

Classification of 2D Torus Flows

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Abstract

In these notes, we prove a classification theorem which asserts that the 1D foliation induced by a smooth, nonvanishing vector field on the 2D torus is either conjugate to a suspension of an orientation-preserving circle diffeomorphism or contains a Reeb component. We also describe the dynamical picture of such flows. Our discussion follows the main argument of Hector-Hirsch [5], which formulates and proves the theorem in the continuous category.

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1 Nondegenerate torus flows

The aim of these notes is to prove a fundamental classification theorem for two-dimensional nondegenerate torus flows. This classification theorem was used in the recent paper [1] for proving that a two-dimensional periodic, uniformly elliptic system driven by a fractional Brownian motion with Hurst parameter $H > 1/2$ always possesses a Gaussian lower tail.

We will be working on the 2-torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$. A (two-dimensional) *nondegenerate torus flow* is the collection of integral curves of a smooth, nonvanishing vector field W on $\mathbb{T}^2$. The main classification theorem asserts that a nondegenerate torus flow either is conjugate to a *suspension* of a circle diffeomorphism or contains a *Reeb component*. In addition, these two possibilities are mutually exclusive. The definitions of a suspension, Reeb component and conjugation will be made precise later on.

The origin of the above classification theorem went back to H. Kneser [6] in the 1920s. It was explicitly stated and proved in the monograph [5, Proposition 4.3.2] under the C^0 -category. The more recent paper [2] stated a C^r -version which cited the same book without commenting on the transition from the topological to differential setting. In these notes, we provide a relatively self-contained proof of the above classification theorem under the smooth category, following the main argument of [5, Proposition 4.3.2]. The proof here can be easily adapted to the C^r -setting without any difficulty.

1.1 Foliations

We begin with the general definition of a foliation (see [7]).

Definition 1.1. Let M be an n -dimensional differentiable manifold. Let $\mathcal{F}$ be a collection of immersed k -dimensional submanifolds. A chart (U, φ) is said to be *flat* for $\mathcal{F}$ if the following two properties hold true:

- (i) $\varphi(U)$ is a open cube in $\mathbb{R}^n$;
- (ii) for each submanifold $N \in \mathcal{F}$ with nonempty intersection with U , the set $\varphi(N \cap U)$ is a countable union of k -dimensional slices of the form

$$x^{k+1} = c^{k+1}, \dots, x^n = c^n$$

in $\varphi(U)$.

A (*smooth*) k -dimensional *foliation* on M is a collection $\mathcal{F}$ of disjoint, connected, immersed k -dimensional submanifolds such that the union of members of $\mathcal{F}$ is M and for every $x \in M$, there exists a chart around x which is flat for $\mathcal{F}$. Members of a foliation $\mathcal{F}$ are called *leaves* of $\mathcal{F}$.

Definition 1.2. Let $\mathcal{F}, \mathcal{F}'$ be k -dimensional foliations on n -dimensional manifolds M, M' respectively. We say that $\mathcal{F}$ and $\mathcal{F}'$ are *conjugate* (or *isomorphic*) if there exists a diffeomorphism $\Psi : M \rightarrow M'$ such that $\Psi(\mathcal{F}) = \mathcal{F}'$.

1.2 Nondegenerate torus flow as 1D foliation

From now on, we assume that $M = \mathbb{T}^2$. We always fix a smooth, nonvanishing vector field W on $\mathbb{T}^2$. Given $x \in \mathbb{T}^2$, define the flow $t \mapsto \phi_t(x)$ by

$$\frac{d}{dt}\phi_t(x) = W(\phi_t(x)), \quad \phi_0(x) = x.$$

Since $\mathbb{T}^2$ is compact, the flow $\phi_t(x)$ is well-defined for all $t \in \mathbb{R}$. Let $\mathcal{F}_W$ denote the collection of integral curves (flow lines) of W , namely,

$$\mathcal{F}_W := \{\{\phi_t(x)\}_{t \in \mathbb{R}} : x \in \mathbb{T}^2\}.$$

The starting point of the whole discussion is the following basic fact.

Proposition 1.3. *The family $\mathcal{F}_W$ defines a 1D foliation on $\mathbb{T}^2$.*

Proof. It is clear that $\mathcal{F}_W$ is a family of immersed 1D manifolds whose union is $\mathbb{T}^2$. To prove the local flatness property, let $x \in \mathbb{T}^2$. Since $W(x) \neq 0$, one can pick a nonzero vector $v \in T_x \mathbb{T}^2$ that is linearly independent of $W(x)$. Let $\sigma : (-\delta, \delta) \rightarrow \mathbb{T}^2$ be a smooth arc such that $\sigma(0) = x$, $\dot{\sigma}(0) = v$. Define

$$F : \mathbb{R} \times (-\delta, \delta) \rightarrow \mathbb{T}^2, \quad F(t, r) := \phi_t(\sigma(r)).$$

It is clear that $F(0, 0) = x$ and $(dF)_{(0,0)}$ is nonsingular. Therefore, F is a local diffeomorphism from $V := (-\varepsilon, \varepsilon) \times (-\delta', \delta')$ onto $F(V)$ with some small ε and $\delta' < \delta$. The map $F|_V$ provides a flat chart of $\mathbb{T}^2$ around x ; each horizontal arc $s \mapsto (s, r)$ corresponds to a flow line by definition and every flow line inside $F(V)$ must come from a horizontal arc since $W = \partial_s$ in $F(V)$. $\square$

Proposition 1.3 asserts that around every point $x \in \mathbb{T}^2$, one can choose a *flow box* $(U; s, r)$ ($s \in (-\delta_1, \delta_1)$, $r \in (-\delta_2, \delta_2)$) such that $x = (0, 0)$ and $W = \partial_s$ on U . In particular, flow lines of W inside U are precisely the horizontal slices $r = c$ ($c \in (-\delta_2, \delta_2)$). This is a basic but important (technical) fact that will be used for several times.

Remark 1.4. The actual flow $\{\phi_t(x)\}$ carries more information than the foliation $\mathcal{F}_W$ itself; the foliation is determined by the *1D distribution* $\{\text{Span} W(x) : x \in \mathbb{T}^2\}$ while the flow also records the orientation and the strength of the field.

Definition 1.5. The leaf $\mathcal{O}_x := \{\phi_t(x) : t \in \mathbb{R}\}$ is called the *orbit* of W passing through x . An orbit $\mathcal{O}_x$ is said to be *periodic* if it is a compact leaf.

It is a simple exercise that for a periodic orbit $\mathcal{O}_x$, there exist a smallest $T > 0$ such that $\phi_{t+T}(y) = \phi_t(y)$ for all $y \in \mathcal{O}_x$ and $t \in \mathbb{R}$. This number T is called the *period* of $\mathcal{O}_x$ (with respect to W). Note that the period is not a foliation concept; it depends on the actual flow $\phi_t(x)$ or equivalently, on the vector field W (cf. Remark 1.4). Another important feature here is that the vector field W orients every orbit $\mathcal{O}_x$ in a natural way.

1.3 Periodic orbits

In this subsection, we study basic topological and dynamical properties of periodic orbits. This provides some essential tools for the later proof of the main classification theorem.

1.3.1 Transversal intersection and the intersection number

We first recall some notions from intersection theory (see [3, 4]).

Definition 1.6. Let P, Q be two oriented, embedded 1D submanifolds of $\mathbb{T}^2$. We say that P is *transversal* to Q if $T_x P$ and $T_x Q$ (as 1D subspaces of $T_x \mathbb{T}^2$) are linearly independent at every $x \in P \cap Q$. In this case, the *local intersection number* of P and Q at $x \in P \cap Q$ is defined by

$$i_x(P, Q) := \begin{cases} 1, & \text{if } \det(v_x, w_x) > 0; \\ -1, & \text{if } \det(v_x, w_x) < 0, \end{cases}$$

where v_x (respectively, w_x) is any positively oriented vector in $T_x P$ (respectively, $T_x Q$). The *intersection number* of P and Q is defined by

$$I(P, Q) := \sum_{x \in P \cap Q} i_x(P, Q),$$

provided further that $P \cap Q$ is a finite set.

For our purpose, P, Q are always assumed to be smooth, simple and closed curves. Due to compactness, if they intersect transversally their intersection must be a finite set and $I(P, Q)$ is thus well-defined. In particular, $I(P, Q) = 0$ if they do not intersect. It is also known that the intersection number is a topological quantity, i.e. it depends only on the homology classes of P, Q . Let $\{\sigma_1, \sigma_2\}$ be the canonical basis of $H_1(\mathbb{T}^2; \mathbb{Z}) \cong \mathbb{Z}^2$. In other words,

$$\sigma_1 := [(t, 0)_{0 \leq t \leq 1}], \quad \sigma_2 := [(0, t)_{0 \leq t \leq 1}]. \quad (1.1)$$

One has the following formula for the intersection number.

Lemma 1.7. *Let P, Q be smooth, simple and closed curves which intersect transversally and whose homology classes are given by*

$$[P] = p_1 \sigma_1 + p_2 \sigma_2, \quad [Q] = q_1 \sigma_1 + q_2 \sigma_2, \quad (1.2)$$

with $p_i, q_j \in \mathbb{Z}$. Then $I(P, Q) = p_1 q_2 - p_2 q_1$.

Proof. Let

$$P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} : S^1 \rightarrow \mathbb{T}^2$$

be parametrised by $S^1 := \mathbb{R}/\mathbb{Z}$. Consider the map $D : S^1 \times S^1 \rightarrow \mathbb{T}^2$ defined by

$$D(s, t) := P(s) - Q(t) \pmod{\mathbb{Z}}.$$

At each intersection point $(s, t) \in D^{-1}(0)$, by the transversal intersection assumption one has

$$\det((dD)_{(s,t)}) = -\det(\dot{P}(s), \dot{Q}(t)) \neq 0.$$

Therefore, 0 is a regular value of D . It follows that $D^{-1}(0)$ is a zero-dimensional submanifold of $S^1 \times S^1$. Hence, it is a finite set of points, say

$$D^{-1}(0) = \{(s_1, t_1), \dots, (s_k, t_k)\}.$$

There exists a neighbourhood U_k of (s_k, t_k) for each k and a neighbourhood V of 0, such that $D|_{U_k}$ is a diffeomorphism from U_k onto V . In addition, one has

$$\operatorname{sgn} \det ((dD)_{(s_k, t_k)}) = -\operatorname{sgn} \det (\dot{P}(s_k), \dot{Q}(t_k)) = -i_{x_k}(P, Q),$$

where $x_k := P(s_k) = Q(t_k)$.

Let ω be a smooth 2-form that is compactly supported inside V and satisfies $\int_V \omega = 1$. Since $D|_{U_k} : U_k \rightarrow V$ is a diffeomorphism, one has

$$\int_{U_k} D|_{U_k}^* \omega = \operatorname{sgn} \det ((dD)_{(s_k, t_k)}) \int_V \omega = -i_{x_k}(P, Q).$$

Therefore,

$$\int_{S^1 \times S^1} D^* \omega = \sum_{k=1}^m \int_{U_k} D|_{U_k}^* \omega = -I(P, Q).$$

Let $\nu = dx \wedge dy \in \Omega^2(\mathbb{T}^2)$. Since both ω and ν are closed and have the same integral on $\mathbb{T}^2$ (which is equal to one), they differ by an exact form. As a result,

$$\int_{S^1 \times S^1} D^* \nu = \int_{S^1 \times S^1} D^* \omega = -I(P, Q). \quad (1.3)$$

Now it remains to compute the integral of $D^* \nu$.

By using the explicit expressions of D and ν , it is plain to check that

$$D^* \nu = (P'_2(s)Q'_1(t) - P'_1(s)Q'_2(t))ds \wedge dt.$$

Therefore,

$$\begin{aligned} \int_{S^1 \times S^1} D^* \nu &= \int_0^1 \int_0^1 (P'_2(s)Q'_1(t) - P'_1(s)Q'_2(t))dsdt \\ &= \left(\int_0^1 P'_2(s)ds \right) \left(\int_0^1 Q'_1(t)dt \right) - \left(\int_0^1 P'_1(s)ds \right) \left(\int_0^1 Q'_2(t)dt \right). \end{aligned}$$

Let $\tilde{P}, \tilde{Q}$ be some lifts of P, Q on the universal cover $\mathbb{R}^2$. According to the assumption (1.2), one has

$$\begin{aligned} \int_0^1 P'(s)ds &= \int_0^1 \tilde{P}'(s)ds = \tilde{P}(1) - \tilde{P}(0) = (p_1, p_2), \\ \int_0^1 Q'(t)dt &= \int_0^1 \tilde{Q}'(t)dt = \tilde{Q}(1) - \tilde{Q}(0) = (q_1, q_2). \end{aligned}$$

As a consequence,

$$\int_{S^1 \times S^1} D^* \nu = p_2 q_1 - p_1 q_2. \quad (1.4)$$

The conclusion of the lemma follows from the relations (1.3) and (1.4). $\square$

Remark 1.8. Lemma 1.7 remains valid if $P \cap Q = \emptyset$. In this case, $I(P, Q) = p_1q_2 - p_2q_1 = 0$.

1.3.2 Topological properties of periodic orbits

Next, we study periodic orbits of a smooth, nonvanishing vector field W . We first make the following definition.

Definition 1.9. A (continuous) closed curve γ in $\mathbb{T}^2$ is said to be *essential* if its (singular) homology class $[\gamma] \in H_1(\mathbb{T}^2; \mathbb{Z})$ is nonzero. It is *primitive* if its homology class cannot be expressed as $[\gamma] = m\sigma$ for some $\sigma \in H_1(\mathbb{T}^2; \mathbb{Z})$ and $m \geq 2$.

Let $\{\sigma_1, \sigma_2\}$ be the canonical basis of $H_1(\mathbb{T}^2; \mathbb{Z})$ defined by (1.1). Write $[\gamma] = p\sigma_1 + q\sigma_2$ for a closed curve γ . Then γ is primitive if and only if $\gcd(|p|, |q|) = 1$. This is also equivalent to saying that $[\gamma]$ can be extended to a basis of $H_1(\mathbb{T}^2; \mathbb{Z})$.

The main topological properties of periodic orbits are summarised in the following result.

Proposition 1.10. (i) *Every periodic orbit is essential.*

(ii) *Every essential, simple closed curve is primitive. In particular, every periodic orbit is primitive.*

(iii) *For any pair of periodic orbits γ_1, γ_2 , one has $[\gamma_1] = \pm[\gamma_2]$.*

Proof. (i) Let γ be a periodic orbit of W . Suppose on the contrary that $[\gamma] = 0$. Since γ is simple closed, it must bound a topological disk D in this case. According to the Poincaré-Hopf theorem for the disk D (see [8, Chapter 3, Theorem 2]), one has

$$\text{ind}_W(\partial D) = \sum_{z \in \mathcal{Z}(W) \cap \overset{\circ}{D}} \text{ind}_W(z).$$

Here $\text{ind}_W(\partial D)$ denotes the index of W on the boundary ∂D and $\text{ind}_W(z)$ denotes the local index of W at the zero z ($\mathcal{Z}(W)$ denotes the set of zeros of W). It is plain from the definition that $\text{ind}_W(\partial D) = \pm 1$ (the sign depends on the orientation convention of γ). As a result, W must have at least one zero in the interior of D . This is a contradiction. Therefore, $[\gamma] \neq 0$.

(ii) Let γ be an essential, simple closed curve. We claim that $\mathbb{T}^2 \setminus \gamma$ is connected. Suppose on the contrary that $\mathbb{T}^2 \setminus \gamma$ is disconnected. Let $N \cong S^1 \times (-\varepsilon, \varepsilon)$ be a tubular neighbourhood of γ . Then

$$N_- := S^1 \times (-\varepsilon, 0), \quad N_+ := S^1 \times (0, \varepsilon)$$

must be contained in different components of $\mathbb{T}^2 \setminus \gamma$; otherwise, any other component V would intersect this component (because $\gamma \subseteq \bar{V}$) which contradicts disconnectedness. Let U_+ be the component containing N_+ . Since N_- is contained in a different component, $\overline{U_+}$ is a compact surface with boundary γ . As

a consequence, $\gamma = \partial c$ for some singular 2-chain. It follows that $[\gamma] = 0$ which leads to a contradiction. Therefore, $\mathbb{T}^2 \setminus \gamma$ is connected.

Pick two points $x_+ \in N_+$ and $x_- \in N_-$. Draw a smooth, simple curve η from x_+ to x_- inside $\mathbb{T}^2 \setminus \gamma$ (which is possible due to connectedness). Let σ be a simple arc which joins x_+ to x_- inside N . By smoothing the corners and removing loops from $\eta \sqcup \sigma$ if necessary, one obtains a simple closed curve β that has precisely one (transversal) intersection with γ . Therefore,

$$I(\gamma, \beta) = \pm 1. \quad (1.5)$$

Suppose that γ is not primitive. Under the canonical basis $\{\sigma_1, \sigma_2\}$ of $H_1(\mathbb{T}^2; \mathbb{Z})$, one can write $[\gamma] = m(p, q)$ for some $m \geq 2$ and $p, q \in \mathbb{Z}$. Let $[\beta] = (r, s)$. According to Lemma 1.7, one has

$$I(\gamma, \beta) = m(ps - qr).$$

This contradicts the fact that (1.5). Therefore, γ is primitive.

(iii) Let $\gamma_1 = (p, q)$ and $\gamma_2 = (r, s)$ be two periodic orbits. Since γ_1 is primitive by Part (ii), the integers p, q are coprime. Similarly, r, s is also coprime. Since $\gamma_1 \cap \gamma_2 = \emptyset$, one knows from Lemma 1.7 (cf. Remark 1.8) that

$$(p, q) = \frac{a}{b}(r, s)$$

for some coprime pair $(a, b) \in \mathbb{Z}^2$. In particular,

$$ar = bp, \quad as = bq.$$

It follows that $a|p$ and $a|q$ which implies that $a = \pm 1$. Similarly, $b = \pm 1$. Therefore, $[\gamma_1] = \pm[\gamma_2]$. $\square$

1.3.3 The Poincaré map

In this part, we discuss an elementary but rather useful construction: the Poincaré return map. This construction (or its variant) will be used for several times in the proof of the main classification theorem.

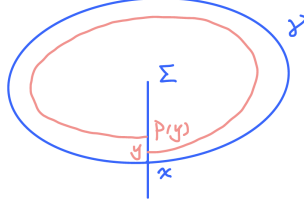
Let $\gamma = \{\phi_t(x)\}_{t \in \mathbb{R}}$ be a periodic orbit of W with period T . Let Σ be a short smooth arc that intersects γ transversally at x . Suppose that $\phi_t(x) \notin \Sigma$ for all $t \in (0, T)$. This is possible by shrinking Σ if necessary; for instance, in a suitable flow box $(U; s, r)$ around x one can assume that $\Sigma = \{s = 0\}$.

We construct a map

$$P : U \subseteq \Sigma \rightarrow \Sigma$$

on a sufficiently small neighbourhood $U \subseteq \Sigma$ of x in the following way. Pick $y \in \Sigma$. Consider the flow line $t \mapsto \phi_t(y)$. Let $\tau(y) > 0$ be the first time that $\phi_t(y)$ returns to Σ . We then define

$$P(y) := \phi_{\tau(y)}(y).$$



Lemma 1.11. *There exists a small neighbourhood $U \subseteq \Sigma$ of x , such that $(\tau(y), P(y))$ is well-defined for all $y \in U$. In addition, both $\tau(y)$, $P(y)$ are smooth on U and one has $\tau(x) = T$, $P(x) = x$.*

Proof. First of all, there exists a neighbourhood $U_0 \subseteq \Sigma$ of x , such that U_0 is represented by the level set $\{\rho = 0\}$ of some smooth function ρ near x . Now consider the function

$$F : U_0 \times (T - \delta_0, T + \delta_0) \rightarrow \mathbb{R}, \quad F(y, t) := \rho(\phi_t(y))$$

where δ_0 is small. Note that $F(x, T) = 0$. In addition, since $T_x \Sigma = \ker(d\rho)_x$ and Σ intersects γ transversally at x , one has

$$\partial_t F(x, T) = \langle (d\rho)_x, W(x) \rangle \neq 0. \quad (1.6)$$

According to the implicit function theorem, there exists $U_1 \subseteq U_0$, $\delta_1 < \delta_0$ and a smooth function $\tau : U_1 \rightarrow (T - \delta_1, T + \delta_1)$ such that $\tau(x) = T$ and

$$F(y, \tau(y)) = 0$$

for all $y \in U_1$.

We claim that by shrinking U_1 to some smaller neighbourhood U of x if necessary, one has

$$\tau(y) = \inf\{t > 0 : \phi_t(y) \in \Sigma\}$$

for all $y \in U$. First of all, according to (1.6) there exists $U_2 \subseteq U_1$ and $\delta_2 < \delta_1$, such that $t \mapsto F(y, t)$ is strictly monotone on $[T - \delta_2, T + \delta_2]$ for all $y \in U_2$. As a result, $\tau(y)$ is the unique $t \in [T - \delta_2, T + \delta_2]$ such that $F(y, t) = 0$. We fix this δ_2 in what follows. Next, by using a flow box argument, there exists $U_3 \subseteq U_2$ and $\varepsilon > 0$, such that $\phi_t(y) \notin \Sigma$ for all $y \in U_3$ and $t \in (0, \varepsilon]$. In addition, since

$$\{\phi_t(x) : \varepsilon \leq t \leq T - \delta_2\} \cap \Sigma = \emptyset,$$

one knows from continuity that $\phi_t(y) \notin \Sigma$ for all y in some smaller $U \subseteq U_3$ and all $t \in [\varepsilon, T - \delta_2]$. Combining the above three scenarios, it follows that $\tau(y)$ is the first return time of $\phi_t(y)$ to Σ for every $y \in U$.

Consequently, the map

$$P : U \rightarrow \Sigma, \quad P(y) := \phi_{\tau(y)}(y)$$

is well-defined. Its smoothness follows immediately from the smoothness of τ and the joint smoothness of $(t, y) \mapsto \phi_t(y)$. This completes the proof of the lemma. $\square$

Definition 1.12. The map P constructed above is known as the *Poincaré map* at $x \in \gamma$ with respect to the transversal Σ .

Since Σ is 1D and $P(x) = x$, the linear map $(dP)_x : T_x \Sigma \rightarrow T_x \Sigma$ is a scalar multiple of the identity, i.e. $(dP)_x = \lambda_\gamma \text{id}$ for some $\lambda_\gamma \in \mathbb{R}$. Although the Poincaré map P depends on the point $x \in \gamma$ and the transversal Σ , it is a remarkable fact that the number λ_γ is intrinsic to the periodic orbit γ .

Proposition 1.13. *The number λ_γ does not depend on the point $x \in \gamma$ or the transversal Σ . In addition, one has $\lambda_\gamma > 0$.*

Proof. We give an alternative intrinsic construction of λ_γ .

(i) Define the function $J_t \in C^\infty(\mathbb{T}^2)$ by

$$\phi_t^* \mu_g = J_t \mu_g,$$

where μ_g denotes the Riemannian volume form on the (oriented) annulus $\mathbb{T}^2$. It is not hard to see that $J_t(x) = \det(d\phi_t)_x$, where the determinant is defined with respect to any positively oriented orthonormal bases of $T_x \mathbb{T}^2$ and $T_{\phi_t(x)} \mathbb{T}^2$. We claim that

$$J_t(x) = \exp \left[\int_0^t (\text{div}_g W)(\phi_s(x)) ds \right], \quad (\text{Liouville's formula}) \quad (1.7)$$

where div_g denotes the Riemannian divergence. To this end, one first recalls the following standard Riemannian geometry fact:

$$\mathcal{L}_W \mu_g = (\text{div}_g W) \mu_g.$$

In addition, by the definition of the Lie derivative, one has

$$\frac{d}{dt}(\phi_t^* \mu_g) = \phi_t^*(\mathcal{L}_W \mu_g).$$

As a consequence,

$$\dot{J}_t(x) = (\text{div}_g W)(\phi_t(x)) J_t(x). \quad (1.8)$$

Since $J_0(x) = 1$, the desired relation (1.7) follows by solving the ODE (1.8).

(ii) Suppose that γ is a periodic orbit of W with period T . According to Part (i), one has

$$\det(d\phi_T)_x = \exp \left[\int_0^T (\text{div}_g W)(\phi_s(x)) ds \right]. \quad (1.9)$$

In particular, $\det(d\phi_T)_x$ is independent of x .

(iii) Under the assumption of Part (ii), let Σ be a transversal curve through $x \in \gamma$. Let $v \in T_x \Sigma$ be a nonzero vector. By the definition of the Poincaré map, one has

$$P(y) = \phi_{\tau(y)}(y), \quad y \in U \subseteq \Sigma.$$

It follows from the definition of λ_γ that

$$\lambda_\gamma v = (dP)_x v = (d\phi_T)_x v + \langle d\tau, v \rangle_x W(x).$$

Also note that $(d\phi_T)_x W(x) = W(x)$ since γ is a periodic orbit. Therefore, the matrix of $(d\phi_T)_x : T_x \mathbb{T}^2 \rightarrow T_x \mathbb{T}^2$ with respect to the basis $\{W(x), v\}$ is given by

$$(d\phi_T)_x = \begin{pmatrix} 1 & -\langle d\tau, v \rangle_x \\ 0 & \lambda_\gamma \end{pmatrix}.$$

This shows that $\det(d\phi_T)_x = \lambda_\gamma$. In particular, λ_γ is independent of x and Σ . The positivity of λ_γ follows directly from the relation (1.9). $\square$

Definition 1.14. The number λ_γ is called the *transverse characteristic number* of γ . A periodic orbit γ is said to be hyperbolic if $\lambda_\gamma \neq 1$.

Remark 1.15. Hyperbolic periodic orbits have strong dynamical stability (exponential attraction of nearby orbits in the case $\lambda_\gamma \in (0, 1)$). Such a stability property is used in the paper [1] to establish a Gaussian lower tail estimate for 2D periodic, uniformly elliptic systems driven by a fractional Brownian motion.

1.3.4 The Poincaré-Bendixson theorem

In general, periodic orbits of W need not always exist. However, if one knows that an end of a nonperiodic orbit is trapped in a compact region *inside a planar domain*, its limit set must be a periodic orbit. This is the content of the *Poincaré-Bendixson theorem*, which provides a general way of identifying periodic orbits.

To state the theorem, we first need to introduce some definitions.

Definition 1.16. Let $x \in \mathbb{T}^2$. The *forward orbit* passing through x is defined by

$$\mathcal{O}_x^+ := \{\phi_t(x) : t \geq 0\}.$$

The *forward limit set* of x is defined by

$$\omega(x) := \bigcap_{T>0} \overline{\{\phi_t(x) : t \geq T\}}.$$

One can define the *backward orbit* $\mathcal{O}_x^-$ and the *backward limit set* $\alpha(x)$ in a similar way. A set $\Lambda \subseteq \mathbb{T}^2$ is *W-invariant* if $\phi_t \Lambda \subseteq \Lambda$ for all $t \in \mathbb{R}$.

By compactness, both $\omega(x)$ and $\alpha(x)$ are nonempty, W -invariant compact subsets of $\mathbb{T}^2$. In addition, a point $y \in \omega(x)$ (respectively, $y \in \alpha(x)$) if and only if $\phi_{t_n}(x) \rightarrow y$ for some sequence $t_n \rightarrow \infty$ (respectively, $t_n \rightarrow -\infty$).

Theorem 1.17 (The Poincaré-Bendixson Theorem). *Let X be a smooth, non-vanishing vector field on a planar domain $U \subseteq \mathbb{R}^2$. Let $x \in U$ be given fixed. Suppose that the flow $\phi_t(x)$ of X exists for all $t \geq 0$ and*

$$\{\phi_t(x) : t \geq 0\} \subseteq K$$

for some compact set $K \Subset U$. Then there exists a periodic orbit $C \subseteq K$ such that $\omega(x) = C$. A similar assertion holds for $\alpha(x)$.

Remark 1.18. The planar assumption is crucial here; it is certainly possible for a vector field on $U = \mathbb{T}^2$ to have no periodic orbits.

Before proving Theorem 1.17, we first state a key lemma which indicates the essential role of the planar assumption.

Lemma 1.19. *Let X be a smooth, nonvanishing vector field on a planar domain $U \subseteq \mathbb{R}^2$. Let γ be a nonperiodic orbit of X which exists in U for all t . Suppose that*

$$V = \{(s, r) : -\delta_1 < s < \delta_2, -\delta_2 < r < \delta_2\}$$

is a local flow box, so that $X = \partial_s$ in V and

$$\Sigma := \{(s, r) : s = 0, -\delta_2 < r < \delta_2\}$$

is the vertical transversal. Let $0 \leq t_1 < t_2 < \dots$ be a sequence of times such that $z_j := \gamma_{t_j} \in \Sigma$ and $\gamma_t \notin \Sigma$ for all $t \in (t_{j-1}, t_j)$. Then either

$$z_1 < z_2 < z_3 < \dots$$

or

$$z_1 > z_2 > z_3 > \dots$$

Here we identify Σ with its parametrisation interval $(-\delta_2, \delta_2)$.

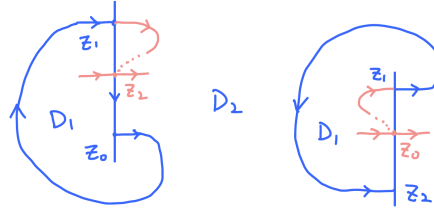
Proof. Let $z_0 < z_1$. Suppose on the contrary that $z_2 < z_1$. There are two cases to discuss.

Case I: $z_2 \in (z_0, z_1)$.

Let Γ_{01} be the concatenation of $\gamma_{(t_0, t_1)} \subseteq \Sigma^c$ and the arc connecting z_1 to z_0 along Σ . Then Γ_{01} is a simple closed curve. According to the Jordan curve theorem, Γ_{01} divides the plane into a bounded domain D_1 and an unbounded domain D_2 . Since $\gamma_{t_1+\varepsilon} \in D_2$ for small ε , one knows by connectedness that $\gamma|_{(t_1, t_2)} \subseteq D_2$. However, since $z_2 \in (z_0, z_1)$, one must have $\gamma_{t_2-\varepsilon} \in D_1$ due to the flow line pattern inside the flow box V . This is a contradiction.

Case II: $z_2 < z_0$.

Let Γ_{12} be the concatenation of $\gamma_{(t_1, t_2)} \subseteq \Sigma^c$ and the arc connecting z_2 to z_1 along Σ . Then Γ_{12} divides the plane into a bounded domain D_1 and an unbounded one D_2 . Note that $\gamma_{t_1-\varepsilon} \in D_1$ for small ε and thus $\gamma_{(t_0, t_1)} \subseteq D_1$. This is a contradiction since $\gamma_{t_0+\varepsilon} \in D_2$ for small ε .



Therefore, one concludes that $z_2 > z_1$. The case when $z_0 > z_1$ is treated in a similar way. $\square$

Lemma 1.19 has the following two important corollaries, which will be used in the proof of Theorem 1.17. A point x is said to be *positively recurrent* (respectively, *negatively recurrent*) if $x \in \omega(x)$ (respectively, if $x \in \alpha(x)$).

Lemma 1.20. *Let X be a smooth, nonvanishing vector field on a planar domain $U \subseteq \mathbb{R}^2$. Let $y \in U$ be a positively recurrent point of X and assume that $\phi_t(y)$ exists in U for all time. Then $\mathcal{O}_y$ is a periodic orbit.*

Proof. Suppose on the contrary that $\mathcal{O}_y$ is a nonperiodic orbit. Let V be a flow box centered at y . Since $y \in \omega(y)$, there exists a sequence $t_n \rightarrow \infty$ such that $\phi_{t_n}(y) \rightarrow y$. Inside the flow box V , one may adjust the time t_n to some s_n (for all large n , say $n > N$) so that

$$s_n \rightarrow \infty, \quad \phi_{s_n}(y) \in \Sigma \text{ are all distinct,} \quad \phi_{s_n}(y) \rightarrow y.$$

Note that it takes a uniformly positive amount of time for the flow line to leave the flow box once it is on Σ . As a result, there are at most finitely many return times to Σ between 0 and s_{N+1} . It follows from Lemma 1.19 that the sequence

$$y = \phi_0(y), \phi_{s_{N+1}}(y), \phi_{s_{N+2}}(y), \dots$$

is monotone on Σ . This contradicts the fact that $\phi_{s_n}(y) \rightarrow y$ as $n \rightarrow \infty$. Therefore, $\mathcal{O}_y$ is a periodic orbit. $\square$

Lemma 1.21. *Let X be a smooth nonvanishing vector field on a planar domain $U \subseteq \mathbb{R}^2$. Let $x \in U$ be given fixed. Suppose that the flow $\phi_t(x)$ of X exists for all $t \geq 0$. Let C be a periodic orbit such that $C \cap \omega(x) \neq \emptyset$. Then $\omega(x) = C$.*

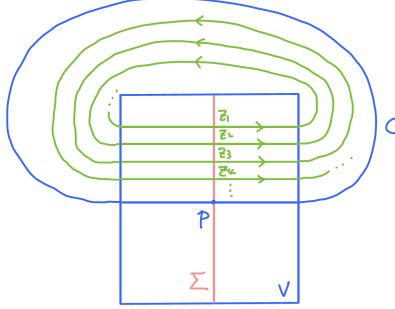
Proof. If $\mathcal{O}_x$ is periodic, one trivially has $\omega(x) = \mathcal{O}_x = C$. We therefore assume that $\mathcal{O}_x$ is nonperiodic. By assumption, let $p \in C \cap \omega(x)$. Then $\phi_{t_n}(x) \rightarrow p$ for some $t_n \rightarrow \infty$. We first claim that $C \subseteq \omega(x)$. Let $q \in C$. One can write $q = \phi_t(p)$ for some $t \geq 0$. It follows that $\phi_{t_n+t}(x) \rightarrow q$, which implies that $q \in \omega(x)$. Therefore, $C \subseteq \omega(x)$.

Conversely, let $q \notin C$. Our aim is to show that $q \notin \omega(x)$. Since C is compact, it suffices to show that

$$\lim_{t \rightarrow \infty} d(\phi_t(x), C) = 0. \tag{1.10}$$

To this end, let V be a flow box centered at p with transversal Σ . Since $\phi_{t_n}(x) \rightarrow p$ and $\mathcal{O}_x$ is nonperiodic, similar to the proof of Lemma 1.20, one can find a sequence $s_n \rightarrow \infty$ such that the following properties hold true:

- (i) $z_n := \phi_{s_n}(x) \in \Sigma$ and $\phi_{s_n}(x) \rightarrow p$;
- (ii) $\phi_t(x) \notin \Sigma$ for $t \in (s_{n-1}, s_n)$;
- (iii) the sequence $z_1, z_2, z_3, \dots$ is monotone on Σ .



Consequently, one has

$$z_n \downarrow p \quad \text{or} \quad z_n \uparrow p \quad \text{on } \Sigma. \quad (1.11)$$

Note that $z_{n+1} = P(z_n)$ for all sufficiently large n , where P is the Poincaré map defined in Section 1.3.3. By the continuity of the Poincaré map, one has

$$\lim_{z \rightarrow p} \sup_{0 \leq t \leq \tau(z)} d(\phi_t(z), C) = 0, \quad (1.12)$$

where $\tau(z)$ is the first return time of $\phi_t(z)$ to Σ . The claim (1.10) is now a simple consequence of the monotone convergence property (1.11) and the one-iteration asymptotics (1.12). $\square$

Remark 1.22. Lemma 1.21 remains valid on the torus $\mathbb{T}^2$. In fact, by cutting $\mathbb{T}^2$ along the periodic orbit C , one obtains an annulus $A \cong S^1 \times [0, 1]$ whose two boundary curves are copies of C and $\mathcal{O}_x \subseteq \text{int} A$ (assuming $\mathcal{O}_x$ is nonperiodic). By thickening A across its boundary to an open annulus U (U is diffeomorphic to a planar domain), one is now reduced to the situation of Lemma 1.21.

We are now in a position to prove the Poincaré-Bendixson theorem.

Proof of Theorem 1.17. Suppose that

$$\mathcal{O}_x^+ \subseteq K \Subset U.$$

Then $\omega(x) \subseteq K$. Since $\omega(x)$ is nonempty, compact and W -invariant, it contains a minimal compact, W -invariant subset (a nonempty, compact W -invariant subset which contains no proper, nontrivial invariant subsets), say C . Let $y \in C$. Then

$$\omega(y) \subseteq C \implies \omega(y) = C$$

since C is minimal. In particular, $y \in \omega(y)$. According to Lemma 1.20, $\mathcal{O}_y$ is a periodic orbit. Hence $\omega(x)$ contains a periodic orbit. It follows from Lemma 1.21 that $\omega(x) = C$. $\square$

Remark 1.23. Here is a simple variant of the Poincaré-Bendixson theorem. Let W be a smooth, nonvanishing vector field on a compact surface M without boundary. Suppose that there exists

$$K \text{ compact, } U \text{ open, } K \Subset U \subseteq M$$

and $t_0 \geq 0$, such that U is diffeomorphic to a planar domain and $\phi_t(x) \in K$ for all $t \geq t_0$. Then $\omega(x)$ is a periodic orbit of W . A similar assertion holds for $\alpha(x)$.

We discuss a few important applications of the Poincaré-Bendixson theorem. Let W be a smooth, nonvanishing vector field on $\mathbb{T}^2$.

Lemma 1.24. *Let $A \cong S^1 \times [0, 1]$ be an embedded annulus in $\mathbb{T}^2$ whose two boundary curves are compact leaves of $\mathcal{F}_W$. Let $x \in \text{int}A$ and assume that $\mathcal{O}_x$ is nonperiodic. Then $\alpha(x)$ and $\omega(x)$ are distinct periodic orbits.*

Proof. Let $U \supseteq A$ be a slightly larger open annulus which is diffeomorphic to a planar domain. According to the Poincaré-Bendixson theorem, both $\alpha(x)$ and $\omega(x)$ are periodic orbits and they are contained in A . Suppose on the contrary that $\alpha(x) = \omega(x) = C$. Let $p \in C$. Then both tails of $\mathcal{O}_x$ are arbitrarily close to C . In particular, there exists $T_0 > 0$ such that $\phi_t(x) \in U$ for all $|t| > T_0$.

Now choose a small flow box centered at p with transversal Σ . Similar to the proof of Lemma 1.20, one can find a two-sided sequence

$$-\infty \downarrow s_{-n} < \cdots < s_{-2} < s_{-1} < s_1 < s_2 < \cdots < s_n \uparrow \infty$$

with the following properties:

- (i) $z_n := \phi_{s_n}(x) \in \Sigma$ and $\phi_t \notin \Sigma$ for all $t \in (s_{n-1}, s_n)$ and for all $n \in \mathbb{Z}$;
- (ii) $z_n \rightarrow p$ as $n \rightarrow \pm\infty$.
- (iii) The sequence $\{z_n\}_{n=-\infty}^{\infty}$ is monotone on Σ .

But the second and third properties force the sequence to be constant, which would imply that $\mathcal{O}_x$ is periodic. This leads to a contradiction. Therefore, $\alpha(x)$ and $\omega(x)$ are distinct periodic orbits. $\square$

Remark 1.25. Lemma 1.24 may fail on the whole torus $\mathbb{T}^2$; it is certainly possible for a nonperiodic orbit to approach the same periodic orbit C in both tails. However, if one cuts $\mathbb{T}^2$ along C to obtain a compact annulus whose two boundary curves are copies of C (and they are considered as distinct periodic orbits), Lemma 1.24 is indeed applicable to this situation.

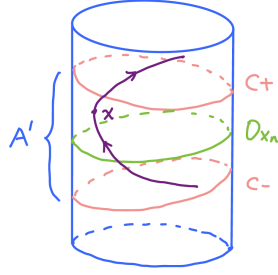
Proposition 1.26. (i) *Let $A \cong S^1 \times [0, 1]$ be an embedded annulus in $\mathbb{T}^2$ whose two boundary curves are compact leaves of $\mathcal{F}_W$. Let $\mathcal{P}_A := \{x \in A : \mathcal{O}_x \text{ is periodic}\}$. Then $\mathcal{P}_A$ is a closed subset of A .*

(ii) *Let $\mathcal{P} := \{x \in \mathbb{T}^2 : \mathcal{O}_x \text{ is periodic}\}$. Then $\mathcal{P}$ is closed.*

Proof. (i) Let $x_n \in \mathcal{P}_A$ and $x_n \rightarrow x$. Suppose on the contrary that $\mathcal{O}_x$ is nonperiodic. According to Lemma 1.24,

$$C_- := \alpha(x), \quad C_+ := \omega(x) \quad (1.13)$$

are distinct periodic orbits on A . Recall from Proposition 1.10 that both $C_\pm$ are essential, simple closed curves. As a result, they bound a subannulus A' and $x \in \text{int}A'$. Since $x_n \rightarrow x$, there is a large n such that the periodic orbit $\mathcal{O}_{x_n}$ is contained inside $\text{int}A'$, hence dividing $\text{int}A'$ into two connected components. The point x lies inside one of these two components. In order for the orbit $\mathcal{O}_x$ to approach $C_\pm$ in both tails (see (1.13)), it must cross the periodic orbit $\mathcal{O}_{x_n}$. Hence $\mathcal{O}_x \cap \mathcal{O}_{x_n} \neq \emptyset$, which is a contradiction. Therefore, $\mathcal{O}_x$ is periodic.



(ii) Let $x_n \in \mathcal{P}$ and $x_n \rightarrow x$. Suppose that $x \notin \mathcal{O}_{x_1}$; otherwise, $\mathcal{O}_x$ is periodic and there is nothing to prove. Cutting $\mathbb{T}^2$ along $\mathcal{O}_{x_1}$ yields a compact annulus A with $x_n \in \mathcal{P}_A$ and $x_n \rightarrow x \in \text{int}A$. The result of Part (i) then implies that $\mathcal{O}_x$ is periodic. $\square$

2 Suspension and Reeb component

In this section, we define the concepts of a suspension and a Reeb component, which are the two basic regimes appearing in the main classification theorem.

2.1 Suspension

We first define the notion of a suspension. Let $S^1 := \mathbb{R}/\mathbb{Z}$ and let $f : S^1 \rightarrow S^1$ be an orientation-preserving diffeomorphism. We define an equivalence relation on $S^1 \times \mathbb{R}$ by identifying

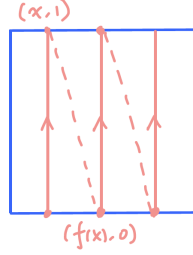
$$(x, t) \sim (f^k(x), t - k) \quad (2.1)$$

for any $(x, t) \in S^1 \times \mathbb{R}$ and $k \in \mathbb{Z}$. Let $\Sigma_f := (S^1 \times \mathbb{R})/f\mathbb{Z}$ be the quotient space under the above identification, whose quotient map is denoted as $\pi_f : S^1 \times \mathbb{R} \rightarrow \Sigma_f$. It is clear that Σ_f is diffeomorphic to the torus $\mathbb{T}^2$. The image of the vertical leaves $\{x\} \times \mathbb{R}$ under π_f , i.e.

$$\mathcal{F}_f := \{\pi_f(\{x\} \times \mathbb{R}) : x \in S^1\},$$

defines a 1D foliation on Σ_f . This foliation is induced by the vector field

$$\partial_t|_{[(x,t)]} := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} [(x, t + \varepsilon)], \quad [(x, t)] \in \Sigma_f.$$



Definition 2.1. The foliation $(\Sigma_f, \mathcal{F}_f)$ is called the *suspension* of the diffeomorphism f . The vector field ∂_t is called the *suspension field*.

Example 2.2. (i) Let $f = \text{id}_{S^1}$. The orbits of ∂_t are the vertical circles and they are all periodic with period one.

(ii) (Rational rotation) Let $f(x) := x + p/q \pmod{\mathbb{Z}}$ where p, q are coprime integers ($q > 0$). Then $f^q = \text{id}_{S^1}$. All orbits are periodic with period q .

(iii) (Irrational rotation) Let $f(x) := x + \alpha \pmod{\mathbb{Z}}$ where $\alpha \notin \mathbb{Q}$. Every orbit is diffeomorphic to $\mathbb{R}$ and is dense in $\mathbb{T}^2$.

(iv) Suppose that f has precisely one fixed point $[a] \in S^1$. Then f can be viewed as a diffeomorphism on $(a, a + 1)$ with no fixed points. Suppose further that $f(x) > x$ for all $x \in (a, a + 1)$. Then there is precisely one periodic orbit

$$C : t \mapsto [(a, t)]$$

which has period one. Every orbit other than C is nonperiodic (hence diffeomorphic to $\mathbb{R}$) and approaches C in both ends.

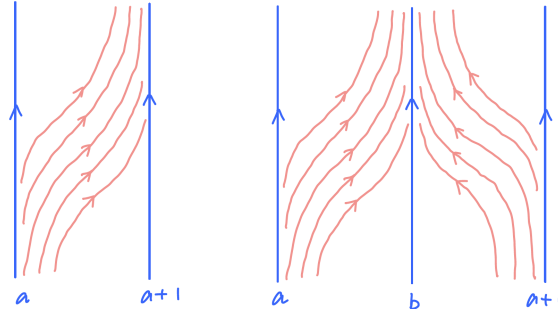


Figure 1: Suspension examples.

(v) Suppose that f has precisely two fixed points $[a], [b] \in S^1$. With suitable choice of representatives, let us assume that $b \in (a, a+1)$. Suppose that $f(x) > x$ on (a, b) and $f(x) < x$ on $(b, a+1)$. Then there are precisely two periodic orbits

$$C_a : t \mapsto [(a, t)], \quad C_b : t \mapsto [(b, t)].$$

All other orbits are nonperiodic. Figure 2.2 sketches the flow lines on the universal cover $\mathbb{R}^2$ for Cases (iv) and (v).

Remark 2.3. Periodic orbits of $(\Sigma_f, \mathcal{F}_f)$ correspond to periodic points of f . A point $x \in S^1$ is a *periodic point* of f if there exists a positive integer q such that $f^q(x) = x$. The smallest such q is called the *period* of x . For an orientation-preserving diffeomorphism $f \in \text{Diff}(S^1)$, the period of a periodic point is intrinsic to f and does not depend on the actual point x .

Let W be a smooth, nonvanishing vector field W on $\mathbb{T}^2$. We give a simple criterion under which the foliation $\mathcal{F}_W$ is conjugate to a suspension (see Definition 1.2).

Definition 2.4. A smooth, simple closed curve Σ is a *complete transversal* if it intersects every orbit of W transversally.

Proposition 2.5. *The foliation $\mathcal{F}_W$ is conjugate to a suspension if and only if there exists a complete transversal.*

Proof. (i) Necessity is obvious. In fact, the horizontal circle

$$S^1 \ni t \mapsto [(t, 0)] \in \Sigma_f$$

is a complete transversal of the suspension foliation. It remains to prove sufficiency.

(ii) Let Σ be a given complete transversal. For each $x \in \Sigma$, we define the first return time

$$\tau(x) := \inf\{t > 0 : \phi_t(x) \in \Sigma\}.$$

We first show that $\tau(x) < \infty$. Suppose on the contrary that $\phi_t(x)$ never meets Σ again at any positive time. We claim that there exists $T_0 > 0$ and a small neighbourhood N of Σ , such that $\phi_t(x) \notin N$ for all $t > T_0$. Indeed, if this were not the case, there would exist a sequence of times $t_n \rightarrow \infty$ such that $\phi_{t_n}(x)$ converges to some $y \in \Sigma$. Draw a flow box $(V; s, r)$ centered at $y = (0, 0)$ so that $\Sigma \cap V = \{s = 0\}$. One can write $\phi_{t_n}(x) = (s_n, r_n)$ under the flow box coordinates and note that $(s_n, r_n) \rightarrow (0, 0)$. Since the flow lines are precisely $s \mapsto (s, r)$ inside V , one has

$$\phi_{-s_n}(\phi_{t_n}(x)) = \phi_{t_n - s_n}(x) = (0, r_n) \in \Sigma$$

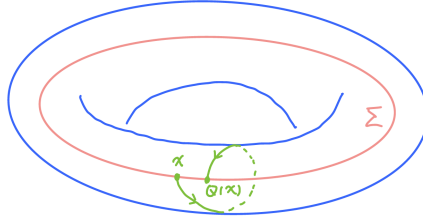
for all large n . As a result, the time $t_n - s_n$ is a positive return time for large n (because $t_n \rightarrow \infty$ and $s_n \rightarrow 0$) which leads to a contradiction. Therefore, $\{\phi_t(x) : t > 0\}$ is contained outside a tubular neighbourhood N of Σ after some

short period of time. Similar to the proof of Proposition 1.10 (i), it can be shown that any complete transversal must be essential. As a result, N^c (as a compact annulus) is contained in a planar domain. By applying the Poincaré-Bendixson theorem, one obtains a periodic orbit that is disjoint from Σ . This contradicts the completeness of Σ . Consequently, $\tau(x) < \infty$.

We define the first return map

$$Q : \Sigma \rightarrow \Sigma, \quad Q(x) := \phi_{\tau(x)}(x). \quad (2.2)$$

By realising a neighbourhood of $Q(x)$ on Σ as a level set $\{\rho = 0\}$ for some smooth function ρ , one can apply the same argument as in the proof of Lemma 1.11 (construction of the Poincaré map) to conclude that $\tau(x)$ is a smooth in x . Therefore, $Q \in C^\infty(\Sigma)$. It is indeed a diffeomorphism since Q^{-1} is defined in the same way by the return-to- Σ map of the inverse flow $-W$. Let $\sigma : S^1 \xrightarrow{\cong} \Sigma$ be an S^1 -parametrisation of Σ . Then $f := \sigma^1 \circ Q \circ \sigma \in \text{Diff}(S^1)$.



(iii) We claim that f is orientation preserving, or more precisely, $f'(r) > 0$ for all r (here we identify $S^1 := \mathbb{R}/\mathbb{Z}$ so that one has the canonical isomorphisms $T_r S^1 \cong T_{f(r)} S^1 \cong \mathbb{R}$). The argument is in spirit similar to the proof of Proposition 1.13 (positivity of the transversal characteristic number).

To this end, let Ω be a volume form on $\mathbb{T}^2$ such that

$$g(r) := \Omega(W(\sigma(r)), \dot{\sigma}(r)) > 0$$

for all $r \in S^1$. This is possible since $\mathbb{T}^2$ is orientable and Σ is transversal to W . Let

$$F(s, r) := \phi_{s\tau(\sigma(r))}(\sigma(r)) \quad (2.3)$$

and define

$$J(s, r) := \Omega(W(F(s, r)), \partial_r F(s, r)).$$

It is plain to see that

$$\begin{aligned} \partial_s F(s, r) &= \tau(\sigma(r))W(F(s, r)), \\ \partial_r F(s, r) &= (d\phi_{s\tau(\sigma(r))})\dot{\sigma}(r) + s\tau'(r)W(F(s, r)). \end{aligned} \quad (2.4)$$

Since $\dot{\sigma}(r)$ and $W(\sigma(r))$ are linearly independent, so are

$$(d\phi_{s\tau(\sigma(r))})\dot{\sigma}(r) \quad \text{and} \quad (d\phi_{s\tau(\sigma(r))})W(\sigma(r)) = W(F(s, r)).$$

As a result,

$$J(s, r) = \Omega(W(F(s, r)), (d\phi_{s\tau(\sigma(r))})\dot{\sigma}(r))$$

has constant sign. Note that $J(0, r) = g(r) > 0$. It follows that

$$J(1, r) = \Omega(W(\sigma \circ f(r)), f'(r)\dot{\sigma}(f(r))) = f'(r)g(f(r)) > 0.$$

Therefore, $f'(r) > 0$ and f is orientation preserving.

Now let Σ_f be the suspension of f . Define $\Psi : \Sigma_f \rightarrow \mathbb{T}^2$ by

$$\Psi([r, t]) := \phi_{t\tau(\sigma(r))}(\sigma(r)), \quad r \in S^1, t \in [0, 1].$$

The definition of Ψ is consistent at the junction $t = 0 = 1 \pmod{\mathbb{Z}}$. In addition, Ψ is a homeomorphism and maps the suspension foliation to $\mathcal{F}_W$. The issue is that Ψ may fail to be a diffeomorphism. This is because

$$\frac{\partial}{\partial t} \Big|_{t=1-} \phi_{t\tau(\sigma(r))}(\sigma(r)) \quad \text{and} \quad \frac{\partial}{\partial t} \Big|_{t=0+} \phi_{t\tau(\sigma \circ f(r))}(\sigma \circ f(r))$$

need not coincide. They coincide precisely when $\tau(\sigma(r)) = \tau(\sigma \circ f(r))$, which needs not be true in general.

The above issue can be fixed by a simple leaf reparametrisation. Let

$$\theta(r, t) := ct + (\tau(\sigma(r)) - c)\beta(t), \quad r \in S^1, t \in [0, 1].$$

Here

$$0 < c < \inf_{r \in S^1} \tau(\sigma(r))$$

and $\beta : [0, 1] \rightarrow [0, 1]$ is smooth, nondecreasing and satisfies

$$\beta = 0 \text{ near } 0, \quad \beta = 1 \text{ near } 1.$$

Then one has

$$\theta(r, 0) = 0, \quad \theta(r, 1) = \tau(\sigma(r)),$$

$$\theta(r, t) = ct \text{ near } t = 0,$$

$$\theta(r, t) = \tau(\sigma(r)) + c(t - 1) \text{ near } t = 1.$$

This ensures that all t -derivatives match at the junction $t = 0 = 1 \pmod{\mathbb{Z}}$. Now we adjust the definition of Ψ to be

$$\hat{\Psi} : \Sigma_f \rightarrow \mathbb{T}^2, \quad \hat{\Psi}([r, t]) := \phi_{\theta(r, t)}(\sigma(r)).$$

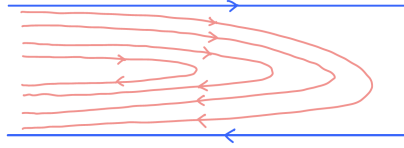
This new map $\hat{\Psi}$ is now easily seen to be a diffeomorphism, which provides a smooth conjugation between Σ_f and $\mathbb{T}^2$. $\square$

2.2 Reeb component

Next, we define the notion of a Reeb component.

Definition 2.6. We say that $\mathcal{F}_W$ contains a Reeb component if there exists an embedded annulus $A \cong S^1 \times [0, 1]$ inside $\mathbb{T}^2$ such that the following two properties are satisfied:

- (i) ∂A is the union of two compact leaves that are in opposite homology classes (under the W -orientation);
- (ii) there are no compact leaves of $\mathcal{F}_W$ in the interior of A .



The following proposition provides a simple criterion for the existence of a Reeb component.

Proposition 2.7. *The foliation $\mathcal{F}_W$ has a Reeb component if and only if there exist two periodic orbits with opposite homology classes.*

Proof. Necessity follows immediate from the definition and Proposition 1.10. We now prove sufficiency. Let C_+, C_- be two periodic orbits with opposite homology classes. Let A be a compact annulus bounded by $C_\pm$. Define $\mathcal{P}_A^\pm$ to be the set of points in A whose orbit is homologous to $C_\pm$. According to Proposition 1.26, it is easily seen that both $\mathcal{P}_A^\pm$ are closed subsets of A (a sequence of periodic orbits with the same homology class cannot converge to a periodic orbit with an opposite direction). Therefore,

$$\inf_{x \in \mathcal{P}_A^+, y \in \mathcal{P}_A^-} d(x, y) > 0.$$

Let $x_* \in \mathcal{P}_A^+$ and $y_* \in \mathcal{P}_A^-$ be two points achieving the above infimum. It follows that there are no periodic orbits between $\mathcal{O}_{x_*}$ and $\mathcal{O}_{y_*}$ inside A . As a consequence, they bound a Reeb component. $\square$

2.3 Mutual exclusiveness of the two regimes

We now show that suspension and Reeb component are mutually exclusive.

Proposition 2.8. *Suppose $\mathcal{F}_W$ is conjugate to a suspension. Then it cannot contain a Reeb component.*

Proof. Let $\Psi : \mathcal{F}_W \rightarrow \Sigma_f$ be a conjugation to a suspension

$$\Sigma_f = \{[x, t] : x \in S^1, t \in \mathbb{R}, (x, t) \sim (f^k(x), t - k)\}.$$

Let $\eta := \Psi^* dt$, where dt is the vertical form on Σ_f defined by

$$\langle dt, \partial_x \rangle = 0, \quad \langle dt, \partial_t \rangle = 1.$$

The image of a periodic orbit γ of W under Ψ is a reparametrisation of a periodic orbit β of the suspension field ∂_t . The period of β is a positive integer q , whose value is uniquely determined by f and is dependent of the choice of γ (cf. Remark 2.3). It follows that

$$\int_{\gamma} \eta = \int_{\beta} dt = q.$$

As a consequence, having two periodic orbits with opposite homology classes is not possible in this case. $\square$

3 The main classification theorem

In this section, we state and prove the main classification theorem for nondegenerate torus flows.

Theorem 3.1. *Let $\mathcal{F}_W$ be the 1D foliation associated with a smooth, nonvanishing vector field W on $\mathbb{T}^2$. Then precisely one of the following two mutually exclusive possibilities holds:*

- (i) $\mathcal{F}_W$ is conjugate to a suspension;
- (ii) $\mathcal{F}_W$ contains a Reeb component.

In what follows, we develop the proof of Theorem 3.1. We will consider two cases separately according to whether W has periodic orbits.

3.1 Case I: No periodic orbits

We first assume that W does not have periodic orbits (in particular, there are no Reeb components). Our aim is to prove that $\mathcal{F}_W$ is conjugate to a suspension in this case. According to Proposition 2.5, it is equivalent to showing that there exists a complete transversal.

Lemma 3.2. *Suppose that there exists a nonperiodic orbit. Then there exists a smooth, simple closed curve Σ that is everywhere transversal to W .*

Proof. Let γ be a nonperiodic orbit. There exists $t_n \rightarrow \infty$ such that γ_{t_n} converges to some y . By drawing a flow box V around y and adjusting t_n if necessary (cf. Proof of Lemma 1.20), one can find $s_0 < s_1$ such that $z_i = \gamma_{s_i} \in \Sigma$ ($i = 0, 1$) and $\gamma_t \notin \Sigma$ for all $t \in (s_0, s_1)$. Here Σ denotes the vertical transversal in V passing through y . Note that $z_0 \neq z_1$.

Let us fix a parametrisation $\sigma : (-c, c) \rightarrow \Sigma$ so that $z_0 = \sigma(0)$ and $z_1 = \sigma(d)$ for some $d \in (0, c)$. Define the first return map $f : (-\delta, \delta) \rightarrow (-c, c)$ by the relation

$$\phi_{\tau(r)}(\sigma(r)) = \sigma(f(r)), \quad r \in (-\delta, \delta),$$

where δ is a sufficiently small number and $\tau(r)$ is the first return time of $\phi_t(\sigma(r))$ to Σ (cf. (2.2)). Note that $f(0) = d$. Consider the function

$$F(s, r) := \phi_{s\tau(r)}(\sigma(r))$$

where $s \in [0, 1]$ and $r \in (-\delta, \delta)$ (cf. (2.3)). It is clear that

$$F(0, r) = \sigma(r), \quad F(1, r) = \sigma(f(r)).$$

By shrinking δ if necessary, the map

$$F : [0, 1] \times (-\delta, \delta) \rightarrow \mathbb{T}^2$$

is injective and

$$F((0, 1) \times (-\delta, \delta)) \cap \Sigma = \emptyset.$$

Similar to the proof of Proposition 2.5, one can show that $f'(r) > 0$ for all $r \in (-\delta, \delta)$. As a result, there exists $\eta \in (-\delta, 0)$ such that $f(\eta) > 0$.

Fix a function $\rho : [0, 1] \rightarrow [\eta, 0]$ such that $\rho(0) = 0$, $\rho(1) = \eta$ and $\rho'(s) < 0$ for all $s \in [0, 1]$. Define the curve

$$\alpha_s := F(s, \rho(s)), \quad 0 \leq s \leq 1.$$

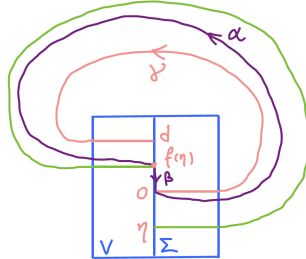
Note that α is a simple curve that joins z_0 to $\sigma(f(\eta))$ outside Σ . By definition, one has

$$\dot{\alpha}_s = \partial_s F(s, \rho(s)) + \rho'(s) \partial_r F(s, \rho(s)).$$

The vector $\partial_s F(s, \rho(s)) \in \text{Span}\{W(\alpha_s)\}$. In addition, since $\rho'(s) < 0$, the vector $\partial_r F(s, \rho(s))$ is linearly independent of $\partial_s F(s, \rho(s))$ (cf. (2.4)). As a consequence, α is everywhere transversal to W . Let $\beta : [1, 2] \rightarrow \Sigma$ be the arc joining $\sigma(f(\eta))$ back to z_0 along Σ . Apparently, β is also everywhere transversal to W . The concatenated curve

$$\zeta_s := \begin{cases} \alpha_s, & 0 \leq s \leq 1; \\ \beta_s, & 1 \leq s \leq 2 \end{cases}$$

is now a simple closed curve, which is smooth and everywhere transversal to W except at the two corners z_0 and $\sigma(f(\eta))$.



It remains to see that the two corners can be smoothed locally without losing the transversal property. Let Ω be a volume form on $\mathbb{T}^2$ such that $\Omega(W(z_0), \dot{\sigma}(0)) > 0$. The function

$$\psi(s) := \Omega(W(\zeta_s), \dot{\zeta}_s)$$

has constant sign on $(0, 1)$ and $(1, 2)$. We claim that

$$\operatorname{sgn}\psi(1-) = \operatorname{sgn}\psi(1+) = \operatorname{sgn}\psi(2-) = \operatorname{sgn}\psi(0+). \quad (3.1)$$

Indeed, since $\rho'(0) < 0$ one has

$$\begin{aligned} \operatorname{sgn}\psi(1-) &= \operatorname{sgn}\psi(0+) = \operatorname{sgn}\Omega(W(z_0), \rho'(0)\dot{\sigma}(0)) \\ &= -\operatorname{sgn}\Omega(W(z_0), \dot{\sigma}(0)). \end{aligned}$$

In addition, by the definition of β one also has

$$\operatorname{sgn}\psi(1+) = \operatorname{sgn}(2-) = -\operatorname{sgn}\Omega(W(z_0), \dot{\sigma}(0))$$

As a consequence, the relation (3.1) holds. This relation shows that at any of the two corners $s_* = 0, 1$, both vectors $\dot{\zeta}_{s_*-}$ and $\dot{\zeta}_{s_*+}$ ($0-$ is the same as $2-$) lie in the cone

$$C_{s_*} := \{v : \Omega(W(\zeta_{s_*}), v) < 0\}.$$

It is clearly possible to smooth the corner ζ_{s_*} locally in a way such that the resulting new smooth arc $\tilde{\zeta}_s$ still satisfies $\Omega(W(\tilde{\zeta}_s), v) < 0$ when s is near s_* . One thus obtains a smooth, simple closed curve which is everywhere transversal to W . $\square$

Proof of Theorem 3.1: No periodic orbits case. Suppose that W has no periodic orbits. According to Lemma 3.2, there exists a smooth, simple closed transversal curve Σ . If Σ does not intersect some orbit γ , the tails of γ will be trapped outside a tubular neighbourhood of Σ . The Poincaré-Bendixson theorem then produces a periodic orbit, which leads to a contradiction (cf. Part (ii) in the proof of Proposition 2.5). Therefore, Σ is complete. It follows from Proposition 2.5 that $\mathcal{F}_W$ is conjugate to a suspension. $\square$

3.2 Case II: At least one periodic orbit

Now we assume that W has at least one periodic orbit. In this case, we aim at showing that

$$\mathcal{F}_W \text{ has no Reeb component} \implies \mathcal{F}_W \text{ is conjugate to a suspension.}$$

The key lemma for this part is stated as follows.

Lemma 3.3. *Let $A \cong S^1 \times [0, 1]$ be an embedded annulus whose boundary consists of two compact leaves C_0 and C_1 . Suppose that A does not contain any Reeb component. Then there exists a smooth, simple curve $c : [0, 1] \rightarrow A$ such that $c_0 \in C_0$, $c_1 \in C_1$ and c is everywhere transversal to W on A .*

Proof. First of all, the periodic orbits C_0, C_1 must have the same homology class; otherwise, there would be a Reeb component inside A according to Proposition 2.7, which contradicts the assumption. Let Ω be a volume form on A such that $\Omega(W(x), v) > 0$ when $x \in C_0$ and $v \in T_x \mathbb{T}^2$ is a transversal vector pointing into A . A vector $v \in T_y \mathbb{T}^2$ is said to be *positively transversal* to W at $y \in A$ if $\Omega(W(y), v) > 0$. A smooth curve c_t in A is *positively transversal* to W if $\Omega(W(c_t), \dot{c}_t) > 0$ for all t . Let us define

$$\mathcal{R} := \{y \in A : \exists \text{ a smooth, simple, positively transversal} \\ c : [0, 1] \rightarrow A \text{ s.t. } c_0 \in C_0, c_1 = y\}.$$

Our aim is to show that $\mathcal{R} \cap C_1 \neq \emptyset$. We divide the argument into several steps.

(i) Let N_1 be an open half-tubular neighbourhood of C_1 in A (a lower collar). We claim that $\mathcal{R} \cap N_1 \neq \emptyset$. Suppose this is true and let $y \in \mathcal{R} \cap N_1$. By definition, there is a simple, positively transversal curve c from C_0 to y . By travelling along the fiber in N_1 , one can easily continue c to the boundary C_1 . The resulting curve is smooth and positively transversal to W except at the junction. Similar to the proof of Lemma 3.2, one can smooth the corner to obtain a smooth, simple, positively transversal curve from C_0 to C_1 . Now it remains to show that $\mathcal{R} \cap N_1 \neq \emptyset$. In what follows, we assume on the contrary that $\mathcal{R} \subseteq A \setminus N_1$.

(ii) We first show that $\mathcal{R} \cap \text{int} A$ is an open and nonempty subset of A . Indeed, any point inside a half-tubular neighbourhood of C_0 in A (an upper collar) is clearly reachable from C_0 . As a result, $\mathcal{R} \cap \text{int} A \neq \emptyset$. Next, let $y \in \mathcal{R} \cap \text{int} A$. Draw a flow box $(V; s, r) \subseteq \text{int} A$ centered at $y = (0, 0)$. We assume that the transversal coordinate r is chosen such that a curve r_t being positively transversal means $\dot{r}_t > 0$. Let c be a smooth, simple and positively transversal arc from C_0 to y . Let $z \in V \cap c$. By positive transversality, one must have $r(z) < 0$. It follows that all points in

$$V' := \{x \in V : r(x) > r(z)\}$$

is reachable from z by a simple positive transversal. By smoothing the corner if necessary, one concludes that $V' \subseteq \mathcal{R}$. Therefore, $\mathcal{R} \cap \text{int} A$ is an open subset of A .

(iii) Define $\partial_A \mathcal{R} := \bar{\mathcal{R}} \setminus \mathcal{R}$. Note that $\partial_A \mathcal{R} \neq \emptyset$ under the current assumption that $\mathcal{R} \subseteq A \setminus N_1$, since $\mathcal{R} \cap \text{int} A$ is open. We claim that $\partial_A \mathcal{R}$ is a single compact leaf of $\mathcal{F}_W$.

To this end, let $y \in \partial_A \mathcal{R}$. Draw a flow box $(V; s, r)$ centered at $y = (0, 0)$, whose transversal coordinate r is oriented in the same way as in Part (ii). The key observation is that

$$V \cap \mathcal{R} = V \cap \{r > 0\}. \quad (3.2)$$

Let $z \in V \cap \{r > 0\}$. Since $y \in \partial_A \mathcal{R}$ and $\mathcal{R}$ is open in $\text{int} A$, there exists $y' \in V \cap \mathcal{R}$ such that $0 < r(y') < r(z)$. It follows that z is reachable by a positive transversal from y' . Upon smoothing the corner at y' , one concludes that $z \in \mathcal{R}$. To prove the other direction, one observes that $V \cap \{r < 0\}$ is not

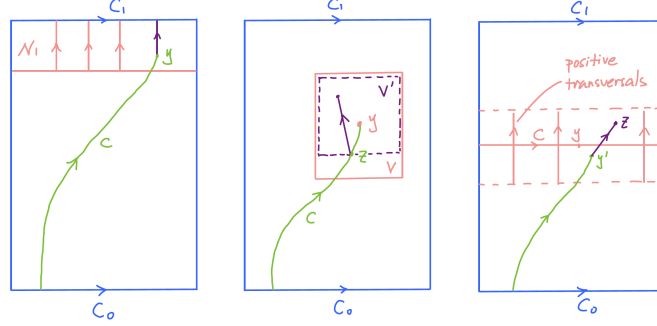


Figure 2: The left, middle and right figures correspond to Part (i), (ii) and (iv) respectively.

reachable; otherwise, y would be reachable which is a contradiction. This also implies that $V \cap \{r = 0\}$ is not reachable, since the only way to reach $V \cap \{r = 0\}$ is through $V \cap \{r < 0\}$. As a consequence, $V \cap \{r \leq 0\} \subseteq \mathcal{R}^c$. The relation (3.2) thus follows. An immediate corollary of (3.2) is that $V \cap \partial_A \mathcal{R} = V \cap \{r = 0\}$. In particular, $\partial_A \mathcal{R}$ is a 1D submanifold of A .

The next observation is that $\partial_A \mathcal{R}$ is a disjoint union of leaves. Indeed, let $x \in \partial_A \mathcal{R}$ and define

$$I_x := \{t \in \mathbb{R} : \phi_t(x) \in \partial_A \mathcal{R}\}.$$

Since $\partial_A \mathcal{R}$ is closed in A , it is obvious that I_x is closed. To prove that I_x is open, let $y = \phi_t(x) \in \partial_A \mathcal{R}$. By applying the local relation (3.2) to y , one sees that $\phi_s(x) \in \partial_A \mathcal{R}$ for s near t (because $V \cap \{r = 0\}$ is a flow line). Therefore, I_x is both open and closed, hence being equal to $\mathbb{R}$.

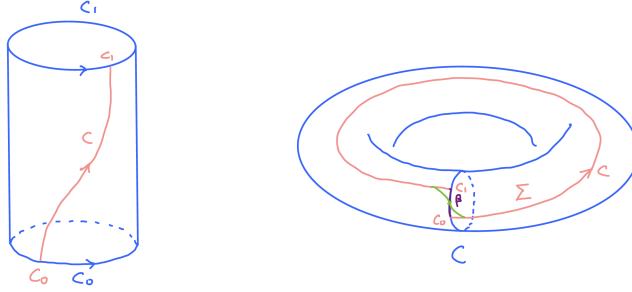
Since $\partial_A \mathcal{R}$ is a closed, embedded 1D submanifold of the compact annulus A , every connected component of $\partial_A \mathcal{R}$ has to be compact. As a consequence, $\partial_A \mathcal{R}$ is a disjoint union of compact leaves. Suppose that there are at least two such leaves, say C', C'' (we assume that C' is closer to C_0). Since a small lower collar of C'' is reachable, any such positive transversal would intersect C' . But $C' \subseteq \partial_A \mathcal{R} \subseteq \mathcal{R}^c$. This leads to a contradiction. Therefore, $\partial_A \mathcal{R}$ has only one connected component, i.e. a compact leaf.

(iv) Let $\partial_A \mathcal{R} = C$ for some compact leaf C according to Part (iii). We claim that C and C_0 are in opposite homology classes. Suppose this is not true. Due to our orientation convention, positively transversal vectors at $y \in C$ are pointing towards the C_1 -side. Let N be a tubular neighborhood of C . Note that C divides N into two halves: N_+ (the component on the C_1 -side) and N_- (the component on the C_0 -side). Let $z \in N_+$ and $y \in C = \partial_A \mathcal{R}$. Then a sufficiently close-to- y element $y' \in N_-$ is reachable by a positive transversal from C_0 . By continuing this transversal towards z inside N in the obvious way (along fiber directions towards C_1), one obtains a positive transversal that will intersect C . This contradicts the fact that $C \subseteq \mathcal{R}^c$. Therefore, C and C_0

must have opposite directions. It follows from Proposition 2.7 that there is a Reeb component inside the annulus bounded by C_0 and C . This contradicts the assumption of the lemma. To summarise, one arrives at the conclusion that the assumption $\mathcal{R} \subseteq A \setminus N_1$ cannot hold. In other words, one must have $\mathcal{R} \cap N_1 \neq \emptyset$. The proof of the lemma is now complete. $\square$

Proof of Theorem 3.1: At least one periodic orbit case. Suppose that $\mathcal{F}_W$ has no Reeb component and there is a periodic orbit C of W . Cutting the torus $\mathbb{T}^2$ along C produces a compact annulus A whose boundary consists of two compact leaves C_0 and C_1 (they are copies of C). Since A does not contain any Reeb component, according to Lemma 3.3, there exists a smooth, simple transversal $c : [0, 1] \rightarrow A$ from $c_0 \in C_0$ to $c_1 \in C_1$. By gluing back the torus along C , the curve c becomes a smooth, simple transversal such that $c_0, c_1 \in C$ and $c_t \notin C$ for all $t \in (0, 1)$. Let β be the arc joining c_1 to c_0 along C .

Similar to the proof of Lemma 3.2, by suitably deforming β along transversal directions (e.g. using the fiber coordinate in a tubular neighbourhood of C) and corner smoothing, one obtains a smooth, simple closed transversal Σ that intersects C precisely once. In particular, $I(\Sigma, C) = \pm 1$.



Suppose that Σ is not complete. The Poincaré-Bendixson theorem then produces a periodic orbit C' which does not intersect Σ . In particular, $I(\Sigma, C') = 0$. But one knows from Proposition 1.10 that $[C'] = \pm[C]$. This leads to a contradiction. Therefore, Σ is a complete transversal. It follows from Proposition 2.5 that $\mathcal{F}_W$ is conjugate to a suspension. $\square$

4 The dynamical picture of nondegenerate torus flows

Let us summarise the dynamical picture of a nondegenerate torus flow. Let W be a smooth, nonvanishing vector field on $\mathbb{T}^2$. Recall from Proposition 1.26 that the set

$$\mathcal{P} := \{x : \mathcal{O}_x \text{ is periodic}\}$$

is closed. As a result, one can decompose $\mathbb{T}^2 \setminus \mathcal{P}$ into an at most countable union of disjoint, connected, open subsets U_α :

$$\mathbb{T}^2 \setminus \mathcal{P} = \bigsqcup_{\alpha} U_{\alpha}.$$

Lemma 4.1. *Suppose that $\mathcal{P} \neq \emptyset, \mathbb{T}^2$. Then each U_α is an open annulus whose boundary consists of two periodic orbits (which may coincide if $\mathcal{P}$ is a single periodic orbit). All orbits inside U_α are nonperiodic.*

Proof. Fix a periodic orbit C_* and cut the torus along C_* . This produces a compact annulus $A \cong S^1 \times [0, 1]$ whose boundary consists of two copies C_0, C_1 of C_* and whose interior is diffeomorphic to $\mathbb{T}^2 \setminus C$ (we view C_0, C_1 as distinct periodic orbits). Let $\mathcal{C}$ be the collection of periodic orbits on A . A given periodic orbit $C \in \mathcal{C} \setminus \{C_0, C_1\}$ divides A into two connected components: $A_-(C)$ (the one containing C_0) and $A_+(C)$ (the one containing C_1). As a convention, $A_-(C_0) = \emptyset$ and $A_+(C_0) = A \setminus C_0$ (respectively, $A_-(C_1) = A \setminus C_1$ and $A_+(C_1) = \emptyset$).

Now let U_α be a connected component of $A \setminus \mathcal{P}$. Given $x \in U_\alpha$, we define

$$\begin{aligned} \mathcal{P}_-(x) &:= \{C \in \mathcal{C} : x \in A_+(C)\}, \\ \mathcal{P}_+(x) &:= \{C \in \mathcal{C} : x \in A_-(C)\} \end{aligned}$$

respectively. Let

$$S_- := \bigcup_{C \in \mathcal{P}_-(x)} C$$

and define

$$d_- := \inf_{y \in S_-} d(x, y).$$

Note that $d_- \in (0, \infty)$ because S_- is nonempty ($C_0 \in S_-$) and $U_\alpha \subseteq \mathcal{P}^c$ is open. Let $y_n \in S_-$ be a sequence such that $d(x, y_n) \rightarrow d_-$. Since A is compact, by passing to a subsequence one can find some z with $y_n \rightarrow z$ and thus $d(x, z) = d_-$. It follows from Proposition 1.26 that $C_-(x) := \mathcal{O}_z$ is a periodic orbit and one has $C_-(x) \in \mathcal{P}_-(x)$. One can identify another extremal periodic orbit $C_+(x) \in \mathcal{P}_+(x)$ in a similar way. The two orbits $C_-(x)$ and $C_+(x)$ bound an annulus whose interior is precisely U_α . By construction, there are no periodic orbits inside U_α . $\square$

Suppose that $\mathcal{P}$ is not a single periodic orbit nor the whole torus. Let $\partial U_\alpha = C_1 \sqcup C_2$ and let $x \in U_\alpha$. According to Lemma 1.24, one has either $\alpha(x) = C_1$, $\omega(x) = C_2$ or $\alpha(x) = C_2$, $\omega(x) = C_1$. In other words, every orbit inside U_α approaches the two distinct boundary leaves in both ends. If C_1, C_2 have opposite homology classes, $\overline{U_\alpha}$ is a Reeb component. In this case, it is clear from the proof of Proposition 2.7 that there is at least another Reeb component inside $\overline{U_\alpha}^c$.

On the other hand, if C_1, C_2 have the same orientation, $\overline{U_\alpha}$ is conjugate to a *suspension annulus*. The notion of a suspension annulus is similar to a suspension, except that one does not close the circle. More precisely, let $f :$

$[0, 1] \rightarrow [0, 1]$ be a diffeomorphism such that $f(0) = 0$ and $f(1) = 1$. Define the quotient space

$$\Xi_f := ([0, 1] \times \mathbb{R}) / \sim$$

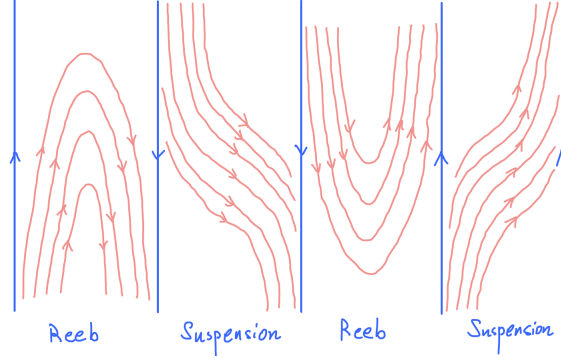
where the equivalence relation $\sim$ is defined by (2.1). The foliation of vertical leafs $t \mapsto [(x, t)]$ is the suspension annulus with respect to f . Topologically, Ξ_f is a compact annulus whose boundary curves are the two periodic leaves

$$t \mapsto [(0, t)], \quad t \mapsto [(1, t)].$$

Suppose that ∂U_α consists of two periodic orbits with the same homology class. By using the same argument as in Section 3.1, one can show that $\overline{U_\alpha}$ is conjugate to a suspension annulus.

If $\mathcal{P} = C$ is a single periodic orbit, the torus $\mathbb{T}^2$ is conjugate to a suspension of some $f \in \text{Diff}^+(S^1)$ with exactly one periodic point. Every nonperiodic orbit approaches C in both tails. If $\mathcal{P} = \emptyset$, every orbit is nonperiodic and none of their tails will be trapped in a compact subset of a planar region. In this case, $\mathcal{F}_W$ is also conjugate to a suspension whose suspension map $f \in \text{Diff}^+(S^1)$ does not have periodic points.

A typical dynamical picture of the foliation $\mathcal{F}_W$ is sketched as follows.



Remark 4.2. It is possible to have a continuum family of periodic orbits, e.g. $\mathcal{P}$ can be a Cantor-like set.

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