant 1$, there exists a finite subset $\mathcal{L}_i \subseteq \mathcal{S}$ such that the following properties hold true.*

(i) Each $\Phi = \varphi c_I \hat{c}_J \in \mathcal{L}_i$ satisfies either

$$|I| + |J| = 4i \quad \& \quad \varphi(y) = 0$$

or $|I| + |J| < 4i$.

(ii) The endomorphism $U^{(i)}(x,y) : \Lambda T_y^* M \to \Lambda T_x^* M$ can be expressed as

$$U^{(i)}(x,y) = \frac{1}{i!}\mathfrak{R}_x^i \circ \tau(x,y) + \sum_{\Phi \in \mathcal{L}_i} \Phi_x \circ \tau(x,y), \quad x \in U. \tag{10.7}$$

Here the operator $\tau(x,y)$ is the parallel transport along the ray from y to x and the subscript x means that the fiberwise endomorphisms are evaluated at the base location x.

Proof. We prove the claim by induction on i. The main tool is the Lichnerowicz formula (see Proposition 8.2), which asserts that

$$-\Delta = \mathfrak{R} - \Big(\Delta^{\Lambda T^*M} + \frac{\mathrm{Scal}}{4}\mathrm{id}\Big). \tag{10.8}$$

(i) Base case.

For $i = 1$, one has from (9.39) and (10.8) that

$$\begin{aligned} U^{(1)}(x,y) =& \frac{1}{\sqrt{j(\mathbf{x})}} \int_0^1 \sqrt{j(v\mathbf{x})}\tau(\mathbf{x}, v\mathbf{x})(-\Delta U^{(0)})(v\mathbf{x}, y)dv \\ =& \frac{1}{\sqrt{j(\mathbf{x})}} \int_0^1 \sqrt{j(v\mathbf{x})}\tau(\mathbf{x}, v\mathbf{x})(\mathfrak{R}U^{(0)})(v\mathbf{x}, y)dv \\ &- \frac{1}{\sqrt{j(\mathbf{x})}} \int_0^1 \sqrt{j(v\mathbf{x})}\tau(\mathbf{x}, v\mathbf{x})\Big[\Big(\Delta^{\Lambda T^*M} + \frac{\mathrm{Scal}}{4}\Big)U^{(0)}\Big](v\mathbf{x}, y)dv \\ =:& A(\mathbf{x}) + B(\mathbf{x}), \end{aligned}$$

where $U^{(0)}(z,y) = j^{-1/2}(z)\tau(z,y)$.

To analyse the $A(\mathbf{x})$-term, since the $\{e_i\}$'s are parallel along rays, one has

$$\begin{aligned}&\tau(\mathbf{x}, v\mathbf{x})(\mathfrak{R}_{\mathbf{z}}U^{(0)})(v\mathbf{x}, y)\\ &= -\frac{1}{8}R_{ijkl}(v\mathbf{x})j^{-1/2}(v\mathbf{x})\tau(\mathbf{x}, v\mathbf{x})(c_ic_j\hat{c}_k\hat{c}_l)(v\mathbf{x})\tau(v\mathbf{x}, y)\\ &= -\frac{1}{8}R_{ijkl}(v\mathbf{x})j^{-1/2}(v\mathbf{x})(c_ic_j\hat{c}_k\hat{c}_l)(\mathbf{x})\tau(\mathbf{x}, y).\end{aligned}$$

It follows that

$$\begin{aligned}A(\mathbf{x}) &= \frac{1}{\sqrt{j(\mathbf{x})}}\int_0^1\Big(-\frac{1}{8}R_{ijkl}(v\mathbf{x})\Big)dv \times (c_ic_j\hat{c}_k\hat{c}_l)(\mathbf{x})\tau(\mathbf{x}, y)\\ &= \mathfrak{R}_{\mathbf{x}}\circ\tau(\mathbf{x}, y) + \varphi_{ijkl}(\mathbf{x})(c_ic_j\hat{c}_k\hat{c}_l)(\mathbf{x})\tau(\mathbf{x}, y).\end{aligned} \tag{10.9}$$

Here

$$\varphi_{ijkl}(\mathbf{x}) \triangleq \frac{1}{\sqrt{j(\mathbf{x})}}\int_0^1\Big(-\frac{1}{8}R_{ijkl}(v\mathbf{x})\Big)dv - \Big(-\frac{1}{8}R_{ijkl}(\mathbf{x})\Big),$$

which vanishes at $\mathbf{x} = 0$ $(x = y)$ at least to the first order. In other words, one can express

$$A(\mathbf{x}) = \mathfrak{R}_{\mathbf{x}}\circ\tau(\mathbf{x}, y) + \sum_{\Phi\in\mathcal{L}_1'}\Phi_{\mathbf{x}}\circ\tau(\mathbf{x}, y) \tag{10.10}$$

for some finite subset $\mathcal{L}_1'$ of $\mathcal{S}$. Elements $\Phi\in\mathcal{L}_1'$ are of the form $\Phi = \varphi c_I\hat{c}_J$ with $\varphi(y) = 0$ and $|I| = |J| = 2$.

To analyse the $B(\mathbf{x})$-term, one first recalls that

$$\Delta^{\Lambda T^*M} = -\big(\bar{\nabla}_{e_i}\bar{\nabla}_{e_i} - \bar{\nabla}_{\nabla_{e_i}e_i}\big).$$

By using the formula (8.25) (as well as the notation (8.18)), one computes explicitly that

$$\begin{aligned}(\bar{\nabla}_{e_i}\bar{\nabla}_{e_i}U^{(0)})(\mathbf{z}, y) = {}& e_i^2(j^{-1/2}(\mathbf{z}))\tau(\mathbf{z}, y) - 2e_i(j^{-1/2}(\mathbf{z}))\Gamma_{il}^k a_l\iota_k\tau(\mathbf{z}, y)\\ &- j^{-1/2}(\mathbf{z})e_i(\Gamma_{il}^k)a_l\iota_k\tau(\mathbf{z}, y) + j^{-1/2}(\mathbf{z})\Gamma_{il}^k\Gamma_{iq}^p a_q\iota_p a_l\iota_k\tau(\mathbf{z}, y)\end{aligned}$$

and

$$(\bar{\nabla}_{\nabla_{e_i}e_i}U^{(0)})(\mathbf{z}, y) = \Gamma_{ii}^j e_j(j^{-1/2}(\mathbf{z}))\tau(\mathbf{z}, y) - j^{-1/2}(\mathbf{z})\Gamma_{ii}^j\Gamma_{jl}^k a_l\iota_k\tau(\mathbf{z}, y).$$

Note that the Christoffel symbols vanish at y. By writing the a_l, ι_k's in terms of the Clifford multipliers, one sees in a similar way leading to (10.10) that

$$B(\mathbf{x}) = \sum_{\Phi\in\mathcal{L}_1''}\Phi_{\mathbf{x}}\circ\tau(\mathbf{x}, y),$$

where $\mathcal{L}_1''$ is a finite subset of $\mathcal{S}$ and every element of $\Phi \in \mathcal{L}_1''$ is of the form $\Phi = \varphi c_I \hat{c}_J$ satisfying either

$$|I| + |J| = 4,\ \ \varphi(y) = 0$$

or $|I| + |J| < 4$ (in fact, $\leqslant 2$).

(ii) *The induction step.*

Suppose that

$$U^{(i)}(x, y) = \frac{1}{i!}\mathfrak{R}_x^i \circ \tau(x, y) + \sum_{\Phi \in \mathcal{L}_i} \Phi_x \circ \tau(x, y).$$

By using the formula (8.25) as before, one finds that

$$(-\Delta U^{(i)})(\mathbf{z}, y) = \frac{1}{i!}\mathfrak{R}_{\mathbf{z}}^{i+1} \circ \tau(\mathbf{z}, y) + \sum_{\Phi \in \mathcal{L}_i'} \Phi_{\mathbf{z}} \circ \tau(\mathbf{z}, y),$$

where $\mathcal{L}_i'$ is a finite subset of $\mathcal{S}$ (determined from $\mathcal{L}_i$) whose elements $\Phi = \varphi c_I \hat{c}_J$ satisfy

$$\text{either } |I| + |J| = 4i + 4,\ \ \varphi(y) = 0 \text{ or } |I| + |J| < 4i + 4.$$

It follows from the formula (9.39) that

$$\begin{aligned} U^{(i+1)}(x, y) =& \frac{1}{\sqrt{j(\mathbf{x})}} \int_0^1 v^i \sqrt{j(\mathbf{x})} \Big(\frac{1}{i!} \tau(\mathbf{x}, v\mathbf{x}) \circ R_{v\mathbf{x}}^{i+1} \circ \tau(v\mathbf{x}, y) \\ &+ \tau(\mathbf{x}, v\mathbf{x}) \circ \sum_{\Phi \in \mathcal{L}_i'} \Phi_{v\mathbf{x}} \circ \tau(v\mathbf{x}, y) \Big) dv \\ =:& \ C(\mathbf{x}) + D(\mathbf{x}). \end{aligned} \tag{10.11}$$

In a similar way as before, one observes that

$$\begin{aligned} \tau(\mathbf{x}, v\mathbf{x}) \circ \mathfrak{R}_{v\mathbf{x}}^{i+1} \circ \tau(v\mathbf{x}, y) =& \Big(-\frac{1}{8} \Big)^{i+1} R_{p_1 p_2 q_1 q_2}(v\mathbf{x}) \cdots R_{p_{2i+1} p_{2i+2} q_{2i+1} q_{2i+2}}(v\mathbf{x}) \\ & (c_{p_1 p_2} \hat{c}_{q_1 q_2} \cdots c_{p_{2i+1} p_{2i+2}} \hat{c}_{q_{2i+1} q_{2i+2}})(\mathbf{x}) \circ \tau(\mathbf{x}, y) \end{aligned} \tag{10.12}$$

and for $\Phi = \varphi c_I \hat{c}_J \in \mathcal{L}_i'$,

$$\tau(\mathbf{x}, v\mathbf{x}) \circ \Phi_{v\mathbf{x}} \circ \tau(v\mathbf{x}, y) = \varphi(v\mathbf{x})(c_I \hat{c}_J)(\mathbf{x}) \circ \tau(\mathbf{x}, y). \tag{10.13}$$

By substituting (10.12, 10.13) into (10.11) and freezing the curvature term at $\mathbf{x}$ in the same way as in (10.9), one concludes that

$$\begin{aligned} U^{(i+1)}(x, y) &= \frac{1}{i!}\mathfrak{R}_{\mathbf{x}}^{i+1} \circ \tau(\mathbf{x}, y) \times \int_0^1 v^i dv + \sum_{\Phi \in \mathcal{L}_{i+1}} \Phi_{\mathbf{x}} \circ \tau(\mathbf{x}, y) \\ &= \frac{1}{(i+1)!}\mathfrak{R}_{\mathbf{x}}^{i+1} \circ \tau(\mathbf{x}, y) + \sum_{\Phi \in \mathcal{L}_{i+1}} \Phi_{\mathbf{x}} \circ \tau(\mathbf{x}, y) \end{aligned}$$

where $\mathcal{L}_{i+1}$ is some finite subset of $\mathcal{S}$ whose elements $\Phi = \varphi c_I \hat{c}_J$ satisfy either

$$|I| + |J| = 4(i+1), \ \varphi(y) = 0$$

or $|I| + |J| < 4(i+1)$. This completes the induction step. □

10.2.3 Proof of Patodi's local index formula

We now proceed to prove Theorem 10.2. The main philosophy is that only the $\mathfrak{R}$-part in the decomposition (10.7) of $U^{(m)}(y,y)$ will produce a nonzero supertrace, which in fact yields the Pfaffian of the curvature.

First of all, according to Proposition 10.2, one can write

$$U^{(m)}(y,y) = \frac{1}{m!}\mathfrak{R}_y^m + \sum_{\Phi \in \mathcal{L}'} \Phi_y,$$

where $\mathcal{L}'$ is a finite subset of $\mathcal{S}$; elements of $\mathcal{S}$ are of the form $\Phi = \varphi c_I \hat{c}_J$ where either $|I| + |J| < 4m$ or $|I| + |J| = 4m$ and $\varphi(y) = 0$. . It follows from Corollary 10.1 that $\mathrm{str}[\Phi_y] = 0$ for all such Φ's. As a result, one has

$$\begin{aligned}
\frac{1}{(4\pi)^m}\mathrm{str}\big(U^{(m)}(y,y)\big)dy =& \frac{(-1)^m}{(4\pi)^m m! 8^m} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2m-1} i_{2m} j_{2m-1} j_{2m}} \\
& \times \mathrm{str}\big[c_{i_1 i_2}\hat{c}_{j_1 j_2} \cdots c_{i_{2m-1} i_{2m}} \hat{c}_{j_{2m-1} j_{2m}}\big] dy \\
=& \frac{(-1)^m}{(2\pi)^m 2^{4m} m!} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2m-1} i_{2m} j_{2m-1} j_{2m}} \\
& \times \mathrm{str}\big[c_{i_1 \cdots i_{2m}} \hat{c}_{j_1 \cdots j_{2m}}\big] dy, \qquad (10.14)
\end{aligned}$$

where the curvature coefficients and Clifford multipliers are all evaluated at y.

According to Corollary 10.1 again, for the summands on the RHS of (10.14) to be nonzero, the words $(i_1, \cdots, i_{2m})$ and $(j_1, \cdots, j_{2m})$ must be permutations of

$(1,\cdots,2m)$. More explicitly, letting $\{\theta^i\}$ be the dual of $\{e_i\}$ one has

$$\begin{aligned}
&\frac{1}{2^{4m}m!}R_{i_1i_2j_1j_2}\cdots R_{i_{2m-1}i_{2m}j_{2m-1}j_{2m}}\mathrm{str}\big[c_{i_1\cdots i_{2m}}\hat{c}_{j_1\cdots j_{2m}}\big]dy\\
&=\frac{(-1)^{m(2m+1)}}{2^{2m}m!}\sum_{\sigma,\tau\in\mathcal{S}_{2m}}\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\times R_{\sigma(1)\sigma(2)\tau(1)\tau(2)}\\
&\qquad\cdots R_{\sigma(2m-1)\sigma(2m)\tau(2m-1)\tau(2m)}\theta^1\wedge\cdots\wedge\theta^{2m}\\
&=\frac{(-1)^m}{2^{2m}m!}\sum_{\sigma\in\mathcal{S}_{2m}}\mathrm{sgn}(\sigma)\Big(\sum_{\tau\in\mathcal{S}_{2m}}R_{\sigma(1)\sigma(2)\tau(1)\tau(2)}\cdots\\
&\qquad R_{\sigma(2m-1)\sigma(2m)\tau(2m-1)\tau(2m)}\theta^{\tau(1)}\wedge\cdots\wedge\theta^{\tau(2m)}\Big)\\
&=\frac{(-1)^m}{2^{2m}m!}\sum_{\sigma\in\mathcal{S}_{2m}}\mathrm{sgn}(\sigma)\Big(\sum_{j_1,\cdots j_{2m}=1}^{2m}R_{i_1i_2j_1j_2}\cdots R_{i_{2m-1}i_{2m}j_{2m-1}j_{2m}}\\
&\qquad\theta^{j_1}\wedge\cdots\wedge\theta^{j_{2m}}\Big).
\end{aligned}\tag{10.15}$$

On the other hand, recall that the local curvature matrix F with respect to $\{e_i\}$ is defined by

$$F^i_j(X,Y)\triangleq\langle R(X,Y)e_j,e_i\rangle.$$

It is easily checked that

$$F^i_j=\frac{1}{2}R_{ijkl}\theta^k\wedge\theta^l.\tag{10.16}$$

By substituting (10.16) into (10.15), one obtains that

$$\begin{aligned}
&\frac{1}{2^{4m}m!}R_{i_1i_2j_1j_2}\cdots R_{i_{2m-1}i_{2m}j_{2m-1}j_{2m}}\mathrm{str}\big[c_{i_1\cdots i_{2m}}\hat{c}_{j_1\cdots j_{2m}}\big]dy\\
&=\frac{(-1)^m}{2^m m!}\sum_{\sigma\in\mathcal{S}_{2m}}\mathrm{sgn}(\sigma)F^{\sigma(1)}_{\sigma(2)}\wedge\cdots\wedge F^{\sigma(2m-1)}_{\sigma(2m)}.
\end{aligned}$$

In view of the formula (7.12), the RHS is exactly $\mathrm{Pf}(R^{\mathrm{LC}})$. By further substituting it back into (10.14), one obtains the desired formula (10.5) of Patodi.

Now the proof of Theorem 10.2 and hence of the CGB theorem is complete.

11 A superconnection approach

In this final chapter, we discuss a beautiful approach of [ZF22] to the CGB theorem which is based on Quillen's superconnection technique (cf. [Qui85, MQ86]). This approach has a similar spirit to the Chern-Weil perspective and it relies on a Chern-Weil type theorem for superconnections. Unlike Chern's original argument where localisation of the Pfaffian integral is based on the exactness of the lifted Euler form on the sphere bundle, the localisation here will be a consequence of a simple deformation of a superconnection on the exterior algebra bundle. This deformation leaves the Chern character invariant (as a consequence of the Chern-Weil theorem) but annihilates the contribution of the integral outside the zeros of the vector field. In the limit of the deformation, one picks up the indices of the vector field at its zeros, hence yielding the Euler characteristic of the manifold. At a conceptual level, the main strategy is summarised as follows. Let M be a closed, oriented, $2m$-dimensional Riemannian manifold.

1. The Pfaffian $\mathrm{Pf}(F_{\nabla^M})$ of the Levi-Civita connection ∇^M is equal to the so-called *Chern character form* $\mathrm{ch}(\Lambda_{\mathbb{C}}T^*M, \nabla^{\Lambda_{\mathbb{C}}T^*M})$ associated with the induced *superconnection* $\nabla^{\Lambda_{\mathbb{C}}T^*M}$ on the complexified exterior algebra bundle $\Lambda_{\mathbb{C}}T^*M$. More precisely, one has (see Proposition 11.2)

$$\mathrm{ch}(\Lambda_{\mathbb{C}}T^*M, \nabla^{\Lambda_{\mathbb{C}}T^*M}) = \frac{1}{\pi^m}\mathrm{Pf}(F_{\nabla^M}).$$

2. As a consequence of a Chern-Weil type theorem for superconnections (see Theorem 11.1), the cohomology class of the Chern character form is independent of the superconnection.

3. Consider the following deformation of the superconnection (see (11.16)):

$$\mathbf{A}_T \triangleq \nabla^{\Lambda_{\mathbb{C}}T^*M} + \sqrt{\frac{2\pi}{i}}Tc(V), \quad T > 0.$$

   Here V is a vector field on M with isolated zeros and $c(V)$ is the fiberwise Clifford action by V on $\Lambda_{\mathbb{C}}T^*M$. It follows from Facts 1 and 2 that

$$\frac{(-1)^m}{(2\pi)^m}\int_M \mathrm{Pf}(F_{\nabla^M}) = \frac{(-1)^m}{(2\pi)^m}\int_M \mathrm{ch}(\Lambda_{\mathbb{C}}T^*M, \mathbf{A}_T)$$

   for all $T > 0$.

4. The key observation is that when T becomes large, the contribution from the part outside the zeros of V becomes negligible, due to a factor $e^{-T^2|V|^2}$ coming from the expression of $\mathrm{ch}(\Lambda_{\mathbb{C}}T^*M, \mathbf{A}_T)$ (see (11.20)). On the other

hand, the integral near each zero p of V is just a simple Gaussian integral (see (11.23)); on a small neighbourhood U_p of p one can modify the metric to be Euclidean (due to the Chern-Weil theorem) and modify the vector field to be linear (which does not change the local indices of V) . After evaluating the localised integral in the large T limit, one obtains precisely the index of V at p (see (11.24)). To summarise, one has

$$\begin{aligned}(-1/2)^m \int_M \mathrm{ch}(\Lambda_{\mathbb{C}} T^*M, \mathbf{A}_T) &= \lim_{T\to\infty} \sum_{p\in \text{zeros of } V} (-1/2)^m \int_{U_p} \mathrm{ch}(\Lambda_{\mathbb{C}} T^*M, \mathbf{A}_T) \\ &= \sum_{p\in \text{zeros of } V} \mathrm{ind}_p(V) = \chi(M),\end{aligned}$$

where the last equality follows from the Poincaré-Hopf index theorem. This is exactly the CGB formula.

11.1 Super vector spaces

We begin with some basic definitions. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 11.1. A *super ($\mathbb{K}$-)vector space* is a $\mathbb{K}$-vector space V equipped with a linear endomorphism $\tau : V \to V$ (a *superstructure*) such that $\tau^2 = \mathrm{id}$.

A super vector space is just a usual vector space V equipped with a $\mathbb{Z}_2$-grading, i.e. a decomposition $V = V_+ \oplus V_-$. Indeed, given a superstructure τ one defines $V_\pm$ to be the (± 1)-eigenspaces of τ. Conversely, given such a decomposition the associated superstructure $\tau : V \to V$ is defined by $\tau|_{V_\pm} \triangleq \pm\mathrm{id}$. Therefore, a superstructure and a $\mathbb{Z}_2$-grading are essentially the same thing. Given $v \in V_\pm$, we define $\deg(v) \triangleq 0$ if $v \in V_+$ and $\deg(v) \triangleq 1$ if $v \in V_-$. Note that any vector space V is a super vector space with trivial grading $V_+ = V, V_- = \{0\}$.

Definition 11.2. A *super ($\mathbb{K}$-)algebra* is a $\mathbb{K}$-algebra A such that:

(i) A is a super vector space with $\mathbb{Z}_2$-grading $A = A_+ \oplus A_-$;

(ii) $A_\pm \cdot A_\pm \subseteq A_+$ and $A_\pm \cdot A_\mp \subseteq A_-$.

Example 11.1. A basic example of a superalgebra is the algebra $\mathrm{End}(V)$ of endomorphisms over a super vector space (V, τ). One defines its $\mathbb{Z}_2$-grading by

$$\mathrm{End}_+(V) \triangleq \{A \in \mathrm{End}(V) : \tau A = A\tau\} = \{A \in \mathrm{End}(V) : AV_\pm \subseteq V_\pm\},$$

$$\mathrm{End}_-(V) \triangleq \{A \in \mathrm{End}(V) : \tau A = -A\tau\} = \{A \in \mathrm{End}(V) : AV_\pm \subseteq V_\mp\}.$$

It is readily checked that the above grading gives rise to a superalgebra structure over $\mathrm{End}(V)$. We always work with this induced superalgebra structure.

Definition 11.3. Let (V, τ) be a super vector space. The *supertrace* of $A \in \mathrm{End}(V)$ is defined as $\mathrm{str}[A] \triangleq \mathrm{tr}[\tau A]$.

It is immediate from the definition that $\mathrm{str}[A] = 0$ if $A \in \mathrm{End}_-(V)$ while

$$\mathrm{str}[A] = \mathrm{tr}[A|_{V_+}] - \mathrm{tr}[A|_{V_-}] \tag{11.1}$$

if $A \in \mathrm{End}_+(V)$. Just like the commutativity of the usual trace, the supertrace is also "super commutative". To explain this, we first introduce the notion of super Lie bracket.

Definition 11.4. The *super Lie bracket* $[A, B]_{\mathrm{s}}$ of $A, B \in \mathrm{End}(V)$ is defined in the following way. For $A, B \in \mathrm{End}_\pm(V)$, one defines

$$[A, B]_{\mathrm{s}} \triangleq AB - (-1)^{\deg(A)\deg(B)} BA. \tag{11.2}$$

For general A, B, one extends the definition (11.2) through bilinearity.

Lemma 11.1. $\mathrm{str}\big[[A, B]_{\mathrm{s}}\big] = 0$ *for all* $A, B \in \mathrm{End}(V)$.

Proof. Left as an exercise. □

Let (V, σ) and (W, τ) be super vector spaces. Their tensor product $V \otimes W$ admits a natural superstructure defined by $\sigma \otimes \tau$, or equivalently, by the $\mathbb{Z}_2$-grading

$$(V \otimes W)_+ \triangleq (V_+ \otimes W_+) \oplus (V_- \otimes W_-),\ (V \otimes W)_- \triangleq (V_+ \otimes W_-) \oplus (V_- \otimes W_+).$$

The corresponding super vector space, which is denoted as $V \hat{\otimes} W$, is called the *super tensor product* between V and W. For superalgebras A, B, their super tensor product $A \hat{\otimes} B$ admits a superalgebra structure whose product $*$ is induced by

$$(a_1 \hat{\otimes} b_1) * (a_2 \hat{\otimes} b_2) \triangleq (-1)^{\deg(b_1)\deg(a_2)} a_1 a_2 \hat{\otimes} b_1 b_2$$

for $a_i \in A_\pm$ and $b_j \in B_\pm$. Here $\hat{\otimes}$ is merely a notation (algebraically it is the same as $\otimes$) where the "hat" records the underlying superstructure.

Example 11.2. Suppose that V, W are super vector spaces. We view $A \hat{\otimes} B \in \mathrm{End}(V) \hat{\otimes} \mathrm{End}(W)$ as an element of $\mathrm{End}(V \hat{\otimes} W)$ through the action defined by

$$(A \hat{\otimes} B)(v \hat{\otimes} w) \triangleq (-1)^{\deg(B)\deg(v)} (Av) \hat{\otimes} (Bw).$$

Under this identification, one can show that $\mathrm{End}(V) \hat{\otimes} \mathrm{End}(W)$ and $\mathrm{End}(V \hat{\otimes} W)$ are isomorphic as superalgebras.

Example 11.3. Another basic example of a superalgebra is the exterior algebra ΛV^* where V is a given vector space. A commonly used superstructure is the *parity grading* defined by $\Lambda V^* = \Lambda^{\text{even}} V^* \oplus \Lambda^{\text{odd}} V^*$ (exterior covectors of even / odd degree). If V, W are two vector spaces, there is a canonical superalgebra isomorphism between $\Lambda V^* \hat{\otimes} \Lambda W^*$ and $\Lambda(V \oplus W)^*$ with respect to the parity grading. This isomorphism is induced by

$$\alpha \hat{\otimes} \beta \mapsto \alpha \wedge \beta, \quad \alpha \in \Lambda V^*, \beta \in \Lambda W^*.$$

In Section 11.2 below, we will introduce another superstructure on ΛV^* that is needed for the superconnection proof of CGB.

Let V be a super vector space. Consider the superalgebra $\Lambda V^* \hat{\otimes} \text{End} V$ and the super vector space $\Lambda V^* \hat{\otimes} V$. In a similar way as before, we define the action of the superalgebra $\Lambda V^* \hat{\otimes} \text{End}(V)$ on the super vector space $\Lambda V^* \hat{\otimes} V$ by setting

$$(\alpha \hat{\otimes} T)(\beta \hat{\otimes} v) \triangleq (-1)^{\deg(T)\deg(\beta)} (\alpha \wedge \beta) \hat{\otimes} (Tv) \tag{11.3}$$

for homogeneous elements and extending the relation through linearity.

11.2 The chirality grading

Let V be a real, oriented, Euclidean space with dimension $2m$. Consider the *complexified exterior algebra* $\Lambda_{\mathbb{C}} V^*$. This is by definition the algebra of $\mathbb{C}$-valued alternating multilinear functionals on V. We are going to construct a new superstructure on $\Lambda_{\mathbb{C}} V^*$ that is different from the parity grading. This superstructure will play an essential role in the superconnection proof of CGB later on. Recall that for each $v \in V$, the Clifford multiplier $c(v) \in \text{End}(\Lambda_{\mathbb{C}} V^*)$ is defined by $c(v) \triangleq a(v) - \iota(v)$, where v^* is the metric dual of v, $a(v)$ is the exterior product by v^* and $\iota(v)$ is the interior product by v (see (8.16)). Let $\{\varepsilon_1, \cdots, \varepsilon_{2m}\}$ be a PONB of V. We will also continue to use the notation

$$c(\varepsilon_I) \triangleq c(\varepsilon_{i_1}) \cdots c(\varepsilon_{i_k})$$

for $I = (i_1, \cdots, i_k)$.

Definition 11.5. The *chirality operator* is defined by

$$\hat{\tau} \triangleq i^m c(\varepsilon_1) \cdots c(\varepsilon_{2m}) \in \text{End}(\Lambda_{\mathbb{C}} V^*).$$

Lemma 11.2. *$\hat{\tau}$ does not depend on the choice of the positive ONB. In addition, one has $\hat{\tau}^2 = \text{id}$.*

Proof. Let $\mathcal{C}$ be the $\mathbb{C}$-subalgebra of $\mathrm{End}(\Lambda_{\mathbb{C}}V^*)$ generated by the family

$$\left\{c(\varepsilon_{i_1})\cdots c(\varepsilon_{i_k}) : 1 \leqslant i_1 < \cdots < i_k \leqslant 2m\right\}.$$

This family is linearly independent and thus $\dim \mathcal{C} = 2^{2m} = \dim \Lambda_{\mathbb{C}}V^*$. Define a linear map $\phi : \mathcal{C} \to \Lambda_{\mathbb{C}}V^*$ by

$$\phi(c(\varepsilon_I)) \triangleq c(\varepsilon_I)1 = \varepsilon_{i_1}^* \wedge \cdots \wedge \varepsilon_{i_k}^*, \quad I = (i_1 < \cdots < i_k).$$

Here 1 is the unit of $\Lambda_{\mathbb{C}}V^*$ and $v^* \in V^*$ means the metric dual of $v \in V$. It is clear that ϕ is surjective and thus an isomorphism by dimension comparison.

We claim that

$$\phi(c(v_1)\cdots c(v_r)) = c(v_1)\cdots c(v_r)1 \tag{11.4}$$

for all $v_1, \cdots, v_r \in V$. By linearity, it suffices to consider the case when $(v_1, \cdots, v_r) = (\varepsilon_{i_1}, \cdots, \varepsilon_{i_r})$ but without the index monotonicity condition. In this case, by using the Clifford relation (8.20) one can write

$$c(\varepsilon_{i_1})\cdots c(\varepsilon_{i_r}) = \sigma c(\varepsilon_J),$$

where $\sigma = \pm 1$, $s \leqslant r$ and $J = (j_1 < \cdots < j_s)$. It follows that

$$\phi(c(\varepsilon_{i_1})\cdots c(\varepsilon_{i_r})) = \sigma\phi(c(\varepsilon_J)) = \sigma c(\varepsilon_J)1 = c(\varepsilon_{i_1})\cdots c(\varepsilon_{i_r})1.$$

This gives the relation (11.4).

Now suppose that $\{\eta_1, \cdots, \eta_{2m}\}$ is another PONB. Then $\eta_i^* = a_i^j \varepsilon_j^*$ for some $a = (a_i^j) \in \mathrm{SO}(2m)$. According to the relation (11.4), one has

$$\begin{aligned}\phi(c(\eta_1)\cdots c(\eta_{2m})) &= c(\eta_1)\cdots c(\eta_{2m})1 = \eta_1^* \wedge \cdots \wedge \eta_{2m}^* \\ &= (\det a)\varepsilon_1^* \wedge \cdots \wedge \varepsilon_{2m}^* = \phi(c(\varepsilon_1)\cdots c(\varepsilon_{2m})).\end{aligned}$$

This implies

$$c(\eta_1)\cdots c(\eta_{2m}) = c(\varepsilon_1)\cdots c(\varepsilon_{2m})$$

since ϕ is an isomorphism. Therefore, $\hat{\tau}$ is independent of the choice of PONB. The fact that $\hat{\tau}^2 = \mathrm{id}$ follows easily from (8.20). □

Definition 11.6. The $\mathbb{Z}_2$-grading $\Lambda_{\mathbb{C}}V^* = E_+ \oplus E_-$ induced by $\hat{\tau}$ is called the *chirality grading.*

The following key property of $\hat{\tau}$ will be used in the sequel. We view $\mathrm{End}(\Lambda_{\mathbb{C}}V^*)$ as a superalgebra whose superstructure is induced by the chirality grading (cf. Example 11.1). We also recall that $\hat{c}(v) = a(v^*) + \iota(v)$ (cf. Definition 8.11).

Lemma 11.3. *For all* $I = (i_1 < \cdots < i_k)$ *and* $J = (j_1 < \cdots < j_l)$*, one has*

$$\operatorname{str}\big[c(\varepsilon_I)\hat{c}(\varepsilon_J)\big] = \begin{cases} (-i)^m 2^{2m}, & \text{if } I = (1, \cdots, 2m), J = \emptyset; \\ 0, & \text{otherwise.} \end{cases} \tag{11.5}$$

Proof. By the definition of the supertrace, one has

$$\operatorname{str}\big[c(\varepsilon_I)\hat{c}(\varepsilon_J)\big] = \operatorname{tr}\big[\hat{\tau}c(\varepsilon_I)\hat{c}(\varepsilon_J)\big] = i^m \operatorname{tr}\big[c(\varepsilon_1)\cdots c(\varepsilon_{2m})c(\varepsilon_I)\hat{c}(\varepsilon_J)\big].$$

According to the Clifford relation (8.20), the RHS is nonzero iff $I = (1, \cdots, 2m)$ and $J = \emptyset$. Its exact value is given by (11.5). □

To conclude this subsection, we give an explicit description of $\hat{\tau}$ as well as the chirality grading $\Lambda_{\mathbb{C}}V^* = E_+ \oplus E_-$ in terms of the Hodge star operator.

Proposition 11.1. *Let* $* : \Lambda_{\mathbb{C}}V^* \to \Lambda_{\mathbb{C}}V^*$ *be the Hodge star operator with respect to the dual Euclidean structure over* V^* *(cf. Definition 8.1 with its obvious extension over* $\mathbb{C}$*). For each* k*, we define* $\lambda_{m,k} \triangleq i^{m-k(k-1)}$*. Then* $\hat{\tau}|_{\Lambda^k_{\mathbb{C}}} = \lambda_{m,k}* : \Lambda^k_{\mathbb{C}}V^* \xrightarrow{\cong} \Lambda^{2m-k}_{\mathbb{C}}V^*$*. In addition, the chirality grading over* $\Lambda_{\mathbb{C}}V^*$ *is given by*

$$E_+ = \bigoplus_{k=0}^{m-1} \big\{\alpha + \lambda_{m,k} * \alpha : \alpha \in \Lambda^k_{\mathbb{C}}V^*\big\} \bigoplus \big\{\alpha \in \Lambda^m_{\mathbb{C}}V^* : *\alpha = \lambda^{-1}_{m,m}\alpha\big\},$$

$$E_- = \bigoplus_{k=0}^{m-1} \big\{\alpha - \lambda_{m,k} * \alpha : \alpha \in \Lambda^k_{\mathbb{C}}V^*\big\} \bigoplus \big\{\alpha \in \Lambda^m_{\mathbb{C}}V^* : *\alpha = -\lambda^{-1}_{m,m}\alpha\big\}.$$

Proof. Given $i_1 < \cdots < i_k$, one has

$$\begin{aligned} \hat{\tau}(\varepsilon^*_{i_1} \wedge \cdots \wedge \varepsilon^*_{i_k}) &= i^m c(\varepsilon_1)\cdots c(\varepsilon_{2m})\varepsilon^*_{i_1} \wedge \cdots \wedge \varepsilon^*_{i_k} \\ &= i^m \operatorname{sgn}(J, I)c(\varepsilon_J)c(\varepsilon_I)\varepsilon^*_{i_1} \wedge \cdots \wedge \varepsilon^*_{i_k} \quad (J \triangleq I^c) \\ &= i^m i^{k(k+1)} \operatorname{sgn}(J, I)c(\varepsilon_J)1 \\ &= i^{m+k(k+1)}(-1)^{k(2m-k)} \operatorname{sgn}(I, J)\varepsilon^*_{j_1} \wedge \cdots \wedge \varepsilon^*_{j_{2m-k}} \\ &= \lambda_{m,k} * (\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_k}). \end{aligned}$$

This shows that $\hat{\tau}|_{\Lambda^k_{\mathbb{C}}} = \lambda_{m,k}*$. In particular, for each $k \leqslant m$ the subspace $E_{k,2m-k} \triangleq \Lambda^k_{\mathbb{C}}V^* \oplus \Lambda^{2m-k}_{\mathbb{C}}V^*$ is invariant under $\hat{\tau}$. For any

$$\alpha = \sum_{k=0}^{m} \alpha_k \in \bigoplus_{k=0}^{m} E_{k,2m-k} \in \Lambda_{\mathbb{C}}V^*,$$

one can now write

$$\alpha = \sum_{k=0}^{m} \frac{\alpha_k + \hat{\tau}(\alpha_k)}{2} + \sum_{k=0}^{m} \frac{\alpha_k - \hat{\tau}(\alpha_k)}{2} \in E_+ \oplus E_-.$$

This gives the desired $\mathbb{Z}_2$-grading. □

11.3 Super vector bundles and superconnections

We now carry the previous constructions to a vector bundle. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (in the latter case, differential forms are understood as complex-valued).

Definition 11.7. A *super ($\mathbb{K}$-)vector bundle* E over a manifold M is a vector bundle (E, M, π) which admits a fiberwise superstructure (equivalently, $\mathbb{Z}_2$-grading) $E = E_+ \oplus E_-$ that varies smoothly over M. The notion of a *super ($\mathbb{K}$-)algebra bundle* is defined in a similar way.

Let (E, M, π) be a super vector bundle. The bundle $\mathrm{End}(E)$ is the usual endomorphism bundle whose fiberwise superstructure is given by Example 11.1. The space of E-sections (respectively, $\mathrm{End}(E)$-sections) is denoted as $\Gamma(E)$ (respectively, $\Gamma(\mathrm{End}(E))$). Note that a vector bundle is a super vector bundle under the trivial grading $E = E \oplus \{0\}$.

One defines the tensor product between super vector or algebra bundles as usual where the superstructure is induced in a fiberwise manner. As usual, ΛT^*M denotes the exterior bundle over M but viewed as a super algebra bundle under the parity grading. We shall also consider the bundles $\Lambda T^*M \hat{\otimes} E$ (as a super vector bundle) and $\Lambda T^*M \hat{\otimes} \mathrm{End}(E)$ (as a super algebra bundle); the $\mathbb{Z}_2$-grading is given by

$$(\Lambda T^*M \hat{\otimes} E)_+ = \Lambda^{\mathrm{even}} T^*M \otimes E_+ \bigoplus \Lambda^{\mathrm{odd}} T^*M \otimes E_-,$$

$$(\Lambda T^*M \hat{\otimes} E)_- = \Lambda^{\mathrm{even}} T^*M \otimes E_- \bigoplus \Lambda^{\mathrm{odd}} T^*M \otimes E_+$$

and similarly for $\Lambda T^*M \hat{\otimes} \mathrm{End}(E)$. Sections of ΛT^*M, $\Lambda T^*M \hat{\otimes} E$ and $\Lambda T^*M \hat{\otimes} \mathrm{End}(E)$ are denoted by $\Omega(M)$, $\Omega(M, E)$ and $\Omega(M, \mathrm{End}(E))$ respectively. Note that $\Omega(M, E)$ is an $\Omega(M)$-module under exterior multiplication by differential forms on M.

There is a canonical fiberwise $\Omega(M, \mathrm{End}(E))$-action on $\Omega(M, E)$ which is induced by

$$(\alpha \hat{\otimes} S)(\beta \otimes e) \triangleq (-1)^{\deg(S)\deg(\beta)} (\alpha \wedge \beta) \hat{\otimes} S(e)$$

for $\alpha, \beta \in \Omega(M)$, $S \in \Gamma(\mathrm{End}(E))$ and $e \in \Gamma(E)$. It can be shown that

$$\Xi(\alpha \wedge \xi) = (-1)^{\deg T \deg \alpha} \alpha \wedge \Xi(\xi)$$

for all $\Xi \in \Omega(M, \mathrm{End}(E))$, $\alpha \in \Omega(M)$ and $\xi \in \Omega(M, E)$.

Definition 11.8. The *supertrace operator* $\mathrm{str} : \Omega(M, \mathrm{End}(E)) \to \Omega(M)$ is defined by taking supertrace along each fiber, i.e. the unique linear operator such that

$$\mathrm{str}\big[\alpha \hat{\otimes} S\big] = \alpha \mathrm{str}[S]$$

for all $\alpha \in \Omega(M)$ and $S \in \Gamma(\mathrm{End}(E))$.

The commutativity of supertrace extends naturally to the bundle context. To see this, we first define the notion of super Lie bracket.

Definition 11.9. Let $\Xi, \Theta \in \Omega(M, \mathrm{End}(E))$. Their *super Lie bracket* is defined by

$$[\Xi, \Theta]_{\mathrm{s}} \triangleq \Xi\Theta - (-1)^{\deg \Xi \deg \Theta} \Theta\Xi \in \Omega(M, \mathrm{End}(E))$$

for homogeneous elements and extended bilinearly.

Lemma 11.4. *One has* $\mathrm{str}\big[[\Xi, \Theta]_s\big] = 0$ *for all* $\Xi, \Theta \in \Omega(M, \mathrm{End}(E))$.

Proof. Working locally, one can assume that $\Xi = \alpha\hat{\otimes}S$ and $\Theta = \beta\hat{\otimes}T$. By definition,

$$\begin{aligned}[\Xi, \Theta]_{\mathrm{s}} &= (\alpha\hat{\otimes}S)(\beta\hat{\otimes}T) - (-1)^{(\deg \alpha + \deg S)(\deg \beta + \deg T)}(\beta\hat{\otimes}T)(\alpha\hat{\otimes}S) \\ &= (-1)^{\deg S \deg \beta}(\alpha \wedge \beta)\hat{\otimes}ST \\ &\quad - (-1)^{(\deg \alpha + \deg S)(\deg \beta + \deg T)}(-1)^{\deg T \deg \alpha}\beta \wedge \alpha\hat{\otimes}TS \\ &= (-1)^{\deg S \deg \beta}(\alpha \wedge \beta)\hat{\otimes}(ST - (-1)^{\deg S \deg T}TS) \\ &= (-1)^{\deg S \deg \beta}(\alpha \wedge \beta)\hat{\otimes}[S, T]_{\mathrm{s}}.\end{aligned}$$

It follows from Lemma 11.1 that

$$\mathrm{str}\big[[\Xi, \Theta]_{\mathrm{s}}\big] = (-1)^{\deg S \deg \beta}(\alpha \wedge \beta) \cdot \mathrm{str}\big[[S, T]_{\mathrm{s}}\big] = 0.$$

□

We now introduce the notion of a superconnection on a super vector bundle (cf. Definition 6.15).

Definition 11.10. Let (E, M, π) be a super vector bundle. A *superconnection* on E is a linear operator $\mathbf{A} : \Omega(M, E) \to \Omega(M, E)$ which satisfies the following two properties.

(i) $\mathbf{A}$ has odd parity, i.e. $\mathbf{A}((\Omega(M, E)_{\pm}) \subseteq \Omega(M, E)_{\mp}$.

(ii) For any $\alpha \in \Omega(M)$ and $\xi \in \Omega(M, E)$, one has

$$\mathbf{A}(\alpha \wedge \xi) = d\alpha \wedge \xi + (-1)^{\deg \alpha}\alpha \wedge \mathbf{A}(\xi). \tag{11.6}$$

Remark 11.1. It is clear from the relation (11.6) that $\mathbf{A}$ is uniquely determined by its evaluation on $\Gamma(E)$.

Example 11.4. Let $\nabla^{E_{\pm}} : \Gamma(E_{\pm}) \to \Omega^1(M, E_{\pm})$ be a connection on $E_{\pm}$. We use the same notation to denote its induced action on $\Omega(M, E_{\pm})$. Let $S \in \Gamma(\mathrm{End}_{-}(E))$ whose action on $\Omega(M, E)$ is induced by

$$S(\alpha \wedge \xi) = (-1)^{\deg \alpha}\alpha \wedge S(\xi)$$

for all $\alpha \in \Omega(M)$ and $\xi \in \Omega(M,E)$. Define

$$\mathbf{A} \triangleq \nabla^{E_+} \oplus \nabla^{E_-} + S.$$

Then $\mathbf{A}$ is a superconnection.

A key concept associated with a superconnection is its curvature.

Definition 11.11. The *curvature* of a superconnection $\mathbf{A}$ is the operator $\mathbf{A}^2 : \Omega(M,E) \to \Omega(M,E)$.

Lemma 11.5. *$\mathbf{A}^2(\alpha \wedge \xi) = \alpha \wedge \mathbf{A}^2(\xi)$ for all $\alpha \in \Omega(M)$ and $\xi \in \Omega(M,E)$.*

Proof. It is enough to consider the case when $\xi \in \Gamma(E)$. By the definition of $\mathbf{A}$, one has

$$\begin{aligned}\mathbf{A}^2(\alpha \wedge \xi) =& \mathbf{A}\big(d\alpha \wedge \xi + (-1)^{\deg\alpha}\alpha \wedge \mathbf{A}(\xi)\big) \\ =& (-1)^{1+\deg\alpha} d\alpha \wedge \mathbf{A}(\xi) + (-1)^{\deg\alpha} d\alpha \wedge \mathbf{A}(\xi) \\ & + (-1)^{\deg\alpha}(-1)^{\deg\alpha}\alpha \wedge \mathbf{A}^2(\xi) \\ =& \alpha \wedge \mathbf{A}^2(\xi).\end{aligned}$$

The result thus follows. □

Lemma 11.5 shows that the curvature operator $\mathbf{A}^2$ is tensorial (i.e. it acts fiberwisely). In particular, it can be identified as an element of $\Omega(M, \mathrm{End}(E))_+$ which is denoted as $F_{\mathbf{A}}$. If $\mathbf{A} = \nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$, one has $F_{\nabla^E} \in \Omega^2(M, \mathrm{End}_+(E))$ and it coincides with the curvature tensor defined by (B.20), namely,

$$F_{\mathbf{A}}|_{\Gamma(E)}(X,Y)\xi = \nabla^E_X \nabla^E_Y \xi - \nabla^E_Y \nabla^E_X \xi - \nabla^E_{[X,Y]}\xi$$

for all $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(E)$. A simple way to see this is to write $\mathbf{A} = d + A$ locally where A is the local connection matrix.

The curvature satisfies an obvious Bianchi's identity (cf. Theorem 6.4 and (6.38)). To state this, we first generalise the super Lie bracket to allow the input of a superconnection. Given a superconnection $\mathbf{A}$ and $\Xi \in \Omega(M, \mathrm{End}(E))_\pm$, we define

$$[\mathbf{A}, \Xi]_{\mathrm{s}} \triangleq \mathbf{A}\Xi - (-1)^{\deg \Xi}\Xi\mathbf{A} \tag{11.7}$$

and extend it by linearity to all $\Xi \in \Omega(M, \mathrm{End}(E))$. By working locally, it is routine to check that

$$\Xi(\alpha \wedge \xi) = (-1)^{\deg\alpha \deg\Xi}\alpha \wedge \Xi(\xi)$$

and

$$[\mathbf{A}, \Xi]_{\mathrm{s}}(\alpha \wedge \xi) = (-1)^{(1+\deg\Xi)\deg\alpha}\alpha \wedge [\mathbf{A}, \Xi]_{\mathrm{s}}(\xi).$$

In particular, $[\mathbf{A}, \Xi]_{\mathrm{s}}$ is tensorial and $[\mathbf{A}, \Xi]_{\mathrm{s}} \in \Omega(M, \mathrm{End}(E))$.

Lemma 11.6 (Bianchi Identity)**.** *One has* $[\mathbf{A}, (F_{\mathbf{A}})^k]_{\mathrm{s}} = 0$ *for all* $k \geqslant 1$.

Proof. Since $\deg(F_{\mathbf{A}})^k = 0$ ($F_{\mathbf{A}} = \mathbf{A}^2$ has even parity), it follows from the definition (11.7) that

$$[\mathbf{A}, (F_{\mathbf{A}})^k]_{\mathrm{s}} = [\mathbf{A}, \mathbf{A}^{2k}] = \mathbf{A}\mathbf{A}^{2k} - \mathbf{A}^{2k}\mathbf{A} = 0.$$

This gives the desired relation. □

11.4 The Chern-Weil theorem for superconnections

In this section, we are going to establish a Chern-Weil type theorem for superconnections. Basically, the theorem says that *the supertrace of any polynomial (more generally, any entire function) of the curvature is a closed form over M and its cohomology class is independent of the underlying superconnection.*

Let (E, M, π) be a super vector bundle and let $\mathbf{A}$ be a superconnection on E with curvature $F_{\mathbf{A}}$. Let

$$f(x) = a_0 + a_1 x + \cdots + a_k x^k$$

be a polynomial. We define

$$f(F_{\mathbf{A}}) \triangleq a_0 + a_1 F_{\mathbf{A}} + \cdots + a_k (F_{\mathbf{A}})^k \in \Omega^{\mathrm{even}}(M, \mathrm{End}_+(E)) \oplus \Omega^{\mathrm{odd}}(M, \mathrm{End}_-(E)).$$

Note that $\mathrm{str}[f(F_{\mathbf{A}})] \in \Omega^{\mathrm{even}}(M)$ because $\mathrm{str}[S] = 0$ for any $S \in \Gamma(\mathrm{End}_-(V))$.

Theorem 11.1 (The Chern-Weil Theorem for Superconnections)**.** *(i) The form* $\mathrm{str}[f(F_{\mathbf{A}})]$ *is closed.*

(ii) Let $\tilde{\mathbf{A}}$ *be another superconnection with curvature* $F_{\tilde{\mathbf{A}}}$*. Then one has*

$$\mathrm{str}\big[f(F_{\tilde{\mathbf{A}}})\big] - \mathrm{str}\big[f(F_{\mathbf{A}})\big] = d\omega$$

for some $\omega \in \Omega^{\mathrm{odd}}(M)$*. In particular, the cohomology class* $[\mathrm{str}[f(F_{\mathbf{A}})]]$ *is independent of the superconnection* $\mathbf{A}$.

The proof of Theorem 11.1 relies on the following crucial observation.

Lemma 11.7. *Let* $\mathbf{A}$ *be a superconnection and* $\Xi \in \Omega(M, \mathrm{End}(E))$*. Then*

$$\mathrm{str}\big[[\mathbf{A}, \Xi]_{\mathrm{s}}\big] = d\mathrm{str}[\Xi]. \tag{11.8}$$

Proof. If $\tilde{\mathbf{A}}$ is another superconnection, one knows that $\tilde{\mathbf{A}} - \mathbf{A} \in \Omega(M, \mathrm{End}(E))$. According to Lemma 11.1 (applied fiberwisely), $\mathrm{str}\big[[\tilde{\mathbf{A}} - \mathbf{A}, \Xi]_{\mathrm{s}}\big] = 0$. In other words, the relation (11.8) is independent of the superconnection and we may choose any one for convenience.

Since the relation (11.8) is local, one can just work locally with respect to some frame field $\{s_\alpha^\pm\}$ over some $U\subseteq M$ (here $\{s_\alpha^+\}$ trivialises $E_+|_U$ and $\{s_\alpha^-\}$ trivialises $E_-|_U$ respectively). Take $\mathbf{A}=d$ on U. This means that the local connection matrix is zero and thus $\mathbf{A}(fs_\alpha^\pm)=df\hat{\otimes}s_\alpha^\pm$ for any $f\in C^\infty(U)$ (in particular, $\mathbf{A}s_\alpha^\pm=0$). We also assume that $\Xi=\omega\hat{\otimes}T$ where $\omega\in\Omega(U)$ and $T\in\Gamma(\mathrm{End}(E|_U))$. In this case, one has

$$d\mathrm{str}[\Xi]=d(\omega\mathrm{str}[T])=\mathrm{str}[T]d\omega+(-1)^{\deg\omega}\omega\wedge d\mathrm{str}[T]. \tag{11.9}$$

On the other hand, according to the definition (11.7) of the super Lie bracket,

$$\begin{aligned}[\mathbf{A},\Xi]_{\mathrm{s}}s_\alpha^\pm&=\mathbf{A}\big((\omega\hat{\otimes}T)(s_\alpha^\pm)\big)-(-1)^{\deg\omega+\deg T}(\omega\hat{\otimes}T)(\mathbf{A}s_\alpha^\pm)\\&=\mathbf{A}\big(\omega\hat{\otimes}T(s_\alpha^\pm)\big)=d\omega\hat{\otimes}T(s_\alpha^\pm)+(-1)^{\deg\omega}\omega\wedge\mathbf{A}\big(T(s_\alpha^\pm)\big).\end{aligned} \tag{11.10}$$

To compute the supertrace of $[\mathbf{A},\Xi]_{\mathrm{s}}$, we consider two cases separately.

(i) *Case I: T has odd parity.*

In this case, one can write

$$T(s_\alpha^+)=A_\alpha^\beta s_\beta^-,\ T(s_\alpha^-)=B_\alpha^\beta s_\beta^+$$

for some $A_\alpha^\beta,B_\alpha^\beta\in C^\infty(U)$ and therefore,

$$\mathbf{A}\big(T(s_\alpha^+)\big)=dA_\alpha^\beta\hat{\otimes}s_\beta^-,\ \mathbf{A}\big(T(s_\alpha^-)\big)=dB_\alpha^\beta\hat{\otimes}s_\beta^+.$$

It follows from the formula (11.10) that $[\mathbf{A},\Xi]_{\mathrm{s}}$ has odd parity as a fiberwise endomorphism over E_x ($x\in U$). In particular, $\mathrm{str}\big[[\mathbf{A},\Xi]_{\mathrm{s}}\big]=0$. But one also has $\mathrm{str}[\Xi]=\omega\mathrm{str}[T]=0$ in this case. As a result, the relation (11.8) holds.

(ii) *Case II: T has even parity.*

In this case, one can write

$$T(s_\alpha^+)=C_\alpha^\beta s_\beta^+,\ T(s_\alpha^-)=D_\alpha^\beta s_\beta^-.$$

By using the formula (11.1), one has

$$\mathrm{str}[T]=\sum_\alpha C_\alpha^\alpha-\sum_\beta D_\beta^\beta. \tag{11.11}$$

It follows from (11.10), (11.11) and (11.9) that

$$\begin{aligned}\mathrm{str}\big[[\mathbf{A},\Xi]_{\mathrm{s}}\big]&=\mathrm{str}[T]d\omega+(-1)^{\deg\omega}\omega\wedge\Big(\sum_\alpha dC_\alpha^\alpha-\sum_\beta dD_\beta^\beta\Big)\\&=\mathrm{str}[T]d\omega+(-1)^{\deg\omega}\omega\wedge d\mathrm{str}[T]=d\mathrm{str}[\Xi].\end{aligned}$$

This gives the desired relation (11.8). □

Proof of Theorem 11.1. (i) According to Lemma 11.7 and Bianchi's identity (see Lemma 11.6), one has

$$d\mathrm{str}\big[f(F_{\mathbf{A}})\big] = \mathrm{str}\big[[\mathbf{A}, f(F_{\mathbf{A}})]_{\mathrm{s}}\big] = 0.$$

(ii) Similar to the proof of Theorem 7.1, for each $t \in [0,1]$ we define the superconnection $\mathbf{A}_t \triangleq (1-t)\mathbf{A} + t\tilde{\mathbf{A}}$ and denote $F_t \triangleq \mathbf{A}_t^2$ as its curvature. Then one has

$$\begin{aligned} \frac{d}{dt}\mathrm{str}\big[f(F_t)\big] &= \sum_{l=0}^{k} a_l \mathrm{str}\Big[\frac{d}{dt}F_t^l\Big] =_1 \sum_{l=1}^{k} a_l \mathrm{str}\Big[\frac{dF_t}{dt}\cdot lF_t^{l-1}\Big] = \mathrm{str}\Big[\frac{dF_t}{dt}f'(F_t)\Big] \\ &= \mathrm{str}\Big[\mathbf{A}_t\frac{d\mathbf{A}_t}{dt}f'(F_t) + \frac{d\mathbf{A}_t}{dt}\mathbf{A}_t f'(F_t)\Big] \\ &=_2 \mathrm{str}\Big[\mathbf{A}_t\frac{d\mathbf{A}_t}{dt}f'(F_t) + \frac{d\mathbf{A}_t}{dt}f'(F_t)\mathbf{A}_t\Big] \\ &=_3 \mathrm{str}\Big[\Big[\mathbf{A}_t, \frac{d\mathbf{A}_t}{dt}f'(F_t)\Big]_{\mathrm{s}}\Big] =_4 d\mathrm{str}\Big[\frac{d\mathbf{A}_t}{dt}f'(F_t)\Big]. \end{aligned} \tag{11.12}$$

To reach $=_1$, we used the the supercommutativity of str together with the fact that F_t has even parity. To reach $=_2$, we used the commutativity between $\mathbf{A}_t$ and $f'(F_t)$. To reach $=_3$, we used the fact that $\frac{d\mathbf{A}_t}{dt}f'(F_t) \in \Omega(M, \mathrm{End}(E))_-$. The last equality $=_4$ comes from Lemma 11.7. By integrating (11.12) over $t \in [0,1]$, one concludes that

$$\mathrm{str}\big[f(F_{\tilde{\mathbf{A}}})\big] - \mathrm{str}\big[f(F_{\mathbf{A}})\big] = \int_0^1 \mathrm{str}\big[f(F_t)\big]dt = d\Big(\int_0^1 \mathrm{str}\Big[\frac{d\mathbf{A}_t}{dt}f'(F_t)\Big]dt\Big).$$

The result thus follows by setting

$$\omega \triangleq \int_0^1 \mathrm{str}\Big[\frac{d\mathbf{A}_t}{dt}f'(F_t)\Big]dt \in \Omega^{\mathrm{odd}}(M).$$

□

Remark 11.2. The theorem remains valid if M is compact and f is an entire function (e.g. $f(z) = e^{\lambda z}$ with $\lambda \in \mathbb{C}$).

Remark 11.3. The theorem of course applies to case when E is not graded (i.e. $E_- = \{0\}$) and $\mathbf{A} = \nabla^E$. In this case, $F_{\mathbf{A}} \in \Omega^2(M, \mathrm{End}(E))$ and thus $(F_{\mathbf{A}})^k = 0$ for all $k > \dim M/2$. As a result, the theorem remains valid for all formal power series f (without any convergence assumption).

11.5 The Chern character of a superconnection

In this section, we define the Chern character of a complex vector bundle and show how it is related to the Euler class we defined in Section 7.3.3. In this section, $\mathbb{K} = \mathbb{C}$.

Definition 11.12. Let $(\mathcal{E}, M, \pi)$ be a super $\mathbb{C}$-vector bundle over a manifold M. Let $\mathbf{A}$ be a superconnection on $\mathcal{E}$. The *Chern character form* associated with $\mathbf{A}$ is defined by

$$\operatorname{ch}(\mathcal{E}, \mathbf{A}) \triangleq \operatorname{str}\Big[\exp\big(\frac{i}{2\pi} F_{\mathbf{A}}\big)\Big] \in \Omega^{\text{even}}_{\mathbb{C}}(M).$$

The cohomology class of $\operatorname{ch}(\mathcal{E}, \mathbf{A})$ is known as the *Chern character* of $\mathcal{E}$ and is denoted by $\operatorname{ch}(\mathcal{E})$.

Remark 11.4. The Chern character $\operatorname{ch}(\mathcal{E})$ is independent of the superconnection $\mathbf{A}$ as a consequence of Theorem 11.1.

We now restrict to a situation that will be relevant to the CGB theorem. Let E be a real, oriented, Euclidean vector bundle of rank $2m$ over a manifold M. Define $\mathcal{E} \triangleq \Lambda_{\mathbb{C}} E^*$ (the complexified exterior bundle) and view it as a super $\mathbb{C}$-vector bundle with respect to the *chirality grading* induced from the fiberwise Euclidean structure on E (see Definition 11.6). Let ∇^E be a connection on E which is compatible with the metric. Let $\nabla^{\mathcal{E}} : \Omega^*(M, \mathcal{E}) \to \Omega^{*+1}(M, \mathcal{E})$ be the induced connection on $\mathcal{E}$.

Lemma 11.8. *The connection $\nabla^{\mathcal{E}}$ respects the chirality grading, i.e. $\nabla^{\mathcal{E}}_X(\Gamma(\mathcal{E}_\pm)) \subseteq \Gamma(\mathcal{E}_\pm)$ for all $X \in \Gamma(TM)$. In particular, it admits a splitting $\nabla^{\mathcal{E}} = \nabla^{\mathcal{E},+} \oplus \nabla^{\mathcal{E},-}$ on $\mathcal{E}_+ \oplus \mathcal{E}_-$ and is therefore a superconnection on $\mathcal{E}$.*

Proof. By definition, it suffices to show that

$$[\nabla^{\mathcal{E}}_X, \hat{\tau}] \triangleq \nabla^{\mathcal{E}}_X \circ \hat{\tau} - \hat{\tau} \circ \nabla^{\mathcal{E}}_X = 0, \tag{11.13}$$

where $\hat{\tau} \in \Gamma(\operatorname{End}(\mathcal{E}))$ is the chirality grading on $\mathcal{E}$. Let $x \in M$ be given fixed and we prove (11.13) at x. To this end, consider a local PONF $\{e_i\}$ of E which is parallel along geodesic rays emitting from x. For any $\xi \in \Gamma(\mathcal{E})$, one sees from the relation (8.28) that

$$\begin{aligned}&\nabla^{\mathcal{E}}_X \circ c(e_1)\big(c(e_2)\cdots c(e_{2m})\xi\big) - c(e_1) \circ \nabla^{\mathcal{E}}_X\big(c(e_2)\cdots c(e_{2m})\xi\big)\\ &= c(\nabla^E_X e_1)\big(c(e_2)\cdots c(e_{2m})\xi\big),\end{aligned}$$

which equals zero at x since $(\nabla^E_X e_i)(x) = 0$ by the construction of $\{e_i\}$ (vanishing Christoffel symbols at x). Therefore, $\nabla^{\mathcal{E}}_X$ commutes with $c(e_1)$ at x. Arguing inductively, it follows that $\nabla^{\mathcal{E}}_X$ commutes with $c(e_1)\cdots c(e_{2m})$, hence commuting with $\hat{\tau}$. □

The following result relates the Chern character form of $\nabla^{\mathcal{E}}$ to the Pfaffian of the curvature of ∇^E. In particular, the Chern character form provides a new representative of the Euler class. Recall that $F_{\mathcal{E}}$ and F_E are the curvatures of $\nabla^{\mathcal{E}}$ and ∇^E respectively.

Proposition 11.2. *One has*

$$\mathrm{ch}(\mathcal{E}, \nabla^{\mathcal{E}}) = \frac{1}{\pi^m}\mathrm{Pf}(F_{\nabla^E}).$$

The proof of Proposition 11.2 relies on the following representation of $R^{\mathcal{E}}$ in terms of the Clifford multipliers. Let $F = (F_i^j)$ denote the curvature matrix of ∇^E with respect to $\{e_\alpha\}$, namely,

$$F_{\nabla^E}(X,Y)e_i = F_i^j(X,Y)e_j, \quad X, Y \in \Gamma(TM).$$

Note that F is a local $\mathfrak{so}(2m)$-valued 2-form.

Lemma 11.9. $F_{\nabla^{\mathcal{E}}} = \frac{1}{4}F_i^j(c(e_i)c(e_j) - \hat{c}(e_i)\hat{c}(e_j))$.

Proof. For $X, Y \in \Gamma(TM)$, let $R(X,Y)^*$ be the induced (fiberwise) endomorphism on $\mathcal{E}$ (see (8.21)). According to Lemma 8.13,

$$\begin{aligned} F_{\nabla^{\mathcal{E}}}(X,Y) &= -R(X,Y)^* = -F_i^j a(e_i)\iota(e_j) \\ &= -\frac{1}{4}F_i^j(\hat{c}_i + c_i)(\hat{c}_j - c_j) \quad (c_i \triangleq c(e_i), \hat{c}_i \triangleq \hat{c}(e_i)) \\ &= \frac{1}{4}F_i^j\big(c_ic_j - \hat{c}_i\hat{c}_j\big) - \frac{1}{4}F_i^j(c_i\hat{c}_j - \hat{c}_ic_j). \end{aligned} \tag{11.14}$$

By using the relation (8.20), one has

$$F_i^j c_i\hat{c}_j = -F_i^j\hat{c}_jc_i = F_j^i\hat{c}_jc_i = F_i^j\hat{c}_ic_j.$$

Therefore, the second term on the RHS of (11.14) vanishes and the result thus follows. ☐

Proof of Proposition 11.2. According to the definition of the Chern character form, the supertrace formula (11.5) and Lemma 11.9, one has

$$\begin{aligned} \mathrm{ch}(\mathcal{E}, \nabla^{\mathcal{E}}) &= \mathrm{str}\Big[\exp\big(\frac{i}{2\pi}F_{\nabla^{\mathcal{E}}}\big)\Big] \\ &= \frac{i^m}{m!(2\pi)^m4^m}F_{i_1}^{i_2}\wedge\cdots\wedge F_{i_{2m-1}}^{i_{2m}}\mathrm{str}\big[c(e_{i_1})\cdots c(e_{i_{2m}})\big] \\ &= \frac{i^m}{m!(2\pi)^m4^m}\times(-1)^mF_{i_2}^{i_1}\wedge\cdots\wedge F_{i_{2m}}^{i_{2m-1}} \\ &\qquad\times(-i)^m2^{2m}\mathrm{sgn}(i_1,\cdots,i_{2m}) \\ &= \frac{(-1)^m}{m!\pi^m2^m}\mathrm{sgn}(i_1,\cdots,i_{2m})F_{i_2}^{i_1}\wedge\cdots\wedge F_{i_{2m}}^{i_{2m-1}} = \frac{1}{\pi^m}\mathrm{Pf}(F_{\nabla^E}), \end{aligned}$$

where the last equality follows from (7.12).

$\square$

11.6 Superconnection proof of CGB

We are now in a position to give the superconnection proof of CGB. In what follows, let M be a closed, oriented, $2m$-dimensional Riemannian manifold. We take $E \triangleq TM$ and $\mathcal{E} \triangleq \Lambda_{\mathbb{C}}T^*M$ in the previous discussion. Let ∇ denote the Levi-Civita connection and let $\nabla^{\mathcal{E}}$ be the induced superconnection on $\mathcal{E}$ with respect to the chirality grading.

Let V be a smooth vector field on M with isolated and nondegenerate zeros. The set of zeros of V is denoted as $\mathcal{Z}(V)$. On a normal chart $(U_p; x^i)$ around each $p \in \mathcal{Z}(V)$, we will assume that the Riemannian metric is flat (i.e. $g = (dx^1)^2 + \cdots + (dx^{2m})^2$) and the vector field V is given by

$$V(x) = A^i_j(p)x^j \frac{\partial}{\partial x^j} \tag{11.15}$$

where $(A^j_i(p))_{1\leqslant i,j\leqslant 2m}$ is some nondegenerate matrix. Indeed, one can use a bump function to modify the original metric into Euclidean on each U_p; this will not affect the form of the CGB because the cohomology class of $\mathrm{Pf}(F_{\nabla^M})$ is independent of the metric. For the vector field V, since p is an isolated zero, one has

$$V(x) = A^i_j(p)x^j \frac{\partial}{\partial x^i} + O(|\mathbf{x}|^2)$$

on U_p. In a similar way, one can use a bump function to modify V on U_p by only keeping its linear part.

To prove CGB, the essential idea is to consider the following *deformation of superconnections*:

$$\mathbf{A}_T \triangleq \nabla^{\mathcal{E}} + \lambda T c(V), \quad T > 0, \tag{11.16}$$

where λ is chosen by $\frac{i}{2\pi}\lambda^2 = 1$ (so that $\frac{i}{2\pi}\lambda^2 c(V)^2 = -|V|^2$). The point is that when T becomes large, the geometry of $\mathbf{A}_T$ outside $\mathcal{Z}(V)$ in the integral of the Chern character class gets forgotten and the whole-space integral is being localised around the zeros of V. As a result, one picks up the corresponding indices in the limit.

First of all, note that $c(V) \in \Gamma(\mathrm{End}_-(\mathcal{E}))$; this is seen by the relation that $c(V)\hat{\tau} = -\hat{\tau}c(V)$. Extending its action to $\Omega(M, \mathcal{E})$ by setting

$$c(V)(\alpha \wedge \xi) \triangleq (-1)^{\deg \alpha}\alpha \wedge c(V)\xi, \quad (\alpha \in \Omega(M), \xi \in \Omega(M, \mathcal{E}))$$

one easily checks that $\mathbf{A}_T$ defined by (11.16) is indeed a superconnection on $\mathcal{E}$.

Next, we compute the curvature of $\mathbf{A}_T$. To this end, we define a linear operator $c(\nabla V) : \Omega^*(M, \mathcal{E}) \to \Omega^{*+1}(M, \mathcal{E})$ by

$$c(\nabla V)(\xi) \triangleq dx^i \wedge c(\nabla_{\partial_i} V)(\xi), \quad (\xi \in \Omega(M, \mathcal{E})) \tag{11.17}$$

where $\{x^i\}$ is any local coordinates of M. It follows that

$$c(\nabla V)(\alpha \wedge \xi) = \alpha \wedge c(\nabla V)\xi \tag{11.18}$$

for all $\alpha \in \Omega(M)$, $\xi \in \Omega(M, \mathcal{E})$. In particular, $c(\nabla V)$ is tensorial and $c(\nabla V) \in \Omega^1(M, \mathrm{End}_- E)$.

Lemma 11.10. $F_{\mathbf{A}_T} = F_{\nabla^{\mathcal{E}}} - \lambda^2 T^2 |V|^2 + \lambda T c(\nabla V)$.

Proof. By the definition of $\mathbf{A}_T$ and the relation (8.20), one has

$$F_{\mathbf{A}_T} = \mathbf{A}_T^2 = R^{\mathcal{E}} - \lambda^2 T^2 |V|^2 + \lambda T \big(\nabla^{\mathcal{E}} \circ c(V) + c(V) \circ \nabla^{\mathcal{E}}\big).$$

Now it suffices to show that

$$\nabla^{\mathcal{E}} \circ c(V) + c(V) \circ \nabla^{\mathcal{E}} = c(\nabla V). \tag{11.19}$$

The action of both sides on $\Gamma(\mathcal{E})$ coincide and they both satisfy the relation (11.18). The relation (11.19) thus follows. □

Proof of CGB. According to Proposition 11.2, Lemma 11.10 and the Chern-Weil theorem, one has

$$\begin{aligned} \frac{(-1)^m}{(2\pi)^m} \int_M \mathrm{Pf}(F_{\nabla^M}) &= \frac{(-1)^m}{(2\pi)^m} \int_M \mathrm{str}\Big[\exp\big(\frac{i}{2\pi} F_{\mathbf{A}_T}\big)\Big] \\ &= (-1/2)^m \int_M e^{-T^2|V|^2} \mathrm{str}\Big[\exp\big(\frac{i}{2\pi} F_{\nabla^{\mathcal{E}}} + \frac{T}{\lambda} c(\nabla V)\big)\Big] \end{aligned}$$

for all $T > 0$. We write the $\int_M$ integral into two parts:

$$\int_M = \int_{M \setminus \bigcup_{p \in \mathcal{Z}(V)} U_p} + \sum_{p \in \mathcal{Z}(V)} \int_{U_p}.$$

For the first part, since $|V|$ is bounded away from zero on $(\cup_p U_p)^c$ and

$$\mathrm{str}\Big[\exp\big(\frac{i}{2\pi} F_{\nabla^{\mathcal{E}}} + \frac{T}{\lambda} c(\nabla V)\big)\Big] = e^{O(T)},$$

one finds that

$$\lim_{T \to \infty} \int_{M \setminus \bigcup_{p \in \mathcal{Z}(V)} U_p} e^{-T^2|V|^2} \mathrm{str}\Big[\exp\big(\frac{i}{2\pi} F_{\nabla^{\mathcal{E}}} + \frac{T}{\lambda} c(\nabla V)\big)\Big] = 0. \tag{11.20}$$

For the second part, since $F_{\nabla^{\mathcal{E}}} \equiv 0$ on each U_p, one has

$$\begin{aligned} &\operatorname{str}\Big[\exp\big(\frac{i}{2\pi}F_{\nabla^{\mathcal{E}}} + \frac{T}{\lambda}c(\nabla V)\big)\Big] \\ &= \operatorname{str}\Big[\exp\big(\frac{T}{\lambda}c(\nabla V)\big)\Big] = \frac{(T/\lambda)^{2m}}{(2m)!}\operatorname{str}\big[c(\nabla V)^{2m}\big] \end{aligned} \tag{11.21}$$

on U_p. Here the terms with degree $\neq 2m$ in the expansion of the exponential vanish due to the relation (11.5). To compute the above supertrace, one first finds by (11.17) and (11.15) that

$$\begin{aligned} c(\nabla V)^{2m} &= dx^{j_1} \wedge c(\nabla_{\partial_{j_1}} V) \circ \cdots \circ dx^{j_{2m}} \wedge c(\nabla_{\partial^{j_{2m}}} V) \\ &= A^{i_1}_{j_1}(p) \cdots A^{i_{2m}}_{j_{2m}}(p) dx^{j_1} \wedge c(\partial_{i_1}) \circ \cdots \circ dx^{j_{2m}} \wedge c(\partial_{i_{2m}}) \\ &= (-1)^m A^{i_1}_{j_1}(p) \cdots A^{i_{2m}}_{j_{2m}}(p) dx^{j_1} \wedge \cdots \wedge dx^{j_{2m}} \wedge c(\partial_{i_1}) \cdots c(\partial_{i_{2m}}) \\ &= (-1)^m \operatorname{sgn}(i_1, \cdots, i_{2m}) \operatorname{sgn}(j_1, \cdots, j_{2m}) A^{i_1}_{j_1}(p) \cdots A^{i_{2m}}_{j_{2m}}(p) \\ &\quad \times dx^1 \wedge \cdots \wedge dx^{2m} \wedge c(\partial_1) \cdots c(\partial_{2m}) \\ &= (-1)^m (2m)! \det A(p) \cdot dx^1 \wedge \cdots \wedge dx^{2m} \wedge c(\partial_1) \cdots c(\partial_{2m}). \end{aligned}$$

It follows again from (11.5) that

$$\begin{aligned} \operatorname{str}\big[c(\nabla V)^{2m}\big] &= (-1)^m (2m)! \det A(p) \cdot (-i)^m 2^{2m} dx^1 \wedge \cdots \wedge dx^{2m} \\ &= i^m 2^{2m} (2m)! \det A(p) \cdot dx^1 \wedge \cdots \wedge dx^{2m}. \end{aligned} \tag{11.22}$$

Let us assume that $U_p = \{|\mathbf{x}| < \varepsilon\}$. By substituting (11.22) into (11.21), one obtains that

$$(-1/2)^m \int_{U_p} e^{-T^2|V|^2} \operatorname{str}\Big[\exp\big(\frac{\sqrt{-1}}{2\pi}F_{\nabla^{\mathcal{E}}} + \frac{T}{\lambda}c(\nabla V)\big)\Big]$$

$$= (-1/2)^m (T/\lambda)^{2m} i^m 2^{2m} \det A(p) \int_{\{\mathbf{x}:|\mathbf{x}|<\varepsilon\}} e^{-T^2|A(p)\cdot\mathbf{x}|^2} dx^1 \wedge \cdots \wedge dx^{2m}$$

$$= \frac{1}{\pi^m} \det A(p) \cdot \int_{\{\mathbf{y}:|\mathbf{y}|<T\varepsilon\}} e^{-|A(p)\cdot\mathbf{y}|^2} dy^1 \wedge \cdots \wedge dy^{2m} \tag{11.23}$$

$$\to \frac{1}{\pi^m} \det A(p) \cdot \frac{\pi^m}{|\det A(p)|} = \operatorname{sgn}(\det A) = \operatorname{ind}_p(V) \tag{11.24}$$

as $T \to \infty$.

As a consequence, one concludes that

$$\begin{aligned} \frac{(-1)^m}{(2\pi)^m} \int_M \operatorname{Pf}(F_{\nabla^M}) &= \lim_{T\to\infty} (-1/2)^m \Big(\int_{M \setminus \bigcup_{p\in\mathcal{Z}(V)} U_p} + \sum_{p\in\mathcal{Z}(V)} \int_{U_p} \Big) \\ &\qquad e^{-T^2|V|^2} \operatorname{str}\Big[\exp\big(\frac{\sqrt{-1}}{2\pi}F_{\nabla^{\mathcal{E}}} + \frac{T}{\lambda}c(\nabla V)\big)\Big] \\ &= \sum_{p\in\mathcal{Z}(V)} \operatorname{ind}_p(V) = \chi(M). \end{aligned}$$

This completes the proof of the CGB theorem.

□

Appendix A Algebraic topology

In this appendix, we review some basic concepts and tools from algebraic topology that are used in the present notes. The main references for this part are [Hat22, Spa66]. Unless otherwise stated Λ is always assumed to be a principal ideal domain (PID). The most important examples are $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/2\mathbb{Z}$.

A.1 Basic tools from homological algebra

We discuss two basic tools from homological algebra that are frequently used in these notes: the Snake Lemma and the Five Lemma.

A.1.1 Chain complex and exact sequence

We first recall some definitions.

Definition A.1. A *chain complex* (*of* Λ*-modules*) is a sequence

$$\cdots \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0, \tag{A.1}$$

where C_k is a Λ-module and $\partial_k : C_k \to C_{k-1}$ is a (Λ-module) homomorphism such that $\partial_k \circ \partial_{k+1} = 0$ for every k. The chain complex (A.1) is denoted as (C_*, ∂_*) in short. Elements of the submodule

$$Z_k(C_*) \triangleq \ker \partial_k = \{\sigma \in C_k : \partial_k \sigma = 0\}$$

are called *k-cycles*. Elements of the submodule

$$B_k(C_*) \triangleq \mathrm{Im}\partial_{k+1} = \{\partial_{k+1}\sigma : \sigma \in C_{k+1}\}$$

are called *k-boundaries*. The *k-th homology group* of (C_*, ∂_*) *with coefficients in* Λ is the quotient module defined by

$$H_k(C_*; \Lambda) \triangleq \frac{Z_k(C_*)}{B_k(C_*)}.$$

Remark A.1. One can also consider a chain complex indexed by $\mathbb{Z}$. In the present notes, the index set is always taken to be $\mathbb{N}$.

Definition A.2. A *chain map* between two chain complexes (C_*, ∂_*^C) and (D_*, ∂_*^D) is a sequence of homomorphisms $f_k : C_k \to D_k$ ($k \in \mathbb{N}$) such that $f_{k-1} \circ \partial_k^C = \partial_k^D \circ f_k$ for every k. We write

$$(C_*, \partial_*^C) \xrightarrow{f_*} (D_*, \partial_*^D)$$

as a shorthand notation.

Note that a chain map f_* induces a homomorphism (still denoted as f_k by abuse of notation)

$$f_k : H_k(C_*; \Lambda) \to H_k(D_*; \Lambda), \ f_k([\sigma]) \triangleq [f_k(\sigma)] \tag{A.2}$$

on homology for every k.

Definition A.3. An *exact sequence of* Λ*-modules* is a sequence (either finite or infinite)

$$\cdots \xrightarrow{f_{k+2}} C_{k+1} \xrightarrow{f_{k+1}} C_k \xrightarrow{f_k} C_{k-1} \xrightarrow{f_{k-1}} \cdots,$$

where C_k is a Λ-module and f_k is a homomorphism, such that $\mathrm{Im} f_{k+1} = \ker f_k$ for every k. A *short exact sequence of* Λ*-modules* is an exact sequence of the form

$$0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0.$$

In other words, C, D, E are Λ-modules and f, g are homomorphisms such that f is injective, g is surjective and $\mathrm{Im} f = \ker g$.

Definition A.4. A *short exact sequence of chain complexes* refers to the following sequence

$$0 \to (C_*, \partial_*^C) \xrightarrow{f_*} (D_*, \partial_*^D) \xrightarrow{g_*} (E_*, \partial_*^E) \to 0, \tag{A.3}$$

where C_*, D_*, E_* are chain complexes, f_*, g_* are chain maps and at every fixed degree n the sequence (A.3) is a short exact sequence of Λ-modules.

Throughout the rest, we will omit the PID Λ to ease notation and all homology groups are understood over Λ.

A.1.2 The Snake Lemma

The *Snake Lemma* asserts that a short exact sequence of chain complexes induces a long exact sequence on homology. This is an important technique for computing (co)homology groups of a space in general.

Lemma A.1. *Let*

$$0 \to (C_*, \partial_*^C) \xrightarrow{f_*} (D_*, \partial_*^D) \xrightarrow{g_*} (E_*, \partial_*^E) \to 0$$

be a short exact sequence of chain complexes. There exists a long exact sequence

$$\begin{aligned} \cdots \to H_k(C_*) \xrightarrow{f_k} H_k(D_*) \xrightarrow{g_k} H_k(E_*) \xrightarrow{\bar{\partial}_k} H_{k-1}(C_*) \\ \xrightarrow{f_{k-1}} H_{k-1}(D_*) \to \cdots \to H_0(D_*) \xrightarrow{g_0} H_0(E_*) \to 0 \end{aligned}$$

on homology. Here f_k, g_k are the induced homomorphisms on the homology groups defined by (A.2). The map $\bar{\partial}_k$ is the so-called connecting homomorphism which is *constructed as follows. Let* $[\sigma] \in H_k(E_*)$ *and pick any representative* $\sigma \in E_k$*. Let* $\tau \in D_k$ *be such that* $g_k(\tau) = \sigma$*. Pick* $\theta \in Z_{k-1}(C_*)$ *such that* $f_{k-1}(\theta) = \partial_k^D(\tau)$*. Then one defines* $\bar{\partial}_k([\sigma]) \triangleq [\theta]$*. The map* $\bar{\partial}_k$ *does not depend on the choices of* σ, τ, θ *and is a well-defined homomorphism on homology.*

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_{k+1} \arrow[r, "f_{k+1}"] \arrow[d, "\partial^C_{k+1}"'] & D_{k+1} \arrow[r, "g_{k+1}"] \arrow[d, "\partial^D_{k+1}"'] & E_{k+1} \arrow[r] \arrow[d, "\partial^E_{k+1}"'] & 0 \\
0 \arrow[r] & C_k \arrow[r, "f_k"] \arrow[d, "\partial^C_k"'] & D_k \arrow[r, "g_k"] \arrow[d, "\partial^D_k"'] & E_k \arrow[r] \arrow[d, "\partial^E_k"'] \arrow[dll, dashed, "\bar{\partial}_k"] & 0 \\
0 \arrow[r] & C_{k-1} \arrow[r, "f_{k-1}"] \arrow[d, "\partial^C_{k-1}"'] & D_{k-1} \arrow[r, "g_{k-1}"] \arrow[d, "\partial^D_{k-1}"'] & E_{k-1} \arrow[r] \arrow[d, "\partial^E_{k-1}"'] & 0 \\
0 \arrow[r] & C_{k-2} \arrow[r, "f_{k-2}"] \arrow[d] & D_{k-2} \arrow[r, "g_{k-2}"] \arrow[d] & E_{k-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

A.1.3 The Five Lemma

The *Five Lemma* is a useful tool for establishing isomorphisms between homology groups.

Lemma A.2. *Consider the following commutative diagram*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5.
\end{tikzcd}$$

Here both rows are exact sequences of Λ*-modules and* f_i *(*$i = 1, \cdots, 5$*) are homomorphisms. Suppose that* f_1, f_2, f_4, f_5 *are all isomorphisms. Then* f_3 *is also an isomorphism.*

Remark A.2. In the above discussion, there is nothing special about the downward indexing convention. One can also index the sequence by

$$\cdots \xrightarrow{f^{k-1}} C^k \xrightarrow{f^k} C^{k+1} \xrightarrow{f^{k+1}} C^{k+2} \xrightarrow{f^{k+2}} \cdots .$$

Such a sequence is called a *cochain complex*. Elements of the kernel (respectively, image) of the homomorphism are called *cocycles* (respectively, *coboundaries*). The *k-th cohomology group* of (C^*, f^*) is defined to be the quotient space of k-cocycles by k-coboundaries in the same way as in Definition A.1.

A.2 Singular (co)homology

Let $k \in \mathbb{N}$. The *standard k-simplex* is the subset of $\mathbb{R}^{k+1}$ defined by

$$\Delta^k \triangleq \big\{(v_0, \cdots, v_k) : v_i \geqslant 0,\ \sum_{i=0}^{k} v_i = 1\big\}.$$

A.2.1 Singular chain complex and homology

Definition A.5. Let X be a topological space. A *singular k-simplex* in X is a continuous map $\sigma : \Delta^k \to X$. For $0 \leqslant i \leqslant k$, *the i-th face of σ* is the singular $(k-1)$-simplex defined by $\sigma \circ \iota_i : \Delta^{k-1} \to X$, where $\iota_i : \Delta^{k-1} \to \Delta^k$ is the inclusion map defined by

$$\iota_i(v_0, \cdots, v_{k-1}) \triangleq (v_0, \cdots, v_{i-1}, 0, v_i, \cdots, v_{k-1}).$$

We will simply denote $\sigma \circ \iota_i$ as $\sigma|_{[v_0,\cdots,\widehat{v_i},\cdots,v_k]}$. In a similar way, one can also define the *front i-face* $\sigma|_{[v_0,\cdots,v_i]}$(respectively, the *back i-face* $\sigma|_{[v_{k-i},\cdots,v_k]}$) of σ as a singular i-simplex.

Definition A.6. The *k-th singular chain group of X* is the free Λ-module $C_k(X;\Lambda)$ generated by the singular k-simplices in X. In other words, every element of $C_k(X;\Lambda)$ is uniquely expressed as

$$\xi = \sum_{\sigma:\ \text{singular } k\text{-simplex}} \lambda_\sigma \sigma$$

where λ_σ is nonzero for at most finitely many σ's. Elements of $C_k(X;\Lambda)$ are called *singular k-chains*. The chain complex $(C_*(X;\Lambda), \partial_*)$ is called the *singular chain complex of X* with coefficients in Λ.

Definition A.7. The *boundary* of a singular k-simplex σ is the singular $(k-1)$-chain defined by

$$\partial_k \sigma = \sum_{i=0}^{k} (-1)^i \sigma|_{[v_0,\cdots,\widehat{v_i},\cdots,v_k]}.$$

By linearity, this extends to a Λ-module homomorphism $\partial_k : C_k(X;\Lambda) \to C_{k-1}(X;\Lambda)$ which is called *the boundary map*. Sometimes we will omit the subscript k when no confusion is caused.

The singular homology groups of X is defined as follows.

Definition A.8. A *singular k-cycle* is an element $\sigma \in C_k(X;\Lambda)$ with zero boundary, i.e. $\partial\sigma = 0$. A *singular k-boundary* is an element $\sigma \in C_k(X;\Lambda)$ of the form $\sigma = \partial\tau$ for some $\tau \in C_{k+1}(X;\Lambda)$. The submodule of singular k-cycles (respectively, k-boundaries) is denoted as $Z_k(X;\Lambda)$ (respectively, $B_k(X;\Lambda)$). The *k-th singular homology group of X with coefficients in* Λ is the k-th homology group of the singular chain complex $(C_*(X;\Lambda), \partial_*)$ defined by

$$H_k(X;\Lambda) \triangleq \frac{Z_k(X;\Lambda)}{B_k(X;\Lambda)}.$$

A.2.2 Singular cohomology

By dualising the singular chain complex, one can define the singular cohomology in an analogous way.

Definition A.9. The *singular cochain complex of X* is the sequence

$$C^0(X;\Lambda) \xrightarrow{\delta^0} C^1(X;\Lambda) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{k-1}} C^k(X;\Lambda) \xrightarrow{\delta^k} C^{k+1}(X;\Lambda) \xrightarrow{\delta^{k+1}} \cdots .$$

Here $C^k(X;\Lambda) \triangleq \mathrm{Hom}(C_k(X;\Lambda);\Lambda)$ is the dual Λ-module of $C_k(X;\Lambda)$. The *coboundary map* δ^k is defined to be the dual of ∂_{k+1}, i.e.

$$(\delta^k\varphi)(\sigma) = \varphi(\partial_{k+1}\sigma)$$

for all $\varphi \in C^k(X;\Lambda)$ and $\sigma \in C_{k+1}(X;\Lambda)$.

It is easily seen that $\delta^{k+1} \circ \delta^k = 0$. In particular, $(C^*(X;\Lambda), \delta^*)$ is indeed a cochain complex. We sometimes omit the superscript of the coboundary map when no confusion is caused.

Definition A.10. The submodule of *singular k-cocycles* and *k-coboundaries*, denoted as $Z^k(X;\Lambda)$ and $B^k(X;\Lambda)$ respectively, are defined in the same way as Definition A.8 by using the coboundary map instead. The *k-th singular cohomology group of X with coefficients in* Λ is the k-th cohomology group of the cochain complex $(C^*(X;\Lambda), \delta^*)$ defined by

$$H^k(X;\Lambda) \triangleq \frac{Z^k(X;\Lambda)}{B^k(X;\Lambda)}.$$

There is an obvious homomorphism

$$f_k : H^k(X;\Lambda) \to \mathrm{Hom}(H_k(X;\Lambda);\Lambda)$$

which is induced by the natural pairing between $C^k(X;\Lambda)$ and $C_k(X;\Lambda)$. In general, f_k needs not be an isomorphism. However, one has the following result.

Proposition A.1. *Suppose that either $H_{k-1}(X;\Lambda)$ is a free Λ-module or Λ is a field. Then f_k is an isomorphism.*

Remark A.3. The general relation between cohomology and homology is described by the so-called *universal coefficient theorem* which involves the torsion functor. We will not elaborate this result here.

A.2.3 Cup product

There is a ring structure (multiplication) on cohomology which is defined by the so-called cohomology cup product.

Definition A.11. The *cup product* between $\varphi \in C^k(X;\Lambda)$ and $\psi \in C^l(X;\Lambda)$ is the singular $(k+l)$-cochain $\varphi \cup \psi \in C^{k+l}(X;\Lambda)$ such that

$$(\varphi \cup \psi)(\sigma) \triangleq \varphi(\sigma|_{[v_0,\cdots,v_k]})\psi(\sigma|_{[v_k,\cdots,v_{k+l}]})$$

for any singular $(k+l)$-simplex σ.

The coboundary map δ satisfies the following Leibniz rule with respect to the cup product:

$$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^{\deg \varphi}\varphi \cup \delta\psi. \tag{A.4}$$

According to the relation (A.4), $\varphi \cup \psi$ is a cocycle if both φ and ψ are. As a consequence, the cup product descends to a bilinear operation

$$\cup : H^k(X;\Lambda) \times H^l(X;\Lambda) \to H^{k+l}(X;\Lambda)$$

on cohomology. This is known as the *cohomology cup product*. It can be shown that $\cup$ is associative and satisfies the following sign-commutativity relation:

$$a \cup b = (-1)^{kl} b \cup a \tag{A.5}$$

for any $a \in H^k(X;\Lambda)$ and $b \in H^l(X;\Lambda)$.

A.3 Relative (co)homology

From now on, we will omit the PID Λ to ease notation. All modules and (co)homology groups are understood over Λ unless otherwise stated. Let X be a topological space and let $A \subseteq X$. Define the quotient module

$$C_k(X,A) \triangleq \frac{C_k(X)}{C_k(A)}.$$

The boundary map descends to (still denoted as) $\partial_k : C_k(X,A) \to C_{k-1}(X,A)$. This gives rise to the so-called *relative singular chain complex* $(C_*(X,A), \partial_*)$ of the pair (X,A). The submodule of relative k-cycles (respectively, relative k-boundaries) is denoted as $Z_k(X,A)$ (respectively, $B_k(X,A)$). Geometrically, a relative cycle is a singular chain whose boundary takes values in A.

Definition A.12. The *k-th relative singular homology group* of the pair (X, A) is the k-th homology group of $(C_*(X, A), \partial_*)$ defined by

$$H_k(X, A) \triangleq \frac{Z_k(X, A)}{B_k(X, A)}.$$

There is an obvious short exact sequence of chain complexes:

$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0, \tag{A.6}$$

where the second arrow is the inclusion and the third arrow is the quotient map. According to the Snake Lemma (Lemma A.1), there is an induced long exact sequence

$$\begin{aligned} \cdots \to H_{k+1}(X, A) \to H_k(A) \to H_k(X) \to H_k(X, A) \\ \to H_{k-1}(A) \to \cdots \to H_0(X) \to H_0(X, A) \to 0. \end{aligned} \tag{A.7}$$

on homology.

Now consider $A \subseteq B \subseteq X$. There is also an obvious short exact sequence of chain complexes

$$0 \to C_k(B, A) \to C_k(X, A) \to C_k(X, B) \to 0, \tag{A.8}$$

where the second arrow is the inclusion homomorphism and the third one is the quotient map. For the same reason, there is also an induced long exact sequence

$$\begin{aligned} \cdots \to H_{k+1}(X, B) \to H_k(B, A) \to H_k(X, A) \to H_k(X, B) \\ \to H_{k-1}(B, A) \to \cdots \to H_0(X, A) \to H_0(X, B) \to 0 \end{aligned} \tag{A.9}$$

on relative homology. One can easily write down the connecting homomorphisms of the above two exact sequences from the Snake Lemma.

By dualising the above consideration, one can easily obtain a relative version of cohomology. Let $A \subseteq X$. The *k-th relative singular cochain group of the pair* (X, A) is defined by

$$C^k(X, A) \triangleq \mathrm{Hom}(C_k(X, A); \Lambda).$$

The coboundary map δ^k is defined by the dual of the boundary map in the same way as in the absolute case. It is useful to note that $C^k(X, A)$ can be identified with the submodule of $C^k(X)$ consisting of singular k-cochains that annihilate singular k-chains in A. The *relative singular cohomology of the pair* (X, A), which is denoted as $H^*(X, A)$, is the cohomology of the cochain complex $(C^*(X, A), \delta^*)$. There is a canonical homomorphism

$$H^*(X, A) \to \mathrm{Hom}(H_*(X, A); \Lambda),$$

which is an isomorphism in the case when Λ is a field.

Similar to the homology case, there is a short exact sequence of cochain complexes (the dual of (A.6)) which gives rise to a long exact sequence on cohomology (the one with all members in (A.7) replaced by cohomology and all arrows reversed). There is also a long exact sequence

$$\begin{aligned}0 &\to H^0(X,B) \to H^0(X,A) \to \cdots \to H^{k-1}(B,A)\\ &\to H^k(X,B) \to H^k(X,A) \to H^k(B,A) \to H^{k+1}(X,B) \to \cdots\end{aligned} \tag{A.10}$$

of relative cohomology that is dual to (A.9).

Cup product on relative cohomology

The cup product on relative cohomology requires some extra care. Let X be a topological space and let $A, B \subseteq X$. Define

$$\hat{C}^k(X;A,B) \triangleq C^k(X,A) \cap C^k(X,B)$$

for each $k \in \mathbb{N}$. This is the submodule of $C^k(X)$ consisting of those singular cochains that annihilate simplices in A or in B. Note that $\hat{C}^*(X;A,B)$ is also a cochain complex whose coboundary map is the dual of the boundary map. The cohomology of $\hat{C}^*(X;A,B)$ is denoted as $\hat{H}^*(X;A,B)$.

The cup product on absolute cochains induces a bilinear operator

$$\hat{\cup} : C^k(X,A) \times C^l(X,B) \to \hat{C}^{k+l}(X;A,B)$$

on relative cochains, which descends to a product

$$\hat{\cup} : H^k(X,A) \times H^l(X,B) \to \hat{H}^{k+l}(X;A,B) \tag{A.11}$$

on cohomology. The following result allows one to obtain an actual cup product which takes values in the relative cohomology group $H^{k+l}(X, A \cup B)$.

Proposition A.2. *Suppose that both A and B are relatively open in $A \cup B$. Then the canonical inclusion*

$$C^k(X, A \cup B) \to \hat{C}^k(X;A,B)$$

induces an isomorphism

$$H^k(X, A \cup B) \cong \hat{H}^k(X;A,B) \tag{A.12}$$

on cohomology.

Under the assumption of Proposition A.2, by composing the product $\hat{\cup}$ defined by (A.11) with the inverse of the isomorphism (A.12), one obtains an actual cup product

$$\cup : H^k(X,A) \times H^l(X,B) \to H^{k+l}(X, A \cup B) \tag{A.13}$$

on relative cohomology.

Remark A.4. The consideration of $\hat{C}^*(X;A,B)$ is needed because for $\varphi \in C^k(X,A)$ and $\psi \in C^l(X,B)$, their cup product $\varphi \cup \psi$ needs not annihilate simplices in $A \cup B$ (but it clearly annihilates simplices that lie entirely in A or in B). This is not needed if $A = B$ or if one of them is the empty set.

A.4 Functorial properties and homotopy invariance

Let $f : (X,A) \to (Y,B)$ be a morphism between pairs of topological spaces, i.e. $f : X \to Y$ is a continuous map and $f(A) \subseteq B$. Then f induces homomorphisms

$$f_* : H_k(X,A) \to H_k(Y,B), \quad f^* : H^k(Y,B) \to H^k(X,A)$$

on both homology and cohomology.

Two morphisms $f, g : (X,A) \to (Y,B)$ between pairs are said to be *homotopic*, which is denoted as $f \simeq g$, if there exists a continuous map $F : X \times [0,1] \to Y$ such that $F(\cdot,0) = f$, $F(\cdot,1) = g$ and $F(A,t) \subseteq B$ for all $t \in [0,1]$. In this case, one has $f_* = g_*$ and $f^* = g^*$ on homology and cohomology respectively.

Two pairs (X,A) and (Y,B) are said to be *homotopy equivalent* if there exist morphisms $f : (X,A) \to (Y,B)$ and $g : (Y,B) \to (X,A)$ such that

$$g \circ f \simeq \mathrm{id}_{(X,A)}, \quad f \circ g \simeq \mathrm{id}_{(Y,B)}.$$

In this case, the map f_* (respectively, f^*) is an isomorphism on homology (respectively, cohomology).

Remark A.5. The above discussion contains the absolute case by taking $A = B = \emptyset$ as a special situation.

A subset $A \subseteq X$ is said to be a *deformation retract* of X if there exists a continuous map

$$F : X \times [0,1] \to X$$

such that

$$F(\cdot,0) = \mathrm{id}_X; \quad F(x,1) \in A; \;\; F(y,t) = y$$

for all $x \in X$, $y \in A$ and $t \in [0,1]$. In this case, the inclusion induces an isomorphism between $H^*(A)$ and $H^*(X)$ whose inverse is r_* where $r(\cdot) \triangleq F(\cdot,1) : X \to A$.

Let X be a topological space and let $A \subseteq B \subseteq X$. Suppose that A is a deformation retract of B (or more generally that the inclusion $i : A \to B$ is a homotopy equivalence). Then $i_* : H_k(X, A) \cong H_k(X, B)$ is an isomorphism for every k. This is easily seen by applying the Five Lemma to the long exact sequences (A.7) of the pairs (X, A) and (X, B), noting that $H_*(A) \cong H_*(B)$. In a similar way, one also has $i^* : H^k(X, B) \cong H^k(X, A)$ for every k.

A.5 Excision and the Mayer-Vietoris sequence

We present two useful tools for computing (co)homology groups: excision and the Mayer-Vietoris sequence. Let X be a topological space.

A.5.1 Excision for relative (co)homology

Intuitively, excision means that cutting away a part that lies entirely inside the subspace A of X does not affect the relative (co)homology of the pair (X, A).

Theorem A.1 (Excision)**.** *Let $Z \subseteq A \subseteq X$. Suppose that $\bar{Z} \subseteq \mathring{A}$ (the interior of A). Then the inclusion of pairs*

$$(X\backslash Z, A\backslash Z) \hookrightarrow (X, A)$$

induces an isomorphism on relative homology:

$$H_k(X\backslash Z, A\backslash Z; \Lambda) \cong H_k(X, A; \Lambda)$$

as well as an isomorphism on relative cohomology:

$$H^k(X, A; \Lambda) \cong H^k(X\backslash Z, A\backslash Z; \Lambda)$$

for every $k \in \mathbb{N}$.

A.5.2 The Mayer-Vietoris sequence

A standard methodology in algebraic topology is "divide and conquer". To summarise this philosophy, one divides a complicated space into simple pieces whose topological invariants are known or easy to compute. Then one reconstructs invariants of the original space from the pieces and their overlaps. An important technique to implement this philosophy is the so-called Mayer-Vietoris sequence.

We first discuss the result for absolute (co)homology. Let $X = U \cup V$ where U, V are open subsets of X. Let $\hat{C}_k(U + V)$ denote the free submodule of $C_k(X)$ generated by those singular k-simplices that lie entirely in U or in V. There is a short exact sequence

$$0 \to C_*(U \cap V) \xrightarrow{i_*} C_*(U) \oplus C_*(V) \xrightarrow{j_*} \hat{C}_*(U + V) \to 0 \qquad \text{(A.14)}$$

of chain complexes, where

$$i_*(\sigma) \triangleq (-\sigma, \sigma), \quad j_*(\sigma, \tau) \triangleq \sigma + \tau.$$

By the Snake Lemma, there is an induced long exact sequence

$$\begin{aligned} \cdots \to H_{k+1}(\hat{C}_*(U+V)) \to H_k(U \cap V) \to H_k(U) \oplus H_k(V) \to H_k(\hat{C}_*(U+V)) \\ \to H_{k-1}(U \cap V) \to \cdots \to H_0(U) \oplus H_0(V) \to H_0(\hat{C}_*(U+V)) \to 0. \end{aligned}$$

on homology. On the other hand, a key fact is that the inclusion

$$\hat{C}_k(U+V) \to C_k(X)$$

induces an isomorphism

$$H_k(\hat{C}_*(U+V)) \cong H_k(X)$$

on homology for every k. By using this isomorphism, one thus obtains the following long exact sequence

$$\begin{aligned} \cdots \to H_{k+1}(X) \to H_k(U \cap V) \to H_k(U) \oplus H_k(V) \to H_k(X) \to \\ \to H_{k-1}(U \cap V) \to \cdots \to H_0(U) \oplus H_0(V) \to H_0(X) \to 0. \end{aligned} \tag{A.15}$$

This is known as the *Mayer-Vietoris sequence for homology.* The cohomology sequence is obtain in a similar way through duality. The resulting long exact sequence is

$$\begin{aligned} 0 \to H^0(X) \to H^0(U) \oplus H^0(V) \to \cdots \to H^{k-1}(U \cap V) \\ \to H^k(X) \xrightarrow{j^*} H^k(U) \oplus H^k(V) \xrightarrow{i^*} H^k(U \cap V) \to H^{k+1}(X) \to \cdots, \end{aligned}$$

where

$$j^*(a) \triangleq (a|_U, a|_V), \quad i^*(a,b) \triangleq b|_{U\cap V} - a|_{U \cap V}.$$

This is known as the *Mayer-Vietoris sequence for cohomology.*

Next, we state a version for the relative (co)homology that was used in the main text. Let U, V be given open subsets of X. Define the quotient module

$$\hat{C}_k(X; U, V) \triangleq \frac{C_k(X)}{\hat{C}_k(U+V)},$$

where $\hat{C}_k(U+V)$ is the free submodule defined previously. The following short exact sequence

$$0 \to C_*(X, U \cap V) \xrightarrow{\text{inclusion}} C_*(X,U) \oplus C_*(X,V) \xrightarrow{\text{back}-\text{front}} \hat{C}_*(X; U, V) \to 0$$

induces a long exact sequence on relative homology. Through the isomorphism

$$H_k(\hat{C}_*(X;U,V)) \cong H_k(X,U\cup V),$$

this long exact sequence reads

$$\begin{aligned} \cdots \to H_{k+1}(X,U\cup V) &\to H_k(X,U\cap V) \to H_k(X,U)\oplus H_k(X,V) \\ &\to H_k(X,U\cup V) \to \cdots \to H_0(X,U)\oplus H_0(X,V) \to H_0(X,U\cup V) \to 0. \end{aligned} \tag{A.16}$$

The relative cohomology sequence is obtained in a similar way, which reads

$$\begin{aligned} 0 \to H^0(X,U\cup V) &\to H^0(X,U)\oplus H^0(X,V) \to \cdots \to H^k(X,U\cup V) \\ &\to H^k(X,U)\oplus H^k(X,V) \to H^k(X,U\cap V) \to H^{k+1}(X,U\cup V) \to \cdots . \end{aligned} \tag{A.17}$$

Example A.1. Let D^n ($n \geqslant 2$) be the closed unit ball in $\mathbb{R}^n$ with boundary $S^{n-1} = \partial D^n$ (the unit sphere). Then

$$H_k(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}) \cong H_k(D^n, S^{n-1}) \cong \begin{cases} \Lambda, & \text{if } k=n; \\ 0, & \text{otherwise.} \end{cases} \tag{A.18}$$

Indeed, one has

$$H_*(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}) \cong H_*(D^n, D^n\backslash\{0\}) \cong H_*(D^n, S^{n-1}),$$

where the first isomorphism follows from excision and the second one comes from the fact that S^{n-1} is a deformation retract of $D^n\backslash\{0\}$ (along radii). The second isomorphism of (A.18) can be easily obtained by using the long exact sequence (A.7), noting the standard facts that

$$H_k(D^n) \cong \begin{cases} \Lambda, & \text{if } k=0; \\ 0, & \text{otherwise,} \end{cases} \qquad H_k(S^{n-1}) \cong \begin{cases} \Lambda, & \text{if } k=0 \text{ or } n-1; \\ 0, & \text{otherwise.} \end{cases} \tag{A.19}$$

The first part of (A.19) is trivial because D^n deformation retracts to a single point. The second part follows from applying the Mayer-Vietoris sequence to an open cover $\{U,V\}$ of S^{n-1} by two spherical caps whose intersection is a tubular neighbourhood of the equator. The cohomology version of (A.18) holds for the same reason.

A.6 The Künneth theorem

Let (X,A) and (Y,B) be pairs of topological spaces where A is an open subset of X and B is an open subset of Y.

Definition A.13. The *cohomology cross product* is the bilinear map

$$\times : H^k(X,A) \times H^l(Y,B) \to H^{k+l}(X \times Y, A \times Y \cup X \times B)$$

defined by

$$a \times b \triangleq (\pi_X^* a) \cup (\pi_Y^* b), \quad a \in H^k(X,A), b \in H^l(Y,B),$$

where

$$\pi_X : (X \times Y, A \times Y \cup X \times B) \to (X,A)$$
$$\pi_Y : (X \times Y, A \times Y \cup X \times B) \to (Y,B)$$

are the projections as pairs and $\cup$ is the relative cup product defined in (A.13).

According to the definition and the relation (A.5), one has

$$(a \times b) \cup (c \times d) = (-1)^{\deg b \times \deg c}(a \cup c) \times (b \cup d) \tag{A.20}$$

for any homogeneous classes $a, c \in H^*(X,A)$ and $b, d \in H^*(Y,B)$.

Theorem A.2 (The Künneth Theorem)**.** *Let Λ be a field. Then the cross product defines an isomorphism*

$$\times : \bigoplus_{i=0}^{k} H^i(X,A;\Lambda) \otimes H^{k-i}(Y,B;\Lambda) \to H^k(X \times Y, A \times Y \cup X \times B; \Lambda)$$

for every $k \in \mathbb{N}$.

Remark A.6. If $B = \emptyset$, there is no need to assume that A is open (and vice versa).

A.7 Equivalence between simplicial and singular homologies

Let K be a simplicial complex whose topological point set is denoted as $|K|$. The aim of this subsection is to prove that the simplicial homology of $|K|$, which is defined in Section 4.7.1 and denoted as $H_*^{\mathrm{sim}}(K)$, is isomorphic to the singular homology $H_*(|K|)$. The argument is a nice application of the topological tools we reviewed so far. Although the de Rham theorem in Section 4.7 was discussed over $\mathbb{R}$, the identification between simplicial and singular (co)homologies holds over any coefficient PID Λ. We therefore work over Λ but will omit this symbol to ease notation.

Let $(C_*^{\mathrm{sim}}(K), \partial_*^{\mathrm{sim}})$ denote the simplicial chain complex of K. By definition, $C_k^{\mathrm{sim}}(K)$ is the free Λ-module generated by the k-simplices of K. There is a homomorphism $\iota_k : C_k^{\mathrm{sim}}(K) \to C_k(|K|)$ for every k which is obtained by linearly extending the definition

$$\iota_k([v_0, \cdots, v_k]) \triangleq \left[\sigma_s : \Delta^k \ni (t_0, \cdots, t_k) \mapsto \sum_{j=0}^{k} t_j v_j \in |K|\right]$$

on any k-simplex $[v_0, \cdots, v_k]$. It is plain to check that ι_k commutes with the boundary maps, i.e.

$$\iota_k \circ \partial^{\text{sim}}_{k+1} = \partial_k \circ \iota_{k+1},$$

where ∂^{sim} is the boundary map on simplicial chains (see Definition 4.9). As a result, ι_k descends to a homomorphism $(\iota_k)_* : H^{\text{sim}}_k(K) \to H_k(|K|)$ on homology.

Theorem A.3. *The map $(\iota_k)_*$ is an isomorphism for every k.*

The proof of Theorem A.3 relies on the following lemma. A topological pair (X, A) (X is a topological space and $A \subseteq X$) is said to be a *good pair* if A is a deformation retract of an open neighbourhood V.

Lemma A.3. *Let (X, A) be a good pair. Then the quotient map $q : (X, A) \to (X/A, A/A)$ induces an isomorphism*

$$q_* : H_k(X, A) \to H_k(X/A, A/A) \tag{A.21}$$

for every $k \in \mathbb{N}$. Here X/A is the quotient topological space (and thus A/A is a point in X/A).

Proof. Let V be an open neighbourhood of A which deformation retracts to A. Consider the following commutative diagram

$$\begin{tikzcd}
H_k(X, A) \arrow[r] \arrow[d, "q_*"] & H_k(X, V) \arrow[d, "q_*"] & H_k(X \setminus A, V \setminus A) \arrow[l] \arrow[d, "q_*"] \\
H_k(X/A, A/A) \arrow[r] & H_k(X/A, V/A) & H_k\big((X/A) \setminus (A/A), (V/A) \setminus (A/A)\big). \arrow[l]
\end{tikzcd}$$

The left two horizontal maps are isomorphisms because A is a deformation retract of V. The right two horizontal maps are isomorphisms due to excision. The right vertical map is an isomorphism because the quotient map q restricts to a homeomorphism on $X \backslash A$. Therefore, the left vertical map (A.21) is also an isomorphism. ☐

Remark A.7. By using the long exact sequence for the pair $(X/A, A/A)$ and noting that A/A is just a single point, it is easily seen that

$$H_k(X/A) \cong H_k(X/A, A/A)$$

for all $k > 0$ and $H_0(X/A) \cong H_0(X/A, A/A) \oplus \Lambda$.

Proof of Theorem A.3. We divide the argument into the following major steps.

(i) Let K^n denote the n-skeleton of K and let $|K^n|$ be its topological point set. For every $k \in \mathbb{N}$, we define $H_k^{\text{sim}}(K^n, K^{n-1})$ to be the homology of the relative simplicial chain complex

$$C_k^{\text{sim}}(K^n, K^{n-1}) \triangleq \frac{C_k^{\text{sim}}(K^n)}{C_k^{\text{sim}}(K^{n-1})}.$$

Similar to the singular case, there is a long exact sequence

$$\begin{aligned} \cdots \to H_{k+1}^{\text{sim}}(K^n, K^{n-1}) \to H_k^{\text{sim}}(K^{n-1}) \to H_k^{\text{sim}}(K^n) \to H_k^{\text{sim}}(K^n, K^{n-1}) \\ \to H_{k-1}^{\text{sim}}(K^{n-1}) \to \cdots \to H_0(K^n) \to H_0(K^n, K^{n-1}) \to 0. \end{aligned} \tag{A.22}$$

(ii) The homomorphism $(\iota_k)_*$ induces a homomorphism (still denoted as)

$$(\iota_k)_* : H_k^{\text{sim}}(K^n, K^{n-1}) \to H_k(|K^n|, |K^{n-1}|) \tag{A.23}$$

on relative homology. One has the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_{k+1}^{\text{sim}}(K^n, K^{n-1}) \arrow[r] \arrow[d, "\iota_*"] & H_k^{\text{sim}}(K^{n-1}) \arrow[r] \arrow[d, "\iota_*"] & H_k^{\text{sim}}(K^n) \arrow[r] \arrow[d, "\iota_*"] & H_k^{\text{sim}}(K^n, K^{n-1}) \arrow[r] \arrow[d, "\iota_*"] & \cdots \\
\cdots \arrow[r] & H_{k+1}(|K^n|, |K^{n-1}|) \arrow[r] & H_k(|K^{n-1}|) \arrow[r] & H_k(|K^n|) \arrow[r] & H_k(|K^n|, |K^{n-1}|) \arrow[r] & \cdots ,
\end{tikzcd} \tag{A.24}$$

where the top row is the long exact sequence (A.22) and the bottom row is the long exact sequence (A.7) for singular homology.

Note that the theorem is trivial for K^0. If one can show that the map (A.23) is an isomorphism for every $k > 0$, the theorem would follow immediately by applying the Five Lemma to the diagram (A.24) and induction on n. In view of Lemma A.3 and Remark A.7, it therefore remains to prove that

$$(\bar{\iota}_k)_* : H_k^{\text{sim}}(K^n, K^{n-1}) \to H_k(K^n / K^{n-1})$$

is an isomorphism for every $k > 0$. Here $(\bar{\iota}_k)_*$ is the composition of $(\iota_k)_*$ and the canonical isomorphisms in Lemma A.3 and Remark A.7.

(iii) It is easily seen from definition that $H_n^{\text{sim}}(K^n, K^{n-1})$ is the free Λ-module generated by the n-simplices of K and

$$H_k^{\text{sim}}(K^n, K^{n-1}) = 0$$

for $k \neq n$. On the other hand, K^n / K^{n-1} is homeomorphic to the topological wedge sum

$$\bigvee_{s:n\text{-simplex}} [s]/\partial[s].$$

It follows from (A.18) that

$$\begin{aligned} H_k^{\mathrm{sim}}(K^n, K^{n-1}) &\cong \bigoplus_{s:n\text{-simplex}} H_k([s]/\partial[s]) \\ &\cong \bigoplus_{s:n\text{-simplex}} H_k([s], \partial[s]) \cong \begin{cases} \bigoplus_{s:n\text{-simplex}} \Lambda, & \text{if } k = n; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore,

$$H_k^{\mathrm{sim}}(K^n, K^{n-1}) \cong H_k(K^n/K^{n-1}) = 0$$

if $k \neq n$. Under the aforementioned isomorphisms, the map $(\bar{\iota}_n)_*$ sends a generator of $H_n^{\mathrm{sim}}(K^n, K^{n-1})$ (an n-simplex $[s]$) to a generator of $H_n(K^n/K^{n-1})$ (represented by the same n-simplex as the generator of $H_n([s], \partial[s])$). This proves that $(\bar{\iota}_n)_*$ is an isomorphism.

□

Remark A.8. The isomorphism between the two cohomologies are obtained in a similar way. This follows immediately from duality in the case when Λ is a field.

Appendix B Differential geometry

In this appendix, we review basic concepts and tools from differential geometry that are used in the present notes. The main references for this part are [CCL00, KN63, War83]. We always assume that M is an n-dimensional differentiable manifold.

B.1 The Lie bracket and Lie derivative

The tangent (respectively, cotangent) bundle over M is denoted as TM (respectively, T^*M). Smooth sections of TM are called (*smooth vector fields*) on M and the space of vector fields is denoted as $\Gamma(TM)$. A vector field X induces a first order differential operator (directional derivative along X) $X : C^\infty(M) \to C^\infty(M)$ in the following way. Let $f \in C^\infty(M)$. Given $x \in M$, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve such that $\gamma_0 = x$ and $\dot{\gamma}_0 = X_x$. Then one defines

$$(Xf)(x) \triangleq \frac{d}{dt}\Big|_{t=0} f(\gamma_t).$$

This above expression does not depend on the choice of the curve γ_t representing the vector X_x. The operator X satisfies

$$X(fg) = fXg + gXf \tag{B.1}$$

for all $f, g \in C^\infty(M)$. It can be shown that every linear operator on $C^\infty(M)$ satisfying (B.1) arises from a vector field.

Definition B.1. The *Lie bracket* between two vector fields X, Y is the vector field $[X, Y]$ corresponding to the first order differential operator $XY - YX$.

The Lie bracket is consistent with smooth maps between manifolds. More precisely, let $F : M \to N$ be a smooth map between manifolds M and N. Let X, Y (respectively, $\tilde{X}, \tilde{Y}$) be vector fields on M (respectively, on N). Suppose that

$$\tilde{X}_{F(x)} = (dF)_x X_x, \ \tilde{Y}_{F(x)} = (dF)_x Y_x$$

for all $x \in M$. Then one has

$$[\tilde{X}, \tilde{Y}]_{F(x)} = (dF)_x [X, Y]_x \tag{B.2}$$

for all $x \in M$.

There is a notion of differentiation for tensor fields which relies only on the differential structure: the Lie derivative. Recall that an (r, s)-tensor field ξ is a (smooth) section of the tensor bundle

$$T^r_s M \triangleq \underbrace{TM \otimes \cdots \otimes TM}_{r} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{s}. \tag{B.3}$$

In other words, ξ_x is an element of $(T_xM)^{\otimes r} \otimes (T_x^*M)^{\otimes s}$ for each $x \in M$ and $x \mapsto \xi_x$ is smooth.

Let X be a vector field on M. The ODE

$$\frac{d\varphi_t(x)}{dt} = X(\varphi_t(x)), \;\; \varphi_0(x) = x \tag{B.4}$$

induces a flow of (local) diffeomorphisms

$$U \times (-\varepsilon, \varepsilon) \ni (y,t) \mapsto \varphi_t(y) \in M$$

where U is a neighbourhood of any $x \in M$ and ε is small. If M a closed manifold, the ODE (B.4) admits a unique solution for all time. In this case, $\{\varphi_t : t \in \mathbb{R}\}$ is a flow of (global) diffeomorphisms.

Now let ξ be an (r,s)-tensor field and let X be a vector field on M. Given $x \in M$, we define $(\Phi_t^*\xi)_x$ to be the pullback of ξ along the flow of X. More precisely, for monomials

$$\xi = X_1 \otimes \cdots \otimes X_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s \tag{B.5}$$

with $X_i \in \Gamma(TU), \alpha^j \in \Gamma(T^*U)$ on some neighbourhood U of x, we define

$$\begin{aligned}(\Phi_t^*\xi)_x =& (d\varphi_{-t})_{\varphi_t(x)}\big((X_1)_{\varphi_t(x)}\big) \otimes \cdots \otimes (d\varphi_{-t})_{\varphi_t(x)}\big((X_r)_{\varphi_t(x)}\big) \\ &\otimes \varphi_t^*\big(\alpha^1_{\varphi_t(x)}\big) \otimes \cdots \otimes \varphi_t^*\big(\alpha^s_{\varphi_t(x)}\big)\end{aligned} \tag{B.6}$$

where $\{\varphi_t : t \in (-\varepsilon, \varepsilon)\}$ is the flow of (local) diffeomorphisms induced by X near x. The definition (B.6) extends linearly to a general ξ which can always be expressed as a linear combination of monomials (B.5).

Definition B.2. The *Lie derivative* of ξ with respect to X is the (r,s)-tensor field defined by

$$(\mathcal{L}_X\xi)_x \triangleq \lim_{t \to 0} \frac{(\Phi_t^*\xi)_x - \xi_x}{t}, \quad x \in M.$$

For $f \in C^\infty(M)$, one has $\mathcal{L}_X f = Xf$. For a vector field Y, one has $\mathcal{L}_X Y = [X,Y]$. In general, the Lie derivative is characterised by the Leibniz rules

$$(\mathcal{L}_X\alpha)(Y) = X(\alpha(Y)) - \alpha(\mathcal{L}_X Y),$$

$$\mathcal{L}_X(\xi \otimes \eta) = (\mathcal{L}_X\xi) \otimes \eta + \xi \otimes (\mathcal{L}_X\eta)$$

for $\alpha \in \Gamma(T^*M)$, $X, Y \in \Gamma(TM)$ and tensor fields ξ, η. In contrast to the covariant derivative which will be introduced later on, the Lie derivative is *not* tensorial in X; the tensor $(\mathcal{L}_X\xi)_x$ depends on the values of X near x.

B.2 Exterior algebra

Let V be a finite dimensional vector space over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. The *k-th exterior power* of V is the vector subspace of the tensor product space $V^{\otimes k}$ defined by $\Lambda^k V \triangleq A_k(V^{\otimes k})$. Here $A_k : V^{\otimes k} \to V^{\otimes k}$ is the antisymmetrisation induced by

$$A_k(v_1 \otimes \cdots \otimes v_k) \triangleq \frac{1}{k!}\sum_{\sigma \in \mathcal{S}_k} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)},$$

where $\mathcal{S}_k$ denotes the permutation group of order k. By identifying $V^{\otimes k}$ with the space $\mathcal{L}^k(V^*)$ of k-linear functionals on V^* (V^* is the dual of V), elements of $\Lambda^k V$ are precisely the alternating ones, i.e. those $\xi \in \mathcal{L}^k(V^*)$ satisfying

$$\xi(v^{*1}, \cdots, v^{*k}) = 0, \quad v^{*1}, \cdots, v^{*k} \in V^*$$

whenever $v^{*i} = v^{*j}$ for some $i \neq j$. As a convention, we set $\Lambda^0 V \triangleq \mathbb{F}$.

Definition B.3. Let $\xi \in \Lambda^k V$ and $\eta \in \Lambda^l V$. Their *exterior product* is defined by

$$\xi \wedge \eta \triangleq \frac{(k+l)!}{k!l!} A_{k+l}(\xi \otimes \eta) \in \Lambda^{k+l} V.$$

The exterior product is associative and satisfies the following relation:

$$\xi \wedge \eta = (-1)^{kl} \eta \wedge \xi \tag{B.7}$$

for $\xi \in \Lambda^k V$ and $\eta \in \Lambda^l V$. The *exterior algebra* over V is the $\mathbb{F}$-algebra defined by

$$\Lambda V \triangleq \bigoplus_{k \geqslant 0} \Lambda^k V$$

where the product structure is the exterior product extended bilinearly to ΛV.

Suppose that $\dim V = n$ and let $\{\varepsilon_1, \cdots, \varepsilon_n\}$ be a basis of V. Then $\Lambda^k V = \{0\}$ for all $k > n$. For $1 \leqslant k \leqslant n$, the family

$$\{\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_k} : 1 \leqslant i_1 < \cdots < i_k \leqslant n\}$$

provides a basis of $\Lambda^k V$. In particular, ΛV is a 2^n-dimensional $\mathbb{F}$-vector space.

Now let $F : V \to W$ be an $\mathbb{F}$-linear map. By identifying $\Lambda^k V^*$ with the space of alternating k-linear functionals on V, the map F induces a linear map $F^* : \Lambda^k W^* \to \Lambda^k V^*$ which is defined by

$$(F^*\xi)(v_1, \cdots, v_k) \triangleq \xi(F(v_1), \cdots, F(v_k)) \tag{B.8}$$

for $\xi \in \Lambda^k W^*$ and $v_1, \cdots, v_k \in V$.

B.3 Exterior derivative and differential forms

From now on, we take $\mathbb{F} = \mathbb{R}$. The exterior algebra bundle over M is defined by

$$\Lambda T^*M \triangleq \bigcup_{x \in M} \Lambda T_x^*M = \{(x, \xi) : x \in M, \xi \in \Lambda T_x^*M\}.$$

The space ΛT^*M admits a canonical differential structure which makes it into a $(2^n + n)$-dimensional manifold. It is also a vector bundle over M with rank 2^n (see Definition B.14 for the concept of a vector bundle). Similarly, one can also define the k-th exterior bundle $\Lambda^k T^*M$ which is the subbundle of ΛT^*M consisting of exterior covectors of order k.

Definition B.4. A (*differential*) *k-form* on M is a smooth section of $\Lambda^k T^*M$. In other words, it is an assignment

$$\alpha : M \ni x \mapsto \alpha_x \in \Lambda^k T_x^*M$$

which depends on x smoothly. The space of k-forms is denoted as $\Omega^k(M)$. As a convention, a 0-form is a smooth function on M. A *differential form* on M is a smooth section of ΛT^*M. The space of differential forms is denoted as $\Omega(M)$.

Through the duality perspective, a k-form is an alternating k-linear map

$$\alpha : \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{k} \to C^\infty(M)$$

which is $C^\infty(M)$-linear on each component. For a general differential form α, by definition one can write

$$\alpha = \alpha_0 + \alpha_1 + \cdots + \alpha_n$$

where α_k is a k-form. One can define the exterior product between two differential forms α, β by taking exterior product fiberwise, i.e. $(\alpha \wedge \beta)_x \triangleq \alpha_x \wedge \beta_x$ for $x \in M$. It is clear that $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$. In addition, it is associative and satisfies the relation (B.7) on forms.

Theorem B.1. *There exists a unique linear operator $d : \Omega(M) \to \Omega(M)$ which satisfies the following properties.*

(i) $d(\Omega^k(M)) \subseteq \Omega^{k+1}(M)$.

(ii) For $f \in C^\infty(M)$, df is the differential of f.

(iii) For any $\alpha \in \Omega^k(M)$ and $\beta \in \Omega(M)$, one has

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

(iv) $d^2 f = 0$ for all $f \in C^\infty(M)$.

Definition B.5. The operator d given by Theorem B.1 is called the *exterior derivative operator* on M.

The exterior derivative operator satisfies $d^2 = 0$. From the duality perspective, it is characterised by the relation that

$$\begin{aligned}(d\alpha)(X_0, \cdots, X_k) = &\sum_{i=0}^{k}(-1)^i X_i(\alpha(X_0, \cdots, \hat{X}_i, \cdots, X_k)) \\ &+ \sum_{0\leqslant i<j\leqslant k}(-1)^{i+j}\alpha([X_i, X_j], X_0 \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_k)\end{aligned}$$

for any $\alpha \in \Omega^k(M)$ and $X_0, \cdots, X_k \in \Gamma(TM)$. In particular,

$$(d\alpha)(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) \tag{B.9}$$

for any 1-form α and smooth vector fields X, Y.

Suppose that $F : M \to N$ is a smooth map between two manifolds. For each $k \in \mathbb{N}$, it induces a map (pullback of differential forms) $F^* : \Omega^k(N) \to \Omega^k(M)$ which is defined through (B.8) in a fiberwise manner, namely,

$$(F^*\alpha)_x(v_1, \cdots, v_k) \triangleq \alpha_{F(x)}((dF)_x v_1, \cdots, (dF)_x v_k), \quad x \in M, v_i \in T_x M. \tag{B.10}$$

The map F^* is consistent with the exterior product and the exterior derivative:

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta), \ F^*(d\alpha) = d(F^*\alpha)$$

for all $\alpha, \beta \in \Omega(N)$.

B.4 Interior product and Cartan's formula

The Lie derivative $\mathcal{L}_X$ restricts to a differential operator $\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$ on forms, which can be computed explicitly through *Cartan's magic formula.* To state this formula, we first recall the definition of the interior product.

Definition B.6. Let $X \in \Gamma(TM)$. The *interior product* by X is the unique linear operator $\iota(X) : \Omega^*(M) \to \Omega^{*-1}(M)$ which satisfies the following properties.

(i) $\iota(X)\alpha = \alpha(X)$ for $\alpha \in \Omega^1(M)$.

(ii) For any $\alpha \in \Omega^k(M)$ and $\beta \in \Omega(M)$, one has

$$\iota(X)(\alpha \wedge \beta) = (\iota(X)\alpha) \wedge \beta + (-1)^k \alpha \wedge (\iota(X)\beta).$$

As a convention, we set $\iota(X)f = 0$ for $f \in C^\infty(M)$.

From a duality perspective, the interior product is given by

$$(\iota(X)\alpha)(Y_1, \cdots, Y_{k-1}) = \alpha(X, Y_1, \cdots, Y_{k-1}) \tag{B.11}$$

for $\alpha \in \Omega^k(M)$ and $Y_1, \cdots, Y_{k-1} \in \Gamma(TM)$. It is easily seen from (B.11) that $\iota(X)$ acts fiberwise. Cartan's formula for the Lie derivative is stated as follows.

Proposition B.1. *$\mathcal{L}_X = d \circ \iota(X) + \iota(X) \circ d$ for all $X \in \Gamma(TM)$.*

B.5 Partition of unity and good cover

A basic tool in differential geometry is partition of unity. It often allows one to patch local constructions to the global.

Definition B.7. A *partition of unity* on M is a family $\{\varphi_i : i \in \mathcal{I}\}$ of smooth functions which satisfy the following properties.

(i) $0 \leqslant \varphi_i \leqslant 1$ for all $i \in \mathcal{I}$.

(ii) The family $\{\mathrm{supp}\varphi_i : i \in \mathcal{I}\}$ is locally finite in the sense that for any $x \in M$, there exists some neighborhood U of x which intersects at most finitely many $\mathrm{supp}\varphi_i$'s.

(iii) $\sum_{i \in I} \varphi_i(x) = 1$ for all $x \in M$.

The following theorem contains two versions of the existence of a partition of unity. Both versions are useful depending on the context.

Theorem B.2. *Let $\{U_\alpha : \alpha \in \mathcal{A}\}$ be an open cover of M.*

(i) There exists a countable partition of unity $\{\varphi_i : i \in \mathbb{N}\}$ such that each $\mathrm{supp}\varphi_i$ is compact and contained in $U_{\alpha(i)}$ for some $\alpha(i) \in \mathcal{A}$. We say that $\{\varphi_i\}$ is subordinated to the cover $\{U_\alpha\}$.

(ii) There exists a partition of unity $\{\varphi_\alpha : \alpha \in \mathcal{A}\}$ such that $\mathrm{supp}\varphi_\alpha \subseteq U_\alpha$ for every α and there are at most countably many nonzero φ_α's among them. We say that $\{\varphi_\alpha\}$ is subordinated to the cover $\{U_\alpha\}$ with the same index.

Another useful fact for localisation is the existence of a good (open) cover for any smooth manifold M. Recall that a *good cover* of M is a (countable) family $\{U_\alpha : \alpha \in \mathcal{A}\}$ of open sets such that $M = \cup_{\alpha \in \mathcal{A}} U_\alpha$ and any nonempty finite intersection $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ among them is diffeomorphic to $\mathbb{R}^n$. The following result can be found in [BT82].

Theorem B.3. *Any smooth manifold M admits a good cover. If M is compact, one can choose a good cover consisting of finitely many open sets.*

B.6 Integration and Stokes' theorem

Top forms can be integrated on an oriented manifold. We first recall the definition of orientability.

Definition B.8. An n-manifold is said to be *orientable* if there exists an everywhere nonvanishing n-form $\omega \in \Omega^n(M)$. Once such an ω is given, we say that M is *oriented* by ω (or ω defines an *orientation* on M).

An orientation ω induces an orientation on each tangent space T_xM; indeed, a basis $\{\varepsilon_1, \cdots, \varepsilon_n\}$ of T_xM is claimed to be *positive* if $\omega(\varepsilon_1, \cdots, \varepsilon_n)_x > 0$. An equivalent definition of orientability is the existence of an atlas (i.e. an open cover consisting of local charts) on M such that the Jacobian of coordinate transformation between different charts always has positive determinant. Such an atlas is called an *oriented atlas* and charts in it are called *oriented charts*.

Remark B.1. The notion of orientability extends naturally to the case with boundary. In this case, an oriented atlas is defined in the same way but with charts now being homeomorphic to open subsets of the upper half space

$$\mathbb{R}^n_+ \triangleq \{(x^1, \cdots, x^n) : x^n \geqslant 0\}.$$

Next, we also need to recall the definition of a regular domain.

Definition B.9. An open subset $D \subseteq M$ is called a *regular domain* if for any $x \in \partial D \triangleq \bar{D} \backslash D$, there exists a local chart (U, φ) around x such that $\varphi(x) = 0$ and

$$\varphi(U \cap \bar{D}) = \varphi(U) \cap \mathbb{R}^n_+.$$

Let M be an oriented n-manifold and let D be a regular domain. The definition of the integral $\int_D \alpha$ for any *compactly supported* n-form α is summarised as follows.

(i) Suppose for now $\text{supp}\alpha$ is contained in some oriented chart $(U, \varphi; x^i)$. One defines

$$\int_D \alpha \triangleq \int_{\varphi(U \cap D)} a dx^1 \cdots dx^n,$$

where $\alpha = a dx^1 \wedge \cdots \wedge dx^n$ with some $a \in C_c^\infty(U)$.

(ii) In general, let $\mathcal{U} = \{U_i : i \in \mathcal{I}\}$ be an oriented atlas of M and let $\{\varphi_i : i \in \mathcal{I}\}$ be a partition of unity subordinated to $\mathcal{U}$. One then defines

$$\int_D \alpha \triangleq \sum_{i \in \mathcal{I}} \int_D \varphi_i \alpha, \tag{B.12}$$

where $\int_D \varphi_i \alpha$ is defined by (i) since $\varphi_i \alpha$ is now compactly supported in the oriented chart U_i.

Definition B.10. The real number defined by (B.12) is called the integral of α over D.

An basic result about the integration of forms is *Stokes' theorem*. Before stating this theorem, we first need to recall the notion of induced boundary orientation. Let D be a regular domain. By Definition B.9, a local chart of M around each $x \in \partial D$ restricts to a chart of ∂D around x by setting $x^n = 0$. By varying $x \in \partial D$, this gives rise to an atlas (hence a differential structure) on ∂D which makes it into an $(n-1)$-dimensional manifold. In addition, the inclusion $i : \partial D \to M$ is an embedded submanifold. Now suppose that M is oriented. By requiring the aforementioned charts around each $x \in \partial D$ to be consistent with the M-orientation, one obtains an oriented atlas (hence an orientation) on ∂D. This is known as the induced orientation on ∂D from D.

Stokes' theorem is stated as follows. Note that if ω is a compactly supported $(n-1)$-form on M, the pullback $i^*\omega$ by the inclusion is a compactly supported $(n-1)$-form on ∂D.

Theorem B.4 (Stokes' theorem). *Let α be a compactly supported $(n-1)$-form on M. Then one has*

$$\int_D d\alpha = \int_{\partial D} i^*\alpha$$

for any regular domain D.

A typical situation is when M is a compact manifold without boundary. In this case, by taking $D = M$ (so $\partial D = \emptyset$) one concludes that

$$\int_M d\alpha = 0$$

for any $\alpha \in \Omega^{n-1}(M)$.

Remark B.2. If the regular domain D is compact, there is no need to assume that α is compactly supported for the definition of $\int_D \alpha$ as well as Stokes' theorem to be valid.

B.7 The de Rham cohomology

A k-form $\alpha \in \Omega^k(M)$ is said to be

(i) *closed* if $d\alpha = 0$;

(ii) *exact* if $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(M)$.

Let $Z^k(M)$ (respectively, $B^k(M)$) denote the space of closed (respectively, exact) k-forms. As a convention, we set $B^0(M) \triangleq \{0\}$. Since $d^2 = 0$, it is obvious that exact forms are closed (i.e. $B^k(M) \subseteq Z^k(M)$). However, the converse needs not be true in general; in fact, the discrepancy between closed and exact forms encode topological information about the manifold M which motivates the definition of the de Rham cohomology.

Definition B.11. For each $k \in \mathbb{N}$, the k-th *de Rham cohomology group with real coefficients* is the real vector space $H^k_{\mathrm{dR}}(M)$ defined by

$$H^k_{\mathrm{dR}}(M) \triangleq \frac{Z^k(M)}{B^k(M)} = \frac{\ker\big[d : \Omega^k(M) \to \Omega^{k+1}(M)\big]}{\mathrm{Im}\big[d : \Omega^{k-1} \to \Omega^k(M)\big]}.$$

Elements of $H^k_{\mathrm{dR}}(M)$ are called *de Rham cohomology classes* of degree k.

Remark B.3. One can define the de Rham cohomology group $H^k_{\mathrm{dR}}(M;\mathbb{C})$ with complex coefficients in the same way by replacing the real-valued forms with complex-valued ones. It is apparent that $H^k_{\mathrm{dR}}(M;\mathbb{C}) = H^k_{\mathrm{dR}}(M) \otimes_{\mathbb{R}} \mathbb{C}$.

The following result has been used implicitly for several times in the main text.

Proposition B.2. *Suppose that M is a compact manifold without boundary. Then* $\dim H^k_{\mathrm{dR}}(M)$ *is finite dimensional for every k.*

One often considers the *de Rham cohomology ring*

$$H^*_{\mathrm{dR}}(M) \triangleq \bigoplus_{k \geqslant 0} H^k_{\mathrm{dR}}(M),$$

where the sum indeed truncates at $\dim M$ since $H^k_{\mathrm{dR}}(M) = \{0\}$ for all $k > \dim M$. As suggested by its name, H^*_{dR} admits a ring structure induced by the exterior product on forms:

$$a \cdot b \triangleq [\alpha \wedge \beta]$$

for $a = [\alpha]$ and $b = [\beta]$ in $H^*_{\mathrm{dR}}(M)$. According to (B.7), one has

$$a \cdot b = (-1)^{kl} b \cdot a$$

for $a \in H^k_{\mathrm{dR}}(M)$ and $b \in H^l_{\mathrm{dR}}(M)$.

A smooth map $F : M \to N$ induces a degree-preserving ring homomorphism $F^* : H^*_{\mathrm{dR}}(N) \to H^*_{\mathrm{dR}}(M)$ by setting $F^*[\alpha] \triangleq [F^*\alpha]$, where $F^*\alpha$ is the pullback of forms defined by (B.10).

A basic property of the de Rham cohomology is that it is a (smooth) homotopy invariant. We first recall the following definition.

Definition B.12. Two smooth maps $f, g : M \to N$ between manifolds M, N are *smoothly homotopic*, which is denoted as $f \simeq g$, if there exists a smooth map $F : M \times (-\varepsilon, 1+\varepsilon) \to N$ with some $\varepsilon > 0$ such that $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$. Two manifolds M, N are *smoothly homotopic* if there exist smooth maps $f : M \to N$ and $g : N \to M$ such that $g \circ f \simeq \mathrm{id}_M$ and $f \circ g \simeq \mathrm{id}_N$.

Theorem B.5. *Let $f, g : M \to N$ be two smooth maps between manifolds M, N. Suppose that f, g are smoothly homotopic (i.e. there exists a smooth map $F : M \times (-\varepsilon, 1+\varepsilon) \to N$ such that $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$). Then $f^* = g^* : H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$ for all $k \geqslant 0$. As a consequence, the de Rham cohomology rings of two manifolds that are smoothly homotopic are isomorphic.*

Most of the results for the singular cohomology have their counterparts for the de Rham cohomology. This is not surprising from the perspective of the de Rham theorem (see Theorem 4.5). One has seen a few such examples in the main text (e.g. Poincaré duality, Thom isomorphism etc.). We conclude this section with a useful tool for computing the de Rham cohomology: the Mayer-Vietoris sequence. This is the differential counterpart for the same result in singular cohomology.

Let U, V be open subsets of M such that $U \cup V = M$. There is a short exact sequence of cochain complexes (of differential forms):

$$0 \to \Omega^*(M) \xrightarrow{i^*} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j^*} \Omega^*(U \cap V) \to 0. \tag{B.13}$$

Here the maps i^*, j^* are defined by

$$i^*(\omega) \triangleq (\omega|_U, \omega|_V), \quad j^*(\omega, \tau) \triangleq (\tau|_{U\cap V} - \omega|_{U\cap V}).$$

According to the Snake Lemma (Lemma A.1), the sequence (B.13) induces a long exact sequence

$$\begin{aligned} 0 \to H^0_{\mathrm{dR}}(M) \to H^0_{\mathrm{dR}}(U) \oplus H^0_{\mathrm{dR}}(V) \to \cdots \to H^{k-1}_{\mathrm{dR}}(U \cap V) \\ \to H^k_{\mathrm{dR}}(M) \to H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \to H^k_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta^k} H^{k+1}_{\mathrm{dR}}(M) \to \cdots \end{aligned} \tag{B.14}$$

on cohomology. The connecting homomorphism δ^k can be constructed in the following explicit way. Let $\{\varphi_U, \varphi_V\}$ be a partition of unity subordinated to the cover $\{U, V\}$ (Theorem B.2 (ii)). Let $[\omega] \in H^k_{\mathrm{dR}}(U \cap V)$. The pair of forms

$$(-d(\varphi_V \omega), d(\varphi_U \omega)) \in \Omega^{k+1}(U) \oplus \Omega^{k+1}(V)$$

agree on $U \cap V$ and hence they patch to a global form $\tau \in \Omega^{k+1}(U \cup V)$. Then $\delta^k[\omega] = [\tau]$. Note that τ is supported in $U \cap V$.

B.8 Sard's theorem

We recall a classical result in differential topology which is used in the proof of Lemma 2.4. Recall that a subset A of a manifold N is *a null set* if $\varphi(A \cap U)$ has zero Lebesgue measure in $\varphi(U)$ for any coordinate chart (U, φ).

Definition B.13. Let $F : M \to N$ be a smooth map between two manifolds M, N. A point $x \in M$ is a *critical point* of F if $\operatorname{rank}((dF)_x) < \dim N$. A point $y \in N$ is a *critical value* of F if $y = F(x)$ for some critical point $x \in M$. A point $y \in N$ is a *regular value* of F if it is not a critical value.

Theorem B.6 (Sard's Theorem)**.** *Let $F : M \to N$ be a smooth map between two manifolds M, N. Then the set of critical values of F is a null set in N.*

B.9 Connections on vector bundles

We begin with the definition of a vector bundle. Let V be a finite dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition B.14. A $\mathbb{K}$-*vector bundle with fiber V* consists of a triple $(E, M, \pi; V)$ which satisfies the following properties.

(i) E, M are both manifolds and $\pi : E \to M$ is a smooth surjective submersion.

(ii) The fiber $E_x \triangleq \pi^{-1}(x)$ is a $\mathbb{K}$-vector space for every $x \in M$.

(iii) There exists an open cover $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ of M (a *local trivialisation*) such that the following properties hold true.

(a) On each U_α, there is a diffeomorphism

$$\psi_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times V, \ \psi_\alpha(u) = (\pi(u), \varphi_\alpha(u))$$

such that the map $\varphi_\alpha|_{E_x} : E_x \to V$ is a $\mathbb{K}$-linear isomorphism for every $x \in U_\alpha$.

(b) On each overlap $U_{\alpha\beta} \triangleq U_\alpha \cap U_\beta$ (provided this is nonempty), the map

$$\psi_\beta \circ \psi_\alpha^{-1} : U_{\alpha\beta} \times V \to U_{\alpha\beta} \times V$$

is of the form

$$\psi_\beta \circ \psi_\alpha^{-1}(x, v) = (x, \rho_{\beta\alpha}(x)v),$$

where $\rho_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}(V; \mathbb{K})$ is a smooth map.

The bundle E is called the *total space*, the manifold M is called the *base manifold* and the map π is called the *bundle projection*. The dimension of V is called the *rank* of E. The family of maps $\{\rho_{\beta\alpha}\}$ in Definition B.14 (iii) are called the *transition functions* with respect to the local trivialisation $\mathcal{U}$. They satisfy the following consistency conditions:

$$\rho_{\gamma\alpha}(x) = \rho_{\gamma\beta}(x)\rho_{\beta\alpha}(x), \quad x \in U_\alpha \cap U_\beta \cap U_\gamma \tag{B.15}$$

provided that the above intersection is nonempty.

Given an open cover $\mathcal{U} = \{U_\alpha\}$ of M and a family of transition functions $\rho_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}(V;\mathbb{K})$ satisfying (B.15), one can construct a vector bundle in the following way. Define the total space

$$E \triangleq \Big(\bigsqcup_{\alpha} (\{\alpha\} \times U_\alpha \times V) \Big)_{\sim}$$

where $\sqcup$ means disjoint union and the equivalence relation $\sim$ is defined by

$$(\alpha, x, v) \sim (\beta, y, w) \iff y = x, w = \rho_{\beta\alpha}(x)v.$$

The space E admits a natural differential structure. In addition, with the projection π defined in the obvious way, $(E, M, \pi; V)$ is a vector bundle over M with fiber V. The family $\{U_\alpha, \psi_\alpha, \rho_{\beta\alpha}\}$ with $\psi_\alpha([(\alpha, x, v)]) \triangleq (x, v)$ is a local trivialisation of this bundle.

Definition B.15. A (*smooth*) *section* of a vector bundle E is a smooth map $s : M \to E$ such that $s(x) \in E_x$ for every $x \in M$. The space of sections of E is denoted as $\Gamma(E)$.

Example B.1. Both the tangent bundle TM and the exterior algebra bundle ΛT^*M are vector bundles over M with rank n and 2^n respectively ($n = \dim M$). Sections of TM are vector fields and sections of ΛT^*M are differential forms.

Definition B.16. A *Euclidean* (respectively, *Hermitian*) *metric* on a real (respectively, complex) vector bundle E is a fiberwise Euclidean (respectively, Hermitian) inner product $\langle \cdot, \cdot \rangle_x$ on each fiber E_x that varies smoothly in x. A real vector bundle E is *orientable* if there exists a local trivialisation whose transition functions all have positive determinants. An *orientation* on E is a choice of such a local trivialisation. This also gives rise to a consistent way of assigning to each vector space E_x ($x \in M$) an orientation in the vector space sense.

Suppose V, W are both finite dimensional $\mathbb{K}$-vector spaces. One can define the spaces

$$V^* : \text{dual space},\ \mathrm{End}(V) : \text{endomorphisms over } V,\ V \oplus W : \text{direct sum},$$

$$V \otimes W : \text{tensor product},\ \mathrm{End}(V, W) : \text{endomorphisms from } V \text{ to } W.$$

Carrying these fiberwise to a vector bundle, one can construct the following new bundles from the old:

$$E^* \triangleq \bigsqcup_{x \in M} E_x^*,\ \mathrm{End}(E) \triangleq \bigsqcup_{x \in M} \mathrm{End}(E_x), E \oplus F \triangleq \bigsqcup_{x \in M} E_x \oplus F_x, \tag{B.16}$$

$$E \otimes F \triangleq \bigsqcup_{x \in M} E_x \otimes F_x,\ \mathrm{End}(E, F) \triangleq \bigsqcup_{x \in M} \mathrm{End}(E_x, F_x). \tag{B.17}$$

It is routine to justify that they are vector bundles and to write down the corresponding transition functions from the ones on E, F. An example we saw before is the (r, s)-tensor bundle $T^r_s M$ (see (B.3)).

Another useful construction is the pullback of a vector bundle. Let $(E, M, \pi; V)$ be a vector bundle and let $f : N \to M$ be a smooth map. One can define a new vector bundle by

$$f^*E \triangleq \{(n, u) \in N \times E : f(n) = \pi(u)\}. \tag{B.18}$$

This is a vector bundle with base manifold N whose fiber at each $n \in N$ is $E_{f(n)}$. The bundle f^*E is called the *pullback of E by f*.

Let $(E, M, \pi; V)$ be a vector bundle. An *E-valued k-form* on M is a smooth assignment $M \ni x \mapsto \alpha_x$ where α_x is an E_x-valued alternating k-linear map on $T_x M$. The space of E-valued k-forms on M is denoted as $\Omega^k(M, E)$. The space of E-valued forms (with mixed degrees) is denoted as $\Omega(M, E)$.

Next, we give the definition of a connection on a vector bundle from the covariant derivative viewpoint (an intrinsic way of differentiating sections). A deeper and more fundamental approach to connection theory is through the perspective of principal bundles, which is developed in Section 6.5 in the main text.

Definition B.17. A *connection* (or a *covariant derivative*) on a vector bundle $(E, M, \pi; V)$ is a linear operator

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E),\ (X, s) \mapsto \nabla_X s$$

which satisfies the following three properties:

(i) $\nabla_{(fX+Y)} s = f \nabla_X s + \nabla_Y s$;

(ii) $\nabla_X (s + t) = \nabla_X s + \nabla_X t$;

(iii) $\nabla_X (fs) = (Xf)s + f \nabla_X s$

for any $X, Y \in \Gamma(TM)$, $s, t \in \Gamma(E)$ and $f \in C^\infty(M)$. One can equivalently view

$$\nabla : \Gamma(E) \to \Gamma\big(T^*M \otimes E\big) = \Omega^1(M, E) \tag{B.19}$$

by setting $\langle \nabla s, X \rangle \triangleq \nabla_X s$.

The value of $\nabla_X s$ at $x \in M$ depends only on the value of X_x and the values of s near x (in fact, along the direction of X_x). One can therefore define a notion of covariant derivative along a smooth curve in M. More precisely, let $\gamma : I \to M$ be a smooth curve and let s be a smooth section on γ (i.e. $s_t \in E_{\gamma_t}$ which varies smoothly in t). The covariant derivative of s along γ is defined by

$$\frac{Ds}{dt} \triangleq (\nabla_{\dot{\gamma}_t} s)(t).$$

A section s on γ is said to be *parallel* if $\frac{Ds}{dt} = 0$ for all t. This equation is essentially a first order linear ODE. Fix $t_0 \in I$ and $v \in E_{t_0}$. As a result, there exists a unique parallel section s on γ such that $s_{t_0} = v$.

Definition B.18. The parallel section s defined as above is called the *parallel transport* of v along γ.

Now we define the curvature of a connection.

Definition B.19. The *curvature tensor* of a connection ∇ is the map

$$F : \Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

$$F(X,Y)s \triangleq \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \tag{B.20}$$

for $X, Y \in \Gamma(TM)$ and $s \in \Gamma(E)$.

It can be shown that F is tensorial in all components (i.e. $F(fX, gY)(hs) = fghF(X,Y)s$ for any smooth functions f, g, h). In addition, it is antisymmetric in (X,Y). As a result, one can regard $F \in \Omega^2(M, \mathrm{End}(E))$.

In practice, one often works with local representations of connection and curvature. A *local frame field* of E is a family $\{e_1, \cdots, e_r\}$ of smooth sections on some open subset $U \subseteq M$ such that $\{e_1(x), \cdots, e_r(x)\}$ is a basis of E_x for each $x \in U$. A local trivialisation $\psi : \pi^{-1}U \to U \times V$ gives rise to a local frame field on U by setting $e_i(x) \triangleq \psi^{-1}(x, \varepsilon_i)$ where $\{\varepsilon_1, \cdots, \varepsilon_r\}$ is a fixed basis of V.

Definition B.20. The *local connection matrix* of ∇ with respect to $\{e_1, \cdots, e_r\}$ is the $\mathfrak{gl}(r; \mathbb{K})$-valued 1-form $A = (A_i^j)_{1 \leqslant i,j \leqslant r}$ on U defined by the relation $\nabla e_i = A_i^j \otimes e_j$. The *local curvature matrix* is the $\mathfrak{gl}(r; \mathbb{K})$-valued 2-form $F = (F_i^j)_{1 \leqslant i,j \leqslant r}$ on U such that $\big(F_i^j(X,Y)_x\big)$ is the matrix of the linear transform $F(X,Y)_x \in \mathrm{End}(E_x)$ with respect to $\{e_1, \cdots, e_r\}$ at every $x \in U$, namely,

$$F(X,Y)e_i = F_i^j(X,Y)e_j$$

for all $X, Y \in \Gamma(TM)$.

The local curvature and connection matrices are related by

$$F_i^j \triangleq dA_i^j + A_i^k \wedge A_k^j,$$

or more concisely,

$$F = dA + A \wedge A$$

under matrix notation.

Suppose that E is a real vector bundle which is equipped with a Euclidean metric $\langle \cdot, \cdot \rangle$. A connection ∇ is *compatible with the metric* (or simply *metric*) if

$$X\langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle \tag{B.21}$$

for all $X \in \Gamma(TM)$ and $s_1, s_2 \in \Gamma(E)$. Under a local ONF $\{e_1, \cdots, e_r\}$, the connection matrix A and curvature matrix F both take values in $\mathfrak{so}(r)$. Similar discussion applies to the Hermitian case, in which case A and F take values in $\mathfrak{u}(r)$.

Finally, let ∇^E and ∇^F be given connections on the vector bundles E and F over M respectively. One can define the following *induced connections* on the new vector bundles (B.16, B.17).

(i) On E^*:

$$\big(\nabla_X^{E^*} s^*\big)(s) \triangleq X(s^*(s)) - s^*\big(\nabla_X^E s\big). \tag{B.22}$$

(ii) On $\mathrm{End}(E)$:

$$\big(\nabla_X^{\mathrm{End}(E)} S\big)(s) \triangleq \nabla_X^E(S(s)) - S\big(\nabla_X^E s\big).$$

(iii) On $E \oplus F$:

$$\nabla_X^{E\oplus F}(s \oplus t) \triangleq \nabla_X^E s \oplus \nabla_X^F t.$$

(iv) On $E \otimes F$:

$$\nabla_X^{E\otimes F}(s \otimes t) \triangleq \nabla_X^E s \otimes t + s \otimes \nabla_X^F t. \tag{B.23}$$

(v) On $\mathrm{End}(E, F)$:

$$\big(\nabla_X^{\mathrm{End}(E,F)} \Phi\big)(s) \triangleq \nabla_X^F(\Phi(s)) - \Phi\big(\nabla_X^E(s)\big).$$

In the above relations, $X \in \Gamma(TM)$, $s^* \in \Gamma(E^*)$, $S \in \Gamma(\mathrm{End}(E))$, $s \in \Gamma(E)$, $t \in \Gamma(F)$ and $\Phi \in \Gamma(\mathrm{End}(E, F))$.

Appendix C Riemannian geometry

In this appendix, we review some basics of Riemannian geometry that are used in the present notes. Standard references for this part are [DOC79, Pet06]. In what follows, M is an n-manifold.

C.1 Riemannian metric, the Levi-Civita connection and the Riemann curvature tensor

We begin with the definition of a Riemannian structure.

Definition C.1. A *Riemannian metric* g on M is an assignment of a positive-definite inner product g_x on each tangent space such that $x \mapsto g_x$ (as a symmetric $(0,2)$-tensor field) is smooth. A manifold equipped with a Riemannian metric is called a *Riemannian manifold.*

A Riemannian metric induces a natural notion of distance which gives rise to a metric structure on M. Let $\gamma : [a,b] \to M$ be a piecewise C^1-curve. The *length* of γ is defined by

$$L(\gamma) \triangleq \int_a^b |\dot{\gamma}_t| dt.$$

Given $x, y \in M$, their *Riemannian distance* is defined by

$$d(x,y) \triangleq \inf\{L(\gamma) : \gamma \text{ is a piecewise } C^1\text{-curve joining } x, y\}.$$

Under the Riemannian distance d, the manifold M becomes a metric space.

To discuss various essential geometric concepts, we shall first recall the following result.

Theorem C.1 (Fundamental Theorem of Riemannian Geometry)**.** *Let (M,g) be a Riemannian manifold. There exists a unique connection ∇ on the tangent bundle TM which satisfies the following two properties.*

(i) (Metric-compatibility) ∇ is compatible with the Riemannian metric g in the sense of (B.21) with $E = TM$.

(ii) (Torsion-freeness) $\nabla_X Y - \nabla_Y X = [X,Y]$ for all $X, Y \in \Gamma(TM)$.

This connection is known as the Levi-Civita connection of (M,g).

By using the formulae (B.22) and (B.23), one can use the Levi-Civita connection to induce a connection on T^*M and $T_s^r M$ respectively (still denoted as ∇). Using the convention (B.19), there is a well-defined covariant derivative operator

$$\nabla^k : T_s^r M \to T_{s+k}^r M \tag{C.1}$$

for every $k \geqslant 1$.

Similarly, the Levi-Civita connection also induces a connection on ΛT^*M (still denoted as ∇). This connection is uniquely characterised by the following three properties.

(i) $\nabla_X f = Xf$ for $f \in C^\infty(M)$ and $X \in \Gamma(TM)$

(ii) $(\nabla_X \alpha)(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y)$ for $\alpha \in \Omega^1(M)$ and $X, Y \in \Gamma(TM)$.

(iii) $\nabla_X(\alpha \wedge \beta) = (\nabla_X \alpha) \wedge \beta + \alpha \wedge (\nabla_X \beta)$ for $\alpha, \beta \in \Omega(M)$ and $X \in \Gamma(TM)$.

For a k-form α, one has

$$(\nabla_X \alpha)(Y_1, \cdots, Y_k) = X\big(\alpha(Y_1, \cdots, Y_k)\big) - \sum_{i=1}^k \alpha\big(Y_1, \cdots, \widehat{\nabla_X Y_i}, \cdots, Y_k\big)$$

for all $X, Y_i \in \Gamma(TM)$.

Definition C.2. The *Riemann curvature tensor* is the curvature tensor of the Levi-Civita connection. In other words, it is the $(1,3)$-tensor field defined by

$$R(X,Y)Z \triangleq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{C.2}$$

for $X, Y, Z \in \Gamma(TM)$, where ∇ is the Levi-Civita connection. One often uses the equivalent $(0,4)$-tensor version defined by

$$R(X,Y,Z,W) \triangleq \langle R(X,Y)Z, W\rangle$$

for $X, Y, Z, W \in \Gamma(TM)$, where $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric.

The Riemann curvature tensor satisfies the following basic symmetries:

$$R(Y,X,Z,W) = -R(X,Y,Z,W) = R(X,Y,W,Z), \tag{C.3}$$

$$R(X,Y,Z,W) = R(Z,W,X,Y), \tag{C.4}$$

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0. \tag{C.5}$$

Definition C.3. (i) Fix $x \in M$. Let σ be a two-dimensional subspace of T_xM. The *sectional curvature* of σ is defined by

$$K(\sigma) \triangleq \frac{R(u,v,v,u)}{\langle u,u\rangle_x \langle v,v\rangle_x - \langle u,v\rangle_x^2},$$

where $\{u, v\}$ is any basis of σ.

(ii) The *Ricci tensor* is the symmetric $(0,2)$-tensor field defined by

$$\mathrm{Ric}(X,Y) \triangleq \mathrm{tr}[Z \mapsto R(Z,X)Y] \tag{C.6}$$

for $X, Y \in \Gamma(TM)$.

(iii) The *scalar curvature* is the smooth function on M defined by $\mathrm{Scal} \triangleq \mathrm{tr}[\mathrm{Ric}]$.

Let $\{e_1, \cdots, e_n\}$ be a local frame field of TM on some open subset $U \subseteq M$ and let $\{\eta_1, \cdots, \eta_n\}$ be its dual. The metric tensor g admits a matrix representation $g_{ij} \triangleq \langle e_i, e_j \rangle$. We use (g^{ij}) to denote its inverse. There are uniquely defined smooth functions Γ_{ij}^k on U which satisfy $\nabla_{e_i} e_j = \Gamma_{ij}^k e_k$. These functions Γ_{ij}^k are called the *Christoffel symbols* of the Levi-Civita connection ∇ with respect to $\{e_1, \cdots, e_n\}$. The local connection matrix is then given by $A_i^j = \Gamma_{ik}^j \eta^k$. The *curvature coefficients* are the smooth functions R_{ijk}^l and R_{ijkl} on U determined by the following relations:

$$R(e_i, e_j)e_k = R_{ijk}^l e_l, \ R_{ijkl} = g_{ip} R_{jkl}^p.$$

Equivalently, one has

$$R_{ijkl} \triangleq \langle R(e_k, e_l)e_j, e_i \rangle. \tag{C.7}$$

The symmetry relations (C.3, C.4, C.5) imply that

$$R_{jikl} = -R_{ijkl} = R_{ijlk}, \ R_{ijkl} = R_{klij}, \ R_{ijkl} + R_{jkil} + R_{kijl} = 0. \tag{C.8}$$

The Ricci tensor is given by

$$\mathrm{Ric}(e_i, e_j) = g^{kl} R_{kilj}.$$

If $\{e_i\}$ is an ONF, one has $\mathrm{Ric}(e_i, e_j) = R_{kikj}$ and $\mathrm{Scal} = R_{ijij}$.

A particular situation is when $U = (U; x^i)$ is a local coordinate chart. In this case, one takes $e_i = \partial_i \triangleq \partial_{x^i}$ and $\eta^j = dx^j$. The Christoffel symbols and curvature coefficients with respect to $\{\partial_i\}$ are explicitly given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \big) = \Gamma_{ji}^k,$$

$$R_{ijkl} = \frac{1}{2} \big(\partial_{kj}^2 g_{il} + \partial_{li}^2 g_{jk} - \partial_{ki}^2 g_{jl} - \partial_{lj}^2 g_{ik} \big) + g_{pq} \big(\Gamma_{jk}^p \Gamma_{il}^q - \Gamma_{jl}^p \Gamma_{ik}^q \big)$$

in terms of the metric tensor g and its derivatives.

We conclude this section with an important result concerning the consistency of the Levi-Civita connection with respect to isometric embeddings. A smooth map $F : (M, g) \to (N, h)$ between two Riemannian manifolds is an *isometric embedding* if F is an embedding in the manifold sense and F preserves the metric structure, namely $F^* h = g$ or equivalently,

$$\langle v, w \rangle_x = \langle (dF)_x v, (dF)_x w \rangle_{F(x)}$$

for all $x \in M$ and $v, w \in T_x M$.

Proposition C.1. *Let $F : (M, g) \to (N, h)$ be an isometric embedding. We view M as a submanifold of N by identifying M with its image $F(M)$ in N. Let X, Y be vector fields defined on a neighbourhood of $x \in M$. Let $\tilde{X}, \tilde{Y}$ be any of their extensions to a neighbourhood of x in N. Then $(\nabla_X Y)(x)$ is the orthogonal projection of $(\tilde{\nabla}_{\tilde{X}} \tilde{Y})(x)$ onto $T_x M$, where ∇ (respectively, $\tilde{\nabla}$) denotes the Levi-Civita connection on M (respectively, on N).*

C.2 The Laplace-Beltrami operator, volume form and Green's formula

There is a geometric counterpart of the Euclidean Laplacian on a Riemannian manifold M. We first recall that the *divergence* of a vector field X is defined by

$$\mathrm{div}X \triangleq \mathrm{tr}\big[Y \mapsto \nabla_Y X\big] \tag{C.9}$$

where ∇ is the Levi-Civita connection.

Definition C.4. The *Laplace-Beltrami operator* $\Delta : C^\infty(M) \to C^\infty(M)$ is defined by

$$\Delta f \triangleq -\mathrm{div}(\nabla f) = -\mathrm{tr}\big[\nabla^* df\big], \tag{C.10}$$

where ∇^* is the induced connection on T^*M so that $\nabla^* df$ is viewed as a symmetric $(0,2)$-tensor field (the *Hessian* of f) .

Remark C.1. Some authors use a different sign convention in the definition of Δ. Here we adopt the minus sign so that Δ becomes a *nonnegative definite* operator. In fact, all notions of Laplacian appearing in the present notes are positive definite.

Under a local chart $(U; x^i)$, one has

$$\Delta f = -\frac{1}{\sqrt{|\det g|}}\partial_i\big(\sqrt{|\det g|}g^{ij}\partial_j f\big) = -g^{ij}\big(\partial^2_{ij} f - \Gamma^k_{ij}\partial_k f\big),$$

where $g_{ij} \triangleq \langle \partial_i, \partial_j \rangle$ and $(g^{ij}) \triangleq (g_{ij})^{-1}$.

Now assume that M is an oriented, n-dimensional Riemannian manifold. The *volume form* on M is the n-form $dx \in \Omega^n(M)$ whose local expression is

$$dx = \sqrt{\det g} dx^1 \wedge \cdots \wedge dx^n \quad \text{on } U,$$

where $(U; x^i)$ is any positively oriented chart. Using the volume form, one can integrate a compactly supported function $f \in C^\infty_c(M)$ on M which is defined as $\int_M f dx$. The divergence of a vector field X is the unique element $\mathrm{div}X \in C^\infty(M)$ such that

$$\int_M f \mathrm{div}X dx = -\int_M \langle \nabla f, X \rangle dx \tag{C.11}$$

for all $f \in C^\infty_c(M)$. The formula (C.11) is known as *Green's formula*. By taking $X = \nabla g$ in (C.11), one obtains that

$$\int_M f \Delta g dx = \int_M \langle \nabla f, \nabla g \rangle dx. \tag{C.12}$$

This is also a useful form of Green's formula. If M is closed, one also has

$$\int_M \mathrm{div}X dx = 0 \tag{C.13}$$

for all vector fields X.

C.3 Geodesics and normal coordinates

Let M be an n-dimensional Riemannian manifold with Levi-Civita connection ∇.

Definition C.5. A smooth curve $\gamma : I \to M$ is a *geodesic* if $\nabla_{\dot{\gamma}_t}\dot{\gamma}_t = 0$ for all $t \in I$.

Under a local chart $(U; x^i)$, the geodesic equation is expressed as

$$\frac{d^2 x_t^i}{dt^2} + \Gamma_{jk}^i \dot{x}_t^j \dot{x}_t^k = 0, \quad i = 1, \cdots, n. \tag{C.14}$$

This is a second order nonlinear ODE which is uniquely solvable for small time but needs not admit global solution for all time. Given $(x, v) \in TM$, there exists a unique geodesic γ defined on some maximal interval $I = (-a, b)$ $(a, b > 0)$ such that $(\gamma_0, \dot{\gamma}_0) = (x, v)$. The manifold M is said to be *complete* if the geodesic is well-defined on $(-\infty, \infty)$ for every initial condition $(x, v) \in TM$. Every closed Riemannian manifold is complete.

We now introduce a particularly useful type of coordinate systems: normal coordinate charts. To this end, we shall first recall the definition of the exponential map. Fix $x \in M$. For each $v \in T_xM$, let $\gamma_v : I_v \to M$ be the geodesic with initial condition (x, v) which is defined on a maximal interval I_v. We set

$$D_x \triangleq \{v \in T_xM : 1 \in I_v\}.$$

There is a well-defined map

$$\exp_x : D_x \to M, \ \exp_x(v) \triangleq \gamma_v(1).$$

This is known as the *exponential map* at x. For each $v \in D_x$, the line segment $(tv)_{0 \leqslant t \leqslant 1}$ is mapped by $\exp_x$ to a geodesic joining x to $\exp_x(v)$. If $|v| < \mathrm{inj}(x)$ (see (C.15) below), this is the unique minimising geodesic and one has $d(x, \exp_x(v)) = |v|$.

The *injectivity radius at* x is the strictly positive number defined by

$$\mathrm{inj}(x) \triangleq \sup\{r > 0 : \exp_x \text{ is a diffeomorphism from } B(r) \text{ onto its image}\}. \tag{C.15}$$

For each $r \in (0, \mathrm{inj}(x))$, $\exp_x$ maps the Euclidean ball $B(r)$ diffeomorphically onto the geodesic ball $B(x, r) \triangleq \{y \in M : d(x, y) < r\}$. The *global injectivity radius of* M is

$$\mathrm{inj}(M) \triangleq \inf_{x \in M} \mathrm{inj}(x) \in [0, \infty].$$

For a closed Riemannian manifold M, it is always true that $\mathrm{inj}(M) \in (0, \infty)$.

The *normal chart* around $y \in M$ is constructed as follows. Fix $r < \mathrm{inj}(y)$ so that

$$\exp_y : V \triangleq \{v \in T_yM : |v| < r\} \to U \triangleq B(y, r)$$

is a diffeomorphism. Let $\{\varepsilon_i : 1 \leqslant i \leqslant n\}$ be a given fixed ONB of $T_y M$ and we identify V with the ε-ball in $\mathbb{R}^n$ through

$$\mathbf{x} = (x^1, \cdots, x^n) \leftrightarrow x^i \varepsilon_i.$$

The normal coordinates on U are then defined by

$$V \ni \mathbf{x} = (x^1, \cdots, x^n) \mapsto \exp_y(x^i \varepsilon_i) \in U.$$

Note that $\mathbf{x} = \mathbf{0}$ corresponds to the point y. On the normal chart U, one often considers the following two local frame fields.

(i) The natural coordinate field $\{\partial_1, \cdots, \partial_n\}$ with respect to the above normal coordinates.

(ii) The parallel ONF $\{e_1, \cdots, e_n\}$ which is defined as follows. For each $x \in U$, we define $e_i(x)$ to be the parallel transport of ε_i with respect to the Levi-Civita connection ∇ along the geodesic ray from y to x.

An important property of normal coordinates is that the Christoffel symbols with respect to $\{\partial_i\}$ all vanish at y. This is essentially due to the fact that geodesics emitting from x correspond to straight lines emitting from $\mathbf{0}$ on V. This property often simplifies local computation (e.g. the vanishing of first order terms in the Taylor expansion of the metric tensor; see Proposition 9.2).

Example C.1. The unit sphere

$$S^n \triangleq \{(x^1, \cdots, x^{n+1}) \in \mathbb{R}^{n+1} : (x^1)^2 + \cdots + (x^{n+1})^2 = 1\}$$

admits a canonical Riemannian metric by restricting the Euclidean metric on $\mathbb{R}^{n+1}$ to each tangent space of S^n. Under this metric, S^n is a complete, Riemannian manifold with constant sectional curvature $K \equiv 1$. With the induced orientation from $\mathbb{R}^{n+1}$ (i.e. a basis $\{v_1, \cdots, v_n\}$ of $T_x M$ is positive if and only if $\{v_1, \cdots, v_n, w\}$ is positive in $\mathbb{R}^{n+1}$ where w is the outer normal at x), the volume form on S^n is given by

$$dx = \sum_{i=1}^{n+1} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}.$$

The total volume of S^n is $2\pi^{(n+1)/2}/\Gamma((n+1)/2)$. Geodesics are the great circles and the global injectivity radius is π.

C.4 Laplacian of a connection on a vector bundle

Let E be a real vector bundle over an n-dimensional Riemannian manifold M. Let ∇^E be a connection on E. Given $s \in \Gamma(E)$, one can define the second order covariant derivative

$$\nabla^{T^*M \otimes E} \nabla^E s \in \Gamma(T^*M \otimes T^*M \otimes E).$$

Here $\nabla^E s \in \Gamma(T^*M \otimes E)$ is understood through (B.19) and $\nabla^{T^*M\otimes E}$ is defined by (B.23) where the differentiation on the T^*M-component is the connection on T^*M induced by the Levi-Civita connection ∇^M on M. The order convention for the two T^*M-components is chosen such that

$$\big(\nabla^{T^*M\otimes E}\nabla^E s\big)(X,Y) = \nabla^E_X \nabla^E_Y s - \nabla^E_{\nabla^M_X Y} s$$

for all $X, Y \in \Gamma(TM)$. Note that the above bilinear form is in general not symmetric in (X,Y); in fact, one has

$$\big(\nabla^{T^*M\otimes E}\nabla^E s\big)(X,Y) - \big(\nabla^{T^*M\otimes E}\nabla^E s\big)(Y,X) = F(X,Y)s$$

where $F(\cdot,\cdot)\cdot$ is the curvature tensor of ∇^E (see (B.20)).

Definition C.6. The *connection Laplacian* associated with ∇^E is defined by

$$\Delta^E_{\mathrm{C}} \triangleq -\mathrm{tr}\big(\nabla^{T^*M\otimes E}\nabla^E\big).$$

More explicitly, one has

$$\Delta^E_{\mathrm{C}} s = -\sum_{i=1}^n \big(\nabla^E_{e_i}\nabla^E_{e_i} s - \nabla^E_{\nabla^M_{e_i} e_i} s\big), \quad s \in \Gamma(E), \tag{C.16}$$

where $\{e_1, \cdots, e_n\}$ is any local ONF of TM.

Remark C.2. The connection Laplacian is just the Laplace-Beltrami operator if $E = M \times \mathbb{R}$.

Now suppose that E is equipped with a Euclidean metric and M is oriented. One can consider the space $L^2(E)$ of square integrable E-sections whose inner product is defined by

$$\langle s, t\rangle_{L^2} \triangleq \int_M \langle s(x), t(x)\rangle_{E_x} dx.$$

The space $L^2(T^*M \otimes E)$ is defined similarly, where the fiberwise inner product is the Hilbert-Schmidt inner product on $T^*_x M \otimes E_x$, namely,

$$\langle \alpha\otimes\xi, \beta\otimes\eta\rangle_{T^*_x M\otimes E_x} = \langle\alpha,\beta\rangle_{T^*_x M}\langle\xi,\eta\rangle_{E_x}$$

for $\alpha, \beta \in T^*_x M$ and $\xi, \eta \in E_x$ (the inner product $\langle\cdot,\cdot\rangle_{T^*_x M}$ is the induced Riemannian metric).

Definition C.7. The *Bochner Laplacian* of a connection ∇^E on E is the operator

$$\Delta^E_{\mathrm{B}} : \Gamma(E) \to \Gamma(E), \quad \Delta^E_{\mathrm{B}} \triangleq (\nabla^E)^*\nabla^E,$$

where one regards $\nabla^E : L^2(E) \to L^2(T^*M \otimes E)$ and $(\nabla^E)^*$ denotes its formal L^2-adjoint.

By definition, given $s \in \Gamma(E)$ the element $\Delta^E_{\mathrm{B}} s$ is the unique smooth section such that

$$\langle \Delta^E_{\mathrm{B}} s, t\rangle_{L^2} = \int_M \langle \nabla^E s, \nabla^E t\rangle_{T^*_x M \otimes E_x} dx$$

for all $t \in \Gamma(E)$ with compact support.

Proposition C.2. *Suppose that ∇^E is compatible with the metric on E. Then $\Delta^E_{\mathrm{B}} = \Delta^E_{\mathrm{C}}$.*

Proof. To ease notation, we will omit the subscripts of the various inner products (which one is used should be clear from the context). Our aim is to show that

$$\int_M \langle \Delta^E_{\mathrm{C}} s, t\rangle dx = \int_M \langle \nabla^E s, \nabla^E t\rangle dx \tag{C.17}$$

for all compactly supported $s, t \in \Gamma(E)$.

To this end, we first claim that

$$\mathrm{tr}\big(\nabla^M \langle \nabla^E s, t\rangle\big) = -\langle \Delta^E_{\mathrm{C}} s, t\rangle + \mathrm{tr}\langle \nabla^E s, \nabla^E t\rangle \tag{C.18}$$

at every $x \in M$. Let $\{e_i\}$ be a PONF of TM and let $\{s_\alpha\}$ be a local ONF of E on some open subset $U \subseteq M$. Locally, by writing $s = \lambda^\alpha s_\alpha$ with $\lambda^\alpha \in C^\infty(U)$ one has

$$\nabla^E s = (d\lambda^\alpha + \lambda^\beta A^\alpha_\beta) \otimes s_\alpha,$$

where A^α_β is the local connection matrix of ∇^E with respect to $\{s_\alpha\}$. Given vector fields X, Y, one has

$$\big(\nabla^M \langle \nabla^E s, t\rangle\big)(X, Y) \;\; = \;\; \big(\nabla^M_X \langle \nabla^E s, t\rangle\big)(Y).$$

By direct computation,

$$\begin{aligned}\nabla^M_X \langle \nabla^E s, t\rangle =& X\langle s_\alpha, t\rangle d\lambda^\alpha + \langle s_\alpha, t\rangle \nabla_X d\lambda^\alpha \\ &+ X(\lambda^\beta \langle s_\alpha, t\rangle) A^\alpha_\beta + \lambda^\beta \langle s_\alpha, t\rangle \nabla_X A^\alpha_\beta.\end{aligned}$$

It follows that

$$\begin{aligned}&\big(\nabla^M \langle \nabla^E s, t\rangle\big)(X, Y) = \big(\nabla^M_X \langle \nabla^E s, t\rangle\big)(Y) \\ &= X\langle s_\alpha, t\rangle Y\lambda^\alpha + \langle s_\alpha, t\rangle \big(XY\lambda^\alpha - (\nabla^M_X Y)\lambda^\alpha \\ &\quad + X(\lambda^\beta\langle s_\alpha, t\rangle) A^\alpha_\beta(Y) + \lambda^\beta \langle s_\alpha, t\rangle (X(A^\alpha_\beta(Y)) - A^\alpha_\beta(\nabla^M_X Y)) \\ &= X\langle s_\alpha, t\rangle Y\lambda^\alpha + \langle s_\alpha, t\rangle XY\lambda^\alpha + X(\lambda^\beta\langle s_\alpha, t\rangle) A^\alpha_\beta(Y) \\ &\quad + \lambda^\beta \langle s_\alpha, t\rangle X(A^\alpha_\beta(Y)) - \langle \nabla^E_{\nabla^M_X Y} s, t\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\langle \nabla^E_X \nabla^E_Y s, t\rangle + \langle \nabla^E_Y s, \nabla^E_X t\rangle \\
&= X\langle \nabla_Y s, t\rangle \quad \text{(by metric-compatibility)} \\
&= X\big((Y\lambda^\alpha + \lambda^\beta A^\alpha_\beta(Y))\langle s_\alpha, t\rangle\big) \\
&= \big(XY\lambda^\alpha + X\lambda^\beta A^\alpha_\beta(Y) + \lambda^\beta X(A^\alpha_\beta(Y))\big)\langle s_\alpha, t\rangle \\
&\quad + \big(Y\lambda^\alpha + \lambda^\beta A^\alpha_\beta(Y)\big)X\langle s_\alpha, t\rangle.
\end{aligned}$$

As a consequence,

$$\big(\nabla^M\langle \nabla^E s, t\rangle\big)(X,Y) = \langle \nabla^M_X \nabla^M_Y s, t\rangle + \langle \nabla^M_Y s, \nabla^M_X t\rangle - \langle \nabla^E_{\nabla^M_X Y} s, t\rangle. \tag{C.19}$$

The claim (C.18) follows by taking $X = Y = e_i$ in (C.19) and summing over i.

To prove the relation (C.17), we recall that

$$\operatorname{tr}\big[\nabla\alpha\big] = \operatorname{div} X \tag{C.20}$$

for any $\alpha \in \Omega^1(M)$ where X is the vector field dual to α under the Riemannian metric. Since $\langle \nabla^E s, t\rangle$ is a 1-form on M, it follows from (C.19), (C.20) and (C.13) that

$$0 = \int_M \operatorname{tr}\big[\nabla^M\langle \nabla^E s, t\rangle\big]dx = -\int_M \langle \Delta^E_{\mathrm{C}} s, t\rangle dx + \int_M \operatorname{tr}\langle \nabla^E s, \nabla^E t\rangle dx.$$

Therefore, one obtains the desired relation (C.17). □

C.5 Sobolev spaces on forms

We recall some basic notions about Sobolev spaces on forms that are used in the proof of the Hodge theorems. There are various equivalent ways of defining Sobolev spaces. We follow [CW93] to choose one which requires the least technical effort. In what follows, let M be a closed and oriented Riemannian manifold.

Definition C.8. Let $s \in \mathbb{N}$. We define an inner product $\langle\cdot,\cdot\rangle_s$ on $\Omega(M)$ by setting

$$\langle \alpha, \beta\rangle_s \triangleq \sum_{k=0}^{s} \int_M \langle \nabla^k\alpha, \nabla^k\beta\rangle_x dx.$$

Here ∇^k is the covariant derivative operator (C.1) induced from the Levi-Civita connection (viewing α, β as tensor fields) and the fiberwise inner product $\langle\cdot,\cdot\rangle_x$ is the Hilbert-Schmidt one on tensor products of $T^*_x M$ (different degrees are claimed to be orthogonal). We also set $\|\alpha\|_s \triangleq \sqrt{\langle \alpha, \alpha\rangle_s}$. The *Sobolev space of order* s over M is the Hilbert space $H_s(\Lambda T^* M)$ defined by the completion of $\Omega(M)$ with respect to the inner product $\langle\cdot,\cdot\rangle_s$.

Remark C.3. One has $H_0(\Lambda T^*M) = L^2(\Lambda T^*M)$. The inner product $\langle \cdot, \cdot \rangle_0$ differs from Definition 8.2 by a multiplicative constant.

Remark C.4. By using a partition of unity, one can define Sobolev spaces (and with any L^p-integrability) on a general manifold M without using any Riemannian structure. All definitions are equivalent (they induce equivalent Sobolev norms) if M is compact. See [War83] for a general discussion.

We conclude this section with two basic theorems about Sobolev spaces. They are natural extensions of the Euclidean results to the manifold setting.

Theorem C.2 (The Sobolev Embedding Theorem)**.** *For each $s \in \mathbb{N}$, one has*

$$H_{n+s}(\Lambda T^*M) \hookrightarrow C^s(\Lambda T^*M).$$

Here $C^s(\Lambda T^*M)$ denotes the space of sections of ΛT^*M which are continuously differentiable up to order s.

Remark C.5. Elements of $H_{n+s}(\Lambda T^*M)$ are equivalence classes (they are only defined modulo dx-null sets). The precise interpretation of the above theorem is that every equivalence class in $H_{n+s}(\Lambda T^*M)$ contains a C^s-representative.

Theorem C.3 (Rellich's Compactness Theorem)**.** *Every bounded sequence $\{\alpha_n\} \subseteq H_1(\Lambda T^*M)$ with respect to $\| \cdot \|_1$ contains a convergent subsequence with respect to $\| \cdot \|_0$.*

References

[All40] C.B. Allendoerfer. The Euler number of a Riemannian manifold. *Amer. J. Math.* 62 (1940): 243–248.

[AW43] C.B. Allendoerfer and A. Weil. The Gauss-Bonnet theorem for Riemannian polyhedra. *Trans. Amer. Math. Soc.* 53 (1943): 101–129.

[AD14] M. Audin and M. Damian. *Morse theory and Floer homology*. Springer, 2014.

[BGV04] N. Berline, E. Getzler and M. Vergne. *Heat kernels and Dirac operators*. Springer, 2004.

[BT82] R. Bott and L.W. Tu. *Differential Forms in Algebraic Topology*. Graduate Texts in Mathematics. Springer, 1982.

[Bre93] G.E. Bredon. *Topology and Geometry*. Graduate Texts in Mathematics. Springer-Verlag, 1993.

[CW93] W.Y. Chen and H. Wu. *Selected topics in Riemannian geometry.* Chinese Edition. Peking University Press, 1993.

[Che44] S.S. Chern. A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds. *Ann. Math.* 45 (4) (1944): 747–752.

[Che45] S.S. Chern. On the curvatura integra in a Riemannian manifold. Ann. of Math. 46 (1945): 674–684.

[Che46] S.S. Chern. Characteristic classes of Hermitian manifolds. Ann. of Math. 47 (1946): 85–121.

[CCL00] S.S. Chern, W.H. Chen and K.S. Lam. *Lectures on Differential Geometry.* World Scientific, 2000.

[DOC79] M.P. do Carmo. *Riemannian geometry.* Birkhäuser, 1979.

[Fen40] W. Fenchel. On total curvatures of Riemannian manifolds. J. London Math. Soc. 15 (1940): 15–22.

[GT77] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order.* Springer-Verlag, 1977.

[GP74] V. Guillemin and A. Pollack. *Differential topology.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1974.

[Hat22] A. Hatcher. *Algebraic topology.* Cambridge University Press, 2022.

[Hop25] H. Hopf. Über die curvatura integra geschlossener hyperflachen. *Mathematische Annalen* 95 (1925): 340–367.

[KN63] S. Kobayashi and K. Nomizu. *Foundations of differential geometry.* Volume I. John Wiley & Sons, 1963.

[MQ86] V. Mathai and D. Quillen. Superconnections, Thom classes, and equivariant differential forms. *Topology* 25 (1) (1986): 85–110.

[MS67] H.P. McKean and I.M. Singer. Curvature and the eigenvalues of the Laplacian. *J. Differential Geometry* 1 (1967): 43–69.

[MS74] J.W. Milnor and J.D. Stasheff. *Characteristic classes.* Princeton University Press, 1974.

[Mun66] J.R. Munkres. Elementary differential topology. Annals of Mathematics Studies. Princeton University Press, 1966.

[Pat71] V.K. Patodi. Curvature and the eigenforms of the Laplace operator. *J. Differential Geometry* 5 (1971): 233–249.

[Pet06] P. Petersen. *Riemannian geometry.* 2nd Edition. Graduate Texts in Mathematics. Springer, 2006.

[Qui85] D. Quillen. Superconnections and the Chern character. *Topology* 24 (1) (1985): 89–95.

[ST67] I.M. Singer and J.A. Thorpe. *Lecture notes on elementary topology and geometry.* Undergraduate Texts in Mathematics. Springer, 1967.

[Spa66] E.H. Spanier. Algebraic topology. Springer-Verlag, 1966.

[War83] F.W. Warner. *Foundations of differentiable manifolds and Lie groups.* Graduate Texts in Mathematics. Springer-Verlag, 1983.

[Wey39] H. Weyl. *The classical groups: their invariants and representations.* Princeton University Press, 1939.

[Yu87] Y. Yu. Local index theorem for signature operators. *Acta Math. Sinica* 3 (4) (1987): 363–372.

[Zha00] W.P. Zhang. Mathai-Quillen's Thom Form and the Gauss-Bonnet-Chern Theorem. *Chinese Quart. J. Math.* 15 (4) (2000): 1–9.

[ZF22] W. Zhang and H. Feng. *Geometry and analysis on manifolds.* Chinese edition. Higher Education Press, 2022.