The Heat Kernel on Hyperbolic Space

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Abstract

In these notes, we discuss the beautiful approach of A. Grigor'yan and M. Noguchi [GN98] for the derivation of the explicit heat kernel formula on the hyperbolic space. Their method is based on solving the wave equation by using Ásgeirsson's mean value theorem, which is in turn a consequence of the Darboux equation whose proof in the hyperbolic space relies on properties of invariant differential operators at the group-theoretic level.

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1 The hyperbolic heat kernel formula

The main purpose of these notes is to give a relatively self-contained derivation of the explicit formula for the heat kernel on the standard hyperbolic space, following the rather beautiful approach of A. Grigor'yan and M. Noguchi [GN98].

Let $M = \mathbb{H}^n$ $(n \geqslant 2)$ be a simply-connected, complete Riemannian manifold with constant curvature $K = -1$. Let Δ denote the Laplace-Beltrami operator on M. The *heat kernel* on M is the fundamental solution $p(x, y, t)$ to the heat equation

$$\frac{\partial p}{\partial t} = \Delta p.$$

Given any initial condition $f : M \to \mathbb{R}$, the solution to the Cauchy problem

$$\frac{\partial u}{\partial t} = \Delta u, \quad u|_{t=0} = f$$

is then given by

$$u(x, t) = \int_M p(x, y, t) f(y) dy,$$

where dy denotes the volume form on M. Due to the homogeneity of M, it is clear that $p(x, y, t)$ depends only on t and the hyperbolic distance between x, y . As a result, one can write $p(x, y, t) = p_n(\rho, t)$ where $\rho \triangleq d(x, y)$ and the subscript n keeps track of the dimension. The following theorem provides an explicit formula for $p_n(\rho, t)$.

Theorem 1.1. *(i) Let $n = 2m + 1$ $(m \geqslant 1)$. Then*

$$p_n(\rho, t) = \frac{(-1)^m}{2^m \pi^m} \cdot \frac{1}{\sqrt{4\pi t}} \Big(\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \Big)^m e^{-m^2 t - \frac{\rho^2}{4t}}. \tag{1.1}$$

(ii) Let $n = 2m + 2$ $(m \geqslant 0)$. Then

$$p_n(\rho, t) = \frac{(-1)^m}{2^{m+5/2} \pi^{m+3/2}} t^{-3/2} e^{-\frac{(2m+1)^2}{4} t} \Big(\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \Big)^m \int_\rho^\infty \frac{s e^{-\frac{s^2}{4t}} ds}{\sqrt{\cosh s - \cosh \rho}}. \tag{1.2}$$

Remark 1.1. According to Theorem 1.1, it is plain to check that $p_n(\rho, t)$ decays like

$$t^{-3/2} e^{-\frac{(n-1)^2}{4} t}$$

as $t \to \infty$. The exponential factor $\frac{(n-1)^2}{4}$ is the bottom spectrum of $-\Delta \geqslant 0$ in $L^2(M)$ (cf. [McK70]); one has

$$\frac{(n-1)^2}{4} = \inf_{f \in C_c^\infty(M)} \frac{\langle -\Delta f, f \rangle_{L^2}}{\langle f, f \rangle_{L^2}}.$$

The polynomial factor $3/2$ has an algebraic nature which accounts for the fact that the isometry group of M is a semisimple Lie group with rank one. For general noncompact symmetric spaces $X = G/K$, one encounters a factor $t^{k/2}$ instead of $t^{3/2}$ where k is the rank of G plus twice the number of positive indivisible roots (cf. [AJ99]).

2 The main idea of A. Grigor'yan and M. Noguchi

Theorem 1.1 is classical and it has several different proofs. We present the approach of Grigor'yan-Noguchi [GN98] as it contains deep geometric-analytic insights that are of fundamental importance. We first explain their essential idea and then develop the relevant details in later sections. Their approach is based on two basic observations.

(i) There is a general (explicit) relation between the heat equation and the wave equation from the perspective of Fourier transform.

(ii) The wave equation on hyperbolic space can be solved explicitly by using the so-called *Ásgeirsson's mean value theorem.*

2.1 The heat-wave relation

To elaborate the first point, suppose that L is a nonpositive elliptic operator on a Riemannian manifold X. Let $f : X \to \mathbb{R}$ be a given initial condition. The solution to the heat equation

$$\frac{\partial v}{\partial t} = Lv, \ v|_{t=0} = f$$

can formally be represented as

$$v(x,t) = (e^{tL} f)(x).$$

Respectively, the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = Lu, \ u|_{t=0} = f, \ \frac{\partial u}{\partial t}|_{t=0} = 0$$

can be written as

$$u(x,t) = \big(\cos(t\sqrt{-L})f\big)(x).$$

The connection between the two equations is inspired by the following elementary Fourier identity:

$$e^{-t\lambda^2} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} \cos(s\lambda) ds.$$

By taking $\lambda = \sqrt{-L}$, one finds that

$$e^{tL} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} \cos(s\sqrt{-L}) ds. \tag{2.1}$$

As a consequence,

$$v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} u(x,s) ds. \tag{2.2}$$

2.2 Ásgeirsson's mean value theorem

In view of the relation (2.2), a reasonable route of proving Theorem 1.1 is to solve the wave equation explicitly. In abstract contexts, there is no reason that this is an easier task. However, on the hyperbolic space $M = \mathbb{H}^n$ (as well as on Euclidean spaces) this is indeed possible due to a mean value theorem of Ásgeirsson.

Theorem 2.1 (Ásgeirsson's mean value theorem)**.** *Let $u \in C^2(M \times M)$ be such that*

$$\Delta_x u(x,y) = \Delta_y u(x,y) \quad \forall (x,y) \in M \times M. \tag{2.3}$$

Then one has

$$\int_{S_r(x_0)} u(x,y_0) d\omega(x) = \int_{S_r(y_0)} u(x_0,y) d\omega(y)$$

for all $r > 0$, where $S_r(x_0) \triangleq \{x \in M : d(x,x_0) = r\}$ and $d\omega(x)$ denotes the volume form on the hyperbolic sphere $S_r(x_0)$.

Let us now demonstrate how Theorem 2.1 easily leads to the explicit solution to the wave equation in the Euclidean space $\mathbb{R}^n$ (Theorem 2.1 is also valid on $\mathbb{R}^n$). Let $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfy

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u, & (x,t) \in \mathbb{R}^n \times \mathbb{R}; \\ u|_{t=0} = u_0, \ \frac{\partial u}{\partial t}|_{t=0} = u_1, \end{cases}$$

where by abuse of notation Δ is the Euclidean Laplacian for now. The function

$$v(x_1, \cdots, x_n, y_1, \cdots, y_n) \triangleq u(x_1, \cdots, x_n, y_1)$$

satisfies the condition (2.3). By taking $x_0 = x$ and $y_0 = 0$ in the theorem, one has

$$\int_{S_r(x)} u_0(z) d\omega(z) = \int_{S_r(0)} u(x, y_1) d\omega(y). \tag{2.4}$$

Let us introduce the *spherical mean operator*

$$(M^r f)(x) \triangleq \frac{1}{A(r)} \int_{S_r(x)} f(z) d\omega(z), \quad f \in C(\mathbb{R}^n),$$

where $A(r) \triangleq \Omega_n r^{n-1}$ is the Euclidean area of $S_r(x)$ and

$$\Omega_n \triangleq \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

denotes the area of the unit sphere in $\mathbb{R}^n$. By disintegrating the right hand side over the hyperspheres $\{y \in S_r(x) : y_1 = s\}$ $(s \in [-r, r])$, the equation (2.4) can be further expressed as

$$\Omega_n r^{n-1} (M^r u_0)(x) = \Omega_{n-1} r \int_{-r}^{r} (r^2 - s^2)^{(n-3)/2} u(x,s) ds.$$

Assume for the moment that $u_1 = 0$. Then $u(x,t)$ is an even function in t and one has

$$\Omega_n r^{n-1}(M^r u_0)(x) = 2\Omega_{n-1} r \int_0^r (r^2 - s^2)^{(n-3)/2} u(x,s) ds.$$

This is an Abel-type equation which can be inverted explicitly. One then recovers $u(x,r)$ from u_0. To treat the more general case, by the superposition principle (i.e. additivity of initial conditions) it is enough to consider the situation where $u_0 = 0$ and $u_1 \neq 0$. Let $w(x,t) \triangleq u_t(x,t)$ in this case. Then w satisfies the wave equation with initial conditions

$$w|_{t=0} = u_1, \ w_t|_{t=0} = u_{tt}|_{t=0} = \Delta u|_{t=0} = \Delta u_0 = 0.$$

This reduces to the previous situation and w can thus be found explicitly from u_1. One then obtains u by integration. We will not give the details since the Euclidean case is not our primary interest here. The hyperbolic situation follows essentially the same lines but with more involved local calculation based on the hyperbolic metric (see Section 5.3).

2.3 The Darboux equation

According to the above discussion, the core ingredient of the program is the proof of Ásgeirsson's mean value theorem. This turns out to be a consequence of the so-called *Darboux equation* for the operator M^r which is stated as follows. Recall that

$$(M^r f)(x) \triangleq \frac{1}{A(r)} \int_{S_r(x)} f(z) d\omega(z), \quad f \in C(M),$$

where $A(r)$ denotes the hyperbolic volume of $S_r(x)$ and $\omega(z)$ is the volume form.

Theorem 2.2. *Let $f \in C^2(M)$. Define*

$$F(x,y) \triangleq (M^{d(o,y)} f)(x), \quad (x,y) \in M \times M,$$

where o is fixed base point in M. Then the following Darboux equation holds:

$$\Delta_x F(x,y) = \Delta_y F(x,y). \tag{2.5}$$

In addition,

$$\Delta M^r f = M^r \Delta f \tag{2.6}$$

for all $r > 0$.

To obtain Ásgeirsson's mean value theorem from the Darboux equation, we use geodesic polar coordinates. With respect to a fixed base point o, the *geodesic polar coordinates* are given by (r, ζ) where r is the distance from o and ζ records a point on the sphere $S_r(o)$. According to Proposition 4.3, the hyperbolic Laplacian can be written as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{A(r)} A'(r) \frac{\partial}{\partial r} + \Delta_{S_r}$$

where Δ_{S_r} is the Laplace-Beltrami operator on the sphere $S_r(o)$.

Let $u \in C^2(M \times M)$ be such that (2.3) holds. Given $r, s > 0$ and $x, y \in M$, we define

$$U(r, s) \triangleq (M_x^r M_y^s u)(x, y),$$

where the subscript denotes the variable with respect to which the spherical mean is taken. By treating s, y as fixed and r, x as variables, one has

$$\begin{aligned} \Delta_r (M_x^r M_y^s u) &= \frac{\partial^2 U}{\partial r^2} + \frac{1}{A(r)} A'(r) \frac{\partial U}{\partial r} \\ &= \Delta_x (M_x^r M_y^s u) \qquad \text{(by (2.5))} \\ &= M_x^r (\Delta_x M_y^s u) \qquad \text{(by (2.6))} \\ &= M_x^r M_y^s \Delta_x u. \end{aligned}$$

Similarly, by switching the roles of (r, x) and (s, y),

$$\Delta_s (M_y^s M_x^r u) = \frac{\partial^2 U}{\partial s^2} + \frac{1}{A(s)} A'(s) \frac{\partial U}{\partial s} = M_y^s M_x^r \Delta_y u.$$

Note that

$$M_x^r M_y^s u(x, y) = \frac{1}{A(r) A(s)} \int_{S_r(x) \times S_s(y)} u(z, w) d\omega(z) d\omega(w) = M_y^s M_x^r u(x, y).$$

Since u satisfies $\Delta_x u = \Delta_y u$ by assumption, it follows that

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{A(r)} A'(r) \frac{\partial U}{\partial r} = \frac{\partial^2 U}{\partial s^2} + \frac{1}{A(s)} A'(s) \frac{\partial U}{\partial s} \quad \text{for all } r, s > 0,$$

where $x, y \in M$ are given fixed. With some analytical effort, the above equation shall imply that $U(r, s) = U(s, r)$ which in particular yields Ásgeirsson's mean value theorem (take $s = 0$). We will complete the argument in Section 5.2.

2.4 Invariance of Laplacian

To complete the program, it remains to establish Theorem 2.2. We first demonstrate this in the Euclidean space $\mathbb{R}^n$, where the argument is substantially simpler and also motivates its hyperbolic counterpart. The key point is the translation and rotation invariance of Laplacian.

Let $f \in C^2(\mathbb{R}^n)$. The orthogonal group

$$O(n) \triangleq \{k \in \mathrm{Mat}(n,n) : k^T k = k k^T = \mathrm{Id}\}$$

is the *isotropy subgroup* at the origin o, i.e. the group of isometries leaving o fixed. Integration on the sphere $S_r(o)$ can be performed over the group $O(n)$:

$$(M^{|y|} f)(x) = \int_{O(n)} f(x + ky) dk,$$

where dk is the normalised Haar measure on $O(n)$ (i.e. $\int_{O(n)} dk = 1$). Let T_x (respectively, R_k) denote the translation by x (respectively, rotation by k). Since Δ is invariant under both translation and rotation, one has

$$\begin{aligned}
\Delta_x M^{|y|} f(x) &= \int_{O(n)} \Delta_x (f(x + ky)) dk \\
&= \int_{O(n)} (\Delta f)(x + ky) dk = (M^{|y|} \Delta f)(x) \implies (2.6) \\
&= \int_{O(n)} (\Delta f)(T_x \circ R_k)(y) dk \\
&= \int_{O(n)} \Delta (f \circ T_x \circ R_k)(y) dk \\
&= \int_{O(n)} \Delta_y (f(x + ky)) dk = \Delta_y M^{|y|} f(x) \implies (2.5).
\end{aligned}$$

This proves Theorem 2.2 for the Euclidean case.

A difficulty of extending the above argument to the hyperbolic case is that one does not have a simple translation operation $x + ky$ on $\mathbb{H}^n$. Translation needs to be understood from the perspective of group action (by isometries of $\mathbb{H}^n$). As a result, one needs to "lift" the above calculation to the group level and make use of the counterpart of Δ on the isometry group. This requires deeper tools from the theory of invariant differential operators on Lie groups, which will be developed in Section 4.1 and Section 4.2. The proof of Theorem 2.2 in the hyperbolic case will be given in Section 5.

In what follows, we develop the above program of Grigor'yan-Noguchi in a mathematically precise way.

3 Basic models of $\mathbb{H}^n$

There are several equivalent models for $\mathbb{H}^n$. We discuss two of them that are related to our study: the hyperboloid and the upper half-space. We only state the relevant properties and will not provide proofs in this part (the reader is referred to [CFK97] for the details).

3.1 The hyperboloid model

The hyperboloid model is convenient for group-theoretic considerations. We equip $\mathbb{R}^{n+1}$ with the Lorentzian metric

$$x * y = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}$$

and define

$$L \triangleq \big\{x \in \mathbb{R}^{n+1} : x * x = -1, x_{n+1} > 0\big\}.$$

Then the Lorentzian metric restricts to a Riemannian metric on L, under which L becomes a model of $\mathbb{H}^n$. Fix the base point $o \triangleq (0, \cdots, 0, 1)^T \in L$. The hyperbolic distance can be computed explicitly through the formula

$$x * y = -\cosh d(x, y), \quad x, y \in L.$$

In particular, under the $\mathbb{R}^{n+1}$ coordinates $S_r(o)$ corresponds to an Euclidean sphere with radius $\sinh r$.

Definition 3.1. The *isometry group* G is the group of diffeomorphisms of L leaving the hyperbolic metric invariant. The *isotropy subgroup* K at o is the subgroup of G that fixes o, i.e. $K = \{g \in G : g \cdot o = o\}$.

The isometry group and isotropy subgroup are described explicitly by the following result.

Theorem 3.1. *A map $g : L \to L$ is an isometry if and only if it is the restriction of a linear transformation defined by a matrix*

$$A = (\mathbf{a}_1, \cdots, \mathbf{a}_n, \mathbf{a}_{n+1})$$

such that

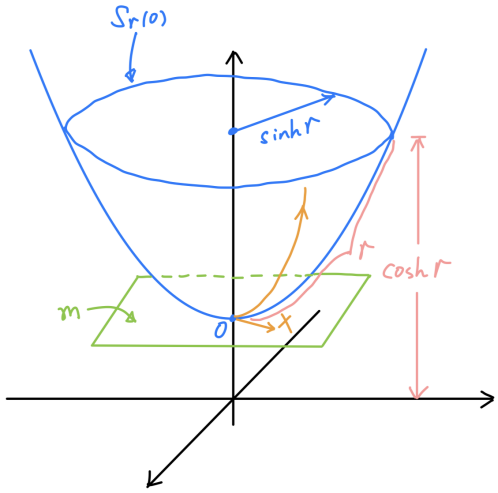
$$\mathbf{a}_i * \mathbf{a}_j = \begin{cases} 0, & i \neq j; \\ 1, & i = j \leqslant n; \\ -1, & i = j = n+1, \end{cases} \qquad A_{n+1}^{n+1} > 0.$$

In addition, $g \in K$ if and only if

$$A = \begin{pmatrix} S & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

with some $S \in O(n)$.

The isometry group of L given by Theorem 3.1 is denoted as $G = O^+(n,1)$. The isotropy subgroup K is naturally identified with $O(n)$. The K-action on L is just rotation with respect to the x_{n+1}-axis. Restricted to $S_r(o)$, it acts as Euclidean isometries and acts *transitively* (i.e. for any $x, y \in S_r(o)$ there exists $k \in K$ such that $y = k \cdot x$). The latter follows from the fact that $O(n)$ acts transitively on Euclidean spheres.



One can form the left coset space

$$G/K \triangleq \{gK : g \in K\}.$$

The group G acts on G/K by left translation:

$$\tau(g)(hK) \triangleq ghK, \quad g \in G, hK \in G/K.$$

The space G/K is canonically isomorphic to L via $gK \leftrightarrow g \cdot o$. The Lie algebra of G and K are given by

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & x \\ x^T & 0 \end{pmatrix} : A \in \mathfrak{so}(n), x \in \mathbb{R}^n \right\}, \ \mathfrak{k} = \{X \in \mathfrak{g} : x = 0\} \simeq \mathfrak{so}(n).$$

The space

$$\mathfrak{m} \triangleq \{X \in \mathfrak{g} : A = 0\} \cong \mathbb{R}^n \tag{3.1}$$

is canonically isomorphic to the tangent space of L at the base point o. Under the identification (3.1), the hyperbolic metric on T_oL coincides with the canonical Euclidean metric on $\mathbb{R}^n$. For any

$$X = \begin{pmatrix} 0 & x \\ x^T & 0 \end{pmatrix} \in \mathfrak{m},$$

the path

$$t \mapsto \exp(tX) \cdot o$$

defines a geodesic L emitting from o along the direction X.

3.2 The upper half-space model

The upper half-space model is convenient for local computations and geodesic properties. Define

$$U \triangleq \{y \in \mathbb{R}^n : y_n > 0\}.$$

The *Lobachevsky metric* on U is defined by

$$ds^2 = \frac{1}{y_n^2}(dy_1^2 + \cdots + dy_n^2).$$

Under the metric ds^2, the upper half-space U becomes a model of $\mathbb{H}^n$. Fix $o = (0, \cdots, 0, 1) \in U$ to be the base point.

Given $y \in U$, we set $\bar{y} \triangleq (y_1, \cdots, y_{n-1}, -y_n)$. The hyperbolic distance is explicitly given by

$$d(x, y) = 2 \tanh^{-1} \frac{|x - y|}{|x - \bar{y}|}. \tag{3.2}$$

In particular,

$$d(o, y) = 2 \tanh^{-1} \sqrt{\frac{|\mathbf{y}^{(n-1)}|^2 + (y_n - 1)^2}{|\mathbf{y}^{(n-1)}|^2 + (y_n + 1)^2}}, \quad y = (\mathbf{y}^{(n-1)}, y_n) \tag{3.3}$$

depends only on the Euclidean norm of $\mathbf{y}^{(n-1)}$ and y_n. Also note that

$$d(0,y) = |\log y_n| \quad \forall y = (0,\cdots,0,y_n).$$

Geodesics in U are given by Euclidean semicircles in U that are perpendicular to the plane $y_n = 0$ at both ends as well as vertical straight lines.

Another advantage of the upper half-space is that disintegration on $S_r(o)$ (with respect to the y_n-variable) is more explicit. This is useful in our later proof of Ásgeirsson's mean value theorem. To derive the disintegration formula, let us introduce a change of variables $y_n = e^s$ ($s \in \mathbb{R}$). The induced Riemannian metric on the plane $P_s : y_n = e^s$ is given by

$$e^{-2s}(dy_1^2 + \cdots + dy_{n-1}^2). \tag{3.4}$$

In particular, the geometry on P_s is Euclidean. We want to write a general spherical integral in the following form:

$$\int_{S_r(o)} f(y)d\omega(y) = \int_{-r}^{r} f(s,\zeta)dl(s) \int_{P_s\cap S_r(o)} d_s\zeta,$$

where $dl(s)$ denotes the arc element along the s direction (the circle $S_r(o)\cap\{y_2 = \cdots = y_{n-1} = 0\}$) and $d_s\zeta$ denotes the volume form on the level set $P_s \cap S_r(o)$.

We first compute the volume form $d_s\zeta$ on $P_s\cap S_r(o)$. Let $r > 0$ and $s \in (-r, r)$. From the relation (3.3), the intersection $P_s \cap S_r(o)$ is the sphere

$$C_s \triangleq \{y = (\mathbf{y}^{(n-1)}, e^s) : |\mathbf{y}^{(n-1)}| = \rho\}$$

where ρ satisfies

$$r = 2\tanh^{-1}\sqrt{\frac{\rho^2 + (e^s-1)^2}{\rho^2 + (e^s+1)^2}}.$$

By solving the above relation, one finds that

$$\rho = \sqrt{2e^s(\cosh r - \cosh s)}. \tag{3.5}$$

In view of the metric (3.4), the volume form on C_s is given by

$$d_s\zeta = (e^{-2s})^{\frac{n-2}{2}} d_s^{\mathrm{eu}}\zeta \tag{3.6}$$

where $d_s^{\mathrm{eu}}\zeta$ denotes the volume form when C_s is viewed as a standard Euclidean sphere of radius ρ in $\mathbb{R}^{n-1}$.

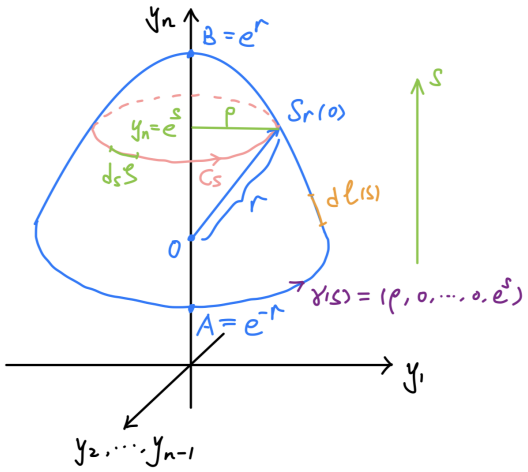


Figure 1: Disintegration on $S_r(o)$.

Next, we compute the arc element $dl(s)$. In Figure 1, the arc from A to B is given by

$$\gamma(s) = (\sqrt{2e^s(\cosh r - \cosh s)}, 0, \cdots, 0, e^s), \quad s \in [-r, r].$$

Under the hyperbolic metric, one finds

$$\begin{aligned} dl(s) = |\dot\gamma(s)|_{\mathrm{hyp}} &= \sqrt{\dot\gamma_n^{-2}(s)\big(\dot\gamma_1^2(s) + \cdots + \dot\gamma_n^2(s)\big)} \\ &= \frac{\sinh r}{\sqrt{2e^s(\cosh r - \cosh s)}} ds. \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), one obtains the following disintegration formula:

$$\int_{S_r(o)} f(y) d\omega(y) = \sinh r \int_{-r}^{r} \frac{e^{-s(n-3/2)} ds}{\sqrt{2(\cosh r - \cosh s)}} \int_{C_s} f(s, \zeta) d_s^{\mathrm{eu}}\zeta. \tag{3.8}$$

The formula further simplifies when f depends only on s. We will come back to this point when it becomes relevant (see (5.6) below).

4 Invariant differential operators

In this section, we develop the invariance properties of the hyperbolic Laplacian at the group-theoretic level that are essential for the proof of the Darboux equation.

4.1 Invariant differential operators on Lie groups

As a starting point, one needs to know how to characterise left invariant differential operators on a Lie group from the Lie algebra perspective. In what follows, let G be a finite dimensional Lie group with Lie algebra $\mathfrak{g}$. The identity of G is denoted as e. Left and right translations by $g \in G$ are denoted as L_g and R_g respectively. The exponential map is denoted as $\exp : \mathfrak{g} \to G$.

4.1.1 Adjoint representations

For each given $g \in G$, there is an associated automorphism $\mathcal{A}_g : G \to G$ defined by

$$\mathcal{A}_g(h) \triangleq ghg^{-1}, \quad h \in G.$$

The differential of $\mathcal{A}_g$ at the identity e is a linear isomorphism over the Lie algebra $\mathfrak{g}$. The group homomorphism

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}), \ \mathrm{Ad}(g) \triangleq (d\mathcal{A}_g)_e$$

is known as the *adjoint representation* of G. Its differential at the identity, denoted as

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}),$$

is a Lie algebra homomorphism which is known as the *adjoint representation* of $\mathfrak{g}$. It is explicitly given by

$$\mathrm{ad}_X(Y) = [X, Y]$$

for all $X, Y \in \mathfrak{g}$.

The exponential map is *natural* in the sense that the following two diagrams are both commutative.

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}(g)} & \mathfrak{g} \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\mathcal{A}_g} & G, \end{array} \qquad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathrm{End}(\mathfrak{g}) \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(\mathfrak{g}). \end{array}$$

In addition, the following relation holds

$$\mathrm{ad}_{\mathrm{Ad}(g)(X)} = \mathrm{Ad}(g) \circ \mathrm{ad}_X \circ \mathrm{Ad}(g^{-1})$$

for all $g \in G$ and $X \in \mathfrak{g}$.

4.1.2 The universal enveloping algebra

Any algebra has a natural Lie algebra structure given by the commutator $[a, b] \triangleq ab - ba$. Conversely, given a Lie algebra $\mathfrak{g}$, there is a unique smallest associative algebra with unit (up to isomorphism) which contains $\mathfrak{g}$ as a Lie subalgebra. Explicitly, this algebra is constructed by $U(\mathfrak{g}) \triangleq T(\mathfrak{g})/I$, where

$$T(\mathfrak{g}) \triangleq \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$$

is the tensor algebra over $\mathfrak{g}$ and I is the ideal generated by the elements

$$X \otimes Y - Y \otimes X - [X, Y] \quad (X, Y \in \mathfrak{g}).$$

Modulo the abstract nonsense, $U(\mathfrak{g})$ consists of finite linear combinations of formal noncommutative products

$$X_1 \cdots X_m \quad (m \geqslant 1,\ X_i \in \mathfrak{g})$$

subject to the relation $XY - YX = [X, Y]$ whenever $X, Y \in \mathfrak{g}$.

Definition 4.1. The algebra $U(\mathfrak{g})$ is called the *universal enveloping algebra* of $\mathfrak{g}$.

4.1.3 Characterisations of left invariant differential operators

The construction of $U(\mathfrak{g})$ is purely algebraic. However, a remarkable fact is that it is precisely the algebra of left invariant differential operators on G.

To understand such a relation, we begin with the following consideration. Any given $X \in \mathfrak{g}$ induces a differential operator $\tilde{X} : C^\infty(G) \to C^\infty(G)$ defined by

$$(\tilde{X} f)(g) \triangleq \frac{d}{dt}\Big|_{t=0} f(g \exp(tX)).$$

Let $D_0(G)$ be the algebra generated by $\{\tilde{X} : X \in \mathfrak{g}\}$. In other words, elements in $D_0(G)$ has the form

$$c_0 + \sum c_{i_1 \cdots i_m} \tilde{X}_{i_1} \cdots \tilde{X}_{i_m}$$

where multiplication is understood as composition. From the definition of $\tilde{X}$, it is clear that elements in $D_0(G)$ are *left invariant*, in the sense that

$$D(f \circ L_g) = (Df) \circ L_g \quad \forall g \in G, f \in C^\infty(G).$$

We now fix a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}$ ($n \triangleq \dim \mathfrak{g}$). Given an n-tuple $\mathbf{m} = (m_1, \cdots, m_n)$ of nonnegative integers, we define $X(\mathbf{m}) \in U(\mathfrak{g})$ to be the $\mathbf{m}$-th coefficient in the formal expansion of

$$(t_1, \cdots, t_n) \mapsto e^{t_1 X_1 + \cdots + t_n X_n}.$$

More explicitly,

$$e^{X(\mathbf{t})} = \sum_{\mathbf{m}} \mathbf{t}^{\mathbf{m}} X(\mathbf{m}),$$

where $\mathbf{t} = (t_1, \cdots, t_n)$, $X(\mathbf{t}) \triangleq t_1 X_1 + \cdots + t_n X_n$ and $\mathbf{t}^{\mathbf{m}} \triangleq t_1^{m_1} \cdots t_n^{m_n}$. Correspondingly, we also define $\tilde{X}(\mathbf{m}) \in D_0(G)$ to be the $\mathbf{m}$-th coefficient in the formal expansion of $e^{t_1 \tilde{X}_1 + \cdots + t_n \tilde{X}_n}$. As a differential operator, one has

$$(\tilde{X}(\mathbf{m}) f)(g) = \frac{1}{m_1! \cdots m_n!} \frac{\partial^{|\mathbf{m}|}}{\partial t_1^{m_1} \cdots \partial t_n^{m_n}} f(g \exp(t_1 X_1 + \cdots + t_n X_n))\Big|_{\mathbf{t}=0}. \quad (4.1)$$

The following result shows that $U(\mathfrak{g})$ and $D_0(G)$ are naturally isomorphic.

Theorem 4.1. *The algebra homomorphism $T : U(\mathfrak{g}) \to D_0(G)$ induced by $X \mapsto \tilde{X}$ is an algebra isomorphism. The family $\{\tilde{X}(\mathbf{m})\}$, where $\mathbf{m}$ runs over all n-tuples of nonnegative integers, defines a basis of $D_0(G)$. The family*

$$\{\tilde{X}_1^{e_1} \cdots \tilde{X}_n^{e_n} : e_1, \cdots, e_n \geqslant 0\}$$

is also a basis of $D_0(G)$.

Proof. We first show that $\{X(\mathbf{m})\}$ linearly spans $U(\mathfrak{g})$. For this purpose, it is enough to show that any given product $X_{i_1} \cdots X_{i_p}$ ($X_i \in \mathfrak{g}$) can be written as a linear combination of $X(\mathbf{m})$ ($|\mathbf{m}| \leqslant p$). This is seen from the following two formal algebraic observations.

(i) Simple combinatorics shows that

$$\sum_{\sigma \in \mathcal{S}_p} X_{i_{\sigma(1)}} \cdots X_{i_{\sigma(p)}} = cX(\mathbf{m})$$

for some constant c, where $\mathcal{S}_p$ denotes the set of permutations of order p and $\mathbf{m}$ is the n-tuple determined by the relation

$$t_{i_1} \cdots t_{i_p} = t_1^{m_1} \cdots t_n^{m_n}.$$

As a result, in order to prove the claim it suffices to show that

$$X_{i_1} \cdots X_{i_p} - X_{i_{\sigma(1)}} \cdots X_{i_{\sigma(p)}} \tag{4.2}$$

is a linear combination of $X(\mathbf{m})$ ($|\mathbf{m}| \leqslant p$) for any given $\sigma \in \mathcal{S}_p$.

(ii) By using the relation $XY - YX = [X, Y]$ ($X, Y \in \mathfrak{g}$), it is easily seen that (4.2) is a linear combination of $X_{j_1} \cdots X_{j_{p-1}}$ with suitable $X_j \in \mathfrak{g}$.

By the above two observations, the claim follows easily by induction.

Next, we claim that $\{\tilde{X}(\mathbf{m})\}$ are linearly independent. To this end, let $\mathbf{m}$ be given fixed. Let $f \in C^\infty(G)$ be a function such that $f(\mathbf{t}) = \mathbf{t}^{\mathbf{m}}$ under the exponential coordinates near e (with respect to the basis $X_1, \cdots, X_n$). According to (4.1), one finds that

$$(\tilde{X}(\mathbf{m})f)(e) = \frac{1}{m_1! \cdots m_n!} \frac{\partial^{|\mathbf{m}|}}{\partial \mathbf{t}^{\mathbf{m}}} \mathbf{t}^{\mathbf{m}} \Big|_{\mathbf{t}=0} \neq 0$$

while

$$(\tilde{X}(\mathbf{m}')f)(e) = 0$$

for all $\mathbf{m}' \neq \mathbf{m}$. This implies the linear independence of $\{\tilde{X}(\mathbf{m})\}$. The first two assertions of the proposition are now a simple matter of linear algebra.

To prove the last assertion, one first notes that

$$X(\mathbf{m}) \in \mathrm{Span}\{X_1^{e_1} \cdots X_n^{e_n} : e_1 + \cdots + e_n \leqslant |\mathbf{m}|\}$$

for any $\mathbf{m}$. This is again a consequence of the relation $XY - YX = [X, Y]$. In addition, one also has

$$X_1^{e_1} \cdots X_n^{e_n} \in \mathrm{Span}\{X(\mathbf{m}) : |\mathbf{m}| \leqslant e_1 + \cdots + e_n\}$$

for any $(e_1, \cdots, e_n)$. This was already showed at the beginning. The proof is now complete by observing that the following two families have the same cardinality

$$\{X_1^{e_1} \cdots X_n^{e_n} : e_1 + \cdots + e_n \leqslant p\}, \quad \{X(\mathbf{m}) : |\mathbf{m}| \leqslant p\}$$

for every given p. □

Next, we proceed to show that $D_0(G)$ consists of all left invariant differential operators on G. We will also give a more useful characterisation of $D_0(G)$.

Definition 4.2. Let M be a differentiable manifold. A *differential operator* on M is a linear map $D : C^\infty(M) \to C^\infty(M)$ such that $D|_{C_c^\infty(M)}$ decreases supports, namely,

$$\mathrm{supp}(Df) \subseteq \mathrm{supp} f$$

for all $f \in C_c^\infty(M)$. Given a diffeomorphism $\phi : M \to M$, we define

$$D^\phi(f) \triangleq D(f \circ \phi) \circ \phi^{-1}.$$

We say that D is ϕ-*invariant* if $D^\phi = D$.

It is a well-known fact from real analysis (cf. [Hel84, Chap. II, Theorem 1.4]) that a differential operator D admits a representation

$$Df = \sum_\alpha c_\alpha \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \tag{4.3}$$

under a local coordinate chart $(U, \{x_i\})$, where the summation is finite and $c_\alpha \in \mathbb{R}$.

Define $D(G)$ to be the algebra of left invariant differential operators on G, i.e. differential operators D in the sense of Definition 4.2 that are L_g-invariant for all $g \in G$. In the mean while, let $S(\mathfrak{g})$ be the unital *commutative* algebra generated by $\mathfrak{g}$. Given a fixed basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}$, $S(\mathfrak{g})$ consists of polynomials in the commutative symbols $X_1, \cdots, X_n$. Let $\lambda : S(\mathfrak{g}) \to D(G)$ be the linear map induced by $X^m \mapsto \tilde{X}^m$ $(X \in \mathfrak{g})$.

Theorem 4.2. *The map λ is a linear isomorphism. In addition, $D_0(G) = D(G)$.*

Proof. Let $\{X_1, \cdots, X_n\}$ be a basis of $\mathfrak{g}$. Note that

$$(\lambda(P)f)(g) = P(\partial_1, \cdots, \partial_n) f(g \exp(t_1 X_1 + \cdots + t_n X_n))\big|_{\mathfrak{t}=0}.$$

for all $P \in S(\mathfrak{g})$ and $f \in C^\infty(G)$.

Injectivity of λ. Let $P \in S(\mathfrak{g})$ be a nonzero element. With respect to a basis $\{X_1, \cdots, X_n\}$, P contains a nonzero top-degree monomial, say $aX_1^{m_1} \cdots X_n^{m_n}$ where $a \neq 0$. Define $f(\mathfrak{t}) \triangleq t_1^{m_1} \cdots t_n^{m_n}$ ($\mathfrak{t}$ small). Then $(\lambda(P)f)(e) \neq 0$ which shows that $\lambda(P) \neq 0$.

Surjectivity of λ. Let $D \in D(G)$. As a differential operator, there exists a polynomial P in n variables, such that (see (4.3))

$$(Df)(e) = P(\partial_1, \cdots, \partial_n) f(\exp(t_1 X_1 + \cdots + t_n X_n))\big|_{t=0}.$$

Since D is left invariant, one has

$$(Df)(g) = P(\partial_1, \cdots, \partial_n) f(g \exp(t_1 X_1 + \cdots + t_n X_n))\big|_{t=0}$$

for all $g \in G$. As a result, $D = \lambda(P)$.

By the definition of λ, it is not hard to see that

$$\lambda(Y_1 \cdots Y_p) = \frac{1}{p!} \sum_{\sigma \in \mathcal{S}_p} \tilde{Y}_{\sigma(1)} \cdots \tilde{Y}_{\sigma(p)}$$

for all $Y_1, \cdots, Y_p \in \mathfrak{g}$. Since $T(U(\mathfrak{g})) = D_0(G)$ (see Theorem 4.1), the surjectivity of λ and the above relation further imply that $D_0(G) = D(G)$. □

4.1.4 The adjoint representation over $D(G)$ and bi-invariant differential operators

We now consider the subspace of *bi-invariant differential operators* on G, i.e. elements $D \in D(G)$ that are also right invariant. As we will see, $D \in D(G)$ is bi-invariant if and only if D commutes with all elements in $D(G)$.

Recall that $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is the adjoint representation of G over $\mathfrak{g}$. By viewing elements in $\mathfrak{g}$ as differential operators, one can compute $(\mathrm{Ad}(g)(X))^{\sim} \in D(G)$ as follows. Given $f \in C^\infty(G)$ and a diffeomorphism ϕ, we denote $f^\phi \triangleq f \circ \phi^{-1}$.

Lemma 4.1. *One has*

$$\big(\mathrm{Ad}(g)(X)\big)^{\sim} = \tilde{X}^{R_{g^{-1}}}$$

for any $g \in G$ and $X \in \mathfrak{g}$.

Proof. This is seen from the following calculation:

$$\begin{aligned}
&\big(\mathrm{Ad}(g)(X)\big)^{\sim} f(x) \\
&= \frac{d}{dt}\Big|_{t=0} f\big(x \exp(t\mathrm{Ad}(g)X)\big) = \frac{d}{dt}\Big|_{t=0} f\big(xg \exp(tX) g^{-1}\big) \\
&= \frac{d}{dt}\Big|_{t=0} f^{R_g}\big(xg \exp(tX)\big) = \big(\tilde{X} f^{R_g}\big)(xg) = \big(\tilde{X} f^{R_g}\big)^{R_{g^{-1}}}
\end{aligned}$$

for all $f \in C^\infty(G)$. □

Lemma 4.1 motivates the following definition. Given $g \in G$ and $D \in D(G)$, we set

$$\mathrm{Ad}(g)(D) \triangleq D^{R_{g^{-1}}}.$$

Note that

$$(\mathrm{Ad}(g)(X))^{\sim} = \mathrm{Ad}(g)(\tilde{X}) \tag{4.4}$$

under this definition. It is a simple unwinding of definitions that $\mathrm{Ad}(g) : D(G) \to D(G)$ is an algebra homomorphism and

$$\mathrm{Ad}(gh)(D) = \mathrm{Ad}(g)\mathrm{Ad}(h)(D) \quad \forall g, h \in G, D \in D(G).$$

Therefore, $\mathrm{Ad} : G \to \mathrm{Aut}_{\mathrm{alg}}(D(G))$ is a group homomorphism. This is also known as the *adjoint representation* of G over $D(G)$. By definition, $D \in D(G)$ is bi-invariant if and only if D is Ad-invariant, i.e. $\mathrm{Ad}(g)(D) = D$ for all $g \in G$.

At the Lie algebra level, let us set

$$\mathrm{ad}_X(D) \triangleq \tilde{X}D - D\tilde{X} \in D(G), \quad X \in \mathfrak{g}, D \in D(G).$$

One easily checks that $\mathrm{ad}_X : D(G) \to D(G)$ is a derivation, i.e.

$$\mathrm{ad}_X(D_1 D_2) = \mathrm{ad}_X(D_1)D_2 + D_1\mathrm{ad}_X(D_2) \tag{4.5}$$

for all $D_1, D_2 \in D(G)$. Define

$$e^{\mathrm{ad}_X}(D) \triangleq \sum_{m=0}^{\infty} \frac{(\mathrm{ad}_X)^m(D)}{m!}, \quad X \in \mathfrak{g}, D \in D(G). \tag{4.6}$$

Note that $(\mathrm{ad}_X)^m D$ is a differential operator whose order is not greater than that of D. As a result, the general terms of the series (4.6) lie in a finite dimensional vector space and there is no issue of convergence.

Proposition 4.1. *The following identity*

$$e^{\mathrm{ad}_X}(D) = \mathrm{Ad}(e^X)(D) \tag{4.7}$$

holds for all $X \in \mathfrak{g}$ and $D \in D(G)$.

Proof. One first observes that

$$\mathrm{ad}_X^m(D_1 D_2) = \sum_{i=0}^{m} \frac{m!}{i!(m-i)!}(\mathrm{ad}_X^i D_1)(\mathrm{ad}_X^{m-i} D_2),$$

which follows from (4.5) and induction. This relation implies that

$$e^{\mathrm{ad}_X}(D_1 D_2) = (e^{\mathrm{ad}_X} D_1)(e^{\mathrm{ad}_X} D_2).$$

As a result, both e^{ad_X} and $\mathrm{Ad}(e^X)$ are algebra homomorphisms over $D(G)$. Since they agree on $\{\tilde{X} : X \in \mathfrak{g}\}$ by the naturality of the exponential map, they must coincide on the entire $D(G)$. □

The following result gives a characterisation of bi-invariant differential operators on G.

Theorem 4.3. *Let G be a connected Lie group and let $D \in D(G)$. Then the following three statements are equivalent:*

(i) D a bi-invariant differential operator;

(ii) $D = \lambda(P)$ where $P \in S(\mathfrak{g})$ is Ad*-invariant in the sense that $\mathrm{Ad}(g)(P) = P$ for all $g \in G$;*

(iii) D commutes with all elements in $D(G)$.

Proof. The equivalence between (i) and (ii) follows from the relations (4.4) and $\lambda(X^m) = \tilde{X}^m$. To prove the equivalence between (i) and (iii), we first claim that

$$\tilde{X}D = D\tilde{X} \iff D^{R_{\exp(tX)}} = D \quad \text{for all } t \in \mathbb{R},$$

where $\tilde{X}$ ($X \in \mathfrak{g}$) is given fixed. Indeed, the " $\Longrightarrow$ " direction follows from

$$\begin{aligned} D^{R_{\exp(tX)}} &= \mathrm{Ad}(e^{-tX})(D) = e^{-t\mathrm{ad}_X}(D) \\ &= \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} (\mathrm{ad}_X)^m(D) = D, \end{aligned} \tag{4.8}$$

while the "$\Longleftarrow$" direction follows from differentiating (4.8).

Note that (ii) holds iff $\tilde{X}D = D\tilde{X}$ for all $X \in \mathfrak{g}$, which is equivalent to saying that

$$D^{R_{\exp(tX)}} = D \tag{4.9}$$

for all $X \in \mathfrak{g}, t \in \mathbb{R}$. Since G is connected, any open neighbourhood of e generates the entire group G. As a result, the relation (4.9) is also equivalent to

$$D^{R_g} = D$$

for all $g \in G$, namely, the bi-invariance of D. □

4.2 Invariant differential operators on factor spaces

Let K be a closed subgroup of G whose Lie algebra is $\mathfrak{k}$. We now consider the factor space G/K, whose projection map is denoted as $\pi : G \to G/K$. A basic example that is relevant to us is $G = O^+(n,1)$, $K = O(n)$ so that $G/K = \mathbb{H}^n$ (see Theorem 3.1).

One can equip G/K with a manifold structure in the following way. Let $\mathfrak{m}$ be a complementary subspace such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$. Let $\{X_1, \cdots, X_r\}$ be a basis of $\mathfrak{m}$. For each $g \in G$, the map

$$(t_1, \cdots, t_r) \mapsto \pi(g \exp(t_1 X_1 + \cdots + t_r X_r))$$

defines a local coordinate chart around the point $gK \in G/K$. We set $o = K$ as the base point.

Let $k \in K$ be fixed. The adjoint representation $\mathrm{Ad}_G(k)$ descends to a linear isomorphism over $\mathfrak{g}/\mathfrak{k}$ (still denoted as $\mathrm{Ad}_G(k)$). On the other hand, the left translation

$$\tau(k) : G/K \to G/K$$

induces a linear isomorphism $d\tau(k)_o$ over the tangent space $T_o(G/K)$. It is routine to check that the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{g}/\mathfrak{k} & \xrightarrow{d\pi} & T_o(G/K) \\ \downarrow{\scriptstyle \mathrm{Ad}_G(k)} & & \downarrow{\scriptstyle d\tau(k)_o} \\ \mathfrak{g}/\mathfrak{k} & \xrightarrow{d\pi} & T_o(G/K) \end{array}$$

In other words, through the identification $d\pi : \mathfrak{g}/\mathfrak{k} \cong T_o(G/K)$ the adjoint action of k is simply the differential of the left translation by k.

Definition 4.3. The factor space G/K is said to be *reductive* if the complementary subspace $\mathfrak{m}$ can be chosen such that $\mathrm{Ad}_G(K)\mathfrak{m} \subseteq \mathfrak{m}$.

If K is compact, then G/K is reductive. To see this, pick an arbitrary inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ and define

$$\langle g, h \rangle' \triangleq \int_K \langle \mathrm{Ad}_G(k)g, \mathrm{Ad}_G(k)h \rangle dk$$

where dk is denotes the normalised Haar measure on K. Then $\mathfrak{m} \triangleq \mathfrak{k}^\perp$ (with respect to $\langle \cdot, \cdot \rangle^\perp$) satisfies Definition 4.3. Note that this condition is satisfied for the hyperbolic space since $K = O(n)$ is compact. Indeed, the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ for the hyperboloid model is canonical and one has $\mathrm{Ad}_G(K)\mathfrak{m} \subseteq \mathfrak{m}$ so that G/K is a reductive factor space.

Let $D(G/K)$ be the algebra of left invariant differential operators over G/K. Define

$$D(G)\mathfrak{k} \triangleq \mathrm{Span}\{D\tilde{X} : D \in D(G), X \in \mathfrak{k}\}$$

and

$$I(\mathfrak{m}) \triangleq \{P \in S(\mathfrak{m}) : \mathrm{Ad}_G(k)P = P \text{ for all } k \in K\}.$$

The following result provides a characterisation of $D(G/K)$ in terms of invariant polynomials on $\mathfrak{m}$.

Theorem 4.4. *Let G/K be a reductive factor space. Let*

$$D_K(G) \triangleq \{u \in D(G) : \mathrm{Ad}_G(k)u = u \text{ for all } k \in K\}$$

be the subalgebra of K-(right-)invariant differential operators over G. Define

$$\mu : D_K(G) \to D(G/K),\ u \mapsto D_u$$

by the relation

$$(D_u f)^{\sim} = u\tilde{f}, \quad f \in C^{\infty}(G/K),$$

where $\tilde{f} \triangleq f \circ \pi \in C^{\infty}(G)$. Then μ is a surjective algebra homomorphism and

$$\ker\mu = D_K(G) \cap D(G)\mathfrak{k}.$$

As a consequence, one has

$$\frac{D_K(G)}{D_K(G) \cap D(G)\mathfrak{k}} \cong D(G/K)$$

In addition, the map $\mu \circ \lambda : I(\mathfrak{m}) \to D(G/K)$ is a linear isomorphism.

Proof of Theorem 4.4

Let $u \in D_K(G)$ be given. The right K-invariance of u implies that D_u is well-defined (i.e. $u\tilde{f}$ is the lifting of a function on G/K). The left invariance of u implies the left invariance of D_u. The fact that μ is an algebra homomorphism is trivial.

To proceed further, we fix a basis $\{X_1, \cdots, X_r\}$ of $\mathfrak{m}$ and extend it to a basis $\{X_1, \cdots, X_r, X_{r+1}, \cdots, X_n\}$ to a basis of $\mathfrak{g}$.

Claim. μ is surjective.

Proof. Let $E \in D(G/K)$. Then there exists $P \in S(\mathfrak{m})$ such that

$$(Ef)(o) = P(\partial_1, \cdots, \partial_r) f(\pi(\exp(x_1X_1 + \cdots + x_rX_r)))|_{x=0}$$

for all $f \in C^\infty(G/K)$. By the left invariance of E,

$$\begin{aligned}(Ef)(g\cdot o) &= E(f^{\tau(g^{-1})})(o)\\ &= P(\partial_{x_1},\cdots,\partial_{x_r}) f^{\tau(g^{-1})}(\pi(\exp(x_1X_1+\cdots+x_rX_r)))|_{x=0}\\ &= P(\partial_{x_1},\cdots,\partial_{x_r})\tilde{f}(\pi(g\exp(x_1X_1+\cdots+x_rX_r)))|_{x=0}.\end{aligned}$$

Taking $g = k \in K$ (so that $k\cdot o = o$), one has

$$\begin{aligned}(Ef)(o) &= P(\partial_{x_1},\cdots,\partial_{x_r})\tilde{f}(\exp(\mathrm{Ad}_G(k)(x_1X_1+\cdots+x_rX_r)))|_{x=0}\\ &= P(\partial_y\cdot A(k))\tilde{f}(\exp(y_1X_1+\cdots+y_rX_r))|_{y=0},\end{aligned} \tag{4.10}$$

where $A(k)$ is the matrix of $\mathrm{Ad}_G(k)$, i.e.

$$\mathrm{Ad}_G(k)(X_i) = \sum_{j=1}^{r} A(k)_i^j X_j, \quad i = 1,\cdots,r$$

and $\partial_y \triangleq (\partial_{y_1},\cdots,\partial_{y_r})$. Since k and f are arbitrary, the relation (4.10) implies that P is $\mathrm{Ad}_G(K)$-invariant, i.e. $P \in I(\mathfrak{m})$. It follows that $u \triangleq \lambda(P) \in D_K(G)$ and

$$(u\tilde{f})(g) = \lambda(P)\tilde{f}(g) = (Ef)(g\cdot o).$$

In particular, $E = \mu(u)$. This shows that μ is surjective. □

The next decomposition is crucial for the characterisation of $\ker\mu$.

Claim. $D(G) = \lambda(S(\mathfrak{m})) \oplus D(G)\mathfrak{k}$.

Proof. We first prove by induction that $D(G) = \lambda(S(\mathfrak{m})) + D(G)\mathfrak{k}$ (without linear independence). Let $D = \tilde{X}$ $(X \in \mathfrak{g})$ and one has

$$\tilde{X} = \tilde{Y} + \tilde{Z},\ Y \in \mathfrak{m}, Z \in \mathfrak{k}$$

according to the Lie algebra decomposition. Suppose that the decomposition is true when the degree of the differential operator is less than $d > 1$. Let

$$D = \lambda(P),\ P = X_1^{e_1}\cdots X_r^{e_r}\cdots X_n^{e_n} \in S(\mathfrak{g})$$

where $e_1+\cdots+e_n = d$. One can write

$$P = X_1^{e_1}\cdots X_r^{e_r}\cdots X_n^{e_{n-1}}Y_n + X_1^{e_1}\cdots X_r^{e_r}\cdots X_n^{e_{n-1}}Z_n \tag{4.11}$$

where $Y_n \in \mathfrak{m}$, $Z_n \in \mathfrak{k}$. As a result, the λ-image of the second term belongs to $D(G)\mathfrak{k}$. In addition, by induction hypothesis

$$\lambda(X_1^{e_1} \cdots X_r^{e_r} \cdots X_n^{e_{n-1}}) \in \lambda(S(\mathfrak{m})) \oplus D(G)\mathfrak{k}.$$

Therefore, the λ-image of the first term in (4.11) belongs to $\lambda(S(\mathfrak{m})) \oplus D(G)\mathfrak{k}$. This completes the induction step.

Next we show that the decomposition is direct. Suppose that $0 \neq P \in S(\mathfrak{m})$. Then there exists $f \in C^\infty(G/K)$ such that

$$\lambda(P)\tilde{f}(e) = P(\partial_1, \cdots, \partial_r)\tilde{f}(\exp(x_1X_1 + \cdots + x_rX_r))|_{x=0} \neq 0.$$

In particular, $\lambda(P) \notin D(G)\mathfrak{k}$ (because $\tilde{f}$ is right K-invariant). □

Claim. $D_K(G) = \lambda(I(\mathfrak{m})) \oplus (D_K(G) \cap D(G)\mathfrak{k})$.

Proof. Let $D \in D_K(G)$ and write $D = D_1 + D_2$ where $D_1 \in \lambda(S(\mathfrak{m}))$, $D_2 \in D(G)\mathfrak{k}$. Then

$$D_1 + D_2 = D = \mathrm{Ad}_G(k)(D) = \mathrm{Ad}_G(k)(D_1) + \mathrm{Ad}_G(k)(D_2)$$

for all $k \in K$. Note that both $\lambda(S(\mathfrak{m}))$ and $D(G)\mathfrak{k}$ are $\mathrm{Ad}_G(K)$-invariant. As a result,

$$\mathrm{Ad}_G(k)(D_i) = D_i \quad (i = 1, 2),$$

which yields the desired decomposition. □

We are now in a position to complete the proof of Theorem 4.4.

Claim. $\ker \mu = D_K(G) \cap D(G)\mathfrak{k}$ and $\mu \circ \lambda : I(\mathfrak{m}) \to D(G/K)$ is a linear isomorphism.

Proof. Suppose that $\mu(u) = D_u = 0$. Write $u = u_1 + u_2$ where $u_1 \in \lambda(I(\mathfrak{m}))$ and $u_2 \in D_K(G) \cap D(G)\mathfrak{k}$. It is clear that $D_{u_2} = 0$ and thus $Du_1 = 0$. This implies that $u_1 = 0$, for otherwise the surjectivity argument yields an $f \in C^\infty(G/K)$ such that $(u_1\tilde{f})(e) \neq 0$ which is a contradiction. Therefore, $u = u_2 \in D_K(G) \cap D(G)\mathfrak{k}$. For the second claim, the surjectivity of $\mu \circ \lambda$ is clear. To show its injectivity, let $\mu(\lambda(P)) = 0$ for some $P \in I(\mathfrak{m})$. Then $\lambda(P) \in D(G)\mathfrak{k}$ as just shown. But we also have $\lambda(P) \in \lambda(S(\mathfrak{m}))$. As a result, $\lambda(P) = 0$ and thus $P = 0$. □

4.3 The Laplace-Beltrami operator on $\mathbb{H}^n$

We now return to the hyperbolic space $M = \mathbb{H}^n = G/K$, where $G = O^+(n, 1)$ and $K = O(n)$ (the hyperboloid model). Let Δ denote the hyperbolic Laplacian.

4.3.1 Characterisation of $D(M)$

We prove a remarkable fact that left invariant differential operators are precisely polynomials of Δ. Recall that $\mathfrak{m}$ is identified with the tangent space

$$T_oM = \{(x_1, \cdots, x_n, 1) : (x_1, \cdots, x_n) \in \mathbb{R}^n\}$$

of the hyperboloid at the base point o and $\mathfrak{k} \cong \mathfrak{so}(n)$. The hyperbolic metric on $\mathfrak{m}$ is the canonical Euclidean metric on the tangent space. As a result, the canonical basis $\{X_1, \cdots, X_n\}$ of $T_o(M)$ is orthonormal.

Theorem 4.5. *The algebra $D(M)$ of left invariant differential operators over M is the algebra of polynomials in Δ, namely,*

$$D(M) = \{p(\Delta) : p \text{ is a polynomial in one variable}\}.$$

Proof. We claim that

$$I(\mathfrak{m}) = \{p(X_1^2 + \cdots + X_n^2) : p \text{ is a polynomial in one variable}\}. \tag{4.12}$$

Indeed, let $P(X_1, \cdots, X_n) \in I(\mathfrak{m})$. One has

$$P(\mathrm{Ad}_G(k)X_1, \cdots, \mathrm{Ad}_G(k)X_n) = P(X_1, \cdots, X_n) \quad \forall k \in K. \tag{4.13}$$

The key observation is that $\mathrm{Ad}_G(k)$ acts on $\mathfrak{m}$ by the Euclidean isometry k (viewed as an element in $O(n)$). Since $O(n)$ acts transitively on Euclidean spheres, the invariance property (4.13) indicates that, as a function in n variables P depends only on $X_1^2 + \cdots + X_n^2$. The claim (4.12) thus follows.

In view of Theorem 4.4, it suffices to show that $\mu(\lambda(X_1^2 + \cdots + X_n^2)) = \Delta$. By left invariance, this boils down to checking that

$$\Delta f(o) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f(\exp(x_1X_1 + \cdots + x_nX_n) \cdot o)|_{x=0} \quad \forall f \in C^\infty(M). \tag{4.14}$$

The relation (4.14) is a consequence of the following basic fact.

(i) The map $(x_1, \cdots, x_n) \mapsto \exp(x_1X_1 + \cdots + x_nX_n) \cdot o$ defines a normal chart around o; the ray

$$t \mapsto \exp(t(x_1X_1 + \cdots + x_nX_n)) \cdot o$$

is a geodesic for every small $x \in \mathfrak{m}$.

(ii) As a result of (i), the Christoffel symbols at o with respect to the normal chart are all zero.

(iii) The equation (4.14) follows from the explicit local expression of Δ. □

4.3.2 The Casimir operator

Although $\lambda(X_1^2 + \cdots + X_n^2)$ provides a natural lifting of Δ onto $D(G)$, it is only right K-invariant but not bi-invariant. The bi-invariance will be needed in the proof of the Darboux equation when one lifts the problem to the group level. There is a canonical bi-invariant differential operator over G which is constructed by means of the Killing form on its Lie algebra.

Definition 4.4. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The *Killing form* on $\mathfrak{g}$ is the symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by

$$B(X, Y) \triangleq \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y), \quad X, Y \in \mathfrak{g}.$$

The Killing form has the following basic properties.

Proposition 4.2. *(i)* $B(\mathrm{ad}_Z X, Y) + B(X, \mathrm{ad}_Z Y) = 0$ *for all* $X, Y, Z \in \mathfrak{g}$.

(ii) $B(\mathrm{Ad}(g)(X), \mathrm{Ad}(g)(Y)) = B(X, Y)$ *for all* $g \in G$ *and* $X, Y \in \mathfrak{g}$.

Proof. (i) This follows from

$$\begin{aligned}[\mathrm{ad}_Z, \mathrm{ad}_X \mathrm{ad}_Y] &= [\mathrm{ad}_Z, \mathrm{ad}_X]\mathrm{ad}_Y + \mathrm{ad}_X[\mathrm{ad}_Z, \mathrm{ad}_Y] \\ &= \mathrm{ad}_{[Z,X]}\mathrm{ad}_Y + \mathrm{ad}_X \mathrm{ad}_{[Z,Y]}\end{aligned}$$

and taking trace on both sides.

(ii) We first establish the following identity:

$$\mathrm{ad}_{\mathrm{Ad}(g)(X)} = \mathrm{Ad}(g) \circ \mathrm{ad}_X \circ \mathrm{Ad}(g^{-1}).$$

Indeed, for any $Y \in \mathfrak{g}$ one has

$$\begin{aligned}\text{R.H.S.}(Y) &= g[X, g^{-1}Yg]g^{-1} \\ &= g(Xg^{-1}Yg - g^{-1}YgX)g^{-1} \\ &= gXg^{-1}Y - YgXg^{-1} \\ &= [\mathrm{Ad}(g)(X), Y] = \text{L.H.S.}(Y).\end{aligned}$$

By using this identity, one finds that

$$\begin{aligned}&B(\mathrm{Ad}(g)(X), \mathrm{Ad}(g)(Y)) \\ &= \mathrm{Tr}\big(\mathrm{Ad}(g) \circ \mathrm{ad}_X \circ \mathrm{ad}_Y \circ \mathrm{Ad}(g^{-1})\big) \\ &= \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) = B(X, Y).\end{aligned}$$

The desired relation thus follows. □

We now further assume that B is nondegenerate (this is true for the hyperbolic case). Let $\{X_1, \cdots, X_n\}$ be a basis of $\mathfrak{g}$. Define

$$\Omega \triangleq \sum_{i,j=1}^{n} g^{ij} \tilde{X}_i \tilde{X}_j \in D(G)$$

where (g^{ij}) is the inverse matrix of

$$g = (g_{ij}) : \ g_{ij} \triangleq B(X_i, X_j).$$

Routine algebra shows that Ω does not depend on the choice of a basis. In addition, Proposition 4.2 (ii) shows that Ω is a bi-invariant differential operator.

Definition 4.5. The differential operator Ω is known as the *Casimir operator* over G.

Remark 4.1. The Casimir operator coincides with the Laplace-Beltrami operator associated with the pseudo-Riemannian structure on G induced by the Killing form B.

In the hyperbolic case, the following result is a direct consequence of Theorem 4.5 and the fact that Ω is quadratic.

Corollary 4.1. *Let Ω be the Casimir operator over $G = O^+(n, 1)$. Then $\mu(\Omega) \in D(M)$ is a scalar multiple of Δ.*

4.3.3 The hyperbolic Laplacian under geodesic polar coordinates

As part of the analysis, we also need the expression of the hyperbolic Laplacian Δ under geodesic polar coordinates. Recall that the geodesic polar coordinates are given by (r, ζ), where r is the hyperbolic distance from o and $\zeta \in S_r(o)$. We use $A(r)$ to denote the area of the hyperbolic sphere $S_r(o)$. It is known that $A(r) = \Omega_n \sinh^{n-1} r$, where Ω_n is the area of the Euclidean sphere in $\mathbb{R}^n$ with unit radius.

Proposition 4.3. *Under geodesic polar coordinates, the hyperbolic Laplacian Δ admits the following expression:*

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{A(r)} A'(r) \frac{\partial}{\partial_r} + \Delta_r, \tag{4.15}$$

where Δ_r denotes the Laplace-Beltrami operator on the sphere $S_r(o)$ with respect to the induced metric.

Proof. The proof relies on several basic Riemannian geometric facts that will be quoted without comments. Let us write the geodesic polar coordinates (locally) as $(\theta_1, \cdots, \theta_{n-1}, \theta_n)$ where $\theta_n = r$. The Riemannian metric is given by

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} d\theta_i d\theta_j, \ \ g_{ij} \triangleq \langle \partial\theta_i, \partial\theta_j \rangle.$$

Note that $\langle \partial r, \partial r \rangle = 1$ and the geodesic rays emitting from o are perpendicular to the sphere $S_r(o)$. As a result, the metric ds^2 becomes

$$ds^2 = dr^2 + \sum_{i,j=1}^{n-1} g_{ij} d\theta_i d\theta_j.$$

Correspondingly, the Laplace-Beltrami operator simplifies to

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{\sqrt{\det g}} \frac{\partial \sqrt{\det g}}{\partial r} \frac{\partial}{\partial r} + \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n-1} \frac{\partial}{\partial \theta_j} \big(g^{ij} \sqrt{\det g} \frac{\partial}{\partial \theta_i} \big), \tag{4.16}$$

where $g \triangleq (g_{ij})_{1 \leqslant i,j \leqslant n}$. By setting $g_* \triangleq (g_{ij})_{1 \leqslant i,j \leqslant n-1}$, one has

$$g = \begin{pmatrix} g_* & 0 \\ 0 & 1 \end{pmatrix} \implies g^{-1} = \begin{pmatrix} g_*^{-1} & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \det g = \det g_*.$$

In particular, the last term in (4.16) is precisely the Laplace-Beltrami operator on $S_r(o)$ (for fixed r).

Now we consider the radial function r. Since Δ is K-invariant and K acts transitively on $S_r(o)$, the function Δr depends only on r. As a result, the function

$$\alpha'(r) \triangleq \frac{\partial}{\partial r} \log \sqrt{\det g} = \frac{1}{\sqrt{\det g}} \frac{\partial \sqrt{\det g}}{\partial r} = \Delta r$$

depends only on r. By integration, one can write

$$\log \sqrt{\det g} = \alpha(r) + \beta(\theta_1, \cdots, \theta_{n-1})$$

with some suitable functions α, β. It follows that

$$\sqrt{\det g} = e^{\alpha(r)} e^{\beta(\theta_1, \cdots, \theta_{n-1})}.$$

On the other hand, one knows that

$$A(r) = \int_{S_r(o)} d\omega = \int_{\{(\theta_1,\cdots,\theta_{n-1})\}} \sqrt{\det g} d\theta_1 \cdots d\theta_{n-1} = Ce^{\alpha(r)}.$$

As a consequence,

$$\frac{A'(r)}{A(r)} = \alpha'(r) = \frac{1}{\sqrt{\det g}} \frac{\partial \sqrt{\det g}}{\partial r}.$$

The equation (4.15) thus follows. □

Remark 4.2. Proposition 4.3 is also valid on $\mathbb{R}^n$ in which case $A(r) = \Omega_n r^{n-1}$. More generally, it is valid on *any symmetric space of rank one*. This assumption ensures that the isotropy subgroup at o acts transitively on $S_r(o)$, which is a crucial fact used in the above proof.

5 Derivation of the hyperbolic heat kernel formula

In this section, we give the complete derivation of the hyperbolic heat kernel formula (Theorem 1.1).

5.1 The hyperbolic Darboux equation

Most of the previous discussions are preparation for the proof the Darboux equation (Theorem 2.2). We now have all the required ingredients and we have also seen the simpler Euclidean argument in Section 2. In what follows, we consider the hyperbolic counterpart, whose proof of course covers the Euclidean case as well.

Proof of Theorem 2.2. Recall that

$$F(x,y) = M^{d(o,y)} f(x).$$

Let $\tilde{f}(g) \triangleq f(g \cdot o)$ and $\tilde{F}(g,h) \triangleq F(g \cdot o, h \cdot o)$ be their liftings onto the group G. As in the Euclidean case, one can write

$$\tilde{F}(g,h) = \int_K \tilde{f}(gkh) dk.$$

Let Ω be the Casimir operator over G. It follows from the bi-invariance of Ω that

$$\begin{aligned}\Omega_g \tilde{F}(g,h) &= \int_K \Omega_g \tilde{f}(gkh)dk \\ &= \int_K (\Omega\tilde{f})(gkh)dk \\ &= \int_K \Omega_h \tilde{f}(gkh)dk = \Omega_h \tilde{F}(g,h).\end{aligned} \tag{5.1}$$

By projecting down to M (cf. Corollary 4.1), one arrives at

$$\Delta_x F(x,y) = \Delta_y F(x,y).$$

The relation

$$\Delta_x M^r f(x) = M^r \Delta_x f(x)$$

follows from (5.1) and the fact that

$$\int_K (\Omega\tilde{f})(gkh)dk = (M^r \Delta_x f)^{\sim}(g)$$

with $r = d(o, h\cdot o)$ being fixed.

□

5.2 Proof of Ásgeirsson's mean value theorem

We now proceed to complete the proof of Theorem 2.1. We have already developed some part of the argument in Section 2. The notation is recalled as follows:

$$U(r,s) \triangleq M_x^r M_y^s u(x,y),$$

where u satisfies $\Delta_x u = \Delta_y u$. For any fixed $x, y \in M$, we have obtained previously that

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{A(r)}A'(r)\frac{\partial U}{\partial r} = \frac{\partial^2 U}{\partial s^2} + \frac{1}{A(s)}A'(s)\frac{\partial U}{\partial s}$$

for all $r, s > 0$. Note that there are no singularities for U or its derivatives with respect to r, s when $r, s \to 0$. This is seen by writing

$$M^r f(x) = \int_K \tilde{f}(gkh)dk$$

where $r = d(o, h\cdot o)$ and $x = g\cdot o$.

To prove Ásgeirsson's mean value theorem, we shall establish the following stronger relation:

$$U(r,s) = U(s,r). \tag{5.2}$$

The theorem is just the special situation when $s = 0$. For this purpose, let us define

$$F(r,s) \triangleq U(r,s) - U(s,r).$$

Then one has

$$\frac{\partial^2 F}{\partial r^2} + \frac{A'(r)}{A(r)}\frac{\partial F}{\partial r} - \frac{\partial^2 F}{\partial s^2} - \frac{A'(s)}{A(s)}\frac{\partial F}{\partial s} = 0.$$

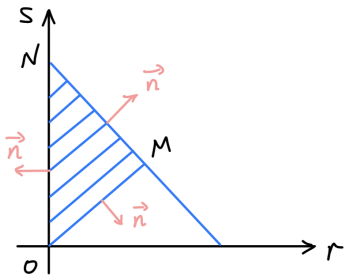
By multiplying both sides by $2A(r)\frac{\partial F}{\partial s}$, one finds that

$$-A(r)\frac{\partial}{\partial s}\Big(\Big(\frac{\partial F}{\partial r}\Big)^2 + \Big(\frac{\partial F}{\partial s}\Big)^2\Big) + 2\frac{\partial}{\partial r}\Big(A(r)\frac{\partial F}{\partial r}\frac{\partial F}{\partial s}\Big) - \frac{2A(r)}{A(s)}A'(s)\Big(\frac{\partial F}{\partial s}\Big)^2 = 0.$$

By applying the 2D Green's formula to the vector field

$$\overrightarrow{X} \triangleq \Big(2A(r)\frac{\partial F}{\partial r}\frac{\partial F}{\partial s}, -A(r)\Big(\Big(\frac{\partial F}{\partial r}\Big)^2 + \Big(\frac{\partial F}{\partial s}\Big)^2\Big)\Big)$$

on the triangle OMN (see the figure below with $N = (0,s)$),



one obtains that

$$\iint_{OMN} \frac{2A(r)}{A(s)}A'(s)\big(\frac{\partial F}{\partial s}\big)^2 dr ds = \int_{\partial OMN} \overrightarrow{X}\cdot\overrightarrow{\mathbf{n}}\, dl, \tag{5.3}$$

where $\overrightarrow{\mathbf{n}}$ denotes the unit outer normal on the boundary of the triangle and dl denotes the arc element. By using the explicit formula for $A(r)$, one can check that this is legal and there are no singularity issues as r or s tends to zero (all the relevant integrals are finite).

On the segment ON, one has $A(r) = 0$. On the segment OM, one has $\overrightarrow{\mathbf{n}} = (1/\sqrt{2}, -1/\sqrt{2})$. It follows that

$$\int_{OM} \overrightarrow{X} \cdot \overrightarrow{\mathbf{n}}\, dl = \frac{1}{\sqrt{2}} \int_{OM} A(r) \big(\frac{\partial F}{\partial r} + \frac{\partial F}{\partial s}\big)^2 dl.$$

Since $F(r, r) = 0$, one knows that $\frac{\partial F}{\partial r} + \frac{\partial F}{\partial s} = 0$ on OM. Therefore,

$$\int_{OM} \overrightarrow{X} \cdot \overrightarrow{\mathbf{n}}\, dl = 0.$$

On the segment MN, one has $\overrightarrow{\mathbf{n}} = (1/\sqrt{2}, 1/\sqrt{2})$ and

$$\int_{MN} \overrightarrow{X} \cdot \overrightarrow{\mathbf{n}}\, dl = -\frac{1}{\sqrt{2}} \int_{MN} A(r) \Big(\frac{\partial F}{\partial r} - \frac{\partial F}{\partial s}\Big)^2 dl.$$

By substituting these relations into (5.3), one obtains that

$$\iint_{OMN} \frac{2A(r)}{A(s)} A'(s) \Big(\frac{\partial F}{\partial s}\Big)^2 dr ds + \frac{1}{\sqrt{2}} \int_{MN} A(r) \Big(\frac{\partial F}{\partial r} - \frac{\partial F}{\partial s}\Big)^2 dl = 0.$$

As a consequence,

$$\frac{\partial F}{\partial s} = 0 \text{ on } OMN \quad \text{and} \quad \frac{\partial F}{\partial r} = \frac{\partial F}{\partial s} \text{ on } MN.$$

Since the point N is arbitrary, one concludes that

$$\frac{\partial F}{\partial r} = \frac{\partial F}{\partial s} = 0$$

for all $s \geqslant r$. Note that $F(r, r) = 0$. As a result, $F(r, s) = 0$, which yields the desired relation (5.2).

The proof of Ásgeirsson's mean value theorem is now complete.

5.3 The heat kernel formula

In Section 2, we have seen in the Euclidean case how Ásgeirsson's mean value theorem can be used to solve the wave equation. In this section, we develop the hyperbolic counterpart of the argument and complete the proof of Theorem 1.1.

5.3.1 Solving the wave equation

There is one subtle point in the hyperbolic situation. The natural wave equation on hyperbolic space is indeed

$$\frac{\partial^2 u}{\partial t^2} = \Big(\Delta + \Big(\frac{n-1}{2}\Big)^2\Big)u. \tag{5.4}$$

As pointed out by S. Helgason [Hel84], if one considers the Lorentzian manifold $\hat{M} \triangleq \mathbb{H}^n \times \mathbb{R}$, the kernel of the operator $\Delta + \big(\frac{n-1}{2}\big)^2 - \frac{\partial^2}{\partial t^2}$ (but not of $\Delta - \frac{\partial^2}{\partial t^2}$!) is conformally invariant. This explains why one should consider (5.4) instead. Note that $(\frac{n-1}{2})^2$ is the bottom spectrum of $-\Delta$ (cf. Remark 1.1), so that

$$L \triangleq \Delta + \big(\frac{n-1}{2}\big)^2 \leqslant 0.$$

We now use Ásgeirsson's mean value theorem to solve the following wave equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = Lu, \quad (x,t) \in \mathbb{H}^n \times \mathbb{R}; \\ u|_{t=0} = u_0, \\ u_t|_{t=0} = 0. \end{cases} \tag{5.5}$$

Although the case $u_t(\cdot, 0) \neq 0$ can also be included, we do not need this in the derivation of the heat kernel formula.

To perform explicit local calculations, we work with the upper half-space model (see Section 3.2)

$$U = \{(y_1, \cdots, y_n) : y_n > 0\}.$$

Recall that the hyperbolic metric is given by

$$ds^2 = \frac{1}{y_n^2}(dy_1^2 + \cdots + dy_n^2).$$

The hyperbolic Laplacian Δ is found to be

$$\Delta = y_n^2\Big(\frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_n^2}\Big) - (n-2)y_n\frac{\partial}{\partial y_n}.$$

By setting $y_n = e^s$, one finds that

$$\Delta = e^{2s}\Big(\frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_{n-1}^2}\Big) + e^{\frac{n-1}{2}s} \circ \Big(\frac{\partial^2}{\partial s^2} - \Big(\frac{n-1}{2}\Big)^2\Big) \circ e^{-\frac{n-1}{2}s},$$

where the functions $e^{\pm\frac{n-1}{2}s}$ are regarded as operators (multiplication by these functions).

Suppose that u is the solution to (5.5). Define

$$v(x_1,\cdots,x_n,y_1,\cdots,y_n) \triangleq e^{\frac{n-1}{2}s}u(x_1,\cdots,x_n,s)$$

where $y_n = e^s$. Then one has

$$\Delta_x v = e^{\frac{n-1}{2}s}\Delta_x u(x_1,\cdots,x_n,s)$$

and

$$\begin{aligned}\Delta_y v &= e^{\frac{n-1}{2}s}\Big(\frac{\partial^2}{\partial s^2} - \Big(\frac{n-1}{2}\Big)^2\Big)u(x_1,\cdots,x_n,s)\\ &= e^{\frac{n-1}{2}s}\Delta_x u(x_1,\cdots,x_n,s).\end{aligned}$$

Therefore, $\Delta_x v = \Delta_y v$.

According to Ásgeirsson's mean value theorem with $x_0 = x$ and $y_0 = o = (0,\cdots 0,1)$, one has

$$\Omega_n \sinh^{n-1} r\cdot(M^r u_0)(x) = \int_{S_r(o)} v(x,y)d\omega(y).$$

By using the disintegration formula (3.8),

$$\begin{aligned}&\int_{S_r(o)} v(x,y)d\omega(y)\\ &= \sinh r\int_{-r}^{r} e^{\frac{n-1}{2}s}u(x,s)\cdot\frac{e^{-s(n-3/2)}}{\sqrt{2(\cosh r-\cosh s)}}\cdot|C_s|_{\mathrm{eu}}ds\\ &= 2^{\frac{n-3}{2}}\Omega_{n-1}\sinh r\cdot\int_{-r}^{r}u(x,s)(\cosh r-\cosh s)^{\frac{n-3}{2}}ds\\ &= 2^{\frac{n-1}{2}}\Omega_{n-1}\sinh r\cdot\int_{0}^{r}u(x,s)(\cosh r-\cosh s)^{\frac{n-3}{2}}ds,\end{aligned}\tag{5.6}$$

where $|C_s|_{\mathrm{eu}}$ denotes the Euclidean area of C_s and we used the observation that u is an even function in t. It follows that

$$\int_0^r u(x,s)(\cosh r-\cosh s)^{\frac{n-3}{2}}ds = \frac{\Omega_n}{\Omega_{n-1}2^{(n-1)/2}}\cdot\sinh^{n-2}r\cdot(M^r u_0)(x).\tag{5.7}$$

One can easily invert this equation to recover $u(x,r)$. To this end, following the notation of [GN98] let us denote

$$(N^r_{m,k}f)(x) \triangleq \Big(\frac{\partial}{\partial\cosh r}\Big)^m\big(\sinh^k r\cdot M^r f(x)\big)$$

where $\frac{\partial}{\partial\cosh r} = \frac{1}{\sinh r}\frac{\partial}{\partial r}$.

Case I: $n \geqslant 3$ is odd.

Let $n = 2m+1$ $(m \geqslant 1)$. By applying $\left(\frac{\partial}{\partial \cosh r}\right)^{\frac{n-3}{2}}$ to the equation (5.7), one obtains that

$$(m-1)! \int_0^r u(x,s)ds = \frac{\Omega_n}{\Omega_{n-1} 2^m} \big(N^r_{\frac{n-3}{2},n-2} u_0\big)(x).$$

Therefore,

$$\begin{aligned} u(x,r) &= \frac{\Omega_n}{\Omega_{n-1} 2^m (m-1)!} \Big(\frac{\partial}{\partial r} N^r_{\frac{n-3}{2},n-2} u_0\Big)(x) \\ &= \frac{1}{(n-2)!!} \Big(\frac{\partial}{\partial r} N^r_{\frac{n-3}{2},n-2} u_0\Big)(x). \end{aligned}$$

Case II: $n \geqslant 2$ is even.

Let $n = 2m+2$ $(m \geqslant 0)$. By applying $\left(\frac{\partial}{\partial \cosh r}\right)^{\frac{n-2}{2}}$ to the equation (5.7) and simplification, one finds that

$$\int_0^r \frac{u(x,s)ds}{\sqrt{\cosh r - \cosh s}} = \frac{\pi}{\sqrt{2}(n-2)!!} \big(N^r_{\frac{n-2}{2},n-2} u_0\big)(x). \tag{5.8}$$

In order to recover $u(x,r)$, one multiplies both sides of (5.8) by $K(t,r)\sinh r$ where

$$K(t,r) \triangleq \frac{1}{\sqrt{\cosh t - \cosh r}}, \quad r < t, \tag{5.9}$$

and integrate over r from 0 to t $(t > 0)$. The left hand side of (5.8) becomes

$$\begin{aligned} &\int_0^t dr \int_0^r K(t,r)K(r,s)u(x,s)ds \\ &= \int_0^t u(x,s)ds \int_s^t K(t,r)K(r,s)dr \\ &= \pi \int_0^t u(x,s)ds, \end{aligned} \tag{5.10}$$

where we used the observation that

$$\begin{aligned} \int_s^t K(t,r)K(r,s)dr &= \int_{\cosh s}^{\cosh t} \frac{d\xi}{\sqrt{\cosh t - \xi}\sqrt{\xi - \cosh s}} \\ &= \int_0^1 \frac{d\eta}{\sqrt{\eta(1-\eta)}} = B(1/2,1/2) = \pi. \end{aligned}$$

On the other hand, the right hand side of (5.8) becomes

$$\frac{\pi}{\sqrt{2}(n-2)!!}\int_0^t K(t,r)\sinh r\big(N^r_{\frac{n-2}{2},n-2}u_0\big)(x)dr. \tag{5.11}$$

It follows from (5.10) and (5.11) that

$$u(x,t)=\frac{1}{\sqrt{2}(n-2)!!}\frac{\partial}{\partial t}\Big(\int_0^t\frac{\sinh r}{\sqrt{\cosh t-\cosh r}}\big(N^r_{\frac{n-2}{2},n-2}u_0\big)(x)\Big)dr.$$

Let us summarise the main conclusion in the proposition below.

Proposition 5.1. *Define*

$$U(x,r)\triangleq\int_{S_r(x)}u_0(y)d\omega(y)=\Omega_n\sinh^{n-1}r\cdot M^r u_0(x).$$

The solution $u(x,t)$ to the hyperbolic wave equation (5.5) is given by the following formula.

(i) If $n\geqslant 3$ is odd, then one has

$$u(x,t)=\frac{1}{\Omega_n(n-2)!!}\frac{\partial}{\partial t}\Big(\Big(\frac{1}{\sinh t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(\frac{U(x,t)}{\sinh t}\Big)\Big). \tag{5.12}$$

(ii) If $n\geqslant 2$ is even, then one has

$$u(x,t)=\frac{1}{\Omega_n\sqrt{2}(n-2)!!}\frac{\partial}{\partial t}\int_0^t\frac{\sinh r}{\sqrt{\cosh t-\cosh r}}\Big(\frac{1}{\sinh r}\frac{\partial}{\partial r}\Big)^{\frac{n-2}{2}}\Big(\frac{U(x,r)}{\sinh r}\Big)dr. \tag{5.13}$$

In both cases, the formulas are stated for $t>0$. For $t<0$, one has $u(x,t)=u(x,-t)$.

Remark 5.1. It can be shown that $U(x,\cdot)\in C^{n-1}(\mathbb{R})$ and $\frac{\partial^k}{\partial t^k}U(x,0)=0$ for $k=0,1,\cdots,n-1$. This fact is used to see that the formulae (5.12) and (5.13) extend continuously to $t=0$. We omit the discussion of this technical point (see [GN98, Lemma 2.4] for the details).

5.3.2 Proof of Theorem 1.1

Recall that

$$e^{t\Delta}=e^{-(n-1)^2t/4}e^{tL},$$

where $L = \Delta + \left(\frac{n-1}{2}\right)^2$. Let $p_\Delta(x,y,t)$ and $p_L(x,y,t)$ denote the heat kernels of Δ and L respectively. It is plain to see that

$$p_\Delta(x,y,t) = e^{-(n-1)^2 t/4} p_L(x,y,t). \tag{5.14}$$

As a consequence, it is sufficient to compute $p_L(x,y,t)$.

Recall from (2.1) that

$$e^{tL} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} \cos(s\sqrt{-L}) ds.$$

Let u be the solution to the wave equation (5.5). Let v be the solution to the heat equation associated with the operator L and $v(x,0) = u_0(x)$. Then one has

$$v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} u(x,s) ds.$$

We again divide the discussion into two cases.

Case I: $n = 2m+1$ $(m \geqslant 1)$.

According to (5.12),

$$v(x,t) = \frac{1}{\Omega_n (n-2)!! \sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} \frac{\partial}{\partial s} \Big(\Big(\frac{1}{\sinh s} \frac{\partial}{\partial s} \Big)^{m-1} \Big(\frac{U(x,s)}{\sinh s} \Big) \Big) ds.$$

By using integration by parts, one can write

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} \frac{\partial}{\partial s} \Big(\Big(\frac{1}{\sinh s} \frac{\partial}{\partial s} \Big)^{\frac{n-3}{2}} \Big(\frac{U(x,s)}{\sinh s} \Big) \Big) ds \\
&= - \int_{-\infty}^{\infty} \Big(\frac{\partial}{\partial s} e^{-\frac{s^2}{4t}} \Big) \Big(\frac{1}{\sinh s} \frac{\partial}{\partial s} \Big)^{m-1} \Big(\frac{U(x,s)}{\sinh s} \Big) ds \\
&= - \int_{-\infty}^{\infty} \Big(\frac{1}{\sinh s} \frac{\partial}{\partial s} e^{-\frac{s^2}{4t}} \Big) \frac{\partial}{\partial s} \Big(\Big(\frac{1}{\sinh s} \frac{\partial}{\partial s} \Big)^{m-2} \Big(\frac{U(x,s)}{\sinh s} \Big) \Big) ds \\
&= (-1)^2 \int_{-\infty}^{\infty} \frac{\partial}{\partial s} \Big(\frac{1}{\sinh s} \frac{\partial}{\partial s} e^{-\frac{s^2}{4t}} \Big) \Big(\frac{1}{\sinh s} \frac{\partial}{\partial s} \Big)^{m-2} \Big(\frac{U(x,s)}{\sinh s} \Big) ds \\
&= (-1)^2 \int_{-\infty}^{\infty} \Big(\frac{1}{\sinh s} \frac{\partial}{\partial s} \Big)^2 \Big(e^{-\frac{s^2}{4t}} \Big) \cdot \frac{\partial}{\partial s} \Big(\Big(\frac{1}{\sinh s} \frac{\partial}{\partial s} \Big)^{m-3} \Big(\frac{U(x,s)}{\sinh s} \Big) \Big) ds \\
&\cdots \\
&= (-1)^{m-1} \int_{-\infty}^{\infty} \Big(\frac{1}{\sinh s} \frac{\partial}{\partial s} \Big)^{m-1} \Big(e^{-\frac{s^2}{4t}} \Big) \cdot \frac{\partial}{\partial s} \Big(\Big(\frac{1}{\sinh s} \frac{\partial}{\partial s} \Big)^{0} \Big(\frac{U(x,s)}{\sinh s} \Big) \Big) ds
\end{aligned}$$

$$= (-1)^m \int_{-\infty}^{\infty} \Big(\frac{1}{\sinh s}\frac{\partial}{\partial s}\Big)^m \Big(e^{-\frac{s^2}{4t}}\Big) \cdot U(x,s)ds$$
$$= (-1)^m \int_{\mathbb{H}^n} \Big(\frac{1}{\sinh \rho}\frac{\partial}{\partial \rho}\Big)^m \big(e^{-\frac{\rho^2}{4t}}\big) \cdot u_0(y)dy,$$

where $\rho \triangleq d(x,y)$ denotes the hyperbolic distance between x and y. It follows that

$$\begin{aligned} v(x,t) &= \int_{\mathbb{H}^n} p_L(x,y,t)u_0(y)dy \\ &= \frac{(-1)^m}{\Omega_n(n-2)!!\sqrt{4\pi t}} \int_{\mathbb{H}^n} \Big(\frac{1}{\sinh \rho}\frac{\partial}{\partial \rho}\Big)^m \Big(e^{-\frac{\rho(x,y)^2}{4t}}\Big) \cdot u_0(y)dy. \end{aligned}$$

Since u_0 is arbitrary, one obtains that

$$p_L(x,y,t) = \frac{(-1)^m}{\Omega_n(n-2)!!} \cdot \frac{1}{\sqrt{4\pi t}} \cdot \Big(\frac{1}{\sinh \rho}\frac{\partial}{\partial \rho}\Big)^m \Big(e^{-\frac{\rho(x,y)^2}{4t}}\Big).$$

In view of (5.14), this gives the desired heat kernel formula (1.1).

Case II: $n = 2m+2$ $(m \geqslant 0)$.

The calculation in this case is similar. Recall that $K(t,r)$ is defined by (5.9). According to (5.13), one has

$$\begin{aligned} v(x,t) =& \frac{1}{\Omega_n\sqrt{2}(n-2)!!\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} \frac{\partial}{\partial s} \int_0^s K(s,\rho)\sinh\rho \\ &\qquad \times \Big(\frac{1}{\sinh \rho}\frac{\partial}{\partial \rho}\Big)^m \Big(\frac{U(x,\rho)}{\sinh\rho}\Big)d\rho ds \\ =& \frac{c_n}{\sqrt{t}} \int_{-\infty}^{\infty} \frac{s}{2t} e^{-\frac{s^2}{4t}} \Big(\int_0^s K(s,\rho)\sinh\rho \Big(\frac{1}{\sinh \rho}\frac{\partial}{\partial \rho}\Big)^m \Big(\frac{U(x,\rho)}{\sinh\rho}\Big)d\rho\Big) ds \\ =& \frac{c_n}{t^{3/2}} \int_0^{\infty} \Big(\int_\rho^{\infty} s e^{-\frac{s^2}{4t}} K(s,\rho)ds\Big) \sinh\rho \Big(\frac{1}{\sinh \rho}\frac{\partial}{\partial \rho}\Big)^m \Big(\frac{U(x,\rho)}{\sinh\rho}\Big)d\rho \\ =& \frac{c_n}{t^{3/2}} \int_0^{\infty} E(\rho)\frac{\partial}{\partial\rho}\Big(\Big(\frac{1}{\sinh \rho}\frac{\partial}{\partial \rho}\Big)^{m-1}\Big(\frac{U(x,\rho)}{\sinh\rho}\Big)\Big)d\rho, \end{aligned}$$

where

$$c_n \triangleq \frac{1}{\Omega_n\sqrt{2}(n-2)!!\sqrt{4\pi}}, \quad E(\rho) \triangleq \int_\rho^{\infty} s e^{-\frac{s^2}{4t}} K(s,\rho)ds.$$

By using integration by parts as in the odd case, one arrives at

$$v(x,t) = \frac{(-1)^m c_n}{t^{3/2}} \int_{\mathbb{H}^n} \Big(\Big(\frac{1}{\sinh \rho}\frac{\partial}{\partial \rho}\Big)^m \int_\rho^{\infty} s e^{-\frac{s^2}{4t}} K(s,\rho)ds\Big) u_0(y)dy.$$

Therefore,

$$p_L(x,y,t) = \frac{(-1)^m c_n}{t^{3/2}} \Big(\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \Big)^m \int_\rho^\infty s e^{-\frac{s^2}{4t}} K(s,\rho) ds.$$

This again leads to the desired heat kernel formula (1.2) in view of (5.14).

Remark 5.2. There is a discrete analogue of the heat kernel asymptotics (cf. Remark 1.1)

$$p(x,y,t) \sim \frac{e^{-(n-1)^2 t/4}}{t^{3/2}} \quad \text{as } t \to \infty.$$

The discrete counterpart of $\mathbb{H}^n$ is the free group $\mathbb{F}^n$ with n generators, i.e. the group of all reduced words over n noncommutative letters. The counterpart of the Brownian motion is the simple random walk and the counterpart of the heat kernel is the transition probability function $p_n(x,y) \triangleq \mathbb{P}(X_n = y | X_0 = x)$ of the walk. Assuming an aperiodicity condition (to avoid $p_n(x,y) = 0$ for some n, x, y), the decay rate of $p_n(x,y)$ is given by

$$p_n(x,y) \sim \frac{R^{-n}}{n^{3/2}} \quad \text{as } n \to \infty.$$

Here $R > 1$ is the *spectral radius* of the generator of the random walk, equivalently, the radius of convergence for the Green function

$$z \mapsto G(z|x,y) \triangleq \sum_{n=0}^\infty p_n(x,y) z^n.$$

This result can be found in [Woe20].

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