Expected Signature on a Riemannian Manifold and Its Geometric Implications

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Abstract

On a compact Riemannian manifold M, we show that the Riemannian distance function $d(x, y)$ can be explicitly reconstructed from suitable asymptotics of the expected signature of Brownian bridge from x to y. In addition, by looking into the asymptotic expansion of the fourth level expected signature of the Brownian loop based at $x \in M$, one can explicitly reconstruct both intrinsic (Ricci curvature) and extrinsic (second fundamental form) curvature properties of M at x. As independent interest, we also derive the intrinsic PDE for the expected Brownian signature dynamics on M from the perspective of the Eells-Elworthy-Malliavin horizontal lifting.

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1 Introduction and outline of main results

In this section, we discuss the main motivation, introduce some background information and outline the main results of the present article.

1.1 The signature transform

The *signature transform* (or simply the signature) of a multidimensional path $\gamma : [0, T] \to \mathbb{R}^d$, which is defined by the formal tensor series

$$S(\gamma) \triangleq \big(1, \gamma_T - \gamma_0, \int_{0<s<t<T} d\gamma_s \otimes d\gamma_t, \cdots, \int_{0<t_1<\cdots<t_n<T} d\gamma_{t_1} \otimes \cdots \otimes d\gamma_{t_m}, \cdots\big)$$

of iterated path integrals, provides an effective summary of the essential information encoded in the original path γ. Such a transformation, as well as its intrinsic form defined in terms of iterated integrals against spatial one-forms, was originally introduced by the geometer K.T. Chen [Che54] in the 1950s for his purpose of constructing a de Rham-type cohomology theory on loop spaces over manifolds. Similar type of iterated integrals (with γ being operator-valued paths) was also used by the physicist F. Dyson [Dys49] to obtain perturbative expansions of Schrödinger equation (Dyson series).

The signature plays a fundamental role in the analysis of rough paths and rough differential equations. Its mathematical properties have been largely developed and better understood over the past two decades by using modern techniques from the rough path theory, which was founded by T. Lyons in his seminal work [Lyo98] in 1998 and has led to far-reaching applications in stochastic analysis. Among others, a basic result is the so-called *signature uniqueness theorem*. It asserts that every rough path is uniquely determined by its signature up to treelike equivalence. This result was already obtained in Chen's originial work [Che58] in 1958 for piecewise smooth paths. It was extended to the bounded variation case by Hambly-Lyons [HL10] in 2010 and further to the general rough path case by

Boedihardjo et al. [BGLY16] in 2016. The importance of the signature uniqueness theorem lies in the fact that it potentially opens a gateway of studying geometric properties of a rough path from its signature. In fact, the reconstruction of a rough path from its signature at variaous quantitative levels (the signature inversion problem) is among the most significant and challenging open problems in rough path theory. Partial progress and several different methods have been developed in the literature over the past years to understand this problem, e.g. Lyons-Xu [LX15] by hyperbolic developments, Lyons-Xu [LX18] by symmetrisation, Chang et al. [CDNX17] by probabilistic sampling, Chang-Lyons [CL19] by tensor insertions, Le Jan-Qian [LQ12] and Geng [Gen17] by iterated integrals against suitable spatial one-forms etc. If one is only interested in some particular quantitative properties of a path instead of recovering the full trajectory, there could be neat inversion formulae such as the relation (1.3) below which reconstructs the length of a smooth path from its signature asymptotics in a rather simple and elegant way.

On the other hand, if the underlying rough path is random (sample paths of a stochastic process such as the Brownian motion) the signature becomes a tensor-valued random variable. In this case, one naturally considers the expectation of the signature (the *expected signature*). There is a probabilistic counterpart of the signature uniqueness theorem due to Chevyrev-Lyons [CL16] in 2016, which asserts that under suitable conditions the law of a stochastic process is uniquely characterised by its expected signature. To some extent, this can be viewed as an infinite dimensional analogue of the classical moment problem for random variables. There has been several works using a PDE approach to compute and analyse the expected signature of Brownian motion [LV04], [LN15], diffusion processes [Ni12] and Lévy processes [FS17]. For a general stochastic process, how one can explicitly and quantitatively reconstruct distributional properties of the process from its expected signature is widely unknown at the moment.

On the applied side, more recently the applications of expected signature in machine learning are gaining attention. Specifically, Sig-MMD or Sig-Wasserstein-1 (Sig-W_1) distance based on the expected signature can serve as the maximum mean discrepancy (MMD) distance for general stochastic processes, which have wide applications in hypothesis testing [CO22], distributional regression [LSD21] and generative models on sequential data [NSW23, NSS21]. Compared with the vanilla MMD with vector-valued time series feature, Sig-MMD demonstrates superior performance in the two-sample hypothesis test of stochastic processes, as illustrated in [CO22]. This metric can be kernelized to facilitate efficient computation [SCF21]. Moreover, the Sig-WGAN model, which utilises Sig-W_1 based discriminator, has empirically shown to improve the accuracy and robustness of traditional GAN models for synthetic time series generation. This approach reduces the min-max game to supervised learning, hence significantly reducing the computational cost and yielding better performance, in particular, for the case of limited data [NSW23, NSS21].

1.2 Extended signature and the basic geometric question

In this article, we take a different perspective of signature inversion. Instead of trying to reconstruct a generic path from the knowledge of its signature, we study signature dymanics in a geometric setting and investigate the following basic question.

Question. *Let M be a Riemannian manifold. By observing the expected signature of certain geometric dynamics on M (e.g. Brownian bridges), can one explicitly recover geometric properties (e.g. Riemannian distance, curvature properties) of the underlying space?*

Summarised in vague terms, the main finding of the present article is that the expected signature of a Brownian bridge on M with lifetime t encode rich geometric information (more precisely, metric and curvature properties) about the underlying manifold M in the asymptotics when $t \to 0^+$. How such information can be extracted explicitly from the expected signature asymptotics is the main focus of the present work. The precise statements of our main results are stated in Theorem 4.1 (reconstruction of Riemannian distance) and Theorem 5.1 (reconstruction of curvature properties) respectively.

Before explaining the essential ideas, one needs to be careful about the notion of signature for manifold-valued paths in the first place. Let $\gamma : [0,T] \to M$ be a smooth path taking values in a differentiable manifold M. In the differential-geometric setting, one cannot give intrinsic meanings to integrals like $\int d\gamma \otimes \cdots \otimes d\gamma$ without any additional structure. In fact, the intrinsic notion of integration along γ is the *line integral against a one-form* on M. More specifically, let $\phi_1, \cdots, \phi_n$ be a given family of smooth one-forms on M. One can consider the iterated line integral

$$\int_{0<t_1<\cdots<t_n<T} \phi_1(d\gamma_{t_1}) \cdots \phi_n(d\gamma_{t_n}),$$

which is globally well-defined due to the natural pairing between cotangent and tangent vectors ($\phi(d\gamma_t)$ is understood as the pairing between $\phi(\gamma_t) \in T^*_{\gamma_t}M$ and

$\dot{\gamma}_t \in T_{\gamma_t}M$). By varying n and the test one-forms $\phi_1, \cdots, \phi_n$, one obtains a family of numbers associated with the given path γ. This collection of numbers, known as the *extended signature* of γ (cf. [LQ12]), uniquely determines γ up to tree-like pieces. This geometric viewpoint is classical and was already well understood back in K.T. Chen's original signature uniqueness theorem [Che58] in the 1950s. The extended signature (iterated line integrals against one-forms) has been used in various contexts, e.g. in reconstructing trajectories from the signature [LQ12, Gen17] and in the study of rough paths on manifolds [CDL15]. In the Euclidean case, the signature is a special case of iterated line integrals (one simply takes $\phi_i = dx^i$) and the extended signature contains exactly the same amount of information as the usual signature. Indeed, it is well known that smooth one-forms can be approximated by polynomials and iterated line integrals against polynomial forms can be expressed as a suitable linear combination of signature coefficients (the shuffle product formula). As a consequence, the extended signature is uniquely determined by the usual signature.

In our study, we will assume that a collection of one-forms $(\phi_1, \cdots, \phi_N)$ on M (equivalently, an $\mathbb{R}^N$-valued one form ϕ) is given fixed and consider the Euclidean signature of the $\mathbb{R}^N$-valued path $\int_0^\cdot \phi(d\gamma_t)$. We call this the ϕ*-signature* of γ (of course, this is just part of the extended signature of γ given by iterated line integrals along combinations of one-forms taken from the family $\phi = (\phi_1, \cdots, \phi_N)$). A basic example that will be of our primary interest is the case when ϕ is given by the exterior derivative of an embedding $F : M \to \mathbb{R}^N$. In this case, the ϕ-signature of γ is just the Euclidean signature of $F(\gamma)$ computed in the ambient space $\mathbb{R}^N$.

1.3 Intrinsic PDE for the expected signature dynamics

Let us now consider a Riemannian manifold M and let ϕ be an $\mathbb{R}^N$-valued one-form on M. For most of the time, $\phi = dF$ where $F : M \to \mathbb{R}^N$ is an *isometric* embedding (this always exists by Nash's isometric embedding theorems). We consider the expected ϕ-signature of Brownian dynamics on M. More specifically, let W_t^x be a Brownian motion on M (i.e. a Markov process generated by $\Delta/2$ where Δ is the Laplace-Beltrami operator) starting at x. Let $\Psi(t, x)$ be the expected value of the ϕ-signature (in the Stratonovich sense) of W^x up to time t.

Our first main result establishes the intrinsic PDE governing the dynamics of $\Psi(t, x)$ (cf. Theorem 3.1).

Theorem. *The function $\Psi(t, x)$ satisfies the following tensor-algebra-valued parabolic*

PDE on M:

$$\frac{\partial\Psi}{\partial t}=\frac{1}{2}\Delta\Psi+\mathrm{Tr}\big(\phi\otimes d\Psi+\frac{1}{2}(\nabla\phi+\phi\otimes\phi)\otimes\Psi\big) \tag{1.1}$$

Here d is the exterior derivative operator and ∇ is the Riemannian connection. The definition of $\mathrm{Tr}(\cdot)$ is explained in the remarks following the statement of Theorem 3.1 below. The PDE for the expected signature of Euclidean Brownian motion (and also of diffusion processes) was first established by Lyons-Ni [LN15] from the Markovian perspective. To derive the PDE (1.1), we take an intrinsic approach from the perspective of the Eells-Elworthy-Malliavin lifting onto the orthornormal frame bundle. It is well known that the lifted Brownian motion Ξ_t on the bundle satisfies a canonical SDE governed by the fundamental horizontal vector fields. As a result, one can write down an SDE for the joint process $(\Xi_t,\tilde{S}_t)$ (cf. (3.10)), where $\tilde{S}_t$ is the Φ-signature of Ξ up to time t and Φ is the pullback of ϕ onto the bundle. Once the SDE for $(\Xi_t,\tilde{S}_t)$ is obtained, it is standard to extract its generator and write down the associated PDE governing the expectation of $\tilde{S}_t$ on the bundle (cf. Lemma 3.4). It then remains to see how the PDE on the bundle gets projected to the intrinsic PDE (1.1) on the base manifold M.

The PDE (1.1) for the Brownian signature dynamics easily yields an associated PDE for the expected signature of Brownian bridge. Let $\{X_s^{t,x,y}:s\in[0,t]\}$ be a Brownian bridge from x to y with lifetime t, i.e. the Brownian motion starting at x conditioned on reaching y at time t. Let $\psi(t,x,y)$ be its expected ϕ-signature.

Theorem. *The function $\psi(t,x,y)$ satisfies the following tensor-algebra-valued parabolic PDE on M:*

$$\begin{aligned}\frac{\partial\psi}{\partial t}=&\frac{1}{2}\Delta_x\psi+\langle\nabla_x\log p(t,x,y),\nabla_x\psi+\phi\otimes\psi\rangle+\mathrm{Tr}(\phi\otimes d_x\psi)\\&+\frac{1}{2}\mathrm{Tr}\big((\nabla\phi+\phi\otimes\phi)\otimes\psi\big),\end{aligned}$$

where $p(t,x,y)$ is the heat kernel on M and the subscript x means differentiating with respect to the x-variable.

As we will see, the function $\psi(t,x,y)$ encodes rich geometric information about the manifold M in the asymptotics as $t\to0^+$. In what follows, we consider the particular case when $\phi=dF$, where $F:M\to E\triangleq\mathbb{R}^N$ is an isometric embedding. We will establish two types of reconstruction results:

(i) [*Reconstruction of Riemannian distance*] Through a suitable procedure of combined asymptotics for $\psi(t,x,y)$ as $t\to0^+$ and $n\to\infty$ (n is the degree of the signature), one can explicitly recover the Riemannian distance between x and y.

(ii) [*Reconstruction of curvature properties*] By considering the asymptotic expansion of the fourth level component of $\psi(t,x,x)$ as $t \to 0^+$, one can explicitly recover the Riemannian metric tensor as well as both intrinsic (Ricci curvature) and extrinsic (the second fundamental form) curvature properties at x. Here we consider the fourth level projection because this can be shown to be the first nonzero component in $\psi(t,x,x)$.

1.4 Reconstruction of Riemannian distance

Recall that $\psi(t,x,y)$ is the expected dF-signature of the Brownian bridge $X^{t,x,y}$ from x to y with lifetime t, where $F : M \to \mathbb{R}^N$ is a given isometric embedding. Our main idea of reconstructing the Riemannian distance $d(x,y)$ is largely inspired by a well-known open problem in rough path theory which we shall first describe.

Let $\gamma : [0,T] \to \mathbb{R}^d$ be a continuous path with bounded total variation. By using the triangle inequality, it is straight forward to see that

$$\big(n!\big\|S_n(\gamma)\big\|\big)^{1/n} \leqslant L(\gamma) \quad \forall n \geqslant 1. \tag{1.2}$$

Here $S_n(\gamma) \in (\mathbb{R}^d)^{\otimes n}$ is the n-th level signature of γ, $\|\cdot\|$ is a suitable tensor norm (e.g. the projective norm) and $L(\gamma)$ is the length of γ. The remarkable (and rather surprising) point is that the simple estimate (1.2) becomes asymptotically sharp as $n \to \infty$. More precisely, it was implicitly conjectured by Hambly-Lyons [HL10] which was made explicit in Chang-Lyons-Ni [CLN18] that under the projective tensor norm $\|\cdot\|_{\text{proj}}$,

$$\lim_{n\to\infty} \big(n!\big\|S_n(\gamma)\big\|_{\text{proj}}\big)^{1/n} = L(\gamma) \tag{1.3}$$

for any tree-reduced (i.e. without treelike pieces) BV path γ. In other words, the length of a tree-reduced path is recovered from its normalised signature asymptotics. The formula (1.3) was established for $\mathcal{C}^3$ paths by Hambly-Lyons [HL10] and $\mathcal{C}^1$ paths by Lyons-Xu [LX15]. It was extended to the two-dimensional BV case by Boedihardjo-Geng [BG22] without any regularity assumption but under a stronger notion of tree-reducedness. A stronger version of (1.3) for $\mathcal{C}^2$-paths was recently obtained by [CMT23] generalising a corresponding result in [HL10]. To our best knowledge, proving (or disproving) the conjectural formula (1.3) for the general BV case remains an open problem.

Returning to the geometric setting, our main idea of reconstructing the Riemannian distance $d(x,y)$ from suitable asymptotics of the expcted dF-signature $\psi(t,x,y)$ of the Brownian bridge $X^{t,x,y}$ can be summarised as follows.

1. When t is small, one is essentially forcing the bridge $X^{t,x,y}$ to travel from x to y by a short amout of time and thus $X^{t,x,y}$ behaves like a minimising geodesic $\gamma^{x,y}$ joining x to y. This property is quantified through the large deviation principle (LDP) of the Brownian bridge established by Hsu [Hsu90].

2. As a consequence of the first point, it is reasonable to expected that $\pi_n\psi(t,x,y)$ (the n-th level projection) is close to the n-th level dF-signature of the geodesic $\gamma^{x,y}$. Note that the latter is just the usual Euclidean signature of $F(\gamma^{x,y})$ in the ambient space $\mathbb{R}^N$.

3. As a consequence of the length conjecture (1.3) (which is a proven fact due to the smoothness of geodesics), the length of $F(\gamma^{x,y})$ in $\mathbb{R}^N$ can be recovered from its normalised signature asymptotics. Note that this length is precisely the Riemannian distance $d(x,y)$ since F is an isometric embedding.

4. In view of the second and third points, one naturally expects that

$$\big(n!\|\pi_n\psi(t,x,y)\|\big)^{1/n} \approx d(x,y)$$

provided that t is small and n is large.

Our main reconstruction result can be roughly stated as follows.

Theorem. *The Riemannian distance $d(x,y)$ can be reconstructed from the following asymptotic formula:*

$$\lim_{n\to\infty}\big(n!\|\pi_n\psi(t_n,x,y)\|\big)^{1/n} = d(x,y) \tag{1.4}$$

provided that $t_n << n^{-6}$.

The precise and quantitative statement of such a result, which requires a few technical assumptions, is given by Theorem 4.1. Although the underlying idea is simple and natural, proving such a relation mathematically requires a substantial amount of non-trivial analysis. For instance, the above second point is not a direct consequence of the LDP since the latter was proved under the uniform (instead of rough path) topology and it is well-known that the signature is not a continuous functional of the driving path with respect to the uniform topology. As a result, one has to establish the signature approximation separately and we do so by using the semimartingale decomposition of Brownian bridge under normal coordinate charts. In particular, one needs to separate out the geodesic component in the decomposition and estimate the remainder effectively.

We should mention a particularly subtle point among several other technical challenges which will all be clear along the development of the proof in Section 4.2. For each fixed level n, as long as the lifetime t_n is chosen to be small enough one naturally has

$$\pi_n \psi(t_n, x, y) \approx n\text{-th level signature of } \gamma^{x,y}.$$

If one applies standard signature moment estimates for semimartingales without much care, one is led to choosing $t_n = o(e^{-Cn})$ which is too small to be practically useful (e.g. in the context of simulating a Brownian bridge with lifetime t_n). A main contribution of our analysis is that the lifetime needs not be exponentially small in n to make the asymptotics (1.4) valid. Indeed, a *polynomial* dependence $t_n = O(n^{-6})$ is sufficient (cf. (4.1) for the quantitative reconstruction estimate with explicit error bound). Improving from exponential to polynomial dependence requires much finer signature estimates for the aforementioned semimartingale decomposition. To achieve this, we make use of a deep result of Kallenberg-Sztencel [KS91] on dimension-free BDG inequalities for Hilbert-space-valued martingales.

1.5 Reconstruction of metric and curvature properties

The asymptotic formula (1.4) provides a way of recovering the Riemannian distance $d(x, y)$ from the expected signature asymptotics of Brownian bridge. This can be viewed as an zeroth order result. It then becomes reasonable to expect that curvature properties would start to appear in the higher order asymptotic expansion of the expected signature function $\psi(t, x, y)$.

To be specific, we consider the case when $x = y$ (the Brownian loop based at x) and look at the function

$$\psi_4(t, x) \triangleq \pi_4 \psi(t, x, x) \in (\mathbb{R}^N)^{\otimes 4}.$$

Due to symmetry considerations, it can be shown that $\pi_i \psi_4(t, x, x) = 0$ for $i = 1, 2, 3$ (see the discussion at the start of Section 5.1). Therefore, $\psi_4(t, x)$ is the first non-trivial component of $\psi(t, x, x)$ which indeed carries rich geometric information. In Propositions 5.9 and 5.16 below, we compute the expansion of $\psi_4(t, x)$ up to order t^3:

$$\psi_4(t, x) = \hat{\Theta}_x t^2 + \hat{\Xi}_x t^3 + O(t^4)$$

with explicit expressions for the tensor coefficients $\hat{\Theta}_x, \hat{\Xi}_x \in (\mathbb{R}^N)^{\otimes 4}$ in terms of metric and curvature coefficients as well as the embedding F. The actual expressions of $\hat{\Theta}_x$ and $\hat{\Xi}_x$ are too complicated to write down here. However, after applying suitable tensor contraction $\mathfrak{C} : (\mathbb{R}^N)^{\otimes 4} \to (\mathbb{R}^N)^{\otimes 2}$ (cf. (5.3) for its precise definition) their expressions are simplified substantially and their intrinsic meanings in terms of metric and curvature properties become clear. The main formulae are given as follows.

Theorem. *The contracted tensors* $\Theta_x \triangleq \mathfrak{C}\hat{\Theta}_x$ *and* $\Xi_x \triangleq \mathfrak{C}\hat{\Xi}_x$*, being viewed as symmetric bilinear forms on* T_xM *(note that* T_xM *is viewed as a subspace of* $\mathbb{R}^N$ *under the embedding* F*), are given by the following formulae:*

$$\Theta_x = \frac{d-1}{24} g_x$$

and

$$\Xi_x = \frac{\mathrm{S}_x - 18d^2 |H_x|^2_{\mathbb{R}^N}}{8640} g_x + \frac{49d-20}{8640} \mathrm{Ric}_x + \frac{(5-4d)d}{480} \langle B_x, H_x \rangle_{\mathbb{R}^N}.$$

Here d *is the dimension of* M*,* g *is the Riemannian metric tensor,* S *is the scalar curvature,* Ric *is the Ricci curvature tensor,* B_x *is the second fundamental form and* H *is the mean curvature field with respect to* F*.*

The precise formulation of the theorem is given by Theorem 5.1 below, where the relevant geometric concepts are recalled in Section 2.1.3. As a consequence, the tensor Θ_x recovers the metric tensor and Ξ_x encodes both intrinsic and extrinsic curvature properties. It is also seen from the formula (5.4) and Remark 5.18 that the tangent space T_xM can be reconstructed from the bilinear form $\Theta_x \in \mathcal{L}(E \times E; \mathbb{R})$. In addition, by modifying the tensor contraction map $\mathfrak{C}$ it is actually possible to recover the quantities

$$\mathrm{Ric}_x, \mathrm{S}_x, \langle B_x, H_x \rangle_{\mathbb{R}^N}, |H_x|_{\mathbb{R}^N}$$

separately. This is a simple linear algebra matter and is explained in Remark 5.23 below.

Comparision with heat kernel asymptotics and potential applications

The aforementioned results should be compared with the classical heat kernel asymptotics in geometric analysis. It was a renowned theorem of Varadhan [Var67] that the Riemannian distance function can be recovered from the small-time asymptotics of the heat kernel:

$$\lim_{t \to \infty} t \log p(t,x,y) = -\frac{1}{2} d(x,y)^2. \tag{1.5}$$

In addition, it was shown by Minakshisundaram-Pleijel [MP49] that the heat kernel admits the following asymptotic expansion:

$$p(t,x,y) \sim \frac{1}{(2\pi t)^{d/2}} e^{-\frac{d(x,y)^2}{2t}} \sum_{k=0}^{\infty} u_k(x,y) t^k \quad \text{as } t \to 0^+,$$

where the coefficients $u_k(x,y)$ are certain C^∞-functions defined in a neighbourhood of the diagonal $\{(x,x) : x \in M\}$. It is known that $u_0(x,x) = 1$, $u_1(x,x) = \mathrm{S}_x/12$ (S_x is the scalar curvature at x) and $u_k(x,x)$ are certain universal polynomial functions of curvature coefficients and their covariant derivatives at x. These asymptotic results indicate that some aspects of the geometry / shape of the underlying space can be inferred from the small-time asymptotics of the heat kernel. On the applied side, the Varadhan asymptotics formula (1.5) provides the theoretical foundation for the so-called *heat method* in manifold learning that are widely used in problems related to learning the shape / structure of high-dimensional data sets (cf. [CWW17, ZCX20, LS23] and the references therein).

Our asymptotic results (Theorems 4.1 and 5.1) provide a different perspective and mechanism for learning the shape of the underlying manifold. It is based on the use of Brownian bridges and their associated signature features instead of the heat kernel. From the practical viewpoint, both the simulation of Brownian bridges on manifolds and the effective computation of path signatures have been largely developed over the past years by various groups (cf. [Jen22], [KL20] and the references therein). We therefore expect that our results will potentially lead to new applications in manifold learning. As a mathematical work on its own, we do not address applications in data sciences in the current article and leave these exciting questions to future work.

2 Some geometric background and a notion of signature on manifolds

We discuss the basic geometric set-up, introduce some necessary geometric tools and define a natural notion of geometric signature that is used in the current work.

2.1 Notions from Riemannian geometry

Throughout the rest of the article, let (M,g) be a d-dimensional, compact, oriented Riemannian manifold without boundary. We choose to focus on the compact case to avoid non-essential technicalities; we do not use global geometric properties (e.g. spectral decomposition) and thus the main results extend naturally to the non-compact case under extra technical conditions. We begin by reviewing some basics on Riemannian geometry. A standard reference for this part is [DoC92].

Convention. Unless otherwise stated, we always adopt Einstein's convention that repeated indices in an expression are summed automatically over their range.

2.1.1 The Laplace-Beltrami operator

A fundamental analytic object on M is the so-called *Laplace-Beltrami operator*. This is a second order differential operator which admits the following local expression:

$$\Delta f = \frac{1}{\sqrt{\det g}}\partial_i\big(\sqrt{\det g}g^{ij}\partial_j f\big). \tag{2.1}$$

Here $g = (g_{ij})$ is the metric tensor and (g^{ij}) is the inverse of g under a local chart. The Laplacian Δ can be alternatively defined as follows. Suppose that M is isometrically embedding inside some Euclidean space $\mathbb{R}^N$ with canonical basis $\{e_1, \cdots, e_N\}$. Let V_i $(1 \leqslant i \leqslant N)$ be the vector field on M defined by orthogonally projecting e_i onto the tangent space at every point of M. Then one has $\Delta = \sum_{i=1}^N V_i^2$, where V_i is equivalently viewed as a differential operator on M (defined by taking directional derivative along V_i).

The Laplace-Beltrami operator Δ is a non-positive definite, unbounded, essentially self-adjoint operator on $L^2(M, dx)$ (dx is the volume measure) which admits a kernel $p(t, x, y)$ known as the *heat kernel*. This is the fundamental solution to the heat equation, i.e. the smallest positive solution to the equation

$$\begin{cases} \partial_t p(t,x,y) = \frac{1}{2}\Delta_x p(t,x,y), \quad t>0, x,y \in M; \\ \lim_{t\downarrow 0} p(t,x,\cdot) = \delta_x, \end{cases} \tag{2.2}$$

where δ_x denotes the Dirac delta function on M. The factor $1/2$ is introduced for its connection with Brownian motion; the Brownian motion B_t is generated by $\Delta/2$ and the heat kernel $p(t, x, y)$ is also the transition density of Brownian motion:

$$\mathbb{P}(B_t \in dy | B_0 = x) = p(t, x, y)dy.$$

2.1.2 The Malliavin-Stroock heat kernel expansion

Small-time expansions of the heat kernel are well studied in the geometric analysis literature (cf. [BGV04] and the references therein). We are going to use the following expansion for the logarithmic derivative of the heat kernel, which was due to Malliavin and Stroock [MS96]. Recall that the *injective radius* at $x \in M$ is the largest $\rho_x > 0$ such that the exponential map $\exp_x : T_xM \to M$ is a diffeomorphism on the metric ball $B_x(\rho) \triangleq \{y : d(x,y) < \rho\}$. Here $d(x,y)$ denotes the Riemannian distance function. The *global injective radius* ρ_M is the minimum of ρ_x over all $x \in M$. This is a finite positive number due to the compactness of M. For example, $\rho_M = \pi$ if $M = S^2$ (the unit sphere in $\mathbb{R}^3$). Define

$$S_M \triangleq \{(x,y) : d(x,y) < \rho_M\}.$$

It is clear that any pair $(x,y) \in S_M$ can be joined by a unique minimising geodesic.

Theorem 2.1. *The following expansion holds uniformly on each compact subset of* S_M :

$$u\nabla_x \log p(u,x,y) \sim \sum_{k=0}^{\infty} \nabla_x G_k(x,y)u^k \quad \text{as } u \searrow 0, \tag{2.3}$$

where ∇_x *denotes the Riemannian gradient with respect to the* x*-variable and* $G_k(x,y)$ *are intrinsic* C^∞*-functions on* S_M*.*

The following explicit expressions will be used in the sequel:

$$G_0(x,y) = -\frac{d(x,y)^2}{2},\ G_1(x,y) = -\frac{1}{2}\log\det(d\exp_y)_{\mathbf{x}}, \quad (x,y) \in S_M. \tag{2.4}$$

Here $\exp_y : T_yM \to M$ denotes the exponential map, $\mathbf{x} \triangleq \exp_y^{-1}(x)$ and $(d\exp_y)_{\mathbf{x}}$ is the differential of $\exp_y$ at $\mathbf{x}$, which is a linear isomorphism between Euclidean spaces $T_{\mathbf{x}}T_yM \cong T_yM$ and T_xM. The functions $G_k(x,y)$ ($k \geqslant 2$) can also be determined explicitly in a recursive manner, but this is not needed for us.

2.1.3 Ricci curvature and the second fundamental form

Many fundamental geometric properties are related to curvature. Recall that the *Riemannian curvature tensor* is defined by

$$R(X,Y)Z \triangleq \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

where ∇ is the Levi-Civita connection and X, Y, Z are smooth vector fields. Note that R is a type-$(1,3)$ tensor field on M and for each $x \in M$, it gives rise to a multilinear map $R_x : T_xM \times T_xM \times T_xM \to T_xM$. The *Ricci curvature tensor* at x is the symmetric bilinear form $\mathrm{Ric}_x : T_xM \times T_xM \to \mathbb{R}$ defined by

$$\mathrm{Ric}_x(v,w) \triangleq \mathrm{Tr}\big[u \mapsto R_x(u,v)w\big], \quad v,w \in T_xM.$$

The *scalar curvature* at x is the trace of Ric_x and is denoted as S_x. Note that Ric is a type-$(0,2)$ tensor field and S is a smooth function on M. All these curvature quantities are *intrinsic* in the sense that their definitions only depend on the metric tensor.

On the other hand, one can also consider extrinsic curvature properties. Let $F : M \to \mathbb{R}^N$ be a given isometric embedding. For each $x \in M$, let $(T_xM)^\perp$ denote the orthogonal complement of T_xM inside $\mathbb{R}^N$ (T_xM is viewed as a subspace of $T_{F(x)}\mathbb{R}^N \cong \mathbb{R}^N$ through the embedding). The *second fundamental form* at x is the vector valued bilinear form defined by

$$B_x(v,w) \triangleq \tilde{\nabla}_{\tilde{X}}\tilde{Y} - \nabla_X Y, \quad v,w \in T_xM. \tag{2.5}$$

Here X, Y are any vector fields on M such that $X_x = v, Y_x = w$ and $\tilde{X}, \tilde{Y}$ are any extensions of X, Y to $\mathbb{R}^N$. The operator $\tilde{\nabla}$ is the Levi-Civita connection in $\mathbb{R}^N$(which is just usual Euclidean differentiation). It can be shown that the expression (2.5) is well-defined (independent of choices of X, Y and their extensions) and B_x is an $(T_xM)^\perp$-valued, symmetric bilinear form on $T_xM \times T_xM$. In other words, B is a type-$(0,2)$ tensor field on M taking values in the normal bundle $(TM)^\perp$. It is also known that $\nabla_X Y$ is the tangential component of $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ (i.e. its projection onto TM). As a result, $B(X,Y)$ can also be defined as the normal component of $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ (i.e. its projection onto $(TM)^\perp$). The *mean curvature vector* at $x \in M$ is the normal vector defined by

$$H_x \triangleq d^{-1}\mathrm{Tr}B_x \in (T_xM)^\perp, \tag{2.6}$$

where d is the dimension of M. By varying x, this gives rise to a smooth section H of the normal bundle $(TM)^\perp$. It can be shown that

$$\Delta F = d \cdot H, \tag{2.7}$$

where Δ is the Laplace-Beltrami operator on M. Note that the second fundamental form is an *extrinsic* curvature quantity as it depends on the embedding F (it describes how M is embedded / curved inside the ambient space).

2.2 Brownian motion and bridge

In this section, we recall the construction of Brownian motion and Brownian bridge on M. The reader is referred to [Hsu02] for more details.

2.2.1 The Eells-Elworthy-Malliavin construction of Brownian motion

The Brownian motion on M could just be defined as the Markov process with generator $\Delta/2$ from the Markovian perspective. However, the following intrinsic (SDE / pathwise) construction by Eells-Elworthy-Malliavin will be useful for our derivation of the intrinsic PDE for the expected signature dynamics in Section 3.

A main difficulty in the intrinsic SDE construction of Brownian motion is that the Laplacian may not always admit a decomposition $\Delta = \sum_i V_i^2$ with intrinsic vector fields V_i. This issue is overcome by lifting the construction to the so-called *orthonormal frame bundle* (OFB) $\mathcal{O}(M)$ over M defined by

$$\mathcal{O}(M) \triangleq \bigcup_{x \in M} \mathcal{F}_x.$$

Here $\mathcal{F}_x$ denotes the set of ONBs of T_pM. A generic element in $\mathcal{O}(M)$ is given by a pair (x, u), where $x \in M$ and $u = (e_1, \cdots, e_d)$ is an ONB of T_xM. The OFB $\mathcal{O}(M)$ is a principal bundle over M with structure group $O(d)$ (the orthogonal group) acting from the right:

$$(x, u) \cdot Q \triangleq (x, u \cdot Q), \quad x \in M, u = (e_1, \cdots, e_d) \in \mathcal{F}_x.$$

In particular, it is a differentiable manifold of dimension

$$D = \dim M + \dim O(d) = \frac{d^2 + d}{2}.$$

Let $\pi : \mathcal{O}(M) \to M$ denote the canonical projection.

An essential point of considering $\mathcal{O}(M)$ is that one can construct intrinsic (horizontal) vector fields on it. Let $1 \leqslant i \leqslant d$ be fixed. Given a point

$$\xi = (x, u = (e_1, \cdots, e_d)) \in \mathcal{O}(M),$$

note that e_i is a tangent vector of M at x. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a curve such that $\gamma_0 = x$ and $\gamma_0' = e_i$. By parallel-translating the ONB u along γ, one obtains a frame of ONBs along γ, more precisely a curve $\xi_t \triangleq (\gamma_t, u_t) \in \mathcal{O}(M)$ where u_t is an ONB of $T_{\gamma_t}M$. Define

$$H_i(\xi) \triangleq \xi_0' \in T_\xi \mathcal{O}(M).$$

Since ξ is arbitrary, one then obtains a global vector field H_i on $\mathcal{O}(M)$. The vector fields $\{H_1, \cdots, H_d\}$ are called *fundamental horizontal vector fields* on $\mathcal{O}(M)$.

It is well known that the second order differential operator $\mathcal{L} \triangleq \sum_{i=1}^d H_i^2$ naturally projects to the Laplacian on M. This motivates the Eells-Elworthy-Malliavin construction of Brownian motion given by the theorem below. The essential idea is to first construct a horizontal Brownian motion on $\mathcal{O}(M)$ by solving an intrinsic SDE associated with the horizontal vector fields $\{H_i\}$ and then to obtain an M-valued Brownian motion through projection.

Theorem 2.2. *Let $x \in M$ and $u = (e_1, \cdots, e_d)$ be a given fixed ONB of T_xM. Let Ξ_t be the solution to the following Stratonovich SDE on $\mathcal{O}(M)$:*

$$\begin{cases} d\Xi_t = \sum_{i=1}^d H_i(\Xi_t) \circ dB_t^i, \\ \Xi_0 = (x, u) \in \mathcal{O}(M), \end{cases} \tag{2.8}$$

where $(B^1, \cdots, B^d)$ denotes a d-dimensional Euclidean Brownian motion. Define $W_t \triangleq \pi(\Xi_t) \in M$. Then the law of W depends only on x but not on the initial frame u. By varying x over M, one obtains a Markovian family whose generator is $\Delta/2$ (thus a Brownian family on M).

2.2.2 The Brownian bridge

Let $x, y \in M$ and $t \in (0, 1]$ be given fixed. The *Brownian bridge* $(X_s^{t,x,y})_{0 \leqslant s \leqslant t}$ from x to y with lifetime t is the Brownian motion starting at x conditional on reaching y at time t:

$$X_s^{t,x,y} = W_s^x|_{W_t^x = y}, \quad 0 \leqslant s \leqslant t,$$

where W^x is a Brownian motion starting at x. Simple calculation shows that this is a (time-inhomogeneous) Markov process with transition density

$$\mathbb{P}\big(X_{s_2}^{t,x,y} \in dz | X_{s_1}^{t,x,y} = w\big) = \frac{p(t - s_2, z, y)p(s_2 - s_1, w, z)}{p(t - s_1, w, y)} dz, \quad 0 \leqslant s_1 < s_2 < t \tag{2.9}$$

and generator

$$(\mathcal{L}f)(w) = \frac{1}{2}\Delta f(w) + \nabla_w \log p(t - s, w, y) \cdot \nabla f(w).$$

Here $p(t, x, y)$ is the heat kernel for $\Delta/2$. One can also use the above Markovian perspective to construct the Brownian bridge mathematically.

If the Brownian motion W^x admits an SDE representation (e.g. under some coordinate chart)

$$dW_s^x = b(W_s^x)ds + \sigma(W_s^x)dB_s, \ W_0^x = x$$

in $\mathbb{R}^d$, then the Brownian bridge $X^{t,x,y}$ has a corresponding SDE representation

$$\begin{cases} dX_s^{t,x,y} = \big(b(X_s^{t,x,y}) + \nabla \log p(t-s, X_s^{t,x,y}, y)\big)ds + \sigma(X_s^{t,x,y})dB_s, & 0 \leqslant s < t; \\ X_0^{t,x,y} = x. \end{cases} \tag{2.10}$$

By abuse of notation the above ∇ is its local expression under the given chart (i.e. the vector $(g^{ij}\partial_j)_{1\leqslant i\leqslant d}$) .

Example 2.3. In the Euclidean case $M = \mathbb{R}^d$, by using the explicit formula for the heat kernel, one finds that

$$\nabla \log p(t-s, x, y) = \frac{y-x}{t-s}.$$

The SDE (2.10) becomes

$$\begin{cases} dX_s^{t,x,y} = -\frac{1}{t-s} X_s^{t,x,y} ds + dB_s, \\ X_0^{t,x,y} = x - y =: \xi. \end{cases}$$

Its solution is explicitly given by

$$X_s^{t,x,y} = \frac{t-s}{t}\xi + (t-s)\int_0^s \frac{dB_u}{t-u}, \quad 0 \leqslant s < t.$$

2.3 The ϕ-signature on manifolds

In this section, we introduce a natural notion of signature for paths on differentiable manifolds (iterated integrals against one-forms). This was first used by K.T. Chen in a geometric setting and also by Le Jan-Qian in the study of the signature uniqueness theorem for Brownian motion (they called it *extended signature*).

We first recall the definition of path signature in the Euclidean space. Let $E = \mathbb{R}^N$ be equipped with the Euclidean metric. The (infinite) tensor algebra over E is the unital algebra

$$T((E)) \triangleq \prod_{n=0}^{\infty} E^{\otimes n} = \big\{a = (a_0, a_1, a_2, \cdots) : a_i \in E^{\otimes i}\ \forall i \in \mathbb{N}\big\},$$

where addition $+$ is just the vector addition, multiplication $\otimes$ is defined by

$$(a \otimes b)_m \triangleq \sum_{k=0}^{m} a_k \otimes b_{m-k} \in (\mathbb{R}^N)^{\otimes m}, \quad a = (a_i), b = (b_i) \in T((E))$$

and the unit is $\mathbf{1} \triangleq (1,0,0,\cdots)$. The *signature* of a smooth path $\gamma:[0,T]\to E$ is the element in $T((E))$ defined by

$$S(\gamma) \triangleq \big(1, \gamma_T-\gamma_0, \int_{0<s<t<T} d\gamma_s\otimes d\gamma_t, \cdots, \int_{0<t_1<\cdots<t_n<T} d\gamma_{t_1}\otimes\cdots\otimes d\gamma_{t_n}, \cdots\big).$$

More generally, one could also define the *signature path* $t\mapsto S_t(\gamma)$ by integrating up to time t instead ($S(\gamma)=S_T(\gamma)$). The definition extends to the rough path setting according to Lyons' extension theorem (cf. [Lyo98, Theorem 2.2.1]). If γ is the Brownian motion, the above iterated integrals are equivalently understood in the Stratonovich sense. It is standard that the signature path $S_t(\gamma)$ satisfies the following differential equation on $T((E))$:

$$dS_t(\gamma) = S_t(\gamma)\otimes d\gamma_t. \tag{2.11}$$

If γ is a smooth path on a differentiable manifold M, one cannot integrate along γ intrinsically without any additional structure; one needs to integrate γ against a one-form. Recall that a (smooth) *one-form* is a smooth section ϕ of the cotangent bundle. In other words, it is a map that assigns to each location $x\in M$ a linear functional $\phi(x)\in T_x^*M$ in a smooth manner. The integral

$$\int_0^t \phi(d\gamma) \triangleq \int_0^t {}_{T^*_{\gamma_s}M}\langle \phi(\gamma_s), \gamma_s'\rangle_{T_{\gamma_s}M} ds$$

clearly has an intrinsic meaning and is called the *line integral* of γ against ϕ.

Example 2.4. Let M be the torus $S^1\times S^1$ and let $\phi=(\alpha,\beta)$ be the two canonical generators of the first homology group of M. Then $\int_0^t \phi(d\gamma)\in\mathbb{R}^2$ describes the winding angles of γ up to time t with respect to the vertical and horizontal circles.

Now let $E=\mathbb{R}^N$ and suppose that ϕ is a smooth E-valued one-form on M. In other words, $\phi(x)$ is an E-valued linear functional on T_xM. Equivalently, one can write $\phi=(\phi^1,\cdots,\phi^N)$ where each ϕ^i is a real-valued one-form.

Definition 2.5. Let $\gamma:[0,T]\to M$ be a smooth path. The *ϕ-signature* of γ up to time t is defined to be

$$S_t^\phi(\gamma) \triangleq S_t\Big(\int_0^\cdot \phi(d\gamma)\Big) \in T((E)).$$

This is the signature up to time t of the E-valued path $\int_0^\cdot \phi(d\gamma)$. We also write $S^\phi(\gamma)\triangleq S_T^\phi(\gamma)$.

Remark 2.6. This notion of signature depends on the choice of ϕ. Indeed, the usual Euclidean signature is a special case of this: the signature of an $\mathbb{R}^N$-valued path is the ϕ-signature with respect to the $\mathbb{R}^N$-valued one-form $\phi = (dx^1, \cdots, dx^N)$.

Remark 2.7. The definition extends naturally to the rough path case. We choose not to introduce any technicalities from general rough path theory since they are not essential for most of our analysis. In our study, γ will either be a smooth path (a geodesic) or an M-valued semimartingale where the Stratonovich calculus for γ is classical (e.g. via embedding M into an ambient Euclidean space and perform the usual stochastic calculus over there). The reader is referred to [Hsu02] for an excellent introduction to stochastic analysis on manifolds.

Example 2.8. An important example which is of our primary interest is the case when M is embedded inside some Euclidean space $\mathbb{R}^N$ and the one-form $\phi = dF$ (F is the embedding map). In this case, one has $\int_0^t \phi(d\gamma_s) = F(\gamma_t) - F(\gamma_0)$ and the ϕ-signature of γ is just the Euclidean signature of $F(\gamma_t)$ in $\mathbb{R}^N$.

3 The expected Brownian signature dynamics

In this section, we consider the expected ϕ-signature of Brownian motion and Brownian bridge on M. We derive intrinsic PDEs describing their dynamics from the perspective of the Eells-Elworthy-Malliavin lifting. Throughout the rest, $E = \mathbb{R}^N$ and ϕ is a given fixed E-valued one-form on M.

3.1 Intrinsic PDE for the expected ϕ-signature

Let $W^x = \{W_t^x : t \geqslant 0\}$ be a Brownian motion in M starting at $x \in M$. Consider the expected ϕ-signature of W^x up to time t defined by

$$\Psi(t,x) \triangleq \mathbb{E}[S_t^{\phi}(W^x)] \in T((E)), \quad (t,x) \in [0,\infty) \times M. \tag{3.1}$$

Note that Ψ is well-defined due to the compactness of M; by embedding M into an Euclidean space V one can view W^x as the solution to an SDE on V with C_b^∞-coefficients and ϕ also extends to a C_b^∞-form on V. We first derive the intrinsic PDE governing the dynamics of $\Psi(t,x)$.

Theorem 3.1. *The function $\Psi(t,x)$ satisfies the following parabolic PDE on M:*

$$\frac{\partial \Psi}{\partial t} = \frac{1}{2}\Delta\Psi + \mathrm{Tr}\big(\phi \otimes d\Psi + \frac{1}{2}(\nabla\phi + \phi\otimes\phi)\otimes\Psi\big) \tag{3.2}$$

with initial condition $\Psi(0,\cdot) = \mathbf{1}$.

Before getting into the proof, some explanation about the notation in (3.2) is needed.

(i) The PDE (3.2) is $T((E))$-valued. Its projection onto the first m degrees becomes a finite dimensional coupled PDE system.
(ii) Since Ψ is an $T((E))$-valued function on M (for fixed time t), $d\Psi$ is a $T((E))$-valued one-form. The product $\phi \otimes d\Psi$ is the $T((E))$-valued bilinear form defined by

$$\phi \otimes d\Psi(U, V) \triangleq \langle \phi, U \rangle \otimes \langle d\Psi, V \rangle$$

where U, V are vector fields on M and the tensor product $\otimes$ is the multiplication in $T((E))$. The trace $\mathrm{Tr}(\phi \otimes d\Psi)$ becomes a $T((E))$-valued function on M. At each point $x \in M$, one has

$$\mathrm{Tr}(\phi \otimes d\Psi)(x) = \sum_{i=1}^{d} \langle \phi, e_i \rangle(x) \otimes \langle d\Psi(t, x), e_i \rangle,$$

where $\{e_1, \cdots, e_d\}$ is any ONB of $T_x M$. The above expression is clearly independent of the choice of the ONB.
(iii) $\nabla\phi$ is the covariant derivative of ϕ with respect to the Levi-Civita connection. The zeroth order term in (3.2) is the $T((E))$-valued function defined by

$$\mathrm{Tr}\big((\nabla\phi + \phi \otimes \phi) \otimes \Psi\big)(x) = \sum_{i=1}^{d} \big(\langle \nabla_{e_i}\phi, e_i \rangle(x) + \langle \phi, e_i \rangle(x) \otimes \langle \phi, e_i \rangle(x)\big) \otimes \Psi(t, x).$$

This expression is also independent of the choice of the ONB $\{e_1, \cdots, e_d\}$ of $T_x M$.

Example 3.2. In the Euclidean case $M = \mathbb{R}^d$ with $\phi = (dx^1, \cdots, dx^d)$, the term $\nabla\phi + \phi \otimes \phi$ appearing in the PDE (3.1) is constant in space. As a result, the PDE must also be spatially homogeneous and thus reduces to the ODE

$$\frac{d\Psi}{dt} = \frac{1}{2} \sum_{i=1}^{d} E_i \otimes E_i \otimes \Psi, \quad \Psi(0, \cdot) = \mathbf{1}, \tag{3.3}$$

where $\{E_1, \cdots, E_d\}$ is the canonical basis of $\mathbb{R}^d$. Its explicit solution is the expected signature of Euclidean Brownian motion:

$$\Psi(t, x) = \exp\big(\frac{t}{2}(E_1 \otimes E_1 + \cdots + E_d \otimes E_d)\big). \tag{3.4}$$

Proof of Theorem 3.1

We now proceed to prove Theorem 3.1. The main observation is that the horizontal Brownian motion on the OFB $\mathcal{O}(M)$ (cf. Theorem 2.2) coupled with its signature process satisfies an intrinsic SDE on the $\mathcal{O}(M)\times T((E))$. A benefit from this is that one can easily identify the generator from the SDE and write down the associated Kolmorogov's forward equation on the bundle $\mathcal{O}(M)$ (Feynman-Kac representation). The PDE on $\mathcal{O}(M)$ then naturally projects to the desired intrinsic PDE (3.2) on the base manifold M. Some technical care is needed to implement this idea precisely and we develop the steps carefully in what follows.

First of all, one has the following standard fact.

Lemma 3.3. *The PDE (3.2) with initial condition* $\Psi(0,\cdot)=\mathbf{1}$ *has a unique smooth solution denoted as* $\hat{\Psi}(t,x)$ *(i.e. every component of* $\hat{\Psi}$ *is smooth in* (t,x)*).*

Proof. Let $f_0\in\mathcal{C}(M)$ and $g\in\mathcal{C}([0,\infty)\times M)$ be given functions. Consider the following inhomogeneous (scalar) Cauchy problem on M:

$$\begin{cases}\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u+g,\\ u(0,\cdot)=f_0.\end{cases}\tag{3.5}$$

Since M is compact, it is standard that the above PDE has a unique solution $u(t,x)$ which is smooth for all positive times. In fact, the solution is explicitly given by

$$u(t,x)=\int_M p(t,x,y)f_0(y)dy+\int_0^t\int_M p(t-s,x,y)g(s,y)dyds.$$

The integrals are well-defined due to the compactness of M and the continuity of f_0,g. The smoothness of u follows from the smoothness of the heat kernel $p(t,x,y)$.

To prove the lemma, it is enough to observe that the PDE (3.2) is completely decoupled into the (scalar) form of (3.5) when one rewrites it in terms of coordinate components of Ψ. In fact, one can solve the PDE (3.5) inductively on the degree of signature where g is given by lower degree components which is presumed to be known by induction. □

Next, we lift the solution $\hat{\Psi}(t,x)$ given by Lemma 3.3 to the bundle $\mathcal{O}(M)$. Let us define

$$f(t,\xi)\triangleq\hat{\Psi}(t,\pi(\xi)),\quad t\geqslant 0,\xi\in\mathcal{O}(M),\tag{3.6}$$

where $\pi:\mathcal{O}(M)\to M$ denotes the bundle projection. We also denote $\Phi\triangleq\pi^*\phi$ as the pullback of ϕ on the bundle. Then one can establish the PDE for $f(t,\xi)$ on $\mathcal{O}(M)$.

Lemma 3.4. *The function $f(t,\xi)$ satisfies the following parabolic PDE:*

$$\frac{\partial f}{\partial t} = \frac{1}{2}\sum_{i=1}^{d} H_i^2 f + \sum_{i=1}^{d}\langle \Phi, H_i\rangle \otimes H_i f + \frac{1}{2}\sum_{i=1}^{d}\big(H_i\langle \Phi, H_i\rangle + \langle \Phi, H_i\rangle^{\otimes 2}\big)\otimes f \quad (3.7)$$

with initial condition $f(0,\xi) = \mathbf{1}$.

Proof. Since $\hat{\Psi}(t,x)$ satisfies the PDE (3.2) on M and f is the pullback of $\hat{\Psi}$ on the bundle $\mathcal{O}(M)$, it is sufficient to prove that the pullback of the right hand side of (3.2) is precisely the right hand side of (3.7). We look at each individual term separately.

(i) The horizontal Laplacian $\sum_{i=1}^{d} H_i^2$ on $\mathcal{O}(M)$ is the pullback of the Laplacian Δ on M (cf. [Hsu02, Proposition 3.1.2]):

$$\sum_{i=1}^{d} H_i^2 f = \sum_{i=1}^{d} H_i^2 \pi^*\hat{\Psi} = \pi^*\Delta\hat{\Psi}.$$

(ii) We claim that

$$\sum_{i=1}^{d}\langle \Phi, H_i\rangle \otimes H_i f = \pi^*\mathrm{Tr}(\phi\otimes d\hat{\Psi}).$$

Indeed, let $\xi = (x,u) \in \mathcal{O}(M)$ with $u = \{e_1, \cdots, e_d\}$. Then one has

$$\langle \Phi, H_i\rangle(\xi) = \langle \pi^*\phi, H_i\rangle(\xi) = \langle \phi, (\pi_*)_\xi H_i\rangle(x) = \langle \phi, e_i\rangle(x).$$

Similarly,

$$(H_i f)(\xi) = \langle df, H_i\rangle(\xi) = \langle d\hat{\Psi}, e_i\rangle(x).$$

It follows that

$$\big(\sum_{i=1}^{d}\langle \Phi, H_i\rangle \otimes H_i f\big)(\xi) = \big(\sum_{i=1}^{d}\langle \phi, e_i\rangle \otimes \langle d\hat{\Psi}, e_i\rangle\big)(x) = \mathrm{Tr}((\phi\otimes d\hat{\Psi}))(x).$$

Note that the trace, as a bilinear form on the inner product space T_xM, is independent of the choice of the ONB u. In a similar way, one also has

$$\sum_{i=1}^{d}\langle \Phi, H_i\rangle^{\otimes 2} = \pi^*\mathrm{Tr}(\phi\otimes\phi).$$

(iii) It remains to show that

$$\sum_{i=1}^{d} H_i\langle \Phi, H_i\rangle = \pi^* \mathrm{Tr}(\nabla\phi). \tag{3.8}$$

As before, let $\xi = (x, u)$ be given fixed with $u = \{e_1, \cdots, e_d\}$. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a curve in M such that $\gamma_0 = x$ and $\gamma_0' = e_i$. Let $\xi_t = (\gamma_t, u_t)$ be the horizontal lifting of γ with $\xi_0 = \xi$. Then one has

$$\begin{aligned} H_i\langle \Phi, H_i\rangle(\xi) &= \frac{d}{dt}\Big|_{t=0}\langle \Phi, H_i\rangle(\xi_t) = \frac{d}{dt}\Big|_{t=0}\langle \phi, (\pi_*)_{\xi_t} H_i\rangle(\gamma_t) \\ &= \frac{d}{dt}\Big|_{t=0}\langle \phi(\gamma_t), e_i(t)\rangle = \langle \frac{D\phi(\gamma_t)}{dt}\Big|_{t=0}, e_i\rangle + \langle \phi(x), \frac{De_i(t)}{dt}\Big|_{t=0}\rangle, \end{aligned}$$

where $e_i(t)$ denotes the i-th component of u_t and $\frac{D}{dt}$ is the covariant derivative along γ. Since ξ_t is horizontal, one has

$$\frac{De_i(t)}{dt} = 0 \quad \forall t.$$

As a result, one obtains that

$$H_i\langle \Phi, H_i\rangle(\xi) = \langle \nabla_{e_i}\phi, e_i\rangle(x)$$

and the claim (3.8) thus follows. □

Now we derive the Feynman-Kac representation for the PDE (3.7) (with no surprise, it is given in terms of the joint process defined by the horizontal Brownian motion and its Φ-signature process). Recall that Ξ^ξ is the horizontal Brownian motion defined by the horizontal SDE (2.8) on the bundle and let $\tilde{S}_t^\Phi(\Xi^\xi) \in T((E))$ denote the signature path of the E-valued path

$$t \mapsto \int_0^t \Phi(\circ d\Xi_s^\xi).$$

Lemma 3.5. *One has the following representation:*

$$f(t, \xi) = \mathbb{E}\big[\tilde{S}_t^\Phi(\Xi^\xi)\big], \quad (t, \xi) \in [0, \infty) \times \mathcal{O}(M). \tag{3.9}$$

Proof. Consider the Markov family

$$(\Xi, \tilde{S}) \triangleq \big\{\big(\Xi_t^\xi, g \otimes \tilde{S}_t^\Phi(\Xi^\xi)\big) : t \geqslant 0, (\xi, g) \in \mathcal{O}(M) \times T((E))\big\}.$$

This family is governed by the Stratonovich SDE (cf. (2.11))

$$\begin{cases} d\Xi_t = \sum_{i=1}^d H_i(\Xi_t) \circ dB_t^i, \\ d\tilde{S}_t = \sum_{i=1}^d \tilde{S}_t \otimes \langle \Phi, H_i \rangle (\Xi_t) \circ dB_t^i \end{cases} \tag{3.10}$$

on the state space $\mathcal{O}(M) \times T((E))$. Let $\mathcal{L}$ be the generator of $(\Xi, \tilde{S})$. It is standard that $\mathcal{L} = \frac{1}{2} \sum_{i=1}^d \mathcal{V}_i^2$ where $\{\mathcal{V}_1, \cdots, \mathcal{V}_d\}$ are the vector fields for the SDE (3.10) and they are viewed as differential operators on $C^\infty\big(\mathcal{O}(M) \times T((E))\big)$. Respectively, we also denote

$$\hat{\mathcal{L}}F \triangleq \frac{1}{2} \sum_{i=1}^d H_i^2 F + \sum_{i=1}^d \langle \Phi, H_i \rangle \otimes H_i F + \frac{1}{2} \sum_{i=1}^d \big(H_i \langle \Phi, H_i \rangle + \langle \Phi, H_i \rangle^{\otimes 2} \big) \otimes F$$

for $F \in C^\infty\big(\mathcal{O}(M) \times T((E))\big)$. Note that the situation here is essentially finite dimensional since one can consistently look at the truncations of the SDE (3.10) and PDE (3.7) up to any fixed level.

Consider the function

$$F(t, (\xi, g)) \triangleq g \otimes f(t, \xi), \tag{3.11}$$

where we recall that $f(t, \xi)$ is defined by (3.6). We claim that $\mathcal{L}F = \hat{\mathcal{L}}F$. Indeed, note that the vector fields for the SDE (3.10) are given by

$$\mathcal{V}_i \triangleq V_i \oplus W_i, \quad 1 \leqslant i \leqslant d,$$

where

$$V_i(\xi, g) = H_i(\xi), \ W_i(\xi, g) = g \otimes \langle \Phi, H_i \rangle (\xi). \tag{3.12}$$

We first compute $\mathcal{V}_i F$. By the definition (3.11) of F, it is clear that

$$(V_i F)(t, (\xi, g)) = g \otimes (H_i f)(t, \xi). \tag{3.13}$$

In addition, by the definition (3.12) of W_i one has (recall $E = \mathbb{R}^N$)

$$(W_i F)(t, (\xi, g)) = \sum_{m \geqslant 1} \sum_{i_1, \cdots, i_m = 1}^N g^{i_1, \cdots, i_{m-1}} \langle \Phi^{i_m}, H_i \rangle (\xi) \partial_{i_1, \cdots, i_m} (g \otimes f(t, \xi)),$$

where $g^{i_1, \cdots, i_{m-1}}$ are the coordinate components of the tensor g, Φ^j is the j-th component of $\Phi = (\Phi^1, \cdots, \Phi^N)$ as an $\mathbb{R}^N$-valued one-form and $\partial_{i_1, \cdots, i_m}$ is the partial derivative in g with respect to the coordinate $g^{i_1,\cdots,i_m}$. For any given word $J=(j_1,\cdots,j_l)$, the J-th component of W_iF is thus given by

$$
\begin{aligned}
&(W_iF)^J(t,(\xi,g))\\
&=\sum_{m\geqslant 1}\sum_{i_1,\cdots,i_m=1}^{N} g^{i_1,\cdots,i_{m-1}}\langle\Phi_{i_m},H_i\rangle(\xi)\partial_{i_1,\cdots,i_m}\big(\sum_{s=1}^{l} g^{j_1,\cdots,j_s} f(t,\xi)^{j_{s+1},\cdots,j_l}\big)\\
&=\sum_{s=1}^{l} g^{j_1,\cdots,j_{s-1}}\langle\Phi_{j_s},H_i\rangle(\xi)f(t,\xi)^{j_{s+1},\cdots,j_l}\\
&=\big(g\otimes\langle\Phi,H_i\rangle\otimes f(t,\xi)\big)^J.
\end{aligned}
$$

In its equivalent tensor form,

$$
(W_iF)(t,(\xi,g))=g\otimes\langle\Phi,H_i\rangle(\xi)\otimes f(t,\xi). \tag{3.14}
$$

Combining (3.13) and (3.14), one finds that

$$
(\mathcal{V}_iF)(t,(\xi,g))=g\otimes\big((H_if)(t,\xi)+\langle\Phi,H_i\rangle(\xi)\otimes f(t,\xi)\big).
$$

By applying $\mathcal{V}_i$ again, one arrives at

$$
\begin{aligned}
\mathcal{V}_i^2F&=g\otimes H_i\big(H_if+\langle\Phi,H_i\rangle\otimes f\big)+g\otimes\langle\Phi,H_i\rangle\otimes\big(H_if+\langle\Phi,H_i\rangle\otimes f\big)\\
&=g\otimes\big(H_i^2f+H_i\langle\Phi,H_i\rangle\otimes f+2\langle\Phi,H_i\rangle\otimes H_if+\langle\Phi,H_i\rangle^{\otimes 2}\otimes f\big).
\end{aligned} \tag{3.15}
$$

This gives the relation $\mathcal{L}F=\hat{\mathcal{L}}F$.

It follows from the above relation and Lemma 3.4 that F satisfies the PDE $\frac{\partial F}{\partial t}=\mathcal{L}F$ with initial condition $F(0,\cdot)=\rho(\cdot)$, where $\rho:\mathcal{O}(M)\times T((E))\to T((E))$ denotes the projection onto the second component. It is now a standard application of the Feynman-Kac representation that

$$
g\otimes f(t,\xi)=F(t,(\xi,g))=\mathbb{E}\big[\rho\big(\Xi_t^\xi,g\otimes\tilde{S}_t^\Phi(\Xi^\xi)\big)\big]=g\otimes\mathbb{E}\big[\tilde{S}_t^\Phi(\Xi^\xi)\big].
$$

By cancelling out the g on both sides, one arrives at the desired relation (3.9). □

Finally, the lemma below completes the proof of Theorem 3.1. Recall that $\Psi(t,x)$ is the expected ϕ-signature of the Brownian motion (cf. (3.1)).

Lemma 3.6. *$\Psi(t,x)=\hat{\Psi}(t,x)$ for all $t\geqslant 0$ and $x\in M$.*

Proof. Fix $x \in M$ and let $\xi \in \pi^{-1}(x)$. According to Theorem 2.2, $W_t \triangleq \pi(\Xi_t^\xi)$ is a Brownian motion starting at x. In addition, by the definition of Φ one has

$$\langle \Phi, \circ d\Xi_t^\xi \rangle = \langle \pi^* \phi, \circ d\Xi_t^\xi \rangle = \langle \phi, \circ \pi_* d\Xi_t^\xi \rangle = \langle \phi, \circ dW_t \rangle.$$

It follows from Lemma 3.5 that

$$\hat{\Psi}(t, x) = f(t, \xi) = \mathbb{E}\big[\tilde{S}_t^\Phi(\Xi^\xi)\big] = \mathbb{E}[S_t^\phi(W)] = \Psi(t, x).$$

Note that the above relation is independent of the choice of $\xi \in \pi^{-1}(x)$ since the law of W is. □

Remark 3.7. The proof of Lemma 3.5 does not use the fact that Φ is the lifting of the one-form ϕ. Indeed, let Φ be any given E-valued one-form on $\mathcal{O}(M)$. For each $\xi \in \mathcal{O}(M)$, let Ξ_t^ξ be the horizontal Brownian motion starting at ξ. Then one can show that the function

$$(t, \xi) \mapsto \mathbb{E}[\tilde{S}_t^\Phi(\Xi^\xi)]$$

satisfies the PDE (3.7).

3.2 The Brownian bridge case

Now let us consider the expected ϕ-signature of the Brownian bridge defined by

$$\psi(t, x, y) \triangleq \mathbb{E}\big[S^\phi(X^{t,x,y})\big], \quad t \geqslant 0, \ x, y \in M,$$

where $X^{t,x,y}$ is the Brownian bridge from x to y with lifetime t. By using the PDE (3.2), it is not hard to derive the associated PDE for $\psi(t, x, y)$.

Theorem 3.8. *For each fixed $y \in M$, the function*

$$(0, \infty) \times M \ni (t, x) \mapsto \psi(t, x, y)$$

satisfies the following PDE:

$$\begin{aligned} \frac{\partial \psi}{\partial t} = & \frac{1}{2} \Delta_x \psi + \langle \nabla_x \log p(t, x, y), \nabla_x \psi + \phi \otimes \psi \rangle + \mathrm{Tr}(\phi \otimes d_x \psi) \\ & + \frac{1}{2} \mathrm{Tr}\big((\nabla \phi + \phi \otimes \phi) \otimes \psi\big), \end{aligned} \tag{3.16}$$

where $p(t, x, y)$ is the heat kernel.

Remark 3.9. The term $\langle \nabla \log p, \phi \otimes \psi \rangle$ is understood as ${}_{TM}\langle \nabla \log p, \phi \rangle_{T^*M} \otimes \psi \in T((E))$.

Proof. By the definition of Brownian bridge, one has

$$\psi(t,x,y) = \frac{\mathbb{E}[S_t^{\phi}(W^x)\delta_y(W_t^x)]}{\mathbb{E}[\delta_y(W_t^x)]} = \frac{\mathbb{E}[S_t^{\phi}(W^x)\delta_y(W_t^x)]}{p(t,x,y)},$$

where W^x is the Brownian motion starting at x. Given any smooth function $f : M \to \mathbb{R}$, exactly the same argument as in the proof of Theorem 3.1 shows that the function

$$(0,\infty) \times M \ni (t,x) \mapsto \mathbb{E}\big[f(W_t^x) S_t^{\phi}(W^x)\big]$$

satisfies the same PDE (3.2). In particular, the function (y being fixed)

$$\bar{\psi}(t,x) \triangleq \mathbb{E}[S_t^{\phi}(W^x)\delta_y(W_t^x)]$$

also satisfies (3.2).

Now let us write $\psi = \bar{\psi}/p$. By using the PDE for $\bar{\psi}$ and the heat equation (2.2) for p, one finds that

$$\begin{aligned}
\frac{\partial \psi}{\partial t} &= \frac{1}{p}\frac{\partial \bar{\psi}}{\partial t} + \bar{\psi}\big(-\frac{1}{p^2}\frac{\partial p}{\partial t}\big) \\
&= \frac{1}{p}\big(\frac{1}{2}\Delta\bar{\psi} + \mathrm{Tr}\big(\phi \otimes d\bar{\psi} + \frac{1}{2}(\nabla\phi + \phi \otimes \phi) \otimes \bar{\psi}\big) - \frac{\bar{\psi}}{2p}\Delta p\big) \\
&= \frac{1}{p}\big(\frac{1}{2}\Delta(p\psi) + \mathrm{Tr}\big(\phi \otimes d(p\psi)\big) + \frac{1}{2}\mathrm{Tr}\big((\nabla\phi + \phi \otimes \phi) \otimes \psi\big) - \frac{\psi}{2p}\Delta p.
\end{aligned}$$

Since

$$\Delta(p\psi) = \Delta p \cdot \psi + p \cdot \Delta\psi + 2\langle \nabla p, \nabla \psi \rangle,$$

it follows that

$$\frac{\partial \psi}{\partial t} = \frac{1}{2}\Delta\psi + \frac{1}{p}\langle \nabla p, \nabla\psi \rangle + \frac{1}{p}\mathrm{Tr}(\phi \otimes \psi dp) + \mathrm{Tr}\big(\phi \otimes d\psi + \frac{1}{2}(\nabla\phi + \phi \otimes \phi) \otimes \psi\big).$$

The result follows by further noting that

$$\mathrm{Tr}(\phi \otimes \psi dp) = \sum_{i=1}^{d} \langle \nabla p, e_i \rangle \phi(e_i) \otimes \psi = \phi(\nabla p) \otimes \psi,$$

where $\{e_1, \cdots, e_d\}$ is any ONB of T_xM and $p^{-1}\nabla p = \nabla \log p$. □

A basic point to make is that rich geometric information about the underlying space M is encoded in the asymptotic behaviour of the function $\psi(t,x,y)$ as $t \to 0^+$. This motivates our study in the next two sections.

4 Signature asymptotics and reconstruction of Riemannian distance

In this section, we develop a method of reconstructing the Riemannian distance $d(x,y)$ from the expected signature of the Brownian bridge $X^{t,x,y}$ through an explicit asymptotics procedure, at least when $d(x,y)$ is not too far from each other.

4.1 The main theorem

Suppose that $F : M \to E = \mathbb{R}^N$ is a given fixed isometric embedding (which always exists by Nash's embedding theorem). For each $n \geqslant 1$, we equip $E^{\otimes n}$ with the Hilbert-Schmidt tensor norm $\|\cdot\|_{\mathrm{HS}}$. In other words, viewing E as an Euclidean space, the space $E^{\otimes n}$ is equipped with an inner product structure induced by

$$\langle v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_n \rangle_{\mathrm{HS}} \triangleq \langle v_1, w_1 \rangle_E \cdots \langle v_n, w_n \rangle_E, \quad v_i, w_j \in E.$$

Recall that $X^{t,x,y}$ is the Brownian bridge from x to y with lifetime t and $S^{dF}(X^{t,x,y})$ is its ϕ-signature with $\phi = dF$. Our main result in this section is stated as follows.

Theorem 4.1. *Let $x, y \in M$ be given fixed such that $d(x,y) < \rho_M/2$ where ρ_M is the global injective radius of M. Then there exist geometric constants C, κ depending on x, y, M, such that*

$$\Big| \big(n! \big\| \mathbb{E}\big[\pi_n S^{dF}(X^{\kappa n^{-6},x,y})\big] \big\|_{\mathrm{HS}} \big)^{1/n} - d(x,y) \Big| \leqslant \frac{C}{n}, \quad n \geqslant 1. \tag{4.1}$$

In particular, with $t_n = o(n^{-6})$ one has

$$\lim_{n\to\infty} \big(n! \big\| \mathbb{E}\big[\pi_n S^{dF}(X^{t_n,x,y})\big] \big\|_{\mathrm{HS}} \big)^{1/n} = d(x,y). \tag{4.2}$$

Remark 4.2. The assumption that $\phi = dF$ with some isometric embedding F plays no essential role; the only relevant fact is that $d(x,y)$ is precisely the Euclidean length of the path $\int_0^\cdot \phi(d\gamma^{x,y})$ ($\gamma^{x,y}$ is the unique minimising geodesic from x to y). For a general ϕ, the theorem remains valid as long as one replaces $d(x,y)$ with the length of the path $\int_0^\cdot \phi(d\gamma^{x,y})$ in E.

Remark 4.3. We expect that the same type of result holds (possibly with a different condition on t_n) even if y is on the cut-locus of x. This may cause extra technical difficulties in the current argument due to the non-uniqueness of minimising geodesics connecting x, y.

4.2 Proof of Theorem 4.1

As we explained in the introduction, the asymptotics formula (4.2) is largely inspired by the length conjecture (which is an actual theorem for smooth paths such as the geodesic $\gamma^{x,y}$) as well as the fact that $X^{t,x,y} \approx \gamma^{x,y}$ when t is small. However, there are several technical challenges to implement such an idea precisely. Essentially, given a degree n one wants to identify a lifetime scale t_n such that the (normalised) n-th level expected signature of the Brownian bridge with lifetime t_n yields the desired limit $d(x,y)$ as $n \to \infty$. If one applies standard signature and semimartingale estimates without much care, one is led to requiring that $t_n = o(e^{-Cn})$, which is way too small to be useful in practice. Improving it from exponential to polynomial scale n^{-6} is the most delicate point in the argument (cf. Section 4.2.2 below).

In what follows, we develop the main ingredients for proving Theorem 4.1 in a precise mathematical way. Basically, we localise the problem on a nice coordinate chart (a normal chart) in which Euclidean stochastic calculus can be applied. The part that the Brownian bridge exits the chart before its lifetime is negligible (due to Hsu's large deviation principle), while the part within the chart yields the main contribution.

Throughout the rest, $\phi \triangleq dF$ and $x, y \in M$ with $d(x,y) < \rho_M/2$ are all given fixed.

4.2.1 Computations under normal chart

We will perform local calculations under normal coordinates. For this purpose, we first collect some useful geometric properties of normal charts.

We fix an orthonormal basis (ONB) $\{e_i(0) : i = 1, \cdots, d\}$ of T_yM. Let $U \subseteq T_yM$ be a fixed open ball centered at the origin with radius $r \leqslant \rho_y$ (the injective radius at y) so that the exponential map $\exp_y : U \to M$ is a diffeomorphism. This gives rise to a local parametrisation

$$\mathbf{x} = (x^1, \cdots, x^d) \mapsto \exp_y\big(x^i e_i(0)\big)$$

of the ball $B(y,r) = \exp_y U =: V$, which is known as the *normal coordinate chart* based at y. As a convention, a bold letter always refers to an Euclidean vector (e.g. $\mathbf{x} = (x^1, \cdots, x^d)$) that is also identified with an element in U, while a regular letter (e.g. x) refers to a point on M. For instance, one can legally write $x = \exp_y(\mathbf{x})$.

The natural coordinate vector fields on V are denoted as $\{\partial_i : i = 1, \cdots, d\}$ as usual. It is useful to introduce another local frame of vector fields $\{e_i : 1, \cdots, d\}$ as follows. Given $\mathbf{x} \in U$, we define $e_i(\mathbf{x})$ to be the parallel translation of $e_i(0)$ along the geodesic $t \mapsto \exp_y(t\mathbf{x})$ to the endpoint $\mathbf{x}$ (at $t=1$). By varying $\mathbf{x}$, one obtains a vector field e_i on V. It is clear that $\{e_1(\mathbf{x}), \cdots, e_d(\mathbf{x})\}$ is an ONB of $T_{\exp_y \mathbf{x}}M$ for every $\mathbf{x} \in U$. Note that $e_i(0) = \partial_i|_{\mathbf{x}=0}$. In general, one can write $e_i = \sigma_i^j \partial_j$ where $(\sigma_i^j)_{1 \leqslant i,j \leqslant d}$ is a smooth matrix-valued function on U that is everywhere invertible.

Recall that the metric tensor under the normal chart V is defined by $g_{ij} \triangleq \langle \partial_i, \partial_j \rangle$. The following fact will be useful to us later on (cf. [BGV04, Proposition 1.27 (iii)]).

Lemma 4.4. *On the normal chart V, one has*

$$x^i g_{ij} = x^j, \ x^i g^{ij} = x^j.$$

It is well known that under the normal chart near the origin, the Riemannian metric g_{ij} agrees with Euclidean metric δ_{ij} up to the second order with error given in terms of curvature coefficients (cf. [BGV04, Proposition 1.28]).

Proposition 4.5. *On the normal chart V, one has*

$$g_{ij}(\mathbf{x}) = \delta_{ij} - \frac{1}{3} R_{ikjl}(\mathbf{0}) x^k x^l + O(|\mathbf{x}|^3) \quad \text{as } \mathbf{x} = (x^1, \cdots, x^d) \to 0,$$

where $R_{ijkl} \triangleq \langle R(\partial_k, \partial_l)\partial_j, \partial_i \rangle$ are the curvature coefficients in V.

We now express $\Delta/2$ on V as

$$\frac{1}{2}\Delta = \frac{1}{2} a^{ij} \partial_{ij}^2 + b^i \partial_i. \tag{4.3}$$

From the local expression (2.1), it is plain to check that

$$a^{ij} = g^{ij}, \ b^i = \frac{g^{ij}}{4 \det g} \partial_j \det g + \frac{1}{2} \partial_j g^{ij}. \tag{4.4}$$

We also recall that $G_0(x,y), G_1(x,y)$ are the first two terms appearing in the Malliavin-Stroock expansion (2.3). The following explicit formulae for the local expressions of these quantities will be needed for us.

Lemma 4.6. *On the normal chart V, one has*

$$a^{ij}(\mathbf{0}) = \delta^{ij}, \ b^i(\mathbf{0}) = 0; \tag{4.5}$$

$$\partial_p a^{ij}(\mathbf{0}) = 0, \ \partial_i b^j(\mathbf{0}) = \frac{1}{3} R_{jppi}, \ \partial_{pp}^2 a^{ij}(\mathbf{0}) = \frac{2}{3} R_{ipjp}; \tag{4.6}$$

$$\nabla_x G_0(\mathbf{x}, \mathbf{0}) = -x^j \partial_j; \tag{4.7}$$

$$\partial_j G_1(\mathbf{0}, \mathbf{0}) = 0, \ \partial^2_{ij} G_1(\mathbf{0}, \mathbf{0}) = \frac{1}{6} R_{pipj}. \tag{4.8}$$

Here R_{ijkl} is the curvature coefficient as in Proposition 4.5 and the derivatives in the last line are taken with respect to the first variable in G_1.

Proof. The curvature coefficients satisfy the following symmetries:

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}. \tag{4.9}$$

The relations (4.5) and (4.6) follow directly from Proposition 4.5 and (4.4). In addition, by the local expression of ∇ and Lemma 4.4 one has

$$\nabla_x G_0(\mathbf{x}, \mathbf{0}) = -x^i g^{ij} \partial_j = -x^j \partial_j.$$

This gives the relation (4.7). Finally, recall that G_1 is defined by (2.4). Standard geometric consideration shows that

$$\det(d \exp_y)_{\mathbf{x}} = \sqrt{\det g(\mathbf{x})} \tag{4.10}$$

for any $\mathbf{x} = \exp_y^{-1}(x) \in U$ $(x \in V)$. The relation (4.8) then easily follows from Proposition 4.5 and (4.10). □

Remark 4.7. For the proof of Theorem 4.1, only (4.5) and (4.7) are relevant. The relations (4.6) and (4.8) will only be needed for the proof of Theorem 5.1 in the next section.

4.2.2 Localisation I: the main contribution

Our analysis is essentially localised on a normal chart around y. Throughout the rest, we take $V \triangleq B(y, \rho_M/2 - \varepsilon)$ where $\varepsilon \in (0, \rho_M/2 - d(x, y))$ is fixed (note that $x \in V$). Define

$$\tau \triangleq \inf\{s \in [0, t] : X_s^{t,x,y} \notin V\}$$

and we set $\tau = t$ if such s does not exist. We shall consider the decomposition

$$\mathbb{E}\big[\pi_n S^\phi(X^{t,x,y})\big] = \mathbb{E}\big[\pi_n S^\phi(X^{t,x,y}); \tau = t\big] + \mathbb{E}\big[\pi_n S^\phi(X^{t,x,y}); \tau < t\big].$$

One naturally expects that the first term provides the main contribution and the second term is negligible. In this section, we establish the main estimate for the first term. The second term will be handled in the next section.

Note that the term $\mathbb{E}\big[\pi_n S^\phi(X^{t,x,y}); \tau = t\big]$ corresponds to the situation where $X^{t,x,y}$ does not leave the normal chart B around y during its lifetime. For this part, one can apply Euclidean stochastic calculus to the SDE (2.10) of the Brownian bridge. We break down the analysis into several steps.

A representation of $X^{t,x,y}$

In the Malliavin-Stroock expansion (2.3), recall that $G_0(x,y) = -d(x,y)^2/2$. By applying Lemma 4.6 on the normal chart V defined previously, one can write (using Euclidean variables on $U \triangleq \exp_y^{-1} V$)

$$\nabla_z \log p(u, \mathbf{z}, \mathbf{0}) = -\frac{\mathbf{z}}{u} + Q(u, \mathbf{z}), \tag{4.11}$$

where ∇_z denotes the Euclidean vector $(g^{ij}\partial_j)_{1\leqslant i\leqslant d}$ and

$$Q(u,\mathbf{z}) \triangleq \sum_{k=0}^{\infty} \nabla_z G_{k+1}(\mathbf{z}, \mathbf{0}) u^k$$

satisfies

$$\sup_{u\in[0,1],\mathbf{z}\in\bar{U}} |Q(u,z)| < \infty.$$

The SDE (2.10) of $X^{t,x,y}$ on V is thus expressed as

$$\begin{cases} dX_s = \big(-\frac{X_s}{t-s} + b(X_s) + Q(t-s, X_s)\big)ds + \sigma(X_s)dB_s, \quad s < \tau; \\ X_0 = \mathbf{x} \triangleq \exp_y^{-1}(x). \end{cases} \tag{4.12}$$

where (σ, b) are local coefficients of $\Delta/2$ satisfying the relation (4.3) with $a = \sigma^T\sigma$. We give a representation of X_s that will be used frequently in the sequel.

Lemma 4.8. *On the event $\{\tau = t\}$, one has*

$$\frac{X_{tr}}{1-r} = \mathbf{x} + t\int_0^{tr} \frac{\sigma(X_u)dB_u}{t-u} + t\int_0^r \frac{1}{1-\eta}\big(b(X_{t\eta}) + Q(t(1-\eta), X_{t\eta})\big)d\eta, \quad r \in [0,1). \tag{4.13}$$

Proof. We introduce the integrating factor $x_s \triangleq t - s$ $(s \in [0,t])$, which satisfies

$$dx_s = -\frac{1}{t-s} x_s ds \in \mathbb{R}.$$

Setting $Z_s \triangleq x_s^{-1} X_s$, one finds that

$$dZ_s = \frac{1}{t-s}\big(b(X_s) + Q(t-s, X_s)\big)ds + \frac{1}{t-s}\sigma(X_s)dB_s.$$

By integrating both sides from 0 to s and noting that $Z_0 = \mathbf{x}/t$, one has

$$\frac{X_s}{t-s} = \frac{\mathbf{x}}{t} + \int_0^s \frac{\sigma(X_u)dB_u}{t-u} + \int_0^s \frac{1}{t-u}\big(b(X_u) + Q(t-u, X_u)\big)du.$$

The result follows by taking $s = tr$. □

A semimartingale decomposition for $\phi(\circ dX_{tr})$

On the event $\{\tau = t\}$, the signature component $\pi_n S^{\phi}(X^{t,x,y})$ is given by

$$\pi_n S^{\phi}(X^{t,x,y}) = \int_{0<s_1<\cdots<s_n<t} \phi(\circ dX_{s_1}) \otimes \cdots \otimes \phi(\circ dX_{s_n})$$
$$= \int_{0<r_1<\cdots<r_n<1} \phi(\circ dX_{tr_1}) \otimes \cdots \otimes \phi(\circ dX_{tr_n}), \qquad (4.14)$$

where X_s satisfies the SDE (4.12). To relate it with the ϕ-signature of the minimising geodesic

$$\gamma_r^{x,y} \triangleq \exp((1-r)\mathbf{x}), \quad 0 \leqslant r \leqslant 1,$$

an important step is to factor out the geodesic component in the semimartingale decomposition of $\phi(\circ dX_{tr})$. We summarise the main structure in the lemma below. Let $(\Gamma_r)_{0\leqslant r\leqslant 1}$ be the E-valued path defined by $\Gamma_r \triangleq \int_0^r \phi(d\gamma^{x,y})$.

Lemma 4.9. *On the event $\{\tau = t\}$, one has the following decomposition of $\phi(\circ dX_{tr})$:*

$$\phi(\circ X_{tr}) = \dot{\Gamma}_r dr + \sqrt{t}\big(A_r^t dr + dM_r^t\big), \quad r \in [0,1), \qquad (4.15)$$

where $\{A_r^t, M_r^t : 0 \leqslant r < 1\}$ are E-valued stochastic processes satisfying the following properties:

(i) $\{A_r^t\}$ is a continuous, $\{\mathcal{F}_{tr}^B\}$-adapted semimartingale such that

$$\|A_r^t\|_p \leqslant \frac{C\sqrt{p}}{\sqrt{1-r}}, \quad \forall r \in [0,1),\ p \geqslant 2, \qquad (4.16)$$

where $\|\cdot\|_p$ denotes the L^p-norm.
(ii) $\{M_r^t\}$ is an $\{\mathcal{F}_{tr}^B\}$-martinagle whose components satisfy

$$\langle M^{t,i}\rangle_r \leqslant Cr, \quad \forall r \in [0,1),\ i = 1, \cdots, N. \qquad (4.17)$$

In the above inequalities, C denotes a geometric constant depending only on M, ϕ and the localisation V.

Proof. We write $\phi = \phi_i dx^i$ on V and also set $\hat{Q}(u,\mathbf{z}) \triangleq b(\mathbf{z}) + Q(u,\mathbf{z})$ to ease notation. According to the SDE (4.12) and the Itô-Stratonovich conversion, one has

$$\phi(\circ dX_s) = -\phi_i(X_s)\frac{X_s^i}{t-s}ds + \big(\phi_i(X_s)\hat{Q}^i(t-s, X_s)$$
$$+ \frac{1}{2}\partial_j\phi_i(X_s)a^{ij}(X_s)\big)ds + \phi_i(X_s)\sigma_\alpha^i(X_s)dB_s^\alpha$$

for $s \in [0,\tau)$. By taking $s = tr$ and substituting (4.13) into the first term, it is easily seen that the decomposition (4.15) holds with

$$A_r^t \triangleq -\phi_i(X_{\tau\wedge tr})\big(\sqrt{t}\int_0^{\tau\wedge tr}\frac{\sigma_\alpha^i(X_u)dB_u^\alpha}{t-u}+\sqrt{t}\int_0^{(\tau/t)\wedge r}\frac{1}{1-\eta}\hat{Q}^i(t(1-\eta),X_{t\eta})d\eta\big) \tag{4.18}$$

$$-\big(\phi_i(X_{\tau\wedge tr})-\phi_i((1-r)\mathbf{x})\big)x^i+\sqrt{t}\big(\phi_i(X_{\tau\wedge tr})\hat{Q}^i(t(1-r),X_{\tau\wedge tr}) + \frac{1}{2}\partial_j\phi_i(X_{\tau\wedge tr})a^{ij}(X_{\tau\wedge tr})\big) \tag{4.19}$$

and

$$M_r^t \triangleq \int_0^{(\tau/t)\wedge r}\phi_i(X_{t\eta})\sigma_\alpha^i(X_{t\eta})dB_\eta^{t,\alpha},$$

where $B_\eta^{t,\alpha} \triangleq B_{t\eta}^\alpha/\sqrt{t}$ is a rescaled Brownian motion. Since $\phi_i, \sigma_\alpha^i \in C_b^\infty(\bar{U})$, it is clear that the bound (4.17) holds.

We now estimate each term of A_r^t. Since $\hat{Q}^i$ is also uniformly bounded on $\bar{U}$, the entire expression of (4.19) is uniformly bounded by some deterministic constant which is in turn trivially enlarged to the form of (4.16). To estimate the first term of (4.18), one notes from the Burkholder-Davis-Gundy (BDG) inequality (cf. [CK91, Theorem A] with BDG constant $2\sqrt{p}$) that

$$\big\|\sqrt{t}\int_0^{\tau\wedge tr}\frac{(\sigma dB)_u^i}{t-u}\big\|_p \leqslant 2\sqrt{p}\cdot\big\|\sqrt{t\int_0^{tr}\frac{C}{(t-u)^2}du}\big\|_p = C'\sqrt{p}\cdot\sqrt{\frac{r}{1-r}},$$

which is dominated by (4.16) since $r \leqslant 1$. The second term of (4.18) is uniformly bounded by $C|\log(1-r)|$ which is also trivially enlarged to (4.16). The desired bound (4.16) thus follows. □

The signature remainder estimate

In the decomposition (4.15), let us denote

$$dY_r^0 \triangleq \dot{\Gamma}_r dr,\ dY_r^1 \triangleq A_r^t dr + dM_r^t,$$

where we omitted the script t to ease notation. Then on the event $\{\tau = t\}$ one can write

$$\int_{0<r_1<\cdots<r_n<1}\phi(\circ dX_{tr_1})\otimes\cdots\otimes\phi(\circ dX_{tr_n}) = \sum_{I=(\omega_1,\cdots,\omega_n)} t^{\frac{n-|I|_0}{2}}\int_{0<r_1<\cdots<r_n<1}\circ dY_{r_1}^{\omega_1}\otimes\cdots\otimes\circ dY_{r_n}^{\omega_n}, \tag{4.20}$$

where the summation is taken over all words $I = (\omega_1, \cdots, \omega_n)$ with $\omega_i = 0, 1$ and $|I|_0$ denotes the number of zero's in I (the number of geodesic components).

Given such a word I, we set

$$J_n(\rho; I) \triangleq \int_{0<r_1<\cdots<r_n<\rho} \circ dY_{r_1}^{\omega_1} \otimes \cdots \otimes \circ dY_{r_n}^{\omega_n}, \quad \rho \in [0, 1]. \tag{4.21}$$

A key step for proving Theorem 4.1 is a proper estimate of $J_n(\rho; I)$ that respects the number of geodesic components in I. The main result for this purpose is stated below. In what follows, $|\cdot|$ denotes the Hilbert-Schmidt tensor norm and $\|\cdot\|_p \triangleq \mathbb{E}[|\cdot|^p; \tau = t]^{1/p}$ denotes the L^p-norm on $\{\tau = t\}$ for tensor-valued random variables.

Proposition 4.10. *There exists a positive constant Λ depending only on M, ϕ, N and the localisation V, such that the following estimate holds true*

$$\big\|J_n(\rho; I)\big\|_p \leqslant \frac{(\Lambda p)^{n-k}}{\sqrt{(n-k)!}} \frac{\|\Gamma\|_{1\text{-}var;[0,\rho]}^k}{k!} (1 - \sqrt{1-\rho})^{\frac{n-k}{2}} \tag{4.22}$$

for any $p \geqslant 2, \rho \in [0, 1]$, any $n \in \mathbb{N}$, $1 \leqslant k \leqslant n$ and any word $I = (\omega_1, \cdots, \omega_n)$ with $|I|_0 = k$.

Remark 4.11. A delicate point in the proof is to make sure that one only introduces an exponential factor like $C^{n-k}\|\Gamma\|_{1\text{-var}}^k$ (as seen in (4.22)) instead of something like C^n. This is the key point for reducing the lifetime scale t_n to polynomial dependence in n in Theorem 4.1; a more "standard" signature estimate would introduce a factor C^n forcing t_n to decay exponentially in order to make (4.2) work.

To prove Proposition 4.10, we first recall a basic result of Kallenberg-Sztencel (cf. [KS91, Theorem 3.1]) which leads to a dimension-free BDG inequality. This also avoids the introduction of C^n when estimating moments of $E^{\otimes n}$-valued martingales.

Lemma 4.12. *Let $S = (S^1, \cdots, S^l)$ be a collection of continuous local martingales. Then there exists a two-dimensional martingale $T = (T^1, T^2)$ defined possibly on an extended filtered probability space, such that*

$$|S| = |T|, \ \langle S \rangle = \langle T \rangle,$$

where

$$|S| \triangleq \sqrt{(S^1)^2 + \cdots + (S^l)^2}, \ \langle S \rangle \triangleq \langle S^1 \rangle + \cdots + \langle S^l \rangle$$

and similarly for T.

Next, we present a lemma that provide the basic estimates for proving Proposition 4.10 by induction.

Lemma 4.13. *There exist positive constants K_1, K_2, K_3 depending only on M, ϕ, N and the localisation V, such that the following estimates hold true for any $p \geqslant 2, n \in \mathbb{N}$ and any word $I = (\omega_1, \cdots, \omega_{n+1})$:*

(i) If $\omega_{n+1} = 0$, then one has

$$\|J_{n+1}(\rho; I)\|_p \leqslant \int_0^\rho \|J_n(\rho; I')\|_p |\dot{\Gamma}_\eta| d\eta, \quad \rho \in [0, 1]. \tag{4.23}$$

(ii) If $\omega_{n+1} = 1$, then one has

$$\|J_{n+1}(\rho; I)\|_p \leqslant K_1 \sqrt{r} \int_0^\rho \|J_n(\eta; I')\|_q \frac{1}{\sqrt{1-\eta}} d\eta + K_2 \sqrt{p} \Big(\int_0^\rho \|J_n(\eta; I')\|_p^2 d\eta \Big)^{1/2} \\ + K_3 \int_0^\rho \|J_{n-1}(\eta; I'')\|_p d\eta, \tag{4.24}$$

where q, r are any positive numbers such that $1/p = 1/q + 1/r$ and I' (resp. I'') is the word obtained by removing the last entry (resp. last two entries) of I.

Proof. If $\omega_{n+1} = 0$, one has

$$J_{n+1}(\rho; I) = \int_0^\rho J_n(\eta; I') \otimes \dot{\Gamma}_\eta d\eta$$

and the estimate (4.23) follows immediately. If $\omega_{n+1} = 1$, one has

$$\begin{aligned} J_{n+1}(\rho; I) &= \int_0^\rho J_n(\eta; I') \otimes A_\eta^t d\eta + \int_0^\rho J_n(\eta; I') \otimes \circ dM_\eta^t \\ &= \int_0^\rho J_n(\eta; I') \otimes A_\eta^t d\eta + \int_0^\rho J_n(\eta; I') \otimes \cdot dM_\eta^t \\ &\quad + \frac{1}{2} \int_0^\rho \cdot dJ_n(\eta; I') \otimes \cdot dM_\eta^t \\ &=: I_n^1(\rho) + I_n^2(\rho) + I_n^3(\rho). \end{aligned}$$

where $\cdot dM$ denotes Itô's integral. We now estimate each term on the right hand side separately. Note that the last term comes from the Itô-Stratonovich correction and is present only when $\omega_n = 1$.

Estimation of I_n^1. According to (4.16) and Young's inequality, one has

$$\|I_n^1(\rho)\|_p \leqslant \int_0^\rho \|J_n(\eta;I')\|_q \|A_\eta^t\|_r d\eta \leqslant K_1\sqrt{r}\int_0^\rho \|J_n(\eta;I')\|_q \frac{1}{\sqrt{1-\eta}} d\eta.$$

Estimation of I_n^2. We are going to applying Lemma 4.12 to the multidimensional martingale

$$I_n^2(\rho) \triangleq \int_0^\rho J_n(\eta;I')\otimes \cdot dM_\eta^t \in (\mathbb{R}^N)^{\otimes(n+1)}.$$

By the definition of the Hilbert-Schmidt norm, one has

$$|I_n^2(\rho)| = \Big(\sum_{i_1,\cdots,i_{n+1}=1}^N \Big(\int_0^\rho J_n(\eta;I)^{i_1,\cdots,i_n} dM_\eta^{t,i_{n+1}}\Big)^2\Big)^{1/2}.$$

According to (4.17), one also has

$$\begin{aligned}\langle I_n^2\rangle_\rho &= \sum_{i_1,\cdots,i_{n+1}=1}^N \int_0^\rho \big(J_n(\eta;I)^{i_1,\cdots,i_n}\big)^2 d\langle M^{t,i_{n+1}}\rangle_\eta \\ &\leqslant NC\int_0^\rho \sum_{i_1,\cdots,i_n=1}^N \big(J_n(\eta;I)^{i_1,\cdots,i_n}\big)^2 d\eta = NC\int_0^\rho \big|J_n(\eta;I)\big|^2 d\eta.\end{aligned}$$

It follows from Lemma 4.12 as well as the 2D BDG-inequality that

$$\|I_n^2(\rho)\|_p \leqslant 4\sqrt{2}\sqrt{p}\|\langle I_n^2\rangle_\rho^{1/2}\|_p \quad \forall p\geqslant 2.$$

Here the constant $4\sqrt{2}\sqrt{p}$ is easily derived from the 1D case whose BDG constant is $2\sqrt{p}$ (cf. [CK91, Theorem A]). Therefore, one finds that

$$\|I_n^2(\rho)\|_p \leqslant 4\sqrt{2NC}\sqrt{p}\Big\|\Big(\int_0^\rho \big|J_n(\eta;I)\big|^2 d\eta\Big)^{1/2}\Big\|_p.$$

In addition, note that

$$\Big\|\Big(\int_0^\rho \big|J_n(\eta;I)\big|^2 d\eta\Big)^{1/2}\Big\|_p = \Big\|\int_0^\rho \big|J_n(\eta;I')\big|^2 d\eta\Big\|_{p/2}^{1/2} \leqslant \Big(\int_0^\rho \big\|J_n(\eta;I')\big\|_p^2 d\eta\Big)^{1/2}.$$

Consequently, one arrives at

$$\|I_n^2(\rho)\|_p \leqslant K_2\sqrt{p}\Big(\int_0^\rho \big\|J_n(\eta;I')\big\|_p^2 d\eta\Big)^{1/2} \tag{4.25}$$

with $K_2 \triangleq 4\sqrt{2NC}$.

<u>*Estimation of I_n^3.*</u> This is only relevant when $\omega_n = \omega_{n+1} = 1$. In this case, by (4.17) one has

$$\big|I_n^3(\rho)\big| \leqslant \frac{1}{2}\int_0^\rho \big|J_{n-1}(\eta; I'')\big| \cdot \big|d\langle M^t, \otimes M^t\rangle_\eta\big| \leqslant K_3 \int_0^\rho \big|J_{n-1}(\eta; I'')\big| d\eta.$$

It follows that

$$\|I_n^3(\rho)\|_p \leqslant K_3 \int_0^\rho \big\|J_{n-1}(\eta; I'')\big\|_p d\eta.$$

By adding up the previous three inequalities, one obtains the desired estimate (4.24) and concludes the proof of the lemma. □

We are now in a position to prove Proposition 4.10.

Proof of Proposition 4.10. We are going to prove the inequality (4.22) by induction on n. Consider first the base case $n = 1$. If $I = (0)$, then one has

$$\big|J_1(\rho; I)\big| \leqslant \int_0^\rho \big|\dot{\Gamma}_\eta\big| d\eta = \|\Gamma\|_{\text{1-var};[0,\rho]}$$

and the estimate (4.10) holds trivially with $n = k = 1$. If $I = (1)$, then

$$J_1(\rho; I) = \int_0^\rho A_\eta^t d\eta + M_\rho^t.$$

The first term is estimated by (4.16) as

$$\Big\| \int_0^\rho A_r^t dr \Big\|_p \leqslant C\sqrt{p} \int_0^\rho \frac{d\eta}{\sqrt{1-\eta}} = 2C\sqrt{p}(1 - \sqrt{1-\rho}).$$

The second term is estimated by the BDG inequality and (4.17) as

$$\|M_\rho^t\|_p \leqslant \sqrt{2NCp}\sqrt{\rho}.$$

It follows that

$$\|J_1(\rho; I)\|_p \leqslant \big(2C(1 - \sqrt{1-\rho}) + \sqrt{2NC}\sqrt{\rho}\big)p.$$

To make the estimate (4.22) work with $n = 1, k = 0$, it suffices to require Λ to satisfy

$$2C(1 - \sqrt{1-\rho}) + \sqrt{2NC}\sqrt{\rho} \leqslant \Lambda\big(1 - \sqrt{1-\rho}\big)^{1/2} \quad \forall \rho \in [0, 1].$$

Such a Λ clearly exists.

In what follows, we fix the choice of

$$\Lambda \triangleq \max\Big\{\sup_{\rho\in[0,1]} \frac{2C(1-\sqrt{1-\rho})+\sqrt{2NC}\sqrt{\rho}}{\big(1-\sqrt{1-\rho}\big)^{1/2}}, 4eK_1+\sqrt{2}K_2+4K_3\Big\}, \qquad (4.26)$$

where K_1, K_2, K_3 are the constants appearing in Lemma 4.13. We just showed the base case $n=1$ and are going to establish the induction step with the same Λ. Suppose that the estimate (4.22) is valid for all words with length $\leqslant n$. Now let $I=(\omega_1,\cdots,\omega_{n+1})$ be a given word of length $n+1$ and let k be the number of zero's in I. We write

$$L_k(\rho) \triangleq \int_{0<r_1<\cdots<r_k<\rho} \big|\dot{\Gamma}_{r_1}\big|\cdots\big|\dot{\Gamma}_{r_k}\big|dr_1\cdots dr_k = \frac{\|\Gamma\|^k_{1\text{-var};[0,\rho]}}{k!}.$$

Case I: $\omega_{n+1}=0$.

According to (4.23) and the induction hypothesis, one has

$$\begin{aligned}
\|J_{n+1}(\rho;I)\|_p &\leqslant \int_0^\rho \frac{(\Lambda p)^{n-k+1}}{\sqrt{(n-k+1)!}}(1-\sqrt{1-\eta})^{\frac{n-k+1}{2}}L_k(\eta)\cdot\big|\dot{\Gamma}_\eta\big|d\eta \\
&\leqslant \frac{(\Lambda p)^{n-k+1}}{\sqrt{(n-k+1)!}}(1-\sqrt{1-\rho})^{\frac{n-k+1}{2}}\int_0^\rho L_k(\eta)\cdot\big|\dot{\Gamma}_\eta\big|d\eta \\
&= \frac{(\Lambda p)^{n-k+1}}{\sqrt{(n-k+1)!}}(1-\sqrt{1-\rho})^{\frac{n-k+1}{2}}L_{k+1}(\rho).
\end{aligned}$$

Case II: $\omega_{n+1}=1$.

In this case, we apply the induction hypothesis to each term on the right hand side of (4.24). Recall that I' (resp. I'') is the word obtained by removing the last entry (resp. last two entries) of I. Firstly, one has

$$\begin{aligned}
\mathrm{I} &\triangleq K_1\sqrt{r}\int_0^\rho \big\|J_n(\eta;I')\big\|_q\frac{1}{\sqrt{1-\eta}}d\eta \\
&\leqslant K_1\sqrt{r}\int_0^\rho \frac{(\Lambda q)^{n-k}}{\sqrt{(n-k)!}}(1-\sqrt{1-\eta})^{\frac{n-k}{2}}L_k(\eta)\frac{1}{\sqrt{1-\eta}}d\eta \\
&\leqslant K_1\sqrt{r}\frac{(\Lambda q)^{n-k}}{\sqrt{(n-k)!}}L_k(\rho)\int_0^\rho(1-\sqrt{1-\eta})^{\frac{n-k}{2}}\frac{d\eta}{\sqrt{1-\eta}}.
\end{aligned}$$

By evaluating the last integral explicitly, it is easily seen that

$$\mathrm{I} \leqslant \frac{4K_1\sqrt{r}(\Lambda q)^{n-k}}{\sqrt{(n-k)!}(n+2-k)}(1-\sqrt{1-\rho})^{\frac{n+1-k}{2}}L_k(\rho). \tag{4.27}$$

We now choose $r = p(n+1-k)$ and $q = \frac{p(n+1-k)}{n-k}$ (so that $1/p = 1/q + 1/r$). It follows that

$$\begin{aligned}\sqrt{r}q^{n-k} &= \sqrt{p(n+1-k)}\big(\frac{p(n+1-k)}{n-k}\big)^{n-k} \\ &= p^{n-k+1/2}\sqrt{n+1-k}\big(1+\frac{1}{n-k}\big)^{n-k} \leqslant e\sqrt{n+1-k}p^{n+1-k}.\end{aligned}$$

By substituting this into (4.27), one obtains that

$$\mathrm{I} \leqslant 4eK_1\frac{\Lambda^{n-k}p^{n+1-k}}{\sqrt{(n+1-k)!}}(1-\sqrt{1-\rho})^{\frac{n+1-k}{2}}L_k(\rho). \tag{4.28}$$

Next, we estimate the second term in (4.24); one has

$$\begin{aligned}\mathrm{II} &\triangleq K_2\sqrt{p}\big(\int_0^\rho \big\|J_n(\eta;I')\big\|_p^2 d\eta\big)^{1/2} \\ &\leqslant K_2\sqrt{p}\big(\int_0^\rho \big(\frac{(\Lambda p)^{n-k}}{\sqrt{(n-k)!}}(1-\sqrt{1-\eta})^{\frac{n-k}{2}}L_k(\eta)\big)^2 d\eta\big)^{1/2} \\ &\leqslant K_2\sqrt{p}\frac{(\Lambda p)^{n-k}}{\sqrt{(n-k)!}}L_k(\rho)\big(\int_0^\rho(1-\sqrt{1-\eta})^{n-k}d\eta\big)^{1/2}.\end{aligned}$$

The last integral is estimated as

$$\begin{aligned}\int_0^\rho(1-\sqrt{1-\eta})^{n-k}d\eta &\leqslant \int_0^\rho(1-\sqrt{1-\eta})^{n-k}\frac{1}{\sqrt{1-\eta}}d\eta \\ &= \frac{2}{n+1-k}(1-\sqrt{1-\rho})^{n+1-k}.\end{aligned}$$

Therefore, one obtains that

$$\begin{aligned}\mathrm{II} &\leqslant K_2\sqrt{p}\frac{(\Lambda p)^{n-k}}{\sqrt{(n-k)!}}L_k(\rho)\times\frac{\sqrt{2}}{\sqrt{n+1-k}}(1-\sqrt{1-\rho})^{\frac{n+1-k}{2}} \\ &\leqslant \sqrt{2}K_2\frac{\Lambda^{n-k}p^{n+1-k}}{\sqrt{(n+1-k)!}}(1-\sqrt{1-\rho})^{\frac{n+1-k}{2}}L_k(\rho). \end{aligned}\tag{4.29}$$

Finally, we estimate the third term in (4.24), which is denoted as

$$\mathrm{III} \triangleq K_3 \int_0^\rho \big\|J_{n-1}(\eta; I'')\big\|_p d\eta.$$

Note that this term comes from Itô-Stratonovich correction and is present only when $\omega_n = \omega_{n+1} = 1$. In particular, there are k geodesic components in I'' for this scenario. As a result, by the induction hypothesis one has

$$\begin{aligned}\mathrm{III} &\leqslant K_3 \int_0^\rho \frac{(\Lambda p)^{n-1-k}}{\sqrt{(n-1-k)!}}(1-\sqrt{1-\eta})^{\frac{n-1-k}{2}} L_k(\eta) d\eta \\ &\leqslant 4K_3 \frac{\Lambda^{n-k} p^{n+1-k}}{\sqrt{(n+1-k)!}}(1-\sqrt{1-\rho})^{\frac{n+1-k}{2}} L_k(\rho). \end{aligned} \tag{4.30}$$

By putting (4.28), (4.29), (4.30) together, one sees that

$$\big\|J_{n+1}(\rho; I)\big\|_p \leqslant (4eK_1 + \sqrt{2}K_2 + 4K_3) \frac{\Lambda^{n-k} p^{n+1-k}}{\sqrt{(n+1-k)!}}(1-\sqrt{1-\rho})^{\frac{n+1-k}{2}} L_k(\rho).$$

In view of the choice of Λ given by (4.26), the right hand side of the above inequality is further bounded by

$$\frac{(\Lambda p)^{n+1-k}}{\sqrt{(n+1-k)!}}(1-\sqrt{1-\rho})^{\frac{n+1-k}{2}} L_k(\rho).$$

This completes the induction step.

□

4.2.3 Localisation II: the remainder

Our aim here is to estimate the term $\mathbb{E}\big[\pi_n S^\phi(X^{t,x,y}); \tau < t\big]$ and the main result is stated as follows.

Proposition 4.14. *There exist constants $C_1, C_2, \delta > 0$ depending only on x, y, M, such that*

$$\big\|\mathbb{E}\big[\pi_n S^\phi(X^{t,x,y}); \tau < t\big]\big\|_{\mathrm{HS}} \leqslant \frac{C_1^n}{\sqrt{n!}} e^{-C_2/t} \tag{4.31}$$

for all $n \geqslant 1$ and $t \in (0, \delta)$.

The main idea of the proof is to estimate the probability of the event $\{\tau < t\}$ and moments of $\pi_n S^\phi(X^{t,x,y})$ separately. For the first part, we rely on Hsu's large deviation principle for the Brownian bridge. For the second part, we make use of several heat kernel estimates in geometric analysis. In what follows, we divide our proof of Proposition 4.14 into several steps.

The exit probability

We first estimate the probability that the Brownian bridge exits the chart V before its lifetime t.

Lemma 4.15. *There are constants $\delta, K > 0$ depending only on x, y, such that*

$$\mathbb{P}(\tau < t) \leqslant e^{-K/t} \quad \forall t \in (0, \delta).$$

Proof. Let $\mathbb{Q}^t_{x,y}$ denote the law of the process $\{X^{t,x,y}_{st} : s \in [0,1]\}$ on the path space

$$\Omega_{x,y} \triangleq \big\{w : [0,1] \to M \big| w \text{ continuous, } w_0 = x, w_1 = y\big\}.$$

It is obvious that

$$\mathbb{P}(\tau < t) \leqslant \mathbb{Q}^t_{x,y}\big(\{w \in \Omega_{x,y} : D(w,y) \geqslant \rho_M/2 - \varepsilon\}\big)$$

where $D(w,y) \triangleq \sup\{d(w_s, y) : s \in [0,1]\}$. According to the large deviation principle for $\{\mathbb{Q}^t_{x,y}\}$ (cf. [Hsu90, Theorem 2.2]), one has

$$\begin{aligned}\overline{\lim_{t\to 0}}\, t \log \mathbb{P}(\tau t) &\leqslant \overline{\lim_{t\to 0}}\, t \log \mathbb{Q}^t_{x,y}\big(\{w \in \Omega_{x,y} : D(w,y) \geqslant \rho_M/2 - \varepsilon\}\big) \\ &\leqslant - \inf_{\substack{w \in \Omega_{x,y} \\ D(w,y) \geqslant \rho_M/2 - \varepsilon}} J_{x,y}(w),\end{aligned}$$

where

$$J_{x,y}(w) \triangleq \begin{cases} \frac{1}{2}\big(\int_0^1 |\dot{w}_s|^2 ds - d(x,y)^2\big), & |\dot{w}| \in L^2([0,1]); \\ \infty, & \text{otherwise.} \end{cases}$$

For those w with $|\dot{w}| \in L^2([0,1])$ and $D(w,y) \geqslant \rho_M/2 - \varepsilon$, one has

$$\int_0^1 |\dot{w}_s|^2 ds \geqslant \Big(\int_0^1 |\dot{w}_s| ds\Big)^2 \geqslant (\rho_M/2 - \varepsilon)^2.$$

It follows that

$$\overline{\lim_{t\to 0}}\, t \log \mathbb{P}(\tau < t) \leqslant \frac{1}{2} d(x,y)^2 - \frac{1}{2}(\rho_M/2 - \varepsilon)^2.$$

The result follows by noting that the right hand side is a negative number. $\square$

Signature moment estimates of Brownian bridge

Next, we estimate moments of $\pi_n S^\phi(X^{t,x,y})$. The main result is stated below. Here we also use the Hilbert-Schmidt norm on tensors.

Lemma 4.16. *There exist positive constants Λ, L depending only on M and ϕ, such that*

$$\left\|\pi_n S^\phi(X^{t,x,y})\right\|_p \leqslant \frac{(\Lambda p)^{n/2}}{\sqrt{n!}} e^{L/pt}$$

for all $p \geqslant 1, n \in \mathbb{N}, t \in (0,1]$ and $x, y \in M$.

The argument is quite similar to the proof of Proposition 4.10. Since the estimate here is global, we take an extrinsic perspective and make use of a well-known estimate for the logarithmic derivative of the heat kernel.

I. Extrinsic construction of the Brownian bridge

Let M be isometrically embedding in some Euclidean space $\mathbb{R}^m$ (which is always possible due to Nash's embedding theorem). A *tubular neighbourhood* of M in $\mathbb{R}^m$ is defined by the normal bundle

$$\mathcal{B}_\varepsilon \triangleq \bigsqcup_{x \in M} B^\perp(x, \varepsilon).$$

Here $B^\perp(x,\varepsilon) \triangleq \{x' \in \mathbb{R}^m : \overrightarrow{xx'} \in (T_x M)^\perp, |x' - x|_{\mathbb{R}^m} < \varepsilon\}$. Since M is compact, it is a standard fact in differential topology (cf. [Lee06, Chap. 10]) that when ε is small, $\mathcal{B}_\varepsilon$ is diffeomorphic to the ε-neighbourhood of M in $\mathbb{R}^m$. We fix such an ε. There is a canonical projection $\pi : \mathcal{B}_\varepsilon \to M$ taking $x \in \mathcal{B}_\varepsilon$ onto its unique closest point $\pi(x)$ in M.

Functions on M can be extended to $\mathbb{R}^m$ by using a bump function on $\mathcal{B}_\varepsilon$. More precisely, let $\eta : [0,\infty) \to \mathbb{R}$ be a smooth bump function such that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ on $[0, \varepsilon/2]$ and $\eta \equiv 0$ on (ε, ∞). Given any function $f \in C^\infty(M)$, we define

$$\bar{f}(x) \triangleq \eta(d_{\mathbb{R}^m}(x, \pi(x))) f(\pi(x)), \quad x \in \mathcal{B}_\varepsilon.$$

It is clear that $\bar{f} \in C_c^\infty(\mathbb{R}^m)$. Vector fields and one-forms on M are extended to $\mathbb{R}^m$ in a similar way. We always use the notation $\bar{\cdot}$ to denote the extended object.

For each $x \in M$ and $1 \leqslant \alpha \leqslant m$, let $V_\alpha(x)$ be the tangent vector to M defined by orthogonally projecting the α-th canonical basis vector of $\mathbb{R}^m$ onto $T_x M$. Then V_α defines a smooth vector field on M and we take its extension $\bar{V}_\alpha$ to $\mathbb{R}^m$ as before.

Note that the Laplacian on M can be expressed as $\Delta_M = \sum_{\alpha=1}^m V_\alpha^2$. Given fixed $t \in (0,1]$ and $x, y \in \mathbb{R}^m$, let $\{\bar{X}_s^{t,x,y} : 0 \leqslant s < t\}$ be the solution to the SDE

$$d\bar{X}_s = \sum_{\alpha=1}^m \bar{V}_\alpha(X_s)dB_s + \bar{\nabla}_x \bar{p}(t-s, \bar{X}_s, y)ds, \quad \bar{X}_0 = x.$$

Here $\bar{\nabla}$ denotes the Euclidean gradient in $\mathbb{R}^m$. Equivalently, $\bar{X}_\cdot^{t,\cdot,y}$ is a Markov family with generator

$$\bar{\mathcal{L}}_s^{t,y} = \frac{1}{2}\sum_{\alpha=1}^m \bar{V}_\alpha^2 + \bar{\nabla}_x \bar{p}(t-s, \cdot, y) \cdot \bar{\nabla}.$$

The following fact is of no surprise.

Lemma 4.17. *Suppose that $x, y \in M$. Then with probability one, the process $\bar{X}^{t,x,y}$ lives on M for all time. In this case, $\bar{X}_\cdot^{t,x,y}$ has the same law as the Brownian bridge from x to y on M with lifetime t.*

Proof. Consider the function $F(z) \triangleq d_{\mathbb{R}^m}(z, M)^2$ on $\mathbb{R}^m$. For $x \in \mathcal{B}_{2\varepsilon}$, one has $F(x) = 0$ iff $x \in M$. Define

$$\sigma \triangleq \inf\{s \in [0,t) : \bar{X}_s^{t,x,y} \notin \mathcal{B}_{\varepsilon/2}\}.$$

By our choices of extensions, it is easily seen that $(\bar{\mathcal{L}}_s^{t,y} F)(\bar{X}_s^{t,x,y}) = 0$ on $[0,\sigma]$. It follows from the martingale characterisation that

$$\mathbb{E}\big[F(\bar{X}_{s\wedge\sigma}^{t,x,y})\big] = F(x) = 0 \implies \bar{X}_{s\wedge\sigma}^{t,x,y} \in M \quad \text{a.s.}$$

By continuity, the process lives on M for all time before $\sigma \wedge t$ which also indicates that $\sigma = t$ (i.e. no exiting $\mathcal{B}_{\varepsilon/2}$ occurs). For the second part of the lemma, one first notes that the process

$$s \mapsto F(\bar{X}_s^{t,x,y}) - F(x) - \int_0^s \bar{\mathcal{L}}_r^{t,y} F(\bar{X}_r^{t,x,y})dr$$

is a martingale for all $F \in C_c^2(\mathbb{R}^m)$. In addition, the law of the Brownian bridge $X^{t,x,y}$ on M is characterised by the fact that

$$s \mapsto f(X_s^{t,x,y}) - f(x) - \int_0^s \big(\frac{1}{2}\Delta_M f(X_r^{t,x,y}) + \nabla_x^M p(t-r, X_r^{t,x,y}, y) \cdot \nabla^M f(X_r^{t,x,y})\big)dr$$

is a martingale for all $f \in C^2(M)$. The result follows from the fact that $\bar{X}_r^{t,x,y} \in M$ as well as the relation that

$$\bar{\mathcal{L}}_r^{t,y} \bar{f}\big|_M = \frac{1}{2}\Delta_M f + \nabla_x^M p(t-r, \cdot, y) \cdot \nabla^M f \quad \forall f \in C^2(M).$$

The proof of the lemma is thus complete. □

II. Heat kernel bounds and a moment estimate for the Brownian bridge

Our proof of Lemma 4.16 relies on a few heat kernel estimates which we now recall.

Proposition 4.18. *Let M be a compact Riemannian manifold.*

(i) (cf. [Hsu02, Theorem 5.3.4]) There are positive constants K_1, K_2, K_3 such that

$$K_1 e^{-K_2/t} \leqslant p(t,x,y) \leqslant \frac{K_3}{t^{d-1/2}} e^{-d(x,y)^2/2t} \quad \forall (t,x,y) \in (0,1] \times M \times M. \tag{4.32}$$

(ii) (cf. [ST98, Inequality (1.1)]) There exists a positive constant K_4, such that

$$\big|\nabla_x \log p(t,x,y)\big| \leqslant K_4\Big(\frac{1}{\sqrt{t}} + \frac{d(x,y)}{t}\Big) \quad \forall (t,x,y) \in (0,1] \times M \times M. \tag{4.33}$$

The following moment estimate of the Brownian bridge will also be needed for us. It is not in the sharpest form but is enough for our purpose.

Lemma 4.19. *Let $\beta \in (0, 1/2)$ be a given fixed parameter. There exist positive constants K_5, K_6 such that*

$$\|d(X_s^{t,x,y}, y)\|_p \leqslant K_5 \sqrt{p} e^{K_2/pt} (t-s)^\beta$$

for all $p \geqslant 1$, $s < t \in (0,1]$ and $x, y \in M$. Here K_2 is the same constant appearing in (4.32).

Proof. According to the transition density formula (2.9), one has

$$\begin{aligned}
\mathbb{E}\big[d(X_s^{t,x,y}, y)^p\big] &= \frac{1}{p(t,x,y)} \int_M d(z,y)^p p(s,x,z) p(t-s,z,y) dz \\
&= \frac{1}{p(t,x,y)} \Big(\int_{\{z: d(z,y) \leqslant \sqrt{p}(t-s)^\beta\}} + \int_{\{z: d(z,y) \geqslant \sqrt{p}(t-s)^\beta\}} \Big) \\
&\quad d(z,y)^p p(s,x,z) p(t-s,z,y) dz \\
&\leqslant \sqrt{p}^p (t-s)^{\beta p} + D_M^p \cdot K_1^{-1} e^{K_2/t} \cdot \frac{K_3}{(t-s)^{d-1/2}} e^{-p(t-s)^{2\beta-1}/2},
\end{aligned}$$

where D_M is the diameter of M and we also used the semigroup property as well as the estimate (4.32) to reach the last inequality. One can assume without loss of generality that $K_1 < 1$ and $K_3 > 1$. Since $p \geqslant 1$, one obtains that

$$\big\|d(X_s^{t,x,y}, y)\big\|_p \leqslant \sqrt{p}(t-s)^\beta + D_M K_1^{-1} K_3 e^{K_2/pt} \cdot \frac{e^{-(t-s)^{2\beta-1}/2}}{(t-s)^{d-1/2}}.$$

The desired estimate follows by noting that

$$\frac{e^{-(t-s)^{2\beta-1}/2}}{(t-s)^{d-1/2}} \leqslant C_{d,\beta}(t-s)^{\beta}$$

for all $s < t \in (0,1]$. □

III. Proof of Lemma 4.16

We now proceed to prove Lemma 4.16. The argument is similar to (but not as fine as) the proof of Proposition 4.10 and some repeated calculations will thus be omitted. Recall that $\beta \in (0,1/2)$ is a fixed number appearing in Lemma 4.19. For simplicity, we just write $X = X^{t,x,y}$ (which is also the process $\bar{X} \triangleq \bar{X}^{t,x,y}$ by Lemma 4.17). We postulate the following estimate on signature moments of $\bar{X}$ and will prove it by induction on n:

$$\Big\| \int_{0<r_1<\cdots<r_n<\rho} \bar{\phi}(\circ d\bar{X}_{tr_1}) \otimes \cdots \otimes \bar{\phi}(\circ d\bar{X}_{tr_n}) \Big\|_p \leqslant \frac{(\Lambda p)^{n/2}}{\sqrt{n!}} e^{K_2/pt} \big(1-(1-\rho)^{\beta}\big)^{n/2} \tag{4.34}$$

for all $p \geqslant 1, n \in \mathbb{N}, t \in (0,1]$, $\rho \in [0,1]$. Here K_2 is the same constant appearing in Part II and Λ is a universal constant that will be determined in the induction argument. Lemma 4.16 follows immediately by taking $\rho = 1$. We set

$$J_n(\rho) \triangleq \int_{0<r_1<\cdots<r_n<\rho} \bar{\phi}(\circ d\bar{X}_{tr_1}) \otimes \cdots \otimes \bar{\phi}(\circ d\bar{X}_{tr_n}).$$

Semi-martingale decomposition:

From the extrinsic perspective in Part I, one can write

$$\begin{aligned}\bar{\phi}(\circ d\bar{X}_{tr}) = \sqrt{t}\bar{\phi}_i(\bar{X}_{tr})\bar{V}^i(\bar{X}_{tr})dB_r^t + \frac{t}{2}\partial_j\bar{\phi}_i(\bar{X}_{tr})\bar{a}^{ij}(\bar{X}_{tr})dr \\ + t\bar{\phi}_i(\bar{X}_{tr})\partial_i \log \bar{p}(t(1-r), \bar{X}_{tr}, y)dr, \quad r \in [0,1],\end{aligned} \tag{4.35}$$

where $B_r^t \triangleq B_{tr}/\sqrt{t}$ and $\bar{a} \triangleq \bar{V} \cdot \bar{V}^T$. Note that $\bar{\phi}, \bar{V}, \bar{a} \in C_c^{\infty}$ by our choice of extension. In what follows, we will use C_i to denote constants depending only on M, ϕ, β and the embedding.

Base step:

By using the BDG inequality, one has

$$\Big\|\sqrt{t}\int_0^{\rho} \bar{\phi}_i(\bar{X}_{tr})\bar{V}^i(\bar{X}_{tr})dB_r^t\Big\|_p \leqslant C_1\sqrt{p} \cdot \sqrt{\rho}. \tag{4.36}$$

It is also obvious that

$$\Big\| \int_0^\rho \frac{t}{2} \partial_j \bar{\phi}_i(\bar{X}_{tr}) \bar{a}^{ij}(\bar{X}_{tr}) dr \Big\|_p \leqslant C_2 \rho. \tag{4.37}$$

For the last term of (4.35), by using the Stroock-Turestsky estimate (4.33) and Lemma 4.19, one has

$$\begin{aligned} &\Big\| \int_0^\rho t \bar{\phi}_i(\bar{X}_{tr}) \partial_i \log \bar{p}(t(1-r), \bar{X}_{tr}, y) dr \Big\|_p \\ &\leqslant C_3 t \int_0^\rho K_4 \Big(\frac{1}{\sqrt{t(1-r)}} + \frac{\|d(\bar{X}_{tr}, y)\|_p}{t(1-r)} \Big) dr \\ &\leqslant C_3 K_4 \int_0^\rho \frac{dr}{\sqrt{1-r}} + C_3 K_4 \int_0^\rho \frac{K_5 \sqrt{p} e^{K_2/pt} (t(1-r))^\beta}{1-r} dr. \end{aligned}$$

After evaluating the above integrals explicitly and combining with (4.36, 4.37), one finds that

$$\|J_1(\rho)\|_p \leqslant C_4 \sqrt{p} e^{K_2/pt} \big(1 - (1-\rho)^\beta\big)^{1/2} \tag{4.38}$$

with suitably chosen constant C_4.

Induction step:

Suppose that the induction hypothesis (4.34) holds. According to the decomposition (4.35), one can write

$$\begin{aligned} J_{n+1}(\rho) &= \int_0^\rho J_n(r) \otimes \bar{\phi}(\circ d\bar{X}_{tr}) \\ &= \sqrt{t} \int_0^\rho J_n(r) \otimes \bar{\phi}_i(\bar{X}_{tr}) \bar{V}^i(\bar{X}_{tr}) dB_r^t + \frac{t}{2} \int_0^\rho J_n(r) \otimes \partial_j \bar{\phi}_i(\bar{X}_{tr}) \bar{a}^{ij}(\bar{X}_{tr}) dr \\ &\quad + \frac{t}{2} \int_0^\rho J_{n-1}(r) \otimes \bar{\phi}_i(\bar{X}_{tr}) \otimes \bar{\phi}_j(\bar{X}_{tr}) \bar{a}^{ij}(\bar{X}_{tr}) dr \\ &\quad + t \int_0^\rho J_n(r) \otimes \bar{\phi}_i(\bar{X}_{tr}) \partial_i \log p(t(1-r), \bar{X}_{tr}, y) dr \\ &=: I_n^1(\rho) + I_n^2(\rho) + I_n^3(\rho) + I_n^4(\rho). \end{aligned}$$

By using the dimension-free BDG inequality (the same way leading to (4.25)), one has

$$\|I_n^1(\rho)\|_p \leqslant C_5 \sqrt{p} \Big(\int_0^\rho \|J_n(r)\|_p^2 dr \Big)^{1/2}.$$

By substituting the estimate (4.34) into the right hand side and evaluating the resulting dr-integral explicitly, one obtains that

$$\|I_n^1(\rho)\|_p \leqslant \frac{C_6\Lambda^{n/2}p^{\frac{n+1}{2}}}{\sqrt{(n+1)!}} \cdot e^{K_2/pt}\big(1-(1-\rho)^\beta\big)^{\frac{n+1}{2}}. \tag{4.39}$$

The estimation of $I_n^2(\rho)$ and $I_n^3(\rho)$ follows the same route and one finds that

$$\|I_n^2(\rho)\|_p \vee \|I_n^3(\rho)\|_p \leqslant \frac{C_7(\Lambda p)^{n/2}}{\sqrt{(n+1)!}} \cdot e^{K_2/pt}\big(1-(1-\rho)^\beta\big)^{\frac{n+1}{2}}. \tag{4.40}$$

We now consider $I_n^4(\rho)$. By using the gradient estimate (4.33), one has

$$\|I_n^4(\rho)\|_p \leqslant C_8\big(\int_0^\rho \frac{1}{\sqrt{1-r}}\|J_n(r)\|_p dr + \int_0^\rho \frac{1}{1-r}\|d(\bar{X}_{tr}, y)\cdot J_n(r)\|_p dr\big). \tag{4.41}$$

After applying the induction hypothesis (4.34), the first integral is majorised as

$$\frac{2}{\beta}\frac{(\Lambda p)^{n/2}}{\sqrt{(n+1)!}} e^{K_2/pt}\big(1-(1-\rho)^\beta\big)^{\frac{n+1}{2}}.$$

For the second integral in (4.41), one applies Hölder's inequality

$$\|d(\bar{X}_{tr}, y)\cdot J_n(r)\|_p \leqslant \|d(\bar{X}_{tr}, y)\|_{p_1}\cdot\|J_n(r)\|_{p_2}$$

with $p_1 \triangleq p(n+1)$ and $p_2 \triangleq p(n+1)/n$. After further applying Lemma 4.19 and the induction hypothesis, it follows that

$$\int_0^\rho \frac{1}{1-r}\|d(\bar{X}_{tr}, y)\cdot J_n(r)\|_p dr \leqslant \frac{2K_5\sqrt{e}}{\beta}\cdot\frac{\Lambda^{n/2}p^{\frac{n+1}{2}}}{\sqrt{(n+1)!}}\cdot e^{K_2/pt}\big(1-(1-\rho)^\beta\big)^{\frac{n+1}{2}}.$$

Therefore, one arrives at

$$\|I_n^4(\rho)\|_p \leqslant \frac{C_9\Lambda^{n/2}p^{\frac{n+1}{2}}}{\sqrt{(n+1)!}} \cdot e^{K_2/pt}\big(1-(1-\rho)^\beta\big)^{\frac{n+1}{2}}. \tag{4.42}$$

Putting the estimates (4.39), (4.40), (4.42) together, one concludes that

$$\|J_{n+1}(\rho)\|_p \leqslant (C_6+2C_7+C_9)\cdot\frac{\Lambda^{n/2}p^{\frac{n+1}{2}}}{\sqrt{(n+1)!}} \cdot e^{K_2/pt}\big(1-(1-\rho)^\beta\big)^{\frac{n+1}{2}}.$$

By taking the base step estimate (4.38) into account, one only needs to choose

$$\Lambda = \max\{C_4^2, C_6 + 2C_7 + C_9\}.$$

It then follows that

$$\|J_1(\rho)\|_p \leqslant (\Lambda p)^{1/2} e^{K_2/pt} \big(1 - (1-\rho)^\beta\big)^{1/2}$$

and

$$\|J_{n+1}(\rho)\|_p \leqslant \frac{(\Lambda p)^{\frac{n+1}{2}}}{\sqrt{(n+1)!}} \cdot e^{K_2/pt} \big(1 - (1-\rho)^\beta\big)^{\frac{n+1}{2}}.$$

This finishes the induction argument.

Proof of Proposition 4.14

We now complete the proof of Proposition 4.14. Let $p, q > 1$ be such that $1/p + 1/q = 1$. According to Lemma 4.15, Lemma 4.16 and Hölder's inequality, one has

$$\begin{aligned} \big\|\mathbb{E}\big[\pi_n S^\phi(X^{t,x,y}); \tau < t\big]\big\|_{\mathrm{HS}} &\leqslant \big\|\pi_n S^\phi(X^{t,x,y})\big\|_p \cdot \mathbb{P}(\tau < t)^{1/q} \\ &\leqslant \frac{(\Lambda p)^{n/2}}{\sqrt{n!}} \cdot e^{L/pt} \cdot e^{-K/qt} = \frac{(\Lambda p)^{n/2}}{\sqrt{n!}} e^{-(K/q - L/p)/t}. \end{aligned}$$

One can choose (and fix) p, q so that $K/q - L/p > 0$. The desired estimate (4.31) thus follows with $C_1 \triangleq \sqrt{\Lambda p}$ and $C_2 \triangleq K/q - L/p$.

4.2.4 Completing the proof of Theorem 4.1

Putting together all previously developed ingredients, we are now in a position to prove Theorem 4.1. We will continue to use the notation introduced earlier.

Let us define

$$b_n(t) \triangleq n! \big\|\mathbb{E}\big[\pi_n S^\phi(X^{t,x,y})\big]\big\|_{\mathrm{HS}}, \ L \triangleq \|\Gamma\|_{\text{1-var};[0,1]}.$$

Recall that in the isometric embedding context (i.e. $\phi = dF$) one has $L = d(x, y)$. Our goal is to estimate $b_n(t)^{1/n} - L$. To this end, one writes

$$b_n(t)^{1/n} - L = L\big(\exp\big(\frac{1}{n}\log\frac{b_n(t)}{L^n}\big) - 1\big). \tag{4.43}$$

The proof of Theorem 4.1 will be completed as long as the following lemma is established.

Lemma 4.20. *There exist positive constants κ, K, R_1, R_2, such that*

$$\frac{b_n(t)}{L^n} \in [R_1, R_2]$$

for all $n \geqslant K$ and $t \in (0, \kappa n^{-6}]$.

Proof. One begins by expressing

$$\frac{b_n(t)}{L^n} = \frac{n!\big\|\pi_n S^{\phi}(\gamma^{x,y})\big\|_{\mathrm{HS}}}{L^n} + \frac{b_n(t) - n!\big\|\pi_n S^{\phi}(\gamma^{x,y})\big\|_{\mathrm{HS}}}{L^n}. \tag{4.44}$$

Since $\gamma^{x,y}$ is smooth, the following asymptotics for the first term is a direct consequence of [HL10, Theorem 8] (see also [CMT23, Theorem 22]):

$$\lim_{n\to\infty} \frac{n!\big\|\pi_n S^{\phi}(\gamma^{x,y})\big\|_{\mathrm{HS}}}{L^n} = c(\Gamma), \tag{4.45}$$

where $c(\Gamma) > 0$ is some explicit constant depending on the geometry of Γ.

Our task is now reduced to estimating the second term on the right hand side of (4.44). Trivially, one has

$$\begin{aligned}\Big|\frac{b_n(t) - n!\big\|\pi_n S^{\phi}(\gamma^{x,y})\big\|_{\mathrm{HS}}}{L^n}\Big| &\leqslant \frac{n!}{L^n}\big\|\mathbb{E}\big[\pi_n S^{\phi}(X^{t,x,y})\big] - \pi_n S^{\phi}(\gamma^{x,y})\big\|_{\mathrm{HS}} \\ &=: A_n(t) + B_n(t),\end{aligned} \tag{4.46}$$

where

$$\begin{aligned}A_n(t) &\triangleq \frac{n!}{L^n}\big\|\mathbb{E}\big[\pi_n S^{\phi}(X^{t,x,y}) - \pi_n S^{\phi}(\gamma^{x,y}); \tau < t\big]\big\|_{\mathrm{HS}}, \\ B_n(t) &\triangleq \frac{n!}{L^n}\big\|\mathbb{E}\big[\pi_n S^{\phi}(X^{t,x,y}) - \pi_n S^{\phi}(\gamma^{x,y}); \tau = t\big]\big\|_{\mathrm{HS}}.\end{aligned}$$

According to Proposition 4.14 and Lemma 4.15, one has

$$\begin{aligned}A_n(t) &\leqslant \frac{n!}{L^n}\big\|\mathbb{E}\big[\pi_n S^{\phi}(X^{t,x,y}); \tau < t\big]\big\|_{\mathrm{HS}} + \frac{n!}{L^n}\big\|\pi_n S^{\phi}(\gamma^{x,y})\big\|_{\mathrm{HS}} \cdot \mathbb{P}(\tau < t) \\ &\leqslant \frac{n!}{L^n} \cdot \frac{C_1^n}{\sqrt{n!}} e^{-C_2/t} + \frac{n!}{L^n} \cdot \frac{L^n}{n!} \cdot e^{-C_3/t} \\ &\leqslant \big(\sqrt{n!}(C_1/L)^n + 1\big) e^{-C_4/t}\end{aligned} \tag{4.47}$$

for all $n \geqslant 1$ and $t \in (0, \delta)$ with some positive constant δ.

To estimate $B_n(t)$, we first recall from (4.20) that on the event $\{\tau = t\}$ one has the decomposition

$$\pi_n S^\phi(X^{t,x,y}) = \pi_n S^\phi(\gamma^{x,y}) + \sum_{k=0}^{n-1} \sum_{I \in \mathcal{I}_n(k)} t^{\frac{n-k}{2}} J_n(1; I),$$

where $\mathcal{I}_n(k)$ denotes the set of length-n words I with $|I|_0 = k$ and $J_n(1; I)$ is defined by (4.21). According to Proposition 4.10 and (4.20), one finds that

$$\begin{aligned} B_n(t) &\leqslant \frac{n!}{L^n} \sum_{k=0}^{n-1} \sum_{I \in \mathcal{I}_n(k)} t^{\frac{n-k}{2}} \big\| J_n(1; I) \big\|_2 \\ &\leqslant \frac{n!}{L^n} \sum_{k=0}^{n-1} \sum_{I \in \mathcal{I}_n(k)} \frac{(2\Lambda)^{n-k}}{\sqrt{(n-k)!}} \cdot \frac{L^k}{k!} \cdot t^{\frac{n-k}{2}} = n! \sum_{k=0}^{n-1} \binom{n}{k} \frac{1}{\sqrt{(n-k)!} k!} \big(\frac{2\Lambda\sqrt{t}}{L}\big)^{n-k}. \end{aligned}$$

To analyse the last expression, let us define

$$F_n(x) \triangleq n! \sum_{k=0}^{n-1} \binom{n}{k} \frac{x^{n-k}}{\sqrt{(n-k)!} k!} = \sum_{r=1}^{n} \binom{n}{r} \frac{n! x^r}{(n-r)! \sqrt{r!}}$$

where one takes $x = 2\Lambda\sqrt{t}/L$ in our context. Setting

$$a_r \triangleq \binom{n}{r} \frac{n! x^r}{(n-r)! \sqrt{r!}},$$

one easily sees that

$$\frac{a_{r+1}}{a_r} = x \cdot \frac{(n-r)^2}{(r+1)^{3/2}} \leqslant n^2 x$$

for all $r = 1, \cdots, n-1$. By requiring

$$n^2 x = \frac{2\Lambda n^2 \sqrt{t}}{L} \leqslant 1 \iff t \leqslant \big(\frac{L}{2\Lambda}\big)^2 n^{-4}, \tag{4.48}$$

one concludes that $\{a_r\}_{r=1}^n$ is a decreasing sequence, thus yielding that

$$F_n(x) \leqslant n a_1 = n^3 x = \frac{2\Lambda n^3 \sqrt{t}}{L}.$$

Therefore, one has

$$B_n(t) \leqslant \frac{2\Lambda n^3 \sqrt{t}}{L} \tag{4.49}$$

for all n and t satisfying the relation (4.48).

By substituting the estimates (4.47) and (4.49) into (4.46), one obtains that

$$\Big|\frac{b_n(t)-n!\big\|\pi_n S^{\phi}(\gamma^{x,y})\big\|_{\mathrm{HS}}}{L^n}\Big|\leqslant\big(\sqrt{n!}(C_1/L)^n+1\big)e^{-C_4/t}+\frac{2\Lambda n^3\sqrt{t}}{L}$$

for all n,t provided that (4.48) holds. Note that the right hand side is of constant magnitude if $t=O(n^{-6})$. More specifically, there exist positive constants κ,K such that

$$\Big|\frac{b_n(t)-n!\big\|\pi_n S^{\phi}(\gamma^{x,y})\big\|_{\mathrm{HS}}}{L^n}\Big|\leqslant\frac{1}{2}c(\Gamma) \tag{4.50}$$

for all $n\geqslant K$ and $t\in(0,\kappa n^{-6}]$ (the relation (4.48) is also satisfied with such choice of t). Here $c(\Gamma)$ is the Hambly-Lyons limit appearing in (4.45). Now the conclusion of the lemma follows from the decomposition (4.44), the asymptotics (4.45) and the inequality (4.50) (with enlarged K if necessary). □

Proof of Theorem 4.1. Under the situation of Lemma 4.20, one has

$$\Big|\log\frac{b_n(t)}{L^n}\Big|\leqslant\max\big\{|\log R_1|,|\log R_2|\big\}.$$

for all n,t such that $n\geqslant K,t\leqslant\kappa n^{-6}$. It follows from the relation (4.43) that

$$\big|b_n(t)^{1/n}-L\big|=L\Big|\exp\Big(\frac{1}{n}\log\frac{b_n(t)}{L^n}\Big)-1\Big|\leqslant\frac{C}{n}$$

with a suitable constant C. The proof of Theorem 4.1 is now complete.

□

5 Small-time expansion and its connection with curvature properties

Theorem 4.1 shows that the Riemannian distance can be recovered from suitable signature asymptotics of the Brownian bridge. In this section, we are going to show that curvature properties of the manifold M (both intrinsic and extrinsic) can also be recovered explicitly from the small-time expansion of the signature.

5.1 The main theorem

From now on, we consider the Brownian loop based at $x \in M$ (i.e. the Brownian bridge $X^{t,x,x}$). Recall that an isometric embedding $F : M \to E = \mathbb{R}^N$ is given fixed. For each $n \geqslant 1$, we define (with $\phi = dF$ here)

$$\psi_n(t,x) \triangleq \mathbb{E}\big[\pi_n S^{\phi}(X^{t,x,x})\big] \in E^{\otimes n}, \quad t > 0, x \in M.$$

This is the n-th level expected ϕ-signature of the Brownian loop $X^{t,x,x}$.

Let us denote $S_n \triangleq \pi_n S^{\phi}(X^{t,x,x})$. Since $\int_0^s \phi(dX^{t,x,x}) = F(X_s^{t,x,x}) - F(x)$, it is clear that

$$S_1 = F(X_t^{t,x,x}) - F(x) = F(x) - F(x) = 0 \implies \psi_1(t,x) = 0.$$

From the symmetry of the heat kernel, it is easily seen that $X^{t,x,x}$ has the same law as its time reversal $s \mapsto \overleftarrow{X}_s^{t,x,x} \triangleq X_{t-s}^{t,x,x}$. As a result, the paths $\Gamma_\cdot \triangleq F(X_\cdot^{t,x,x}) - F(x)$ and its reversal $\overleftarrow{\Gamma}$ have the same law. Since the signature of $\overleftarrow{\Gamma}$ is the inverse of the signature of Γ (in the tensor algebra $T((E))$), it follows that

$$S^{\phi}\big(X^{t,x,x}\big)^{-1} \stackrel{\text{law}}{=} S^{\phi}(X^{t,x,x}). \tag{5.1}$$

By taking its second level projection, one has

$$-S_2 + S_1 \otimes S_1 \stackrel{\text{law}}{=} S_2.$$

But one already knows $S_1 = 0$. Therefore, $\psi_2(t,x) = \mathbb{E}[S_2] = 0$. Similarly, the third level projection of (5.1) gives

$$-S_3 = -S_3 + S_1 \otimes S_2 + S_2 \otimes S_1 - S_1^{\otimes 3} \stackrel{\text{law}}{=} S_3 \implies \psi_3(t,x) = \mathbb{E}[S_3] = 0.$$

The above calculation suggests that, in order to extract nontrivial geometric information, one has to at least consider $\psi_4(t,x)$. A similar calculation shows that

$$\psi_4(t,x) = \mathbb{E}[S_4] = \frac{1}{2}\mathbb{E}[S_2 \otimes S_2]. \tag{5.2}$$

This needs not be a zero tensor; indeed it contains the second moment of Lévy areas of the path Γ.

In this section, we study the small-time expansion of the function $\psi_4(t,x)$. We first define a contraction on 4-tensors to reduce dimension. Let $\xi \in E^{\otimes 4}$ be a given tensor. Being viewed as a 4-linear map $\xi : E \times E \times E \times E \to \mathbb{R}$, we define $\mathfrak{C}\xi : E \times E \to \mathbb{R}$ to be the 2-tensor obtained from ξ by taking trace on the $(2,4)$-positions. More precisely,

$$(\mathfrak{C}\xi)(\cdot,\cdot) \triangleq \sum_{\alpha=1} \xi(\cdot, \varepsilon_\alpha, \cdot, \varepsilon_\alpha), \tag{5.3}$$

where $\{\varepsilon_\alpha : \alpha = 1, \cdots, N\}$ is any ONB of E. One also needs to recall the basic curvature quantities introduced in Section 2.1.3. Our main theorem for this part is stated as follows.

Theorem 5.1. *One has the following small-time expansion:*

$$\mathfrak{C}\psi_4(t,x) = \Theta_x \cdot t^2 + \Xi_x \cdot t^3 + O(t^4) \quad \text{as } t \to 0^+.$$

In addition, the coefficients $\Theta_x, \Xi_x \in E^{\otimes 2}$ are symmetric 2-tensors that admit the following explicit representation.

(i) The tensor Θ_x recovers the Riemannian metric tensor g_x. More precisely, one has

$$\Theta_x(v,w) = \frac{d-1}{24} \langle \pi_x v, \pi_x w \rangle_{T_x M}, \quad v, w \in E, \tag{5.4}$$

where $\pi_x : E \to T_x M$ denotes the orthogonal projection onto the tangent space $T_x M$.

(ii) The tensor Ξ_x encodes both intrinsic (Ricci curvature) and extrinsic (second fundamental form) curvature properties of M at x. More precisely, its restriction to $T_x M \times T_x M$ as a symmetric bilinear form on $T_x M$ is explicitly given by

$$\Xi_x|_{T_x M \times T_x M} = \frac{\mathrm{S}_x - 18d^2 |H_x|_E^2}{8640} g_x + \frac{49d - 20}{8640} \mathrm{Ric}_x + \frac{(5-4d)d}{480} \langle B_x, H_x \rangle_E. \tag{5.5}$$

Here g_x is the metric tensor, $\mathrm{Ric}_x, \mathrm{S}_x$ are the Ricci tensor and scalar curvature, B_x, H_x are the second fundamental form and the mean curvature vector, and $\langle \cdot, \cdot \rangle_E$ is the Euclidean inner product in E.

Remark 5.2. It can be shown that the 4-tensor $\psi_4(t,x)$ is anti-symmetric on the first-two components and also on the last-two components. Therefore, the operation (5.3) is basically the only nontrivial way of contraction into a 2-tensor.

To prove Theorem 5.1, we will actually compute the third-order expansion of the 4-tensor

$$\psi_4(t,x) = \hat{\Theta}_x t^2 + \hat{\Xi}_x t^3 + O(t^4) \tag{5.6}$$

with explicit tensor coefficients $\hat{\Theta}_x, \hat{\Xi}_x$ (cf. Propositions 5.9, 5.16 below). By some elementary algebraic manipulation, it is possible to recover all the quantities $\mathrm{Ric}_x, \mathrm{S}_x, \langle B_x, H_x\rangle_E$ and $|H_x|_E$ in (5.5) separately from the coefficients $\hat{\Theta}_x, \hat{\Xi}_x$ (cf. Remark 5.23 for a discussion). But the corresponding formulae are more involved. For simplicity, we choose to present the formula in the form of (5.5) through the contraction $\mathfrak{C}\psi_4(t,x)$.

In the following sections, we develop the proof of Theorem 5.1. Our strategy follows the same spirit as the proof of Theorem 4.1; we perform local calculations in the normal chart around x based on the local SDE representation of $X^{t,x,x}$. The actual calculation turns out to be so involved that it is impossible to achieve entirely by hand. Therefore, our full calculation is computer-assisted (by Wolfram Mathematica). We will develop the explicit analysis for a few representative cases in the computation of the coefficients $\hat{\Theta}_x, \hat{\Xi}_x$ and leave all other parallel cases to the appendix with documented Mathematica codes provided in GitHub.

5.2 Some basic expansions

As in the proof of Theorem 4.1, we will localise the analysis in a normal chart V around x (cf. Section 4.2.1 for the construction with $x = y$ in the current situation). As we have seen in Section 4.2.3, the case that $X^{t,x,x}$ leaves the chart V before its lifetime gives a negligible contribution (it is of order $e^{-C/t}$ which is smaller than any power of t; cf. Proposition 4.14). Therefore, one only needs consider the situation that $X^{t,x,x}$ remains in the chart V for all time (and we will always assume this is the case). In this case, the analysis reduces to Euclidean stochastic calculus.

We will use the relation (5.2) to compute $\psi_4(t,x)$. Namely, we write

$$\psi_4(t,x) = \frac{1}{2}\mathbb{E}[\Pi_t^x \otimes \Pi_t^x], \tag{5.7}$$

where

$$\begin{aligned}\Pi_t^x &\triangleq \int_0^t \Big(\int_0^v \phi(\circ dX_u^{t,x,x})\Big) \otimes \circ\phi\big(\circ\, dX_v^{t,x,x}\big) \\ &= \int_0^1 \Big(\int_0^\rho \phi(\circ dX_{tr}^{t,x,x})\Big) \otimes \circ\phi(\circ dX_{t\rho}^{t,x,x}),\end{aligned}$$

and we made the change of variables $v = t\rho, u = tr$ to normalise the processes on the unit interval. To evaluate the expectation (5.7) (in particular, its expansion in t), one needs to compute the semimartingale decomposition of $\phi(\circ dX_{tr}^{t,x,x})$ in the chart V.

5.2.1 SDE representation of $X^{t,x,x}$

Recall that the SDE representation of $X^{t,x,x}$ in V (in terms of coordinates in $U \triangleq \exp_x^{-1} V$) is given by (4.12) with $Q(u, \mathbf{z})$ defined in terms of the Malliavin-Stroock expansion (2.3) (cf. (4.11)). For our purpose of computing small-time expansion, one needs to extract one more term from $Q(u, \mathbf{z})$, i.e. by writing

$$\nabla_z \log p(u, \mathbf{z}, 0) = -\frac{\mathbf{z}}{u} + \nabla_z G_1(\mathbf{z}, 0) + \bar{Q}(u, \mathbf{z}), \tag{5.8}$$

where $\bar{Q}(u, \mathbf{z}) \triangleq \sum_{k=1}^{\infty} \nabla_z G_{k+1}(\mathbf{z}, \mathbf{0}) u^k$ satisfies

$$\left|\bar{Q}(u, \mathbf{z})\right| \leqslant Cu \quad \forall u \in [0,1], \mathbf{z} \in \bar{U} \tag{5.9}$$

with some geometric constant C depending on the localisation V. Correspondingly, the SDE (4.12) for $X^{t,x,x}$ is rewritten as

$$dX_s = \big(-\frac{X_s}{t-s} + \bar{b}(X_s) + \bar{Q}(t-s, X_s)\big) ds + \sigma(X_s) dB_s, \tag{5.10}$$

where

$$\bar{b}(\mathbf{z}) \triangleq b(\mathbf{z}) + \nabla_z G_1(\mathbf{z}, \mathbf{0}). \tag{5.11}$$

5.2.2 Decomposition of Π_t

By using the SDE (5.10), one can easily compute the semimartingale decomposition of $\phi(\circ dX^{t,x,x})$ and thus an associated decomposition of Π_t. This is summarised in the lemma below. Recall that $\phi = \phi_i dx^i$ and $a = \sigma\sigma^T$.

Lemma 5.3. *Under the chart V, one has*

$$\phi(\circ dX_{tr}) = dI_{tr} + dJ_{tr} + dK_{tr} + dL_{tr},$$

where

$$dI_{tr} \triangleq -\phi_i(X_{tr}) \frac{X_{tr}^i}{1-r} dr, \ dJ_{tr} \triangleq t\varphi(X_{tr}) dr, \tag{5.12}$$

$$dK_{tr} \triangleq \sqrt{t} \phi_i(X_{tr}) \sigma_\alpha^i(X_{tr}) dB_r^{t,\alpha}, \ dL_{tr} \triangleq t\bar{Q}(t(1-r), X_{tr}) dr$$

with

$$\varphi(\mathbf{z}) \triangleq \phi_i(\mathbf{z}) \bar{b}^i(\mathbf{z}) + \frac{1}{2} a^{ij}(\mathbf{z}) \partial_j \phi_i(\mathbf{z}) \tag{5.13}$$

and $B_r^t \triangleq B_{tr}/\sqrt{t}$ being the rescaled Brownian motion. In addition,

$$\Pi_t^x = \int_0^1 \Big(\int_0^\rho dI_{tr} + dJ_{tr} + dK_{tr} + dL_{tr} \Big) \otimes \cdot \big(dI_{t\rho} + dJ_{t\rho} + dK_{t\rho} + dL_{t\rho} \big) + \frac{t}{2} \int_0^1 \phi_i(X_{t\rho}) \otimes \phi_j(X_{t\rho}) a^{ij}(X_{t\rho}) d\rho, \tag{5.14}$$

where the "$\cdot$" in the above double integral means Itô's integral (of course, only the dK-term is relevant).

Proof. This is explicit Itô's calculus based on the semimartingale decomposition (5.10) together with the change of variables $v = t\rho, u = tr$. □

5.2.3 Expansions of $\frac{X_{tr}}{1-r}$ and $f(X_{tr})$

In order to compute $\mathbb{E}[\Pi_t^x \otimes \Pi_t^x]$, in views of the decomposition (5.14) one needs to know suitable expansions of $\frac{X_{tr}}{1-r}$ and $f(X_{tr})$ with respect to t. These are summarised in the lemma below.

Lemma 5.4. *Under the notation in Lemma 5.3, let us define*

$$G_r^t \triangleq \sqrt{t} \int_0^r \frac{1}{1-\eta} \sigma(X_{t\eta}) dB_\eta^t, \ \ H_r^t \triangleq t \int_0^r \frac{1}{1-\eta} \bar{b}(X_{t\eta}) d\eta.$$

Then one has

$$\frac{X_{tr}}{1-r} = G_r^t + H_r^t + \mathcal{E}^t(r) \tag{5.15}$$

and for any $f \in C_b^\infty(\bar{U})$ one has

$$f(X_{tr}) = f(\mathbf{0}) + \partial_k f(\mathbf{0}) \cdot (1-r) \big(G_r^{t,k} + H_r^{t,k} \big) \\ \frac{1}{2} \partial_{kl}^2 f(\mathbf{0}) \cdot (1-r)^2 G_r^{t,k} G_r^{l,k} + \mathcal{E}_f^t(r). \tag{5.16}$$

Here the remainders $\mathcal{E}^t(r), \mathcal{E}_f^t(r)$ satisfy the following estimates:

$$\sup_{0 \leqslant r \leqslant 1} \|\mathcal{E}^t(r)\|_{L^p} \leqslant Ct^2, \ \ \sup_{0 \leqslant r \leqslant 1} \|\mathcal{E}_f^t(r)\|_{L^p} \leqslant Ct^{3/2} \tag{5.17}$$

for all $p \geqslant 1$, $t \in (0,1]$. The above constant C depends on p, f and the localisation V.

Proof. The first expansion (5.15) follows easily from Lemma 4.8; in fact, one explicitly has

$$\mathcal{E}^t(r) = t\int_0^r \frac{1}{1-\eta}\bar{Q}(t(1-\eta), X_{t\eta})d\eta,$$

where we recall that $\bar{Q}$ is defined through (5.8). The first estimate in (5.17) follows immediately from (5.9). The second expansion (5.16) follows from the standard Taylor approximation of f, where

$$\begin{aligned}\big|\mathcal{E}_f^t(r)\big| \leqslant C\|f\|_{C_b^3(\bar{U})}\big[|\mathcal{E}^t(r)| + (1-r)^2\big(|G_r^t| + |H_r^t|\big)\cdot|H_r^t| \\ + (1-r)^3\big(|G_r^t| + |H_r^t| + |\mathcal{E}^t(r)|\big)\big].\end{aligned}$$

Note that

$$\begin{aligned}(1-r)\|G_r^t\|_{L^p} &= (1-r)\Big\|\sqrt{t}\int_0^r \frac{1}{1-\eta}\sigma(X_{t\eta})dB_\eta^t\Big\|_{L^p} \\ &\leqslant C_p\sqrt{t}(1-r)\sqrt{\int_0^r \frac{d\eta}{(1-\eta)^2}} \leqslant C_p'\sqrt{t} \qquad (5.18)\end{aligned}$$

and

$$(1-r)|H_r^t| = (1-r)\Big|t\int_0^r \frac{1}{1-\eta}\bar{b}(X_{t\eta})d\eta\Big| \leqslant Ct(1-r)|\log(1-r)| \leqslant C't.$$

The second estimate in (5.17) thus follows. □

Remark 5.5. The point of Lemma 5.4 is to collect those specific terms in the expansions of $\frac{X_{tr}}{1-r}$ and $f(X_{tr})$ that have order $\leqslant t$ (this is enough for our purpose of computing the expansion of $\psi_4(t,x)$ up to order t^3). Note that the term G_r^t has order $\sqrt{t}$ and the term H_r^t has order t. Both $\mathcal{E}^t(r), \mathcal{E}_f^t(r)$ are remainders that will be ignored in the asymptotic evaluation. We also remark that although G_r^t and H_r^t contain singularities as $r \nearrow 1$, they will always be eliminated after performing integration in (5.7).

5.2.4 Summary of structure and order-counting

The computation of the coefficients $\hat{\Theta}_x, \hat{\Xi}_x$ in (5.6) essentially boils down to selecting different terms in the decomposition (5.14) with specific orders in t. For this purpose, it is useful to recapture the basic structure we already derived. First of all, the decomposition of $\Pi_t^x \otimes \Pi_t^x$ has the structure

$$\begin{aligned}\Pi \otimes \Pi =& \Big[\int \Big(\int dI + dJ + dK + dL\Big) \otimes (dI + dJ + dK + dL) + P\Big] \\ & \otimes \Big[\int \Big(\int dI + dJ + dK + dL\Big) \otimes (dI + dJ + dK + dL) + P\Big],\end{aligned} \tag{5.19}$$

where I, J, K, L are as in Lemma 5.3 and

$$P = P_t \triangleq \frac{t}{2}\int_0^1 \phi_i(X_{t\rho}) \otimes \phi_j(X_{t\rho}) a^{ij}(X_{t\rho}) d\rho$$

is the last term (the Itô-Stratonovich correction) in (5.14).

The key observation is that the individual terms I, J, K, L, P have the following t-orders respectively:

$$I \sim \sqrt{t},\ J \sim t,\ K \sim \sqrt{t},\ L \sim t^2,\ P \sim t. \tag{5.20}$$

This is clear from Lemma 5.3 and Lemma 5.4. As a result, in order to compute $\hat{\Theta}_x$ and $\hat{\Xi}_x$ one only needs to collect and combine terms in the expansion of (5.19) that have total orders of t^2 and t^3 respectively. The principle here is easy and mechanical, but the computation is extremely huge (we have to rely on computer assistance for computing $\hat{\Xi}_x$).

5.3 Computation of the t^2-coefficient $\hat{\Theta}_x$

In view of (5.19) and (5.20), the t^2-term in the expansion of $\psi_4(t, x)$ precisely comes from the following 25 combinations:

$$\begin{gathered}(II;II),(II;IK),(II;KI),(II;KK),\\ (IK;II),(IK;IK),(IK;KI),(IK;KK)\\ (KI;II),(KI;IK),(KI;KI),(KI;KK),\\ (KK;II),(KK;IK),(KK;KI),(KK;KK)\\ (II;P),(IK;P),(KI;P),(KK;P)\\ (P;II),(P;IK),(P;KI),(P;KK),(P;P).\end{gathered} \tag{5.21}$$

Here e.g. $(IK; KI)$ means picking $dI \otimes dK$ in the first line of (5.19) (i.e. the first Π in $\Pi \otimes \Pi$) and $dK \otimes dI$ in the second line of (5.19) (i.e. the second Π in $\Pi \otimes \Pi$); this produces a term $dI \otimes dK \otimes dK \otimes dI$ of order $\sqrt{t}^4 = t^2$. Similarly, $(P; II)$ means picking P in the first Π and $dI \otimes dI$ in the second Π; this produces a term of order $t \times \sqrt{t}^2 = t^2$. The $(P; P)$ means picking P in both Π's which produces a term of order $t \times t = t^2$.

Notation 5.6. In what follows, we write $X_t \overset{2}{=} Y_t$ for random variables X_t, Y_t if $\mathbb{E}[X_t - Y_t] = o(t^2)$ as $t \to 0^+$. In fact, in our problem it will always be the case that $|\mathbb{E}[X_t - Y_t]| \leqslant Ct^3$ (one does not see the $t^{5/2}$-term because the expectation of an Itô integral is zero) and $\|X_t - Y_t\|_{L^p} \leqslant C_p t^{5/2}$. This notation is used to only keep track of terms up to the desired order t^2 and ignore all higher order terms in the expansion.

5.3.1 The $(II;II)$ term

Here we demonstrate the computation of the $(II;II)$-term in detail. According to Lemma 5.3 (the expression of dI_{tr}), the t^2-coefficient of this particular term comes from the expectation of

$$\begin{aligned} A_{II;II} \triangleq & \int_0^1 \Big(\int_0^\rho \phi_i(X_{tr}) \frac{X_{tr}^i}{1-r} dr \Big) \otimes \phi_j(X_{t\rho}) \frac{X_{t\rho}^j}{1-\rho} d\rho \\ & \otimes \int_0^1 \Big(\int_0^\theta \phi_k(X_{t\delta}) \frac{X_{t\delta}^k}{1-\delta} d\delta \Big) \otimes \phi_l(W_{t\theta}) \frac{X_{t\theta}^l}{1-\theta} d\theta. \end{aligned} \tag{5.22}$$

Since we are extracting terms of order t^2, in the expansions (5.15, 5.16) for the ϕ's and X's the only possibility is freezing all the ϕ's at the origin and taking the G^t-term for the X's (recall from (5.18) and Remark 5.5 that G^t is of order $\sqrt{t}$). Namely, using Notation 5.6 one has

$$\begin{aligned} A_{II;II} \overset{2}{=} & \phi_i(\mathbf{0}) \otimes \phi_j(\mathbf{0}) \otimes \phi_k(\mathbf{0}) \otimes \phi_l(\mathbf{0}) \\ & \times \int_0^1 \Big(\int_0^\rho G_r^{t,i} dr \Big) G_\rho^{t,j} d\rho \times \int_0^1 \Big(\int_0^\theta G_\delta^{t,k} d\delta \Big) G_\theta^{t,l} d\theta \\ = & \phi_i(\mathbf{0}) \otimes \phi_j(\mathbf{0}) \otimes \phi_k(\mathbf{0}) \otimes \phi_l(\mathbf{0}) \\ & \times \int_{\substack{0<r<\rho<1 \\ 0<\delta<\theta<1}} G_r^{t,i} G_\rho^{t,j} G_\delta^{t,k} G_\theta^{t,l} dr d\rho d\delta d\theta. \end{aligned} \tag{5.23}$$

To compute the t^2-coefficient of $\mathbb{E}[A_{II;II}]$, one has to evaluate $\mathbb{E}\big[G_r^{t,i} G_\rho^{t,j} G_\delta^{t,k} G_\theta^{t,l}\big]$. This is contained in the lemma below.

Lemma 5.7. *Consider the function*

$$F(a,b,c,d;i,j,k,l) \triangleq \mathbb{E}\big[G_a^{t,i} G_b^{t,j} G_c^{t,k} G_d^{t,l}\big]$$

where $0 \leqslant a \leqslant b \leqslant c \leqslant d \leqslant 1$ *and* i, j, k, l *are arbitrary coordinate indices. Then one has*

$$F(a,b,c,d;i,j,k,l) = t^2\Big[\frac{ab}{(1-a)(1-b)}\big(\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}\big) + \frac{a}{1-a}\frac{c-b}{(1-b)(1-c)}\delta^{ij}\delta^{kl}\Big] + o(t^2) \quad \text{as } t \to 0^+. \tag{5.24}$$

Proof. Since we are only considering the t^2-coefficient and $\sigma(\mathbf{0}) = \mathrm{Id}$ (cf. (4.5) and also note that $a = \sigma^T\sigma$), one has

$$G_a^{t,i}G_b^{t,j}G_c^{t,k}G_d^{t,l} \overset{2}{=} \sqrt{t}^4\Big(\int_0^a \frac{1}{1-\eta}dB_\eta^{t,i}\Big)\Big(\int_0^b \frac{1}{1-\eta}dB_\eta^{t,j}\Big) \times \Big(\int_0^c \frac{1}{1-\eta}dB_\eta^{t,k}\Big)\Big(\int_0^d \frac{1}{1-\eta}dB_\eta^{t,l}\Big).$$

The right hand side is just the product of four Gaussian variables whose covariance function is easily calculated from the relation that

$$\mathbb{E}\Big[\Big(\int_0^\rho \frac{1}{1-\eta}dB_\eta^{t,p}\Big)\Big(\int_0^\theta \frac{1}{1-\eta}dB_\eta^{t,q}\Big)\Big] = \delta^{pq}\int_0^{\rho\wedge\theta}\frac{d\eta}{(1-\eta)^2} = \frac{\rho\wedge\theta}{1-\rho\wedge\theta}\delta^{pq}.$$

The result (5.24) follows immediately from the standard formula for 4-th moments of Gaussian vectors. □

Lemma 5.8. *One has*

$$\mathbb{E}[A_{II;II}] = t^2\phi_i(\mathbf{0})\otimes\phi_j(\mathbf{0})\otimes\phi_k(\mathbf{0})\otimes\phi_l(\mathbf{0}) \times \Big(\frac{1}{4}\delta^{ij}\delta^{kl}+\frac{1}{3}\delta^{ik}\delta^{jl}+\frac{1}{6}\delta^{il}\delta^{jk}\Big) + o(t^2) \quad \text{as } t \to 0^+. \tag{5.25}$$

Proof. This is obtained by further decomposing the integral (5.23) according to the actual orderings of (r,ρ,δ,θ) and applying Lemma 5.7 to each scenario. For instance, the integral over the region $r < \delta < \theta < \rho$ gives

$$\begin{aligned}
&\int_{0<r<\delta<\theta<\rho<1} F(r,\delta,\theta,\rho;i,k,l,j) \\
&= t^2\int_{0<r<\delta<\theta<\rho<1}\Big(\frac{r\delta}{(1-r)(1-\delta)}\epsilon^{ijkl} + \frac{r}{1-r}\frac{\theta-\delta}{(1-\delta)(1-\theta)}\delta^{ik}\delta^{jl}\Big) + o(t^2) \\
&= t^2\Big(\frac{1}{36}\epsilon^{ijkl} + \frac{1}{24}\delta^{ik}\delta^{jl}\Big) + o(t^2),
\end{aligned} \tag{5.26}$$

where $\epsilon^{ijkl} \triangleq \delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}$. Other scenarios corresponding to different orderings of (r,ρ,δ,θ) are obtained from (5.26) by symmetry. Combining and simplifying all these expressions gives the relation (5.25). □

5.3.2 Final result

All the remaining 24 cases in (5.21) are treated by very similar kind of calculations (elementary Itô-calculus). We will not present the separate results here and leave them to Appendix A. The final result is summarised as follows. To ease notation, we denote

$$\phi_{ijkl} \triangleq \phi_i(\mathbf{0}) \otimes \phi_j(\mathbf{0}) \otimes \phi_k(\mathbf{0}) \otimes \phi_l(\mathbf{0}). \tag{5.27}$$

Proposition 5.9. *The t^2-coefficient $\hat{\Theta}_x$ in the expansion of (5.6) is given by*

$$\hat{\Theta}_x = \frac{1}{24}(\phi_{ijij} - \phi_{ijji}) \in E^{\otimes 4}.$$

Proof. By summing up the expressions (A.3–A.16) in Appendix A as well as (5.25), one finds that

$$\mathbb{E}[\Pi_t^x \otimes \Pi_t^x] = \frac{t^2}{12}(\phi_{ijij} - \phi_{ijji}) + o(t^2). \tag{5.28}$$

The result thus follows from the relation (5.7). □

Remark 5.10. Although $\hat{\Theta}_x$ is computed in terms of local coordinates, it is apparently an intrinsic quantity (since $\psi_4(t,x)$ is). Its intrinsic meaning is described as follows. Since ϕ is an E-valued one-form on M, $\phi(x) \otimes \phi(x) \otimes \phi(x) \otimes \phi(x)$ can be viewed as an $E^{\otimes 4}$-valued 4-tensor on T_xM. Consider the $E^{\otimes 4}$-valued bilinear form on $(T_xM)^{\otimes 2}$ defined by

$$\begin{aligned} \mathcal{T}_x :& (T_xM)^{\otimes 2} \times (T_xM)^{\otimes 2} \to E^{\otimes 4} \\ & (u \otimes v, w \otimes z) \mapsto \langle \phi(x), u \rangle \otimes \langle \phi(x), v \rangle \otimes [\langle \phi(x), w \rangle, \langle \phi(x), z \rangle], \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the cotangent-tangent pairing and $[a,b] \triangleq a \otimes b - b \times a$ for $a, b \in E$. Then $\hat{\Theta}_x$ is the trace of $\mathcal{T}_x$ with respect to the Hilbert-Schmidt structure on $(T_xM)^{\otimes 2}$ induced from the Riemannian metric on T_xM. The geometric significance of $\hat{\Theta}_x$ will become clearer in the case when $\phi = dF$ and after applying the contraction operator $\mathfrak{C}$ defined by (5.3) (in fact, $\hat{\Theta}_x$ encodes the entire Riemannian metric tensor at x; cf. equation (5.4) and Remark 5.18 below).

5.4 Computation of the t^3-coefficient $\hat{\Xi}_x$

From Proposition 5.9 and Remark 5.10, the 4-tensor $\hat{\Theta}_x$ encodes information about the metric tensor at the location x (the first fundamental form at x). However, it does not encode any curvature properties as the expression does not involve differentiation of the metric tensor. To recover curvature quantities, one has to look at higher order terms in the expansion.

The basic principle of calculating higher order coefficients in the t-expansion of $\psi_4(t,x)$ is the same as the computation of $\hat{\Theta}_x$, which is in turn based on the decomposition (5.19) and order counting. However, the calculation becomes much lengthier and at some point unmanageable by hand (we thus have to rely on computer assistance). It is quite straight forward that the $t^{5/2}$-coefficient of $\mathbb{E}[\Pi_t^x \otimes \Pi_t^x]$ is exactly zero for the obvious reason that the expected value of an Itô integral is zero. Therefore, our next task is to compute the t^3-coefficient $\hat{\Xi}_x$ of the expansion of $\psi_4(t,x)$. As we will see, $\hat{\Xi}_x$ encodes an interesting amount of explicit information about both intrinsic (Ricci) and extrinsic (second fundamental form) curvature properties at x.

In the decomposition (5.19) of $\Pi_t^x \otimes \Pi_t^x$, we recall that the (lowest) t-orders of the terms I, J, K, L, P are given by (5.20) respectively. Accordingly, we just denote their degrees to be

$$\deg I = \deg K = 0.5, \ \deg J = \deg P = 1, \ \deg L = 2.$$

Note that each of them also contain higher order terms; for instance one can further expand $\phi_i(X_{tr})$ in I (cf. (5.12)) by using (5.16) to get terms in I that have orders $t, t^{3/2}, \cdots$. When one expands the product from (5.19), in order to extract the t^3-term in the asymptotic expansion there are three main scenarios to consider:

1. <u>Combinations with total degree 3</u>. In this case, one extracts leading coefficients and there is no need to further expand any of the individual terms. For instance, consider the combination $JI; P$. This means picking $dJ \otimes dI$ in the first line of (5.19) and P in the second. The resulting total degree is $1 + 0.5 + 1 = 3$. Therefore, one only needs to compute the leading coefficient in the corresponding expectation without further expanding any of the I, J, K's.

2. <u>Combinations with total degree 2.5</u>. In this case, one needs to further extract the $\sqrt{t}$-term in the resulting product (because an extra order of $\sqrt{t}$ is needed to yield a total order of t^3). For instance, consider the combination $JI; IK$ whose total degree is 2.5. One then needs to compute its $\sqrt{t}$-order term which can come from a further $\sqrt{t}$-expansion from any of the I, J, K's; e.g. taking the $\sqrt{t}$-order term

$$\partial_k \varphi(\mathbf{0}) \cdot (1-r)\big(G_r^{t,k} + H_r^{t,k}\big)$$

in the expansion of $\varphi(X_{tr})$ in J (cf. (5.12), (5.13) and (5.16)) will yield such a term and there are several other possibilities.

3. Combinations with total degree 2. In this case, one needs to further extract the t-term in the resulting product. For instance, consider the combination $II;IK$ whose total degree is 2. One then needs to compute its t-order term which can come from many possibilities, e.g. by expanding the first I to order $t^{3/2}$ (note that I itself already has order $\sqrt{t}$) or by expanding the second I and last K each to the order of t but there are many other possibilities as well.

In view of the above three scenarios, we aim at obtaining an expansion of the form

$$\mathbb{E}[\Pi_t^x \otimes \Pi_t^x] = \frac{t^2}{12}(\phi_{ijij} - \phi_{ijji}) + (\mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3)t^3 + o(t^3), \tag{5.29}$$

where the t^2-coefficient in (5.29) was derived in (5.28) and $\mathcal{S}_i$ is the total coefficient to be computed under the above i-th scenario ($i = 1, 2, 3$). We now proceed to derive the explicit formulae for $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$. Since the analysis is quite elementary and mechanical, as before we will only discuss the details in one representative combination in each scenario and leave the case-by-case discussion to the appendix. We first introduce two convenient sets of notation.

Notation 5.11. We write $X_t \overset{3}{=} Y_t$ for random variables X_t, Y_t if $\mathbb{E}[X_t]$ and $\mathbb{E}[Y_t]$ have the same t^3-terms in their t-expansions as $t \to 0^+$.

Notation 5.12. To state the final results in a more compact form, we will introduce notation like

$$\begin{aligned}
\phi_{ijkl} &\triangleq \phi_i(\mathbf{0}) \otimes \phi_j(\mathbf{0}) \otimes \phi_k(\mathbf{0}) \otimes \phi_l(\mathbf{0}),\\
\phi_{i,p|jk|l,q} &\triangleq \partial_p\phi_i(\mathbf{0}) \otimes \phi_j(\mathbf{0}) \otimes \phi_k(\mathbf{0}) \otimes \partial_q\phi_l(\mathbf{0})\\
\phi_{ij|k,pq|l} &\triangleq \phi_i(\mathbf{0}) \otimes \phi_j(\mathbf{0}) \otimes \partial^2_{pq}\phi_k(\mathbf{0}) \otimes \phi_l(\mathbf{0}) \text{ etc.}
\end{aligned}$$

This kind of notation (without the derivatives) was already used in the statement of Proposition 5.9 before. We will also write $\partial_i f \triangleq \partial_i f(\mathbf{0})$.

5.4.1 Total degree $= 3$

All possible combinations in the decomposition (5.19) for this scenario is listed below:

$$\begin{gathered}
(JI;JI), (IJ;JI), (JI;IJ), (IJ;IJ), (JI;JK), (IJ;JK),\\
(JI;KJ), (IJ;KJ), (JK;JI), (KJ;JI), (JK;IJ), (KJ;IJ),\\
(JK;JK), (KJ;JK), (JK;KJ), (KJ;KJ), (JJ;II), (JJ;IK),\\
(JJ;KI), (JJ;KK), (II;JJ), (IK;JJ), (KI;JJ), (KK;JJ)(JJ;P), (P;JJ).
\end{gathered} \tag{5.30}$$

A representative case: $JI;JI$

As a representative case, let us consider the combination $JI;JI$ in (5.30). The resulting product is

$$B_{JI;JI} \triangleq \int_0^1 \big(t\int_0^\rho \varphi(X_{tr})dr\big) \otimes \phi_i(X_{t\rho})\frac{X^i_{t\rho}}{1-\rho}d\rho$$
$$\otimes \int_0^1 \big(t\int_0^\theta \varphi(X_{t\delta})d\delta\big) \otimes \phi_j(X_{t\theta})\frac{X^i_{t\theta}}{1-\theta}d\theta,$$

where φ is the function defined by (5.13). Since $B_{JI;JI}$ already has order t^3, to extract its coefficient the only possibility is freezing $\varphi, \phi_i, \varphi, \phi_j$ at the origin and replacing $\frac{X^i_{t\rho}}{1-\rho}, \frac{X^j_{t\theta}}{1-\theta}$ by their leading terms $G^{t,i}_\rho, G^{t,j}_\theta$ from the expansion (5.15). This leads to

$$B_{JI;JI} \stackrel{3}{=} \varphi(\mathbf{0}) \otimes \phi_i(\mathbf{0}) \otimes \varphi(\mathbf{0}) \otimes \phi_j(\mathbf{0})$$
$$\times \int_0^1 \big(t\int_0^\rho dr\big)G^{t,i}_\rho d\rho \times \int_0^1 \big(t\int_0^\theta d\delta\big)G^{t,j}_\theta d\theta.$$

It is straight forward to check that

$$\mathbb{E}[G^{t,i}_\rho G^{t,j}_\theta] = \frac{\rho\wedge\theta}{1-\rho\wedge\theta}\delta^{ij}t + o(t) \quad \text{as } t\to 0^+. \tag{5.31}$$

Therefore, one has

$$\mathbb{E}\big[\int_0^1 \big(t\int_0^\rho dr\big)G^{t,i}_\rho d\rho \times \int_0^1 \big(t\int_0^\theta d\delta\big)G^{t,j}_\theta d\theta\big]$$
$$= t^2\int_{[0,1]^2} \rho\theta\mathbb{E}[G^{t,i}_\rho G^{t,j}_\theta]d\rho d\theta = \frac{7}{12}\delta^{ij}t^3 + o(t^3).$$

In view of the definition (5.13) of φ and Lemma 4.6, one also has $\varphi(\mathbf{0}) = 1/2\partial_k\phi_k$. As a consequence, with Notation 5.12 in mind one finds that

$$\mathbb{E}[B_{JI;JI}] = \frac{7t^3}{48}\phi_{i,i|j|k,k|j} + o(t^3).$$

Summary of result

The results for all other cases in (5.30) are summarised in Appendix B. By adding up all these expressions, one arrives at the following formula.

Lemma 5.13. *The total t^3-coefficient $\mathcal{S}_1$ (cf. (5.29) for the notation) of $\mathbb{E}[\Pi_t^x \otimes \Pi_t^x]$ coming from all the 26 cases in (5.30) is equal to*

$$\mathcal{S}_1 = \frac{1}{48}\phi_{i,i|j|k,k|j} + \frac{1}{48}\phi_{i|j,j|i|k,k} - \frac{1}{48}\phi_{i,i|jj|k,k} - \frac{1}{48}\phi_{i|j,j|k,k|i}.$$

5.4.2 Total degree $= 2.5$

All possible combinations in the decomposition (5.19) for this scenario include the following 20 cases

$$\begin{array}{c} (JI;P),(IJ;P),(JK;P),(KJ;P),\\ (JI;II),(IJ;II),(JK;II),(KJ;II),\\ (JI;IK),(IJ;IK),(JK;IK),(KJ;IK),\\ (JI;KI),(IJ;KI),(JK;KI),(KJ;KI),\\ (JI;KK),(IJ;KK),(JK;KK),(KJ;KK), \end{array} \tag{5.32}$$

as well as the corresponding 20 cases obtained by swapping the $(1,2)$ and $(3,4)$ tensor slots for each case in (5.32) (e.g. $(P;JI)$ etc.)

A representative case: $JI;P$

We consider one representative case from (5.32): the combination $JI;P$. The resulting product is

$$\begin{aligned} C_{JI;P} \triangleq -t^2 \int_0^1 \Big(\int_0^\rho \varphi(X_{tr})dr\Big) \otimes \phi_i(X_{t\rho})\frac{X^i_{t\rho}}{1-\rho}d\rho \\ \otimes \frac{1}{2}\int_0^1 \phi_k(X_{t\theta}) \otimes \phi_l(X_{t\theta})a^{kl}(X_{t\theta})d\theta. \end{aligned}$$

The order of $C_{JI;P}$ is $t^{2.5}$ and the extra order of $\sqrt{t}$ comes from exactly one of the following expansions:

(i) Take the $\sqrt{t}$-order term (namely, $\partial_p\varphi(\mathbf{0})(1-r)G_r^{t,p}$) in the expansion of $\varphi(X_{tr})$ (and the leading order term in each of the remaining terms in their expansions, namely $\phi_i(\mathbf{0})$, $G_\rho^{t,i}$, $\phi_k(\mathbf{0})$, $\phi_l(\mathbf{0})$ and $a^{kl}(\mathbf{0})$ respectively).

According to Lemma 5.4, the resulting integral for this case is given by

$$\begin{aligned} C^{(1)}_{JI;P} \triangleq -\frac{1}{2}t^2\partial_p\varphi(\mathbf{0}) \otimes \phi_i(\mathbf{0}) \otimes \phi_k(\mathbf{0}) \otimes \phi_l(\mathbf{0})\delta^{kl} \\ \times \int_0^1 \Big(\int_0^\rho (1-r)G_r^{t,p}dr\Big)G_\rho^{t,i}d\rho \times \int_0^1 d\theta. \end{aligned}$$

For $r<\rho$, we recall from the relation (5.31) that

$$\mathbb{E}\big[G_r^{t,p}G_\rho^{t,i}\big]=\frac{tr}{1-r}\delta^{pi}+o(t).$$

As a result, one has

$$\begin{aligned}\mathbb{E}\big[C^{(1)}_{JI;P}\big]&=-\frac{1}{2}t^2\partial_p\varphi(\mathbf{0})\otimes\phi_i(\mathbf{0})\otimes\phi_k(\mathbf{0})\otimes\phi_l(\mathbf{0})\delta^{kl}\\&\quad\times\int_{0<r<\rho<1}(1-r)\big(\frac{tr}{1-r}\delta^{pi}+o(t)\big)drd\rho\\&=-\frac{t^3}{12}\partial_i\varphi(\mathbf{0})\otimes\phi_i(\mathbf{0})\otimes\phi_k(\mathbf{0})\otimes\phi_k(\mathbf{0})+o(t^3).\end{aligned}$$

On the other hand, it is seen from Lemma 4.6 that

$$\partial_i\varphi(\mathbf{0})=\partial_i\bar{b}^j(\mathbf{0})\phi_j(\mathbf{0})+\frac{1}{2}\partial^2_{ij}\phi_j(\mathbf{0}).$$

By using Notation 5.12, one obtains that

$$\mathbb{E}\big[C^{(1)}_{JI;P}\big]=\big(-\frac{1}{12}\partial_i\bar{b}^j\phi_{jikk}-\frac{1}{24}\phi_{j,ij|ikk}\big)t^3+o(t^3).$$

(ii) Take the $\sqrt{t}$-order term in $\phi_i(X_{t\rho})$.

Similar to Case (i), the resulting integral is

$$\begin{aligned}C^{(2)}_{JI;P}&\triangleq-\frac{1}{2}t^2\varphi(\mathbf{0})\otimes\partial_p\phi_i(\mathbf{0})\otimes\phi_k(\mathbf{0})\otimes\phi_k(\mathbf{0})\\&\quad\times\int_0^1\big(\int_0^\rho dr\big)(1-\rho)G_\rho^{t,p}G_\rho^{t,i}d\rho\times\int_0^1d\theta.\end{aligned}$$

As before, with the relation $\varphi(\mathbf{0})=1/2\partial_j\phi_j(\mathbf{0})$ one computes that

$$\begin{aligned}\mathbb{E}\big[C^{(2)}_{JI;P}\big]&=-\frac{1}{2}t^2\varphi(\mathbf{0})\otimes\partial_p\phi_i(\mathbf{0})\otimes\phi_k(\mathbf{0})\otimes\phi_k(\mathbf{0})\\&\quad\times\int_0^1\rho(1-\rho)\big(\frac{t\rho}{1-\rho}\delta^{pi}+o(t)\big)d\rho\times1\\&=-\frac{t^3}{12}\phi_{i,i|j,j|kk}+o(t^3).\end{aligned}$$

(iii) Take the t-order term in $\frac{X^i_{t\rho}}{1-\rho}$ (note that the leading term of $\frac{X_{t\rho}}{1-\rho}$ is G^t_ρ which has order $\sqrt{t}$).

In the expansion (5.15) of $\frac{X_{t\rho}}{1-\rho}$, it is not hard to see from Lemma 4.6 that the term H_ρ^t actually has order $t^{3/2}$ (because $\bar{b}(\mathbf{0})=0$). As a result, this particular case gives zero contribution: $\mathbb{E}[C_{JI;P}^{(3)}]=o(t^3)$.

(iv) <u>Take the $\sqrt{t}$-order term in either $\Phi(X_{t\theta})\triangleq\phi_k(X_{t\theta})\otimes\phi_l(X_{t\theta})a^{kl}(X_{t\theta})$.</u>

The resulting integral for this case is

$$C_{JI;P}^{(4)}\triangleq-\frac{1}{2}t^2\varphi(\mathbf{0})\otimes\phi_i(\mathbf{0})\otimes\partial_p\Phi(\mathbf{0})$$
$$\times\int_0^1\big(\int_0^\rho dr\big)G_\rho^{t,i}d\rho\times\int_0^1(1-\theta)G_\theta^{t,p}d\theta.$$

According to the relation (5.31) and Lemma 4.6, its expectation is given by

$$\begin{aligned}\mathbb{E}\big[C_{JI;P}^{(4)}\big]&=-\frac{1}{2}t^2\times\frac{1}{2}\partial_j\phi_j(\mathbf{0})\otimes\phi_i(\mathbf{0})\otimes\big(\partial_p\phi_k(\mathbf{0})\otimes\phi_k(\mathbf{0})+\phi_k(\mathbf{0})\otimes\partial_p\phi_k(\mathbf{0})\big)\\&\quad\times\int_0^1\int_0^1\rho(1-\theta)\big(\frac{\rho\wedge\theta}{1-\rho\wedge\theta}t\delta^{ip}+o(t)\big)d\rho d\theta\\&=-\frac{t^3}{24}\big(\phi_{j,j|i|k,i|k}+\phi_{j,j|i|k|k,i}\big)+o(t^3).\end{aligned}$$

(v) <u>First take the leading parts of all terms and then expand the expectation to the order of $\sqrt{t}$.</u>

This means taking the $\sqrt{t}$-order term for the function

$$t\mapsto-\frac{1}{2}t^2\varphi(\mathbf{0})\otimes\phi_i(\mathbf{0})\otimes\phi_k(\mathbf{0})\otimes\phi_l(\mathbf{0})\delta^{kl}$$
$$\times\int_0^1\big(\int_0^\rho dr\big)\mathbb{E}\big[G_\rho^{t,i}\big]d\rho\times\int_0^1 d\theta.$$

But this function is identically zero since G_ρ^t is an Itô integral. Therefore, this particular case gives zero contribution: $\mathbb{E}[C_{JI;P}^{(5)}]=0$.

To summarise, by adding up the above results one concludes that the t^3-coefficient of $\mathbb{E}[C_{JI;P}]$ is given by

$$-\frac{1}{12}\partial_i\bar{b}^j\phi_{jikk}-\frac{1}{24}\phi_{j,ij|ikk}-\frac{1}{12}\phi_{i,i|j,j|kk}-\frac{1}{24}\phi_{j,j|i|k,i|k}-\frac{1}{24}\phi_{j,j|i|k|k,i}.$$

Summary of result

The results for all other cases in (5.32) are summarised in Appendix B. By adding up all these expressions, one arrives at the following formula.

Lemma 5.14. *The total t^3-coefficient $\mathcal{S}_2$ (cf. (5.29) for the notation) of $\mathbb{E}[\Pi_t^x \otimes \Pi_t^x]$ coming from all the 40 cases in (5.32) is equal to*

$$\begin{aligned}\mathcal{S}_2 = &-\frac{1}{48}\phi_{i,i|j,k|jk} + \frac{1}{48}\phi_{i,i|j,k|kj} - \frac{1}{48}\phi_{i,j|k,k|ij} + \frac{1}{48}\phi_{i,j|k,k|ji} \\ &- \frac{1}{48}\phi_{i,i|j|k,k|j} + \frac{1}{48}\phi_{i,i|jj|k,k} + \frac{1}{48}\phi_{i|j,j|k,k|i} - \frac{1}{48}\phi_{i|j,j|i|k,k} \\ &- \frac{1}{48}\phi_{ij|i,j|k,k} + \frac{1}{48}\phi_{ij|j,i|k,k} - \frac{1}{48}\phi_{ij|k,k|i,j} + \frac{1}{48}\phi_{ij|k,k|j,i}.\end{aligned}$$

5.4.3 Total degree $= 2$

All possible combinations in the decomposition (5.19) for this scenario is listed below:

$$\begin{gathered}(II;II),(II;IK),(II;KI),(II;KK),\ (IK;II),(IK;IK),(IK;KI),(IK;KK),\\ (KI;II),(KI;IK),(KI;KI),(KI;KK),\ (KK;II),(KK;IK),(KK;KI),(KK;KK),\\ (II;P),(IK;P),(KI;P),(KK;P),\ (P;II),(P;IK),(P;KI),(P;KK),\ (PP).\end{gathered} \tag{5.33}$$

We consider a representative case given by the combination of $II;II$ in the decomposition (5.19). The resulting product is

$$\begin{aligned}D_{II;II} = &\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})\frac{X_{tr}^i}{1-r}dr\Big) \otimes \phi_j(X_{t\rho})\frac{X_{t\rho}^j}{1-\rho}d\rho \\ &\otimes \int_0^1 \Big(\int_0^\theta \phi_k(X_{t\delta})\frac{X_{t\delta}^k}{1-\delta}d\delta\Big) \otimes \phi_l(X_{t\theta})\frac{X_{t\theta}^l}{1-\theta}d\theta.\end{aligned}$$

The order of $D_{II;II}$ is t^2. There are five possible ways of expansion to produce a t^3-term:

(i) Take the t-order term in exactly one of the ϕ's (and the leading order terms for all remaining terms).
(ii) Take the $\sqrt{t}$-order terms in exactly two of the ϕ's.
(iii) Take the $t^{3/2}$-order term in one of the $\frac{W_{tr}}{1-r}$'s.
(iv) First only take the $\sqrt{t}$-order term in exactly one of the ϕ's, then expand the expectation of the resulting integral to an extra order of $\sqrt{t}$.
(v) First take the leading terms for all terms and then expand the expectation of the resulting integral to order t.

The computation here is essentially the same as before. However, the complexity gets substantially higher and we have to rely on computer-assistance (also for all other cases in (5.33)). We directly present the final result for the current scenario; the case-by-case computations are given in Appendix B.

Lemma 5.15. *The total t^3-coefficient $\mathcal{S}_3$ (cf. (5.29) for the notation) of $\mathbb{E}[\Pi_t^x \otimes \Pi_t^x]$ coming from all the 25 cases in (5.33) is equal to*

$$\mathcal{S}_3 = \mathcal{S}_3^{(1)} + \mathcal{S}_3^{(2)} + \mathcal{S}_3^{(3)} + \mathcal{S}_3^{(4)},$$

where the above four quantities are defined by the following expressions respectively:

$$\begin{aligned}\mathcal{S}_3^{(1)} \triangleq & \frac{1}{60}\phi_{i,jk|jik} - \frac{1}{60}\phi_{i.jk|jki} + \frac{1}{120}\phi_{i,kk|jij} - \frac{1}{120}\phi_{i,kk|jji} \\ & - \frac{1}{60}\phi_{i|j,ik|jk} + \frac{1}{60}\phi_{i|j,ik|kj} + \frac{1}{120}\phi_{i|j,kk|ij} - \frac{1}{120}\phi_{i|j,kk|ji} \\ & + \frac{1}{60}\phi_{ij|i,jk|k} + \frac{1}{120}\phi_{ij|i,kk|j} - \frac{1}{60}\phi_{ij|j,ik|k} - \frac{1}{120}\phi_{ij|j,kk|i} \\ & + \frac{1}{120}\phi_{iji|j,kk} - \frac{1}{120}\phi_{ijj|i,kk} - \frac{1}{60}\phi_{ijk|i,jk} + \frac{1}{60}\phi_{ijk|j,ik},\end{aligned}$$

$$
\begin{aligned}
\mathcal{S}_3^{(2)} \triangleq &- \frac{1}{120}\phi_{i,i|j,k|jk} + \frac{1}{120}\phi_{i,i|j,k|kj} - \frac{1}{240}\phi_{i,j|i,k|jk} + \frac{1}{240}\phi_{i,j|j,k|ik} - \frac{1}{240}\phi_{i,j|j,k|ki} \\
&+ \frac{1}{240}\phi_{i,k|i,j|jk} - \frac{1}{240}\phi_{i,k|j,i|jk} + \frac{1}{240}\phi_{i,k|j,i|kj} + \frac{1}{120}\phi_{i,k|j,j|ik} - \frac{1}{120}\phi_{i,k|j,j|ki} \\
&+ \frac{1}{80}\phi_{i,k|j,k|ij} - \frac{1}{80}\phi_{i,k|j,k|ji} - \frac{1}{360}\phi_{i,i|j|j,k|k} + \frac{1}{720}\phi_{i,i|j|k,j|k} + \frac{1}{720}\phi_{i,i|j|k,k|j} \\
&- \frac{1}{360}\phi_{i,j|i|j,k|k} + \frac{1}{180}\phi_{i,j|j|i,k|k} - \frac{1}{360}\phi_{i,j|j|k,i|k} - \frac{1}{360}\phi_{i,j|j|k,k|i} + \frac{1}{720}\phi_{i,k|i|j,j|k} \\
&+ \frac{1}{720}\phi_{i,k|i|j,k|j} + \frac{1}{45}\phi_{i,k|j|i,j|k} + \frac{1}{45}\phi_{i,k|j|i,k|j} - \frac{1}{90}\phi_{i,k|j|j,i|k} - \frac{1}{90}\phi_{i,k|j|j,k|i} \\
&- \frac{1}{90}\phi_{i,k|j|k,i|j} - \frac{1}{90}\phi_{i,k|j|k,j|i} - \frac{1}{720}\phi_{i,i|jj|k,k} + \frac{1}{360}\phi_{i,i|jk|j,k} - \frac{1}{720}\phi_{i,i|jk|k,j} \\
&- \frac{1}{720}\phi_{i,j|ij|k,k} + \frac{1}{360}\phi_{i,j|ji|k,k} - \frac{1}{180}\phi_{i,j|jk|i,k} + \frac{1}{360}\phi_{i,j|jk|k,i} - \frac{1}{720}\phi_{i,k|ij|j,k} \\
&+ \frac{1}{360}\phi_{i,k|ij|k,j} + \frac{1}{90}\phi_{i,k|ji|j,k} + \frac{1}{90}\phi_{i,k|ji|k,j} - \frac{1}{45}\phi_{i,k|jj|i,k} + \frac{1}{90}\phi_{i,k|jj|k,i} \\
&- \frac{1}{45}\phi_{i,k|jk|i,j} + \frac{1}{90}\phi_{i,k|jk|j,i} + \frac{1}{360}\phi_{i|i,j|j,k|k} - \frac{1}{720}\phi_{i|i,k|j,j|k} - \frac{1}{720}\phi_{i|i,k|j,k|j} \\
&- \frac{1}{180}\phi_{i|j,i|j,k|k} + \frac{1}{360}\phi_{i|j,i|k,j|k} + \frac{1}{360}\phi_{i|j,i|k,k|j} + \frac{1}{360}\phi_{i|j,j|i,k|k} - \frac{1}{720}\phi_{i|j,j|k,i|k} \\
&- \frac{1}{720}\phi_{i|j,j|k,k|i} + \frac{1}{90}\phi_{i|j,k|i,j|k} + \frac{1}{90}\phi_{i|j,k|i,k|j} - \frac{1}{45}\phi_{i|j,k|j,i|k} - \frac{1}{45}\phi_{i|j,k|j,k|i} \\
&+ \frac{1}{90}\phi_{i|j,k|k,i|j} + \frac{1}{90}\phi_{i|j,k|k,j|i} + \frac{1}{720}\phi_{i|i,j|j|k,k} + \frac{1}{720}\phi_{i|i,k|j|j,k} - \frac{1}{360}\phi_{i|i,k|j|k,j} \\
&- \frac{1}{360}\phi_{i|j,i|j|k,k} + \frac{1}{180}\phi_{i|j,i|k|j,k} - \frac{1}{360}\phi_{i|j,i|k|k,j} + \frac{1}{720}\phi_{i|j,j|i|k,k} - \frac{1}{360}\phi_{i|j,j|k|i,k} \\
&+ \frac{1}{720}\phi_{i|j,j|k|k,i} + \frac{1}{45}\phi_{i|j,k|i|j,k} - \frac{1}{90}\phi_{i|j,k|i|k,j} - \frac{1}{90}\phi_{i|j,k|j|i,k} - \frac{1}{90}\phi_{i|j,k|j|k,i} \\
&- \frac{1}{90}\phi_{i|j,k|k|i,j} + \frac{1}{45}\phi_{i|j,k|k|j,i} + \frac{1}{120}\phi_{ij|i,j|k,k} + \frac{1}{80}\phi_{ij|i,k|j,k} + \frac{1}{240}\phi_{ij|i,k|k,j} \\
&- \frac{1}{120}\phi_{ij|j,i|k,k} - \frac{1}{80}\phi_{ij|j,k|i,k} - \frac{1}{240}\phi_{ij|j,k|k,i} + \frac{1}{240}\phi_{ij|k,i|j,k} - \frac{1}{240}\phi_{ij|k,i|k,j} \\
&- \frac{1}{240}\phi_{ij|k,j|i,k} + \frac{1}{240}\phi_{ij|k,j|k,i} - \frac{1}{120}\phi_{ij|k,k|i,j} + \frac{1}{120}\phi_{ij|k,k|j,i},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}_3^{(3)} \triangleq &\frac{1}{144}\partial_k \bar{b}^j \phi_{ijik} + \frac{1}{144}\partial_j \bar{b}^k \phi_{ijik} - \frac{1}{144}\partial_k \bar{b}^i \phi_{ijjk} - \frac{1}{144}\partial_i \bar{b}^k \phi_{ijjk} \\
&- \frac{1}{144}\partial_k \bar{b}^j \phi_{ijki} - \frac{1}{144}\partial_j \bar{b}^k \phi_{ijki} + \frac{1}{144}\partial_k \bar{b}^i \phi_{ijkj} + \frac{1}{144}\partial_i \bar{b}^k \phi_{ijkj},
\end{aligned} \tag{5.34}
$$

$$\begin{aligned}\mathcal{S}_3^{(4)} \triangleq & \frac{11}{1440}\partial_{ll}a^{jk}\phi_{ijik} - \frac{11}{1440}\partial_{ll}a^{ik}\phi_{ijjk} - \frac{11}{1440}\partial_{ll}a^{jk}\phi_{ijki} + \frac{11}{1440}\partial_{ll}a^{ik}\phi_{ijkj} \\ & + \frac{1}{120}\partial_{jl}a^{ik}\phi_{ijkl} - \frac{1}{120}\partial_{jk}a^{il}\phi_{ijkl} - \frac{1}{120}\partial_{il}a^{jk}\phi_{ijkl} + \frac{1}{120}\partial_{ik}a^{jl}\phi_{ijkl}. \quad (5.35)\end{aligned}$$

5.4.4 Final result

To summarise, we have obtained the following result.

Proposition 5.16. *The t^3-coefficient $\hat{\Xi}_x$ in the expansion of (5.6) is given by*

$$\hat{\Xi}_x = \frac{1}{2}(\mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3) \in E^{\otimes 4},$$

where $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are explicitly given by Lemmas 5.13, 5.14, 5.15 respectively.

Remark 5.17. Although the expressions we give here are in terms of local quantities, the intrinsic meaning of $\hat{\Xi}_x$ (in particular, its explicit connection with curvature properties) will be clear after suitable geometric reduction (cf. Proposition 5.22 below).

5.5 Completing the proof of Theorem 5.1: geometric reduction

We now specialise in the situation where $\phi = dF$ and $F : M \to E = \mathbb{R}^N$ is an isometric embedding. Despite the complicated expressions of $\hat{\Theta}_x$ and $\hat{\Xi}_x$, in this case they simplify substantially after performing suitable dimension reduction. More precisely, we are going to consider the contraction $\mathfrak{C} : E^{\otimes 4} \to E^{\otimes 2}$ defined by (5.3) (i.e. taking trace over E with respect to the $(2,4)$-position).

5.5.1 Reduction of $\hat{\Theta}_x$

Recall that

$$\Theta_x \triangleq \mathfrak{C}\hat{\Theta}_x \in E^{\otimes 2} \cong \mathcal{L}(E \times E; \mathbb{R})$$

is the corresponding t^2-coefficient of $\mathfrak{C}\psi_4(t,x)$. Since F is an isometric embedding, the metric tensor admits the following local expression under the previous normal chart U:

$$g_{ij}(\mathbf{x}) = \langle \partial_i F(\mathbf{x}), \partial_j F(\mathbf{x}) \rangle_E, \quad \mathbf{x} \in U. \quad (5.36)$$

According to Proposition 4.5, one has $\langle \partial_i F(\mathbf{0}), \partial_j F(\mathbf{0}) \rangle = \delta_{ij}$. In particular, $\{\partial_i F(\mathbf{0})\}_{1 \leqslant i \leqslant d}$ is an ONB of $T_x M$. We also note that $\phi_i(\mathbf{x}) = \partial_i F(\mathbf{x})$ in the current setting.

Proof of Theorem 5.1: expression of Θ_x. According to Proposition 5.9, one has

$$\Theta_x = \frac{1}{24}\mathfrak{C}\big(\partial_i F \otimes \partial_j F \otimes \partial_i F \otimes \partial_j F - \partial_i F \otimes \partial_j F \otimes \partial_j F \otimes \partial_i F\big),$$

where all the derivatives are evaluated at $\mathbf{x} = \mathbf{0}$. Since $\{\partial_i F\}_{1\leqslant i\leqslant d}$ is an ONB of T_xM, one finds that

$$\Theta_x = \frac{1}{24}\sum_{k=1}^{d}\big(\langle \partial_j F, \partial_k F\rangle^2 \partial_i F \otimes \partial_i F \\ - \langle \partial_j F, \partial_k F\rangle\langle \partial_i F, \partial_k F\rangle \partial_i F \otimes \partial_j F\big) = \frac{d-1}{24}\partial_i F \otimes \partial_i F.$$

On the other hand, let $\pi_x : E \to T_xM$ denote the orthogonal projection. Then for any $v, w \in E$, one has

$$\big(\partial_i F \otimes \partial_i F\big)(v, w) = \langle v, \partial_i F\rangle_E \langle w, \partial_i F\rangle_E = \langle \pi_x v, \pi_x w\rangle_{T_xM}.$$

The relation (5.4) thus follows.

□

Remark 5.18. The 2-tensor Θ_x allows one to reconstruct the tangent space T_xM; indeed it is clear from (5.4) that

$$v \in (T_xM)^\perp \iff \Theta_x(v, w) = 0 \ \forall w \in E.$$

As a result, the formula (5.4) shows that the Riemannian metric tensor g can be explicitly reconstructed from the 2-tensor Θ_x.

5.5.2 Reduction of $\hat{\Xi}_x$

Recall that the t^3-coefficient Ξ_x is given by

$$\Xi_x = \mathfrak{C}\hat{\Xi}_x = \frac{1}{2}\big(\mathfrak{C}\mathcal{S}_1 + \mathfrak{C}\mathcal{S}_2 + \mathfrak{C}\mathcal{S}_3\big) \in \mathcal{L}(E \times E; \mathbb{R}). \tag{5.37}$$

To simplify its expression, we are going to write

$$\Xi_x = \Xi_x^T + \Xi_x^\perp,$$

where Ξ_x^T and $\Xi_x^\perp$ are defined by taking tangential and vertical $(2,4)$-traces of $\hat{\Xi}_x$ respectively. Namely, let $\{\varepsilon_1, \cdots, \varepsilon_d\}$ be an ONB basis of T_xM and let $\{\varepsilon_{d+1}, \cdots, \varepsilon_N\}$ be an ONB basis of $(T_xM)^\perp$ respectively. Then one defines

$$\Xi_x^T(\cdot,\cdot) \triangleq \sum_{i=1}^{d}\hat{\Xi}_x(\cdot, \varepsilon_i, \cdot, \varepsilon_i),\ \ \Xi_x^\perp(\cdot,\cdot) \triangleq \sum_{j=d+1}^{N}\hat{\Xi}_x(\cdot, \varepsilon_i, \cdot, \varepsilon_i).$$

In what follows, we shall compute Ξ_x^T, $\Xi_x^\perp$ and obtain their intrinsic forms respectively.

First of all, one has the following basic observation.

Lemma 5.19. *The family $\{\partial_i F(\mathbf{0}) : 1 \leqslant i \leqslant d\}$ is an ONB of $T_x M$. In addition, one has $\partial_{ij}^2 F(\mathbf{0}) \perp T_x M$ for any fixed pair of indices i, j.*

Proof. The first part was already seen before. For the second part, let i, j, k be given fixed indices. Then one has the following relations on the normal chart U:

$$\begin{cases} \partial_k \langle \partial_i F, \partial_j F\rangle_E(\mathbf{x}) = \langle \partial_{ik}^2 F, \partial_j F\rangle_E(\mathbf{x}) + \langle \partial_i F, \partial_{jk}^2 F\rangle_E(\mathbf{x}), \\ \partial_i \langle \partial_j F, \partial_k F\rangle_E(\mathbf{x}) = \langle \partial_{ij}^2 F, \partial_k F\rangle_E(\mathbf{x}) + \langle \partial_j F, \partial_{ik}^2 F\rangle_E(\mathbf{x}), \\ \partial_j \langle \partial_i F, \partial_k F\rangle_E(\mathbf{x}) = \langle \partial_{ij}^2 F, \partial_k F\rangle_E(\mathbf{x}) + \langle \partial_i F, \partial_{jk}^2 F\rangle_E(\mathbf{x}). \end{cases} \tag{5.38}$$

According to (5.36) and Proposition 4.5, the left hand side of (5.38) is zero at $\mathbf{x} = \mathbf{0}$. Therefore,

$$\langle \partial_{ij}^2 F, \partial_k F\rangle_E(\mathbf{0}) = \langle \partial_{ik}^2 F, \partial_j F\rangle_E(\mathbf{0}) = \langle \partial_{jk}^2 F, \partial_i F\rangle_E(\mathbf{0}) = 0.$$

Since $T_x M$ is spanned by $\{\partial_k F(\mathbf{0}) : 1 \leqslant k \leqslant d\}$, one concludes that $\partial_{ij}^2 F(\mathbf{0}) \perp T_x M$. □

Next, we compute intermediate expressions of Ξ_x^T, $\Xi_x^\perp$ in the lemma below. Throughout the rest, unless otherwise stated all quantities are evaluated at $\mathbf{x} = \mathbf{0}$ and this will be omitted to ease notation (e.g. $\partial_{ij}^2 F$ means $\partial_{ij}^2 F(\mathbf{0})$). We will also omit the subscript E for the inner product on E.

Lemma 5.20. *One has*

$$\begin{aligned} \Xi_x^T =& \big(\frac{1}{8640}\mathrm{S}_x + \frac{1}{120}\langle \partial_j F, \partial_{jkk}^3 F\rangle\big)\partial_i F \otimes \partial_i F \\ &+ \big(\frac{d+34}{8640}\mathrm{Ric}_{ij} - \frac{1}{240}\langle \partial_i F, \partial_{jkk}^3 F\rangle - \frac{1}{240}\langle \partial_j F, \partial_{ikk}^3 F\rangle\big)\partial_i F \otimes \partial_j F \\ &+ \frac{d-2}{1440}\partial_{ii}^2 F \otimes \partial_{jj}^2 F + \frac{8d-7}{1440}\partial_{ij}^2 F \otimes \partial_{ij}^2 F \\ &+ \frac{d-1}{240}\big(\partial_i F \otimes \partial_{ijj}^3 F + \partial_{ijj}^3 F \otimes \partial_i F\big), \end{aligned} \tag{5.39}$$

where Ric_{ij} *are the Ricci curvature coefficients and* S_x *is the scalar curvature at* x*. Respectively, one also has*

$$\begin{aligned} \Xi_x^\perp =& \big(\frac{1}{1440}\langle \partial_{jj}^2 F, \partial_{kk}^2 F\rangle + \frac{1}{180}\langle \partial_{jk}^2 F, \partial_{jk}^2\rangle\big)\partial_i F \otimes \partial_i F \\ &- \big(\frac{7}{1440}\langle \partial_{ik}^2 F, \partial_{jk}^2 F\rangle + \frac{1}{720}\langle \partial_{ij}^2 F, \partial_{kk}^2 F\rangle\big)\partial_i F \otimes \partial_j F. \end{aligned} \tag{5.40}$$

Proof. This follows from explicit calculation based on the formulae for $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ provided by Lemmas 5.13, 5.14, 5.15 respectively as well as Lemma 5.19. The curvature quantities appear because there are terms like $\partial_i b, \partial^2_{ij} G_1, \partial^2_{ij} a$ in $\mathcal{S}_3^{(3)}, \mathcal{S}_3^{(4)}$ inside $\mathcal{S}_3$ (cf. (5.34), (5.35) for their definitions and also recall (5.11) for the definition of $\bar{b}$). These quantities are explicitly related to curvature coefficients by Lemma 4.6. We will leave the routine and lengthy details to the patient reader. □

To further simplify the expressions of Ξ_x^T and $\Xi_x^\perp$, we first prepare a lemma which computes the inner products of certain derivatives of F.

Lemma 5.21. *(i) One has*

$$\langle \partial^2_{jj} F, \partial^2_{kk} F \rangle = d^2 |H_x|^2 \tag{5.41}$$

and

$$\langle \partial^2_{jk} F, \partial^2_{jk} F \rangle = -\mathrm{S}_x + d^2 |H_x|^2, \ \langle \partial_j F, \partial^3_{jkk} F \rangle = \frac{2}{3} \mathrm{S}_x - d^2 |H_x|^2, \tag{5.42}$$

where H_x is the mean curvature vector at x (cf. (2.6)).

(ii) One also has

$$\langle \partial^2_{ij} F, \partial^2_{kk} F \rangle = d \cdot \langle B_x(\partial_i F, \partial_j F), H_x \rangle, \tag{5.43}$$

$$\langle \partial_i F, \partial^3_{jkk} F \rangle = \frac{2}{3} \mathrm{Ric}_{ij} - d \cdot \langle B_x(\partial_i F, \partial_j F), H_x \rangle, \tag{5.44}$$

$$\langle \partial^2_{ik} F, \partial^2_{jk} F \rangle = -\mathrm{Ric}_{ij} + d \cdot \langle B_x(\partial_i F, \partial_j F), H_x \rangle, \tag{5.45}$$

where B_x is the second fundamental form at x (cf. (2.5)).

Proof. (i) Let us denote

$$X \triangleq \langle \partial_j F, \partial^3_{jkk} F \rangle, \ Y \triangleq \langle \partial^2_{jk} F, \partial^2_{jk} F \rangle, \ Z \triangleq \langle \partial^2_{jj} F, \partial^2_{kk} F \rangle.$$

Direct calculation shows that

$$\partial^2_{kk} \langle \partial_j F, \partial_j F \rangle = 2X + 2Y, \ \partial^2_{jk} \langle \partial_j F, \partial_k F \rangle = 2X + Y + Z. \tag{5.46}$$

According to Proposition 4.5, one has (at $\mathbf{x} = \mathbf{0}$)

$$\partial^2_{kk} \langle \partial_j F, \partial_j F \rangle = \partial^2_{kk} g_{jj} = -\frac{2}{3} R_{jkjk} = -\frac{2}{3} \mathrm{S}_x$$

and

$$\partial^2_{jk} \langle \partial_j F, \partial_k F \rangle = \partial^2_{jk} g_{jk} = -\frac{1}{3} R_{jkkj} = \frac{1}{3} \mathrm{S}_x.$$

By substituting this back into (5.46), one finds that

$$X = \frac{2}{3}\mathrm{S}_x - Z,\ Y = -\mathrm{S}_x + Z.$$

The relation (5.42) thus follows from the fact that $Z = |\Delta F|^2 = d^2|H_x|^2$ (cf. (2.7)).

(ii) To obtain the relation (5.43), recall that $\partial^2_{kk}F(\mathbf{0}) = d \cdot H_x \in (T_xM)^{\perp}$. It follows that

$$\begin{aligned} \langle \partial^2_{ij}F, \partial^2_{kk}F\rangle &= \langle \tilde{\nabla}^E_{\partial_i F}\partial_j F, d\cdot H_x\rangle \\ &= \langle \tilde{\nabla}^E_{\partial_i F}\partial_j F - \nabla^M_{\partial_i F}\partial_j F, d\cdot H_x\rangle \\ &= \langle B_x(\partial_i F, \partial_j F), d\cdot H_x\rangle, \end{aligned} \tag{5.47}$$

where $\tilde{\nabla}^E$ (respectively, ∇^M) is the Euclidean connection on E (respectively, Riemannian connection on M) and the last equality follows from the definition (2.5) of the second fundamental form. This gives (5.43).

For the relation (5.44), one first writes

$$\langle \partial_i F, \partial^3_{jkk}F\rangle = \partial_j\langle \partial_i F, \partial^2_{kk}F\rangle - \langle \partial^2_{ij}F, \partial^2_{kk}F\rangle. \tag{5.48}$$

The second term is just given by (5.43). By using the relation

$$\langle \partial_i F, \partial^2_{kk}F\rangle = \partial_k g_{ik} - \frac{1}{2}\partial_i g_{kk}$$

as well as Proposition 4.5, the first term on the right hand side of (5.48) is computed as

$$\partial_j\langle \partial_i F, \partial^2_{kk}F\rangle = \frac{2}{3}\mathrm{Ric}_{ij}.$$

The relation (5.44) thus follows.

Similarly, for the last relation (5.45) one first writes

$$\langle \partial^2_{ik}F, \partial^2_{jk}F\rangle = \partial_k\langle \partial_i F, \partial^2_{jk}\rangle - \langle \partial_i F, \partial^3_{jkk}F\rangle. \tag{5.49}$$

The second term is given by (5.44). To compute the first term, one has

$$\langle \partial_i F, \partial^2_{jk}F\rangle = \frac{1}{2}\big(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}\big).$$

It then follows from Proposition 4.5 that (at $\mathbf{x} = \mathbf{0}$)

$$\begin{aligned}\partial_k \langle \partial_i F, \partial^2_{jk} \rangle &= \frac{1}{2}\big(\partial^2_k g_{ij} + \partial^2_{jk} g_{ik} - \partial^2_{ik} g_{jk}\big) \\ &= \frac{1}{2}\big(-\frac{2}{3}\mathrm{Ric}_{ij} + \frac{1}{3}\mathrm{Ric}_{ij} - \frac{1}{3}\mathrm{Ric}_{ij}\big) = -\frac{1}{3}\mathrm{Ric}_{ij}.\end{aligned}$$

By substituting this and (5.44) into (5.49), one obtains the relation (5.45). □

Finally, we consider the restriction of Ξ^T_x and $\Xi^\perp_x$ on $T_xM \times T_xM$ and derive their intrinsic expressions based on Lemma 5.20.

Proposition 5.22. *The restricted* 2*-tensors* $\Xi^T_x, \Xi^\perp_x \in \mathcal{L}(T_xM \times T_xM, \mathbb{R})$ *are given by*

$$\Xi^T_x = \frac{49d - 62}{8640}\mathrm{Ric}_x + \big(\frac{49\mathrm{S}_x}{8640} - \frac{d^2}{120}|H_x|^2\big)g_x - \frac{(d-2)d}{120}\langle B_x, H_x\rangle \tag{5.50}$$

and

$$\Xi^\perp_x = \frac{9d^2|H_x|^2 - 8S_x}{1440}g_x + \frac{7}{1440}\mathrm{Ric}_x - \frac{d}{160}\langle B_x, H_x\rangle. \tag{5.51}$$

Proof. Let $v, w \in T_xM$ be given vectors. We first evaluate $\Xi^T_x(v,w)$ by using (5.39). This is a simple task based on Lemma 5.21 and the following basic relations.

(i) It is obvious that

$$(\partial_i F \otimes \partial_i F)(v,w) = \langle v, \partial_i F\rangle\langle w, \partial_i F\rangle = \langle v, w\rangle.$$

In other words, one has $\partial_i F \otimes \partial_i F = g_x$. In addition, one also has

$$\mathrm{Ric}_{ij}\partial_i F \otimes \partial_j F(v,w) = \mathrm{Ric}(v,w).$$

(ii) Since $v, w \in T_xM$, it is a direct consequence of Lemma 5.19 that

$$(\partial_{ii}F \otimes \partial_{jj}F)(v,w) = (\partial_{ij}F \otimes \partial_{ij}F)(v,w) = 0.$$

(iii) According to Lemma 5.21, one has

$$\langle \partial_i F, \partial^3_{jkk}F\rangle(\partial_i F \otimes \partial_j F)(v,w) = \frac{2}{3}\mathrm{Ric}(v,w) - d\cdot\langle B_x(v,w), H_x\rangle$$

and

$$(\partial_i F \otimes \partial^3_{ijj}F)(v,w) = \frac{2}{3}\mathrm{Ric}(v,w) - d\cdot\langle B_x(v,w), H_x\rangle.$$

By substituting all the above relations as well as (5.42) into (5.39), one obtains the expression (5.50) for Ξ^T_x. The derivation of (5.51) for $\Xi^\perp_x$ is similar and is left to the patient reader. □

Proof of Theorem 5.1: expression of $\Xi_x|_{T_xM\times T_xM}$. The relation (5.16) follows by adding up the two expressions (5.50) and (5.51).

□

Now the proof of Theorem 5.1 is complete.

Remark 5.23. Recall that

$$\mathrm{Tr}g_x = d,\ \mathrm{Tr}\mathrm{Ric}_x = \mathrm{S}_x,\ \mathrm{Tr}B_x = d\cdot H_x,$$

where Tr here means taking trace over T_xM. After taking trace on both (5.50) and (5.51) one obtains a linear system for the variables S_x and $|H_x|^2$. By solving such a system, one obtains the representations of S_x and $|H_x|$ in terms of $\mathrm{Tr}\Xi_x^T$ and $\mathrm{Tr}\Xi_x^\perp$. Once they become known, one can then view (5.50) and (5.51) as a linear system for the (tensor) variables Ric_x and $\langle B_x, H_x\rangle$ (recall that g_x is already known from Θ_x). Solving this system gives corresponding representations of these two tensors. As a consequence, the quantities

$$g_x, \mathrm{Ric}_x, \mathrm{S}_x, \langle B_x, H_x\rangle, |H_x|$$

can all be reconstructed explicitly from the the tensors $\Theta_x, \Xi_x^T, \Xi_x^\perp$. We will not present these formulae here.

Remark 5.24. One can of course consider higher level signatures and higher order expansions. We stopped at level four and order t^3 because (i) the computation is already highly involved and (ii) the encoded curvature properties are simple and rich enough to reveal in this case. Higher order expansions will contain more complicated mixtures of covariant derivatives of the curvature tensor and of the embedding map F, whose information becomes harder to unwind. It is not unreasonable to expect that the entire Riemnannian curvature tensor and the second fundamental form are both encoded in the small-time expansion of the expected signature $\mathbb{E}[S^{dF}(X^{t,x,x})]$. This is an interesting question to investigate, which is not obvious at all due to the complicated nature of the current computation.

Appendix A Remaining cases for t^2-coefficient

Here we present the results for all remaining cases listed in (5.21). We continue to use Notation 5.6 and (5.27). Also recall that $B_r^t \triangleq B_{tr}/\sqrt{t}$ which is again a Brownian motion.

A.1 The $(II;IK)$ and $(IK;II)$ terms

We first consider

$$A_{II;IK} \triangleq -\sqrt{t}\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})\frac{X_{tr}^i}{1-r}dr\Big)\otimes\phi_j(X_{t\rho})\frac{X_{t\rho}^j}{1-\rho}d\rho$$
$$\otimes\int_0^1\Big(\int_0^\theta \phi_k(X_{t\delta})\frac{X_{t\delta}^k}{1-\delta}d\delta\Big)\otimes\phi_l(X_{t\theta})(\sigma(X_{t\theta})dB_\theta^t)^l.$$

By applying the expansions (5.15) and (5.16), one has

$$A_{II;IK} \stackrel{2}{=} -\sqrt{t}^4\phi_{ijkl}\times\int_{0<r<\rho<1}\Big(\int_0^r\frac{dB_u^{t,i}}{1-u}\Big)\Big(\int_0^\rho\frac{dB_v^{t,j}}{1-v}\Big)drd\rho$$
$$\times\int_0^1\Big(\int_0^\theta\Big(\int_0^\delta\frac{dB_\eta^{t,k}}{1-\eta}\Big)d\delta\Big)dB_\theta^{t,l}. \tag{A.1}$$

Computing the expectation of the right hand side of (A.1) is just routine Itô calculus. One finds that

$$\mathbb{E}[A_{II;IK}] = \Big(-\frac{5}{24}\delta^{ik}\delta^{jl}-\frac{1}{24}\delta^{il}\delta^{jk}\Big)\phi_{ijkl}t^2+o(t^2). \tag{A.2}$$

To compute the $(IK;II)$ case, one can make use of the symmetry

$$A_{IK;II} = P(A_{II;IK}),$$

where $P: E^{\otimes 4}\to E^{\otimes 4}$ is the tensor permutation induced by

$$P(v_1\otimes v_2\otimes w_1\otimes w_2)\triangleq w_1\otimes w_2\otimes v_1\otimes v_2,\quad w_i,v_j\in E.$$

By applying this permutation to (A.2), one immediately obtains that (after suitable renaming of indices)

$$\mathbb{E}[A_{IK;II}] = \mathbb{E}[A_{II;IK}] = \Big(-\frac{5}{24}\delta^{ik}\delta^{jl}-\frac{1}{24}\delta^{il}\delta^{jk}\Big)\phi_{ijkl}t^2+o(t^2). \tag{A.3}$$

A.2 The $(II;KI)$ and $(KI;II)$ terms

Target:

$$A_{II;KI} \triangleq -\sqrt{t}\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})\frac{X_{tr}^i}{1-r}dr\Big)\otimes\phi_j(X_{t\rho})\frac{X_{t\rho}^j}{1-\rho}d\rho$$
$$\otimes\int_0^1\Big(\int_0^\theta \phi_k(X_{t\delta})(\sigma(X_{t\delta})dB_\delta^t)^k\Big)\otimes\phi_l(X_{t\theta})\frac{X_{t\theta}^l}{1-\theta}d\theta.$$

Reduction:

$$A_{II;KI} \overset{2}{=} -\sqrt{t}^4 \phi_{ijkl} \times \int_{0<r<\rho<1} \Big(\int_0^r \frac{dB_u^{t,i}}{1-u}\Big)\Big(\int_0^\rho \frac{dB_v^{t,j}}{1-v}\Big) dr d\rho \times \int_0^1 B_\theta^{t,k} \Big(\int_0^\theta \frac{dB_\eta^{t,l}}{1-\eta}\Big) d\theta.$$

<u>*Result*</u>:

$$\mathbb{E}[A_{II;KI}] = \mathbb{E}[A_{KI;II}] = \big(-\frac{1}{2}\delta^{ij}\delta^{kl} - \frac{11}{24}\delta^{ik}\delta^{jl} - \frac{7}{24}\delta^{il}\delta^{jk}\big)\phi_{ijkl} t^2 + o(t^2). \quad \text{(A.4)}$$

A.3 The $(II;KK)$ and $(KK;II)$ terms

Target:

$$A_{II;KK} \triangleq \sqrt{t}^2 \int_0^1 \Big(\int_0^\rho \phi_i(X_{tr}) \frac{X_{tr}^i}{1-r} dr\Big) \otimes \phi_j(X_{t\rho}) \frac{X_{t\rho}^j}{1-\rho} d\rho$$
$$\otimes \int_0^1 \Big(\int_0^\theta \phi_k(X_{t\delta})(\sigma(X_{t\delta}) dB_\delta^t)^k\Big) \otimes \phi_l(X_{t\theta})(\sigma(X_{t\theta}) dB_\theta^t)^l.$$

Reduction:

$$A_{II;KK} \overset{2}{=} \sqrt{t}^4 \phi_{ijkl} \times \int_{0<r<\rho<1} \Big(\int_0^r \frac{dB_u^{t,i}}{1-u}\Big)\Big(\int_0^\rho \frac{dB_v^{t,j}}{1-v}\Big) dr d\rho \times \int_0^1 B_\theta^{t,k} dB_\theta^{t,l}.$$

<u>*Result*</u>:

$$\mathbb{E}[A_{II;KK}] = \mathbb{E}[A_{KK;II}] = \big(\frac{3}{8}\delta^{ik}\delta^{jl} + \frac{1}{8}\delta^{il}\delta^{jk}\big)\phi_{ijkl} t^2 + o(t^2). \quad \text{(A.5)}$$

A.4 The $(IK;KI)$ and $(KI;IK)$ terms

Target:

$$A_{IK;KI} \triangleq \sqrt{t}^2 \int_0^1 \Big(\int_0^\rho \phi_i(X_{tr}) \frac{X_{tr}^i}{1-r} dr\Big) \otimes \phi_j(X_{t\rho})(\sigma(X_{t\rho}) dB_\rho^t)^j$$
$$\otimes \int_0^1 \Big(\int_0^\theta \phi_k(X_{t\delta})(\sigma(X_{t\delta}) dB_\delta^t)^k\Big) \otimes \phi_l(X_{t\theta}) \frac{X_{t\theta}^l}{1-\theta} d\theta.$$

Reduction:

$$A_{IK;KI} \overset{2}{=} \sqrt{t}^4 \phi_{ijkl} \times \int_0^1 \Big(\int_0^\rho \Big(\int_0^r \frac{dB_u^{t,i}}{1-u}\Big) dr\Big) dB_\rho^{t,j} \times \int_0^1 B_\theta^{t,k} \Big(\int_0^\theta \frac{dB_v^{t,l}}{1-v}\Big) d\theta.$$

<u>*Result*</u>:

$$\mathbb{E}[A_{IK;KI}] = \mathbb{E}[A_{KI;IK}] = \big(\frac{1}{4}\delta^{ik}\delta^{jl} + \frac{1}{12}\delta^{il}\delta^{jk}\big)\phi_{ijkl} t^2 + o(t^2). \quad \text{(A.6)}$$

A.5 The $(IK;KK)$ and $(KK;IK)$ terms

Target:

$$A_{IK;KK} \triangleq -\sqrt{t}^3 \int_0^1 \big(\int_0^\rho \phi_i(X_{tr}) \frac{X^i_{tr}}{1-r} dr\big) \otimes \phi_j(X_{t\rho})(\sigma(X_{t\rho}) dB^t_\rho)^j$$
$$\otimes \int_0^1 \big(\int_0^\theta \phi_k(X_{t\delta})(\sigma(X_{t\delta}) dB^t_\delta)^k\big) \otimes \phi_l(X_{t\theta})(\sigma(X_{t\theta}) dB^t_\theta)^l.$$

Reduction:

$$A_{IK;KK} \stackrel{2}{=} -\sqrt{t}^4 \phi_{ijkl} \times \int_0^1 \big(\int_0^\rho \big(\int_0^r \frac{dB^{t,i}_u}{1-u}\big) dr\big) dB^{t,j}_\rho \times \int_0^1 B^{t,k}_\theta dB^{t,l}_\theta.$$

<u>*Result*</u>:

$$\mathbb{E}[A_{IK;KK}] = \mathbb{E}[A_{KK;IK}] = -\frac{1}{4} \delta^{ik} \delta^{jl} \phi_{ijkl} t^2 + o(t^2). \tag{A.7}$$

A.6 The $(KI;KK)$ and $(KK;KI)$ terms

Target:

$$A_{KI;KK} \triangleq -\sqrt{t}^3 \int_0^1 \big(\int_0^\rho \phi_i(X_{tr})(\sigma(X_{tr}) dB^t_r)^i\big) \otimes \phi_j(X_{t\rho}) \frac{X^j_{t\rho}}{1-\rho} d\rho$$
$$\otimes \int_0^1 \big(\int_0^\theta \phi_k(X_{t\delta})(\sigma(X_{t\delta}) dB^t_\delta)^k\big) \otimes \phi_l(X_{t\theta})(\sigma(X_{t\theta}) dB^t_\theta)^l.$$

Reduction:

$$A_{KI;KK} \stackrel{2}{=} -\sqrt{t}^4 \phi_{ijkl} \times \int_0^1 B^{t,i}_\rho \big(\int_0^\rho \frac{dB^{t,j}_r}{1-r}\big) d\rho \times \int_0^1 B^{t,k}_\theta dB^{t,l}_\theta.$$

<u>*Result*</u>:

$$\mathbb{E}[A_{KI;KK}] = \mathbb{E}[A_{KK;KI}] = \big(-\frac{1}{2} \delta^{ik} \delta^{jl} - \frac{1}{4} \delta^{il} \delta^{jk}\big) \phi_{ijkl} t^2 + o(t^2). \tag{A.8}$$

A.7 The $(IK;IK)$ term

Target:

$$A_{IK;IK} \triangleq \sqrt{t}^2 \int_0^1 \Big(\int_0^\rho \phi_i(X_{tr}) \frac{X_{tr}^i}{1-r} dr \Big) \otimes \phi_j(X_{t\rho})(\sigma(X_{t\rho}) dB_\rho^t)^j$$
$$\otimes \int_0^1 \Big(\int_0^\theta \phi_k(X_{t\delta}) \frac{X_{t\delta}^k}{1-\delta} d\delta \Big) \otimes \phi_l(X_{t\theta})(\sigma(X_{t\theta}) dB_\theta^t)^l.$$

Reduction:

$$A_{IK;IK} \stackrel{2}{=} \sqrt{t}^4 \phi_{ijkl} \times \int_0^1 \Big(\int_0^\rho \Big(\int_0^r \frac{dB_u^{t,i}}{1-u} \Big) dr \Big) dB_\rho^{t,j}$$
$$\times \int_0^1 \Big(\int_0^\theta \Big(\int_0^\delta \frac{dB_v^{t,k}}{1-v} \Big) d\delta \Big) dB_\theta^{t,l}.$$

<u>*Result*</u>:

$$\mathbb{E}[A_{IK;IK}] = \frac{1}{6} \delta^{ik} \delta^{jl} \phi_{ijkl} t^2 + o(t^2). \tag{A.9}$$

A.8 The $(KI;KI)$ term

Target:

$$A_{KI;KI} \triangleq \sqrt{t}^2 \int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})(\sigma(X_{tr}) dB_r^t)^i \Big) \otimes \phi_j(X_{t\rho}) \frac{X_{t\rho}^j}{1-\rho} d\rho$$
$$\otimes \int_0^1 \Big(\int_0^\theta \phi_k(X_{t\delta})(\sigma(X_{t\delta}) dB_\delta^t)^k \Big) \otimes \phi_l(X_{t\theta}) \frac{X_{t\theta}^l}{1-\theta} d\theta.$$

Reduction:

$$A_{KI;KI} \stackrel{2}{=} \sqrt{t}^4 \phi_{ijkl} \times \int_0^1 B_\rho^{t,i} \Big(\int_0^\rho \frac{dB_u^{t,j}}{1-u} \Big) d\rho \times \int_0^1 B_\theta^{t,k} \Big(\int_0^\theta \frac{dB_v^{t,l}}{1-v} \Big) d\theta.$$

<u>*Result*</u>:

$$\mathbb{E}[A_{KI;KI}] = \Big(\delta^{ij}\delta^{kl} + \frac{2}{3} \delta^{ik}\delta^{jl} + \frac{1}{2} \delta^{il}\delta^{jk} \Big) \phi_{ijkl} t^2 + o(t^2). \tag{A.10}$$

A.9 The $(KK;KK)$ term

Target:

$$A_{KK;KK} \triangleq \sqrt{t}^4 \int_0^1 \big(\int_0^\rho \phi_i(X_{tr})(\sigma(X_{tr})dB_r^t)^i \big) \otimes \phi_j(X_{t\rho})(\sigma(X_{t\rho})dB_\rho^t)^j$$
$$\otimes \int_0^1 \big(\int_0^\theta \phi_k(X_{t\delta})(\sigma(X_{t\delta})dB_\delta^t)^k \big) \otimes \phi_l(X_{t\theta})(\sigma(X_{t\theta})dB_\theta^t)^l.$$

Reduction:

$$A_{KK;KK} \stackrel{2}{=} \sqrt{t}^4 \phi_{ijkl} \times \int_0^1 B_\rho^{t,i} dB_\rho^{t,j} \times \int_0^1 B_\theta^{t,k} dB_\theta^{t,l}.$$

<u>*Result*</u>:

$$\mathbb{E}[A_{KK;KK}] = \frac{1}{2}\delta^{ik}\delta^{jl}\phi_{ijkl}t^2 + o(t^2). \tag{A.11}$$

A.10 The $(II;P)$ and $(P;II)$ terms

Target:

$$A_{II;P} \triangleq \frac{1}{2}t \int_0^1 \big(\int_0^\rho \phi_i(X_{tr}) \frac{X_{tr}^i}{1-r} dr \big) \otimes \phi_j(X_{t\rho}) \frac{X_{t\rho}^j}{1-\rho} d\rho$$
$$\otimes \int_0^1 \phi_k(X_{t\theta}) \otimes \phi_l(X_{t\theta}) a^{kl}(X_{t\theta}) d\theta.$$

Reduction:

$$A_{II;P} \stackrel{2}{=} \frac{1}{2}t^2 \phi_{ijkl} \times \int_{0<r<\rho<1} \big(\int_0^r \frac{dB_u^{t,i}}{1-u} \big) \big(\int_0^\rho \frac{dB_v^{t,j}}{1-v} \big) dr d\rho \times \delta^{kl}.$$

<u>*Result*</u>:

$$\mathbb{E}[A_{II;P}] = \mathbb{E}[A_{P;II}] = \frac{1}{4}\delta^{ij}\delta^{kl}\phi_{ijkl}t^2 + o(t^2). \tag{A.12}$$

A.11 The $(IK;P)$ and $(P;IK)$ terms

Target:

$$A_{IK;P} \triangleq -\frac{1}{2}t^{3/2} \int_0^1 \big(\int_0^\rho \phi_i(X_{tr}) \frac{X_{tr}^i}{1-r} dr \big) \otimes \phi_j(X_{t\rho})(\sigma(X_{t\rho})dB_\rho^t)^j$$
$$\otimes \int_0^1 \phi_k(X_{t\theta}) \otimes \phi_l(X_{t\theta}) a^{kl}(X_{t\theta}) d\theta.$$

Reduction:

$$A_{IK;P} \overset{2}{=} -\frac{1}{2}t^2\phi_{ijkl} \times \int_0^1 \Big(\int_0^\rho \big(\int_0^r \frac{dB_u^{t,i}}{1-u}\big)dr\Big)dB_\rho^{t,j} \times \delta^{kl}.$$

<u>*Result*</u>:

$$\mathbb{E}[A_{IK;P}] = \mathbb{E}[A_{P;IK}] = o(t^2). \tag{A.13}$$

A.12 The $(KI;P)$ and $(P;KI)$ terms

Target:

$$A_{KI;P} \triangleq -\frac{1}{2}t^{3/2}\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})(\sigma(X_{tr})dB_r^t)^i\Big) \otimes \phi_j(X_{t\rho})\frac{X_{t\rho}^j}{1-\rho}d\rho$$
$$\otimes \int_0^1 \phi_k(X_{t\theta}) \otimes \phi_l(X_{t\theta})a^{kl}(X_{t\theta})d\theta.$$

Reduction:

$$A_{KI;P} \overset{2}{=} -\frac{1}{2}t^2\phi_{ijkl} \times \int_0^1 B_\rho^{t,i}\Big(\int_0^\rho \frac{dB_u^{t,j}}{1-u}\Big)d\rho \times \delta^{kl}.$$

<u>*Result*</u>:

$$\mathbb{E}[A_{KI;P}] = \mathbb{E}[A_{P;KI}] = -\frac{1}{2}\delta^{ij}\delta^{kl}\phi_{ijkl}t^2 + o(t^2). \tag{A.14}$$

A.13 The $(KK;P)$ and $(P;KK)$ terms

Target:

$$A_{KK;P} \triangleq \frac{1}{2}t^2\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})(\sigma(X_{tr})dB_r^t)^i\Big) \otimes \phi_j(X_{t\rho})(\sigma(X_{t\rho})dB_\rho^t)^j$$
$$\otimes \int_0^1 \phi_k(X_{t\theta}) \otimes \phi_l(X_{t\theta})a^{kl}(X_{t\theta})d\theta.$$

Reduction:

$$A_{KK;P} \overset{2}{=} -\frac{1}{2}t^2\phi_{ijkl} \times \int_0^1 B_\rho^{t,i}dB_\rho^{t,j} \times \delta^{kl}.$$

<u>*Result*</u>:

$$\mathbb{E}[A_{KK;P}] = \mathbb{E}[A_{P;KK}] = o(t^2). \tag{A.15}$$

A.14 The $(P;P)$ term

Target:

$$A_{P;P} \triangleq \frac{1}{4}t^2 \int_0^1 \phi_i(X_{t\rho}) \otimes \phi_j(X_{t\rho}) a^{ij}(X_{t\rho}) d\rho$$
$$\otimes \int_0^1 \phi_k(X_{t\theta}) \otimes \phi_l(X_{t\theta}) a^{kl}(X_{t\theta}) d\theta.$$

Reduction:

$$A_{P;P} \stackrel{2}{=} \frac{1}{4}t^2 \phi_{ijkl}\delta^{ij}\delta^{kl}.$$

Result:

$$\mathbb{E}[A_{P;P}] = \frac{1}{4}\delta^{ij}\delta^{kl}\phi_{ijkl}t^2 + o(t^2). \tag{A.16}$$

Appendix B Remaining cases for t^3-coefficient

In this appendix, we summarise the results for all remaining cases in the computation of the t^3-coefficient $\hat{\Xi}_x$. Recall that the function φ is defined by (5.13).

B.1 Total degree $= 3$

Here we discuss all remaining cases listed in (5.30). We continue to use Notation 5.11 and 5.12.

B.1.1 The $(IJ;JI)$ and $(JI;IJ)$ terms

Target:

$$B_{IJ;JI} \triangleq t^2 \int_0^1 \Big(\int_0^\rho \phi_i(X_{tr}) \frac{X_{tr}^i}{1-r} dr \Big) \otimes \varphi(X_{t\rho}) d\rho$$
$$\otimes \int_0^1 \Big(\int_0^\theta \varphi(X_{t\delta}) d\delta \Big) \otimes \phi_j(X_{t\theta}) \frac{X_{t\theta}^j}{1-\theta} d\theta.$$

Reduction:

$$B_{IJ;JI} \stackrel{3}{=} t^3 \phi_i(\mathbf{0}) \otimes \varphi(\mathbf{0}) \otimes \varphi(\mathbf{0}) \otimes \phi_j(\mathbf{0})$$
$$\times \int_{0<r<\rho<1} \Big(\int_0^r \frac{dB_u^{t,i}}{1-u} \Big) dr d\rho \times \int_0^1 \theta \Big(\int_0^\theta \frac{dB_v^{t,j}}{1-v} \Big) d\theta.$$

Result:

$$\mathbb{E}[B_{IJ;JI}] = \frac{1}{24}\phi_{k|i,i|j,j|k}t^3 + o(t^3),$$
$$\mathbb{E}[B_{JI;IJ}] = \frac{1}{24}\phi_{i,i|kk|j,j}t^3 + o(t^3). \quad \text{(by symmetry)}$$

B.1.2 The $(IJ;IJ)$ term

Target:

$$B_{IJ;IJ} \triangleq t^2 \int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})\frac{X^i_{tr}}{1-r}dr\Big) \otimes \varphi(X_{t\rho})d\rho \otimes \int_0^1 \Big(\int_0^\theta \phi_j(X_{t\delta})\frac{X^j_{t\delta}}{1-\delta}d\delta\Big) \otimes \varphi(X_{t\theta})d\theta.$$

Reduction:

$$B_{IJ;IJ} \overset{3}{=} t^3\phi_i(\mathbf{0}) \otimes \varphi(\mathbf{0}) \otimes \phi_j(\mathbf{0}) \otimes \varphi(\mathbf{0}) \times \int_{0<r<\rho<1} \Big(\int_0^r \frac{dB^{t,i}_u}{1-u}\Big)drd\rho \times \int_{0<\delta<\theta<1} \Big(\int_0^\delta \frac{dB^{t,j}_v}{1-v}\Big)d\delta d\theta.$$

Result:

$$\mathbb{E}[B_{IJ;IJ}] = \frac{1}{48}\phi_{k|i,i|k|j,j}t^3 + o(t^3).$$

B.1.3 The $(JI;JK)$ and $(JK;JI)$ terms

Target:

$$B_{JI;JK} \triangleq -t^{5/2} \int_0^1 \Big(\int_0^\rho \varphi(X_{tr})dr\Big) \otimes \phi_i(X_{t\rho})\frac{X^i_{t\rho}}{1-\rho}d\rho \otimes \int_0^1 \Big(\int_0^\theta \varphi(X_{t\delta})d\delta\Big) \otimes \phi_j(X_{t\theta})(\sigma(X_{t\theta})dB^t_\theta)^j.$$

Reduction:

$$B_{JI;JK} \overset{3}{=} -t^3\varphi(\mathbf{0}) \otimes \phi_i(\mathbf{0}) \otimes \varphi(\mathbf{0}) \otimes \phi_j(\mathbf{0}) \times \int_0^1 \rho\Big(\int_0^\rho \frac{dB^{t,i}_u}{1-u}\Big)d\rho \times \int_0^1 \theta dB^{t,j}_\theta.$$

Result:

$$\mathbb{E}[B_{JI;JK}] = -\frac{5}{48}\phi_{i,i|k|j,j|k}t^3 + o(t^3),$$
$$\mathbb{E}[B_{JK;JI}] = -\frac{5}{48}\phi_{i,i|k|j,j|k}t^3 + o(t^3). \quad \text{(by symmetry).}$$

B.1.4 The $(IJ;JK)$ and $(JK;IJ)$ terms

Target:

$$B_{IJ;JK} \triangleq -t^{5/2}\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})\frac{X^i_{tr}}{1-r}dr\Big)\otimes\varphi(X_{t\rho})d\rho \\ \otimes\int_0^1\Big(\int_0^\theta \varphi(X_{t\delta})d\delta\Big)\otimes\phi_j(X_{t\theta})(\sigma(X_{t\theta})dB^t_\theta)^j.$$

Reduction:

$$B_{IJ;JK} \stackrel{3}{=} -t^3\phi_i(\mathbf{0})\otimes\varphi(\mathbf{0})\otimes\varphi(\mathbf{0})\otimes\phi_j(\mathbf{0}) \\ \times\int_{0<r<\rho<1}\Big(\int_0^r \frac{dB^{t,i}_u}{1-u}\Big)drd\rho\times\int_0^1\theta dB^{t,j}_\theta.$$

<u>*Result*</u>:

$$\mathbb{E}[B_{IJ;JK}] = -\frac{1}{48}\phi_{k|i,i|j,j|k}t^3+o(t^3),$$
$$\mathbb{E}[B_{JK;IJ}] = -\frac{1}{48}\phi_{i,i|kk|j,j}t^3+o(t^3). \quad \text{(by symmetry)}$$

B.1.5 The $(JI;KJ)$ and $(KJ;JI)$ terms

Target:

$$B_{JI;KJ} \triangleq -t^{5/2}\int_0^1\Big(\int_0^\rho\varphi(X_{tr})dr\Big)\otimes\phi_i(X_{t\rho})\frac{X^i_{t\rho}}{1-\rho}d\rho \\ \otimes\int_0^1\Big(\int_0^\theta\phi_j(X_{t\delta})(\sigma(X_{t\delta})dB^t_\delta)^j\Big)\otimes\varphi(X_{t\theta})d\theta.$$

Reduction:

$$B_{JI;KJ} \stackrel{3}{=} -t^3\varphi(\mathbf{0})\otimes\phi_i(\mathbf{0})\otimes\phi_j(\mathbf{0})\otimes\varphi(\mathbf{0}) \\ \int_0^1\rho\Big(\int_0^\rho\frac{dB^{t,i}_u}{1-u}\Big)d\rho\times\int_0^1\Big(\int_0^\theta dB^{t,j}_\delta\Big)d\theta.$$

<u>*Result*</u>:

$$\mathbb{E}[B_{JI;KJ}] = -\frac{1}{12}\phi_{i,i|kk|j,j}t^3+o(t^3),$$
$$\mathbb{E}[B_{KJ;JI}] = -\frac{1}{12}\phi_{k|i,i|j,j|k}t^3+o(t^3). \quad \text{(by symmetry)}$$

B.1.6 The $(IJ;KJ)$ and $(KJ;IJ)$ terms

Target:

$$B_{IJ;KJ} \triangleq -t^{5/2}\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})\frac{X^i_{tr}}{1-r}dr\Big)\otimes\varphi(X_{t\rho})d\rho \\ \otimes\int_0^1 \Big(\int_0^\theta \phi_j(X_{t\delta})(\sigma(X_{t\delta})dB^t_\delta)^j\Big)\otimes\varphi(X_{t\theta})d\theta.$$

Reduction:

$$B_{IJ;KJ} \stackrel{3}{=} -t^3\phi_i(\mathbf{0})\otimes\varphi(\mathbf{0})\otimes\phi_j(\mathbf{0})\otimes\varphi(\mathbf{0}) \\ \times\int_{0<r<\rho<1}\Big(\int_0^r\frac{dB^{t,i}_u}{1-u}\Big)drd\rho\times\int_0^1\Big(\int_0^\theta dB^{t,j}_\delta\Big)d\theta.$$

<u>*Result*</u>:

$$\mathbb{E}[B_{IJ;KJ}] = -\frac{1}{24}\phi_{k|i,i|k|j,j}t^3+o(t^3),$$
$$\mathbb{E}[B_{KJ;IJ}] = -\frac{1}{24}\phi_{k|i,i|k|j,j}t^3+o(t^3). \quad \text{(by symmetry)}$$

B.1.7 The $(JK;JK)$ term

Target:

$$B_{JK;JK} \triangleq t^3\int_0^1\Big(\int_0^\rho\varphi(X_{tr})dr\Big)\otimes\phi_i(X_{t\rho})(\sigma(X_{t\rho})dB^t_\rho)^i \\ \otimes\int_0^1\Big(\int_0^\theta\varphi(X_{t\delta})d\delta\Big)\otimes\phi_j(X_{t\theta})(\sigma(X_{t\theta})dB^t_\theta)^j.$$

Reduction:

$$B_{JK;JK} \stackrel{3}{=} t^3\varphi(\mathbf{0})\otimes\phi_i(\mathbf{0})\otimes\varphi(\mathbf{0})\otimes\phi_j(\mathbf{0})\times\int_0^1\rho dB^{t,i}_\rho\times\int_0^1\theta dB^{t,j}_\theta.$$

<u>*Result*</u>:

$$\mathbb{E}[B_{JK;JK}] = \frac{1}{12}\phi_{i,i|k|j,j|k}t^3+o(t^3).$$

B.1.8 The $(JK;KJ)$ and $(KJ;JK)$ terms

Target:

$$
\begin{aligned}
B_{JK;KJ} \triangleq & t^3 \int_0^1 \big(\int_0^\rho \varphi(X_{tr}) dr \big) \otimes \phi_i(X_{t\rho}) (\sigma(X_{t\rho}) dB_\rho^t)^i \\
& \otimes \int_0^1 \big(\int_0^\theta \phi_j(X_{t\delta}) (\sigma(X_{t\delta}) dB_\delta^t)^j \big) \otimes \varphi(X_{t\theta}) d\theta.
\end{aligned}
$$

Reduction:

$$
B_{JK;KJ} = t^3 \varphi(\mathbf{0}) \otimes \phi_i(\mathbf{0}) \otimes \phi_j(\mathbf{0}) \otimes \varphi(\mathbf{0}) \times \int_0^1 \rho dB_\rho^{t,i} \times \int_0^1 \big(\int_0^\theta dB_\delta^{t,j} \big) d\theta.
$$

<u>*Result*</u>:

$$
\begin{aligned}
\mathbb{E}[B_{JK;KJ}] &= \frac{1}{24} \phi_{i,i|kk|j,j} t^3 + o(t^3), \\
\mathbb{E}[B_{KJ;JK}] &= \frac{1}{24} \phi_{k|i,i|j,j|k} t^3 + o(t^3). \quad \text{(by symmetry)}
\end{aligned}
$$

B.1.9 The $(KJ;KJ)$ term

Target:

$$
\begin{aligned}
B_{KJ;KJ} \triangleq & t^3 \int_0^1 \big(\int_0^\rho \phi_i(X_{tr}) (\sigma(X_{tr}) dB_r^t)^i \big) \otimes \varphi(X_{t\rho}) d\rho \\
& \otimes \int_0^1 \big(\int_0^\theta \phi_j(X_{t\delta}) (\sigma(X_{t\delta}) dB_\delta^t)^j \big) \otimes \varphi(X_{t\theta}) d\theta.
\end{aligned}
$$

Reduction:

$$
\begin{aligned}
B_{KJ;KJ} \stackrel{3}{=} & t^3 \phi_i(\mathbf{0}) \otimes \varphi(\mathbf{0}) \otimes \phi_j(\mathbf{0}) \otimes \varphi(\mathbf{0}) \\
& \times \int_0^1 \big(\int_0^\rho dB_r^{t,i} \big) dr \times \int_0^1 \big(\int_0^\theta dB_\delta^{t,j} \big) d\theta.
\end{aligned}
$$

<u>*Result*</u>:

$$
\mathbb{E}[B_{KJ;KJ}] = \frac{1}{12} \phi_{k|i,i|k|j,j} t^3 + o(t^3).
$$

B.1.10 The $(JJ;II)$ and $(II;JJ)$ terms

Target:

$$\begin{aligned}B_{JJ;II} \triangleq & t^2 \int_0^1 \big(\int_0^\rho \varphi(X_{tr})dr\big) \otimes \varphi(X_{t\rho})d\rho \\ & \otimes \int_0^1 \big(\int_0^\theta \phi_i(X_{t\delta})\frac{X^i_{t\delta}}{1-\delta}d\delta\big) \otimes \phi_j(X_{t\theta})\frac{X^j_{t\theta}}{1-\theta}d\theta.\end{aligned}$$

Reduction:

$$\begin{aligned}B_{JJ;II} \stackrel{3}{=} & t^3\varphi(\mathbf{0}) \otimes \varphi(\mathbf{0}) \otimes \phi_i(\mathbf{0}) \otimes \phi_j(\mathbf{0}) \\ & \times \int_0^1 \rho d\rho \times \int_{0<\delta<\theta<1} \big(\int_0^\delta \frac{dB^{t,i}_u}{1-u}\big)\big(\int_0^\theta \frac{dB^{t,j}_v}{1-v}\big)d\delta d\theta.\end{aligned}$$

Result:

$$\mathbb{E}[B_{JJ;II}] = \frac{1}{16}\phi_{i,i|j,j|kk}t^3 + o(t^3),$$
$$\mathbb{E}[B_{JJ;II}] = \frac{1}{16}\phi_{kk|i,i|j,j}t^3 + o(t^3). \quad \text{(by symmetry)}$$

B.1.11 The $(JJ;IK)$, $(IK;JJ)$, $(JJ;KK)$ and $(KK;JJ)$ terms

Target:

$$\begin{aligned}B_{JJ;IK} \triangleq & -t^{5/2} \int_0^1 \big(\int_0^\rho \varphi(X_{tr})dr\big) \otimes \varphi(X_{t\rho})d\rho \\ & \otimes \int_0^1 \big(\int_0^\theta \phi_i(X_{t\delta})\frac{X^i_{t\delta}}{1-\delta}d\delta\big) \otimes \phi_j(X_{t\theta})(\sigma(X_{t\theta})dB^t_\theta)^j.\end{aligned}$$

If one freezes $\varphi(X_{tr})$ and $\varphi(X_{t\rho})$ at the origin, the resulting expression is an Itô integral. The same pattern occurs for the $(JJ;KK)$ term. As a result, one sees that $\mathbb{E}[B_{JJ;IK}], \mathbb{E}[B_{IK;JJ}], \mathbb{E}[B_{JJ;KK}], \mathbb{E}[B_{KK;JJ}]$ are all of order $o(t^3)$.

B.1.12 The $(JJ;KI)$ and $(KI;JJ)$ terms

Target:

$$\begin{aligned}B_{JJ;KI} \triangleq & -t^{5/2} \int_0^1 \big(\int_0^\rho \varphi(X_{tr})dr\big) \otimes \varphi(X_{t\rho})d\rho \\ & \otimes \int_0^1 \big(\int_0^\theta \phi_i(X_{t\delta})(\sigma(X_{t\delta})dB^t_\delta)^i\big) \otimes \phi_j(X_{t\theta})\frac{X^j_{t\theta}}{1-\theta}d\theta.\end{aligned}$$

Reduction:

$$B_{JJ;KI} \stackrel{3}{=} - t^3 \varphi(\mathbf{0}) \otimes \varphi(\mathbf{0}) \otimes \phi_i(\mathbf{0}) \otimes \phi_j(\mathbf{0})$$
$$\times \int_0^1 \rho d\rho \times \int \big(\int_0^\theta dB_\delta^{t,i} \big) \big(\int_0^\theta \frac{dB_u^{t,j}}{1-u} \big) d\theta.$$

<u>*Result*</u>:

$$\mathbb{E}[B_{JJ;KI}] = -\frac{1}{8}\phi_{i,i|j,j|kk} t^3 + o(t^3),$$
$$\mathbb{E}[B_{KI;JJ}] = -\frac{1}{8}\phi_{kk|i,i|j,j} t^3 + o(t^3). \quad \text{(by symmetry)}$$

B.1.13 The $(JJ;P)$ and $(P;JJ)$ terms

Target:

$$B_{JJ;P} \triangleq \frac{1}{2} t^3 \int_0^1 \big(\int_0^\rho \varphi(X_{tr}) dr \big) \otimes \varphi(X_{t\rho}) d\rho$$
$$\otimes \int_0^1 \phi_i(X_{t\theta}) \otimes \phi_j(X_{t\theta}) a^{ij}(X_{t\theta}) d\theta.$$

Reduction:

$$B_{JJ;P} \stackrel{3}{=} \frac{1}{2} t^3 \varphi(\mathbf{0}) \otimes \varphi(\mathbf{0}) \otimes \phi_i(\mathbf{0}) \otimes \phi_j(\mathbf{0}) \times \int_0^1 \rho d\rho \times \delta^{ij} \int_0^1 d\theta.$$

<u>*Result*</u>:

$$\mathbb{E}[B_{JJ;P}] = \frac{1}{16}\phi_{i,i|j,j|kk} t^3 + o(t^3),$$
$$\mathbb{E}[B_{JJ;P}] = \frac{1}{16}\phi_{kk|i,i|j,j} t^3 + o(t^3). \quad \text{(by symmetry)}$$

B.2 Total degree $= 2.5$

Here we present the results for all remaining cases listed in (5.32). The corresponding results for the permuted cases (i.e. interchanging the $(1,2)$ and $(3,4)$ tensor slots) are obtained directly by tensor permutation and will not be displayed here.

The calculation for this part is computer-assisted by Wolfram Mathematica. We only present the final expressions; all source codes and documentation are provided at the link [1]. The displayed tensors are always evaluated at the origin; for instance

$$\partial_i\phi_i \otimes \varphi \otimes \phi_j \otimes \phi_j \triangleq \partial_i\phi_i(\mathbf{0}) \otimes \varphi(\mathbf{0}) \otimes \phi_j(\mathbf{0}) \otimes \phi_j(\mathbf{0}).$$

B.2.1 The $(IJ;P)$ term

Target:

$$-\frac{1}{2}t^2 \int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})\frac{X_{tr}^i}{1-r}dr\Big) \otimes \varphi(X_{t\rho})d\rho \otimes \int_0^1 \phi_k(X_{t\theta})\phi_l(X_{t\theta})a^{kl}(X_{t\theta})d\theta.$$

The extra order of $\sqrt{t}$ comes from one of the following four possibilities: expanding ϕ_i, φ, ϕ_k or ϕ_l. Here the possibilities of expanding $\frac{X_{tr}^i}{1-r}$ or a^{kl} are not considered because the extra order gained from further expanding these two terms is t, which exceeds the needed $\sqrt{t}$.

Expand ϕ_i: the resulting expectation is that

$$-\frac{1}{12}\partial_i\phi_i \otimes \varphi \otimes \phi_j \otimes \phi_j t^3 + o(t^3).$$

In what follows, we will only display the t^3-coefficient and omit the symbols "$\times t^3$" and "$+o(t^3)$".

Expand φ :

$$-\frac{1}{24}\phi_i \otimes \partial_i\varphi \otimes \phi_j \otimes \phi_j.$$

Expand ϕ_k :

$$-\frac{1}{24}\phi_i \otimes \varphi \otimes \partial_i\phi_j \otimes \phi_j.$$

Expand ϕ_l:

$$-\frac{1}{24}\phi_i \otimes \varphi \otimes \phi_j \otimes \partial_i\phi_j.$$

B.2.2 The $(JK;P)$ term

Target:

$$\frac{1}{2}t^{5/2} \int_0^1 \Big(\int_0^\rho \varphi(X_{tr})dr\Big) \otimes \phi_i(X_{t\rho})(\sigma dB^t)^i_\rho \otimes \int_0^1 \phi_k(X_{t\theta})\phi_l(X_{t\theta})a^{kl}(X_{t\theta})d\theta.$$

[1] https://github.com/DeepIntoStreams/ESig_BM_on_Manifold

Expand ϕ_k :

$$\frac{1}{24}\varphi \otimes \phi_i \otimes \partial_i \phi_j \otimes \phi_j.$$

Expand ϕ_l :

$$\frac{1}{24}\varphi \otimes \phi_i \otimes \phi_j \otimes \partial_i \phi_j.$$

Freeze both: the result is 0.

B.2.3 The $(KJ; P)$ term

Target:

$$\frac{1}{2}t^{5/2}\int_0^1 \big(\int_0^\rho \phi_i(X_{tr})(\sigma dB^t)^i_r\big) \otimes \varphi(X_{t\rho})d\rho \otimes \int_0^1 \phi_k(X_{t\theta})\phi_l(X_{t\theta})a^{kl}(X_{t\theta})d\theta.$$

Expand φ :

$$\frac{1}{8}\phi_i \otimes \partial_i\varphi \otimes \phi_j \otimes \phi_j.$$

Expand ϕ_k :

$$\frac{1}{12}\phi_i \otimes \varphi \otimes \partial_i\phi_j \otimes \phi_j.$$

Expand ϕ_l:

$$\frac{1}{12}\phi_i \otimes \varphi \otimes \phi_j \otimes \partial_i\phi_j.$$

Freeze all: the result is 0.

B.2.4 The $(JI; II)$ term

Target:

$$-t\int_0^1 \big(\int_0^\rho \varphi(X_{tr})dr\big)\otimes\phi_j(X_{t\rho})\frac{X^j_{t\rho}}{1-\rho}d\rho\otimes\int_0^1 \big(\int_0^\theta \phi_k(X_{t\delta})\frac{X^k_{t\delta}}{1-\delta}d\delta\big)\otimes\phi_l(X_{t\theta})\frac{X^l_{t\theta}}{1-\theta}d\theta.$$

Expand φ:

$$\partial_i\varphi \otimes \phi_j \otimes \phi_k \otimes \phi_l \times t^3\big(-\frac{25}{432}\delta^{il}\delta^{jk} - \frac{55}{432}\delta^{ik}\delta^{jl} - \frac{1}{12}\delta^{ij}\delta^{kl}\big) + o(t^3).$$

Expand ϕ_j:

$$\varphi \otimes \partial_i\phi_j \otimes \phi_k \otimes \phi_l \times \big(-\frac{19}{216}\delta^{il}\delta^{jk} - \frac{19}{216}\delta^{ik}\delta^{jl} - \frac{1}{6}\delta^{ij}\delta^{kl}\big).$$

Expand ϕ_k:

$$\varphi \otimes \phi_j \otimes \partial_i \phi_k \otimes \phi_l \times \big(-\frac{7}{72}\delta^{il}\delta^{jk} - \frac{35}{144}\delta^{ik}\delta^{jl} - \frac{7}{72}\delta^{ij}\delta^{kl}\big).$$

Expand ϕ_l:

$$\varphi \otimes \phi_j \otimes \phi_k \otimes \partial_i \phi_l \times \big(-\frac{19}{144}\delta^{il}\delta^{jk} - \frac{1}{12}\delta^{ik}\delta^{jl} - \frac{1}{12}\delta^{ij}\delta^{kl}\big).$$

B.2.5 The $(IJ; II)$ term

Target:

$$-t\int_0^1 \big(\int_0^\rho \phi_i(X_{tr})\frac{X^i_{tr}}{1-r}dr\big)\varphi(X_{t\rho})d\rho \otimes \int_0^1 \big(\int_0^\theta \phi_k(X_{t\delta})\frac{X^k_{t\delta}}{1-\delta}d\delta\big)\otimes \phi_l(X_{t\theta})\frac{X^l_{t\theta}}{1-\theta}d\theta.$$

Expand φ:

$$\phi_i \otimes \partial_j \varphi \otimes \phi_k \otimes \phi_l \times \big(-\frac{11}{432}\delta^{il}\delta^{jk} - \frac{17}{432}\delta^{ik}\delta^{jl} - \frac{1}{24}\delta^{ij}\delta^{kl}\big).$$

Expand ϕ_i:

$$\partial_j \phi_i \otimes \varphi \otimes \phi_k \otimes \phi_l \times \big(-\frac{1}{27}\delta^{il}\delta^{jk} - \frac{1}{27}\delta^{ik}\delta^{jl} - \frac{1}{12}\delta^{ij}\delta^{kl}\big).$$

Expand ϕ_k:

$$\phi_i \otimes \varphi \otimes \partial_j \phi_k \otimes \phi_l \times \big(-\frac{1}{16}\delta^{il}\delta^{jk} - \frac{1}{24}\delta^{ik}\delta^{jl} - \frac{1}{24}\delta^{ij}\delta^{kl}\big).$$

Expand ϕ_l:

$$\phi_i \otimes \varphi \otimes \phi_k \otimes \partial_j \phi_l \times \big(-\frac{1}{36}\delta^{il}\delta^{jk} - \frac{1}{16}\delta^{ik}\delta^{jl} - \frac{1}{36}\delta^{ij}\delta^{kl}\big).$$

B.2.6 The $(JK; II)$ term

Target:

$$t^{3/2}\int_0^1 \big(\int_0^\rho \varphi(X_{tr})dr\big)\otimes \phi_i(X_{t\rho})(\sigma dB^t)^i_\rho \otimes \int_0^1 \big(\int_0^\theta \phi_k(X_{t\delta})\frac{X^k_{t\delta}}{1-\delta}d\delta\big)\otimes \phi_l(X_{t\theta})\frac{X^l_{t\theta}}{1-\theta}d\theta.$$

Expand ϕ_k:

$$\varphi \otimes \phi_i \otimes \partial_j \phi_k \otimes \phi_l \times \big(\frac{13}{72}\delta^{il}\delta^{jk} + \frac{1}{18}\delta^{ik}\delta^{jl} + \frac{1}{18}\delta^{ij}\delta^{kl}\big).$$

Expand ϕ_l:

$$\varphi \otimes \phi_i \otimes \phi_k \otimes \partial_j \phi_l \times \big(\frac{1}{18}\delta^{il}\delta^{jk} + \frac{5}{72}\delta^{ik}\delta^{jl} + \frac{1}{18}\delta^{ij}\delta^{kl}\big).$$

Freeze both and expand φ:

$$\partial_j \varphi \otimes \phi_i \otimes \phi_k \otimes \phi_l \times \big(\frac{13}{144}\delta^{il}\delta^{jk} + \frac{1}{48}\delta^{ik}\delta^{jl}\big).$$

Freeze both and expand ϕ_i:

$$\varphi \otimes \partial_j \phi_i \otimes \phi_k \otimes \phi_l \times \big(\frac{7}{72}\delta^{il}\delta^{jk} + \frac{1}{24}\delta^{ik}\delta^{jl}\big).$$

B.2.7 The $(KJ;II)$ term

Target:

$$t^{3/2}\int_0^1 \big(\int_0^\rho \phi_i(X_{tr})(\sigma dB^t)^i_r\big)\otimes\varphi(X_{t\rho})d\rho\otimes\int_0^1 \big(\int_0^\theta \phi_k(X_{t\delta})\frac{X^k_{t\delta}}{1-\delta}d\delta\big)\otimes\phi_l(X_{t\theta})\frac{X^l_{t\theta}}{1-\theta}d\theta.$$

Expand φ:

$$\phi_i \otimes \partial_j \varphi \otimes \phi_k \otimes \phi_l \times \big(\frac{1}{16}\delta^{il}\delta^{jk} + \frac{11}{144}\delta^{ik}\delta^{jl} + \frac{1}{8}\delta^{ij}\delta^{kl}\big).$$

Expand ϕ_k:

$$\phi_i \otimes \varphi \otimes \partial_j \phi_k \otimes \phi_l \times \big(\frac{1}{8}\delta^{il}\delta^{jk} + \frac{1}{12}\delta^{ik}\delta^{jl} + \frac{1}{12}\delta^{ij}\delta^{kl}\big).$$

Expand ϕ_l:

$$\phi_i \otimes \varphi \otimes \phi_k \otimes \partial_j \phi_l \times \big(\frac{1}{18}\delta^{il}\delta^{jk} + \frac{1}{8}\delta^{ik}\delta^{jl} + \frac{1}{18}\delta^{ij}\delta^{kl}\big).$$

Freeze all and expand ϕ_i:

$$\partial_j \phi_i \otimes \varphi \otimes \phi_k \otimes \phi_l \times \big(\frac{5}{72}\delta^{il}\delta^{jk} + \frac{1}{24}\delta^{ik}\delta^{jl}\big).$$

B.2.8 The $(IJ;IK)$ term

Target:

$$t^{3/2}\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})\frac{X^i_{tr}}{1-r}dr\Big)\otimes\varphi(X_{t\rho})d\rho\otimes\int_0^1 \Big(\int_0^\theta \phi_k(X_{t\delta})\frac{X^k_{t\delta}}{1-\delta}d\delta\Big)\otimes\phi_l(X_{t\theta})(\sigma dB^t)^l_\theta.$$

Expand φ:

$$\phi_i\otimes\partial_j\varphi\otimes\phi_k\otimes\phi_l\times\Big(\frac{1}{216}\delta^{il}\delta^{jk}+\frac{1}{54}\delta^{ik}\delta^{jl}\Big).$$

Expand ϕ_i:

$$\partial_j\phi_i\otimes\varphi\otimes\phi_k\otimes\phi_l\times\Big(\frac{1}{108}\delta^{il}\delta^{jk}+\frac{1}{108}\delta^{ik}\delta^{jl}\Big).$$

Freeze both and expand ϕ_k:

$$\phi_i\otimes\varphi\otimes\partial_j\phi_k\otimes\phi_l\times\Big(\frac{1}{48}\delta^{il}\delta^{jk}\Big).$$

Freeze both and expand ϕ_l:

$$\phi_i\otimes\varphi\otimes\phi_k\otimes\partial_j\phi_l\times\Big(\frac{1}{72}\delta^{il}\delta^{jk}\Big).$$

B.2.9 The $(JI;IK)$ term

Target:

$$t^{3/2}\int_0^1 \Big(\int_0^\rho \varphi(X_{tr})dr\Big)\otimes\phi_i(X_{t\rho})\frac{X^i_{t\rho}}{1-\rho}d\rho\otimes\int_0^1 \Big(\int_0^\theta \phi_k(X_{t\delta})\frac{X^k_{t\delta}}{1-\delta}d\delta\Big)\otimes\phi_l(X_{t\theta})(\sigma dB^t)^l_\theta.$$

Expand φ:

$$\partial_j\varphi\otimes\phi_i\otimes\phi_k\otimes\phi_l\times\Big(\frac{17}{216}\delta^{il}\delta^{jk}+\frac{1}{108}\delta^{ik}\delta^{jl}\Big).$$

Expand ϕ_i:

$$\varphi\otimes\partial_j\phi_i\otimes\phi_k\otimes\phi_l\times\Big(\frac{7}{216}\delta^{il}\delta^{jk}+\frac{7}{216}\delta^{ik}\delta^{jl}\Big).$$

Freeze both and expand ϕ_k:

$$\varphi\otimes\phi_i\otimes\partial_j\phi_k\otimes\phi_l\times\Big(\frac{7}{48}\delta^{il}\delta^{jk}\Big).$$

Freeze both and expand ϕ_l

$$\varphi\otimes\phi_i\otimes\phi_k\otimes\partial_j\phi_l\times\Big(\frac{5}{72}\delta^{il}\delta^{jk}\Big).$$

B.2.10 The $(KJ; IK)$ term

Target:

$$-t^2\int_0^1 \big(\int_0^\rho \phi_i(X_{tr})(\sigma dB^t)^i_r\big)\otimes\varphi(X_{t\rho})d\rho\otimes\int_0^1 \big(\int_0^\theta \phi_k(X_{t\delta})\frac{X^k_{t\delta}}{1-\delta}d\delta\big)\otimes\phi_l(X_{t\theta})(\sigma dB^t)^l_\theta.$$

Expand φ:

$$\phi_i\otimes\partial_j\varphi\otimes\phi_k\otimes\phi_l\times\big(-\frac{1}{36}\delta^{ik}\delta^{jl}-\frac{1}{72}\delta^{il}\delta^{jk}\big).$$

Expand ϕ_i:

$$\partial_j\phi_i\otimes\varphi\otimes\phi_k\otimes\phi_l\times\big(-\frac{1}{36}\delta^{il}\delta^{jk}\big).$$

Expand ϕ_k:

$$\phi_i\otimes\varphi\otimes\partial_j\phi_k\otimes\phi_l\times\big(-\frac{1}{24}\delta^{il}\delta^{jk}\big).$$

Expand ϕ_l:

$$\phi_i\otimes\varphi\otimes\phi_k\otimes\partial_j\phi_l\times\big(-\frac{1}{36}\delta^{il}\delta^{jk}\big).$$

B.2.11 The $(JK; IK)$ term

Target:

$$-\delta^{il}t^2\int_0^1 \big(\int_0^\rho \varphi(X_{tr})dr\big)\otimes\phi_i(X_{t\rho})\otimes\big(\int_0^\rho \phi_k(W_{t\eta})\frac{X^k_{t\eta}}{1-\eta}d\eta\big)\otimes\phi_l(X_{t\rho})d\rho.$$

Expand φ:

$$\partial_j\varphi\otimes\phi_i\otimes\phi_k\otimes\phi_l\times\big(-\frac{5}{72}\delta^{il}\delta^{jk}\big).$$

Expand ϕ_i:

$$\varphi\otimes\partial_j\phi_i\otimes\phi_k\otimes\phi_l\times\big(-\frac{1}{18}\delta^{il}\delta^{jk}\big).$$

Expand ϕ_k:

$$\varphi\otimes\phi_i\otimes\partial_j\phi_k\otimes\phi_l\times\big(-\frac{1}{8}\delta^{il}\delta^{jk}\big).$$

Expand ϕ_l:

$$\varphi\otimes\phi_i\otimes\phi_k\otimes\partial_j\phi_l\times\big(-\frac{1}{18}\delta^{il}\delta^{jk}\big).$$

B.2.12 The $(IJ;KI)$ term

Target:

$$t^{3/2}\int_0^1\Big(\int_0^\rho \phi_i(X_{tr})\frac{X^i_{tr}}{1-r}dr\Big)\otimes\varphi(X_{t\rho})d\rho\otimes\int_0^1\Big(\int_0^\theta \phi_k(X_{t\delta})(\sigma dB^t)^k_\delta\Big)\otimes\phi_l(X_{t\theta})\frac{X^l_{t\theta}}{1-\theta}d\theta.$$

Expand φ:

$$\phi_i\otimes\partial_j\varphi\otimes\phi_k\otimes\phi_l\times\Big(\frac{5}{108}\delta^{il}\delta^{jk}+\frac{13}{216}\delta^{ik}\delta^{jl}+\frac{1}{12}\delta^{ij}\delta^{kl}\Big).$$

Expand ϕ_i:

$$\partial_j\phi_i\otimes\varphi\otimes\phi_k\otimes\phi_l\times\Big(\frac{7}{108}\delta^{il}\delta^{jk}+\frac{7}{108}\delta^{ik}\delta^{jl}+\frac{1}{6}\delta^{ij}\delta^{kl}\Big).$$

Expand ϕ_l:

$$\phi_i\otimes\varphi\otimes\phi_k\otimes\partial_j\phi_l\times\Big(\frac{5}{72}\delta^{il}\delta^{jk}+\frac{5}{48}\delta^{ik}\delta^{jl}+\frac{5}{72}\delta^{ij}\delta^{kl}\Big).$$

Freeze all and expand ϕ_k:

$$\phi_i\otimes\varphi\otimes\partial_j\phi_k\otimes\phi_l\times\Big(\frac{1}{24}\delta^{ik}\delta^{jl}+\frac{1}{12}\delta^{ij}\delta^{kl}\Big).$$

B.2.13 The $(JI;KI)$ term

Target:

$$t^{3/2}\int_0^1\Big(\int_0^\rho \varphi(X_{tr})dr\Big)\otimes\phi_i(X_{t\rho})\frac{X^i_{t\rho}}{1-\rho}d\rho\otimes\int_0^1\Big(\int_0^\theta \phi_k(X_{t\delta})(\sigma dB^t)^k_\delta\Big)\otimes\phi_l(X_{t\theta})\frac{X^l_{t\theta}}{1-\theta}d\theta.$$

Expand φ:

$$\partial_j\varphi\otimes\phi_i\otimes\phi_k\otimes\phi_l\times\Big(\frac{19}{108}\delta^{il}\delta^{jk}+\frac{23}{216}\delta^{ik}\delta^{jl}+\frac{1}{6}\delta^{ij}\delta^{kl}\Big).$$

Expand ϕ_i:

$$\varphi\otimes\partial_j\phi_i\otimes\phi_k\otimes\phi_l\times\Big(\frac{31}{216}\delta^{il}\delta^{jk}+\frac{31}{216}\delta^{ik}\delta^{jl}+\frac{1}{3}\delta^{ij}\delta^{kl}\Big).$$

Expand ϕ_l:

$$\varphi\otimes\phi_i\otimes\phi_k\otimes\partial_j\phi_l\times\Big(\frac{13}{72}\delta^{il}\delta^{jk}+\frac{11}{48}\delta^{ik}\delta^{jl}+\frac{13}{72}\delta^{ij}\delta^{kl}\Big).$$

Freeze all and expand ϕ_k:

$$\varphi\otimes\phi_i\otimes\partial_j\phi_k\otimes\phi_l\times\Big(\frac{1}{6}\delta^{ij}\delta^{kl}+\frac{1}{8}\delta^{ik}\delta^{jl}\Big).$$

B.2.14 The $(KJ; KI)$ term

Target:

$$-t^2\int_0^1\Big(\int_0^\rho \phi_i(X_{tr})(\sigma dB^t)^i_r\Big)\otimes\varphi(X_{t\rho})d\rho\otimes\int_0^1\Big(\int_0^\theta \phi_k(X_{t\delta})(\sigma dB^t)^k_\delta\Big)\otimes\phi_l(X_{t\theta})\frac{X^l_{t\theta}}{1-\theta}d\theta.$$

Expand φ:

$$\phi_i\otimes\partial_j\varphi\otimes\phi_k\otimes\phi_l\times\Big(-\frac{1}{9}\delta^{il}\delta^{jk}-\frac{1}{8}\delta^{ik}\delta^{jl}-\frac{1}{4}\delta^{ij}\delta^{kl}\Big).$$

Expand ϕ_l:

$$\phi_i\otimes\varphi\otimes\phi_k\otimes\partial_j\phi_l\times\Big(-\frac{5}{36}\delta^{il}\delta^{jk}-\frac{5}{24}\delta^{ik}\delta^{jl}-\frac{5}{36}\delta^{ij}\delta^{kl}\Big).$$

Freeze both and expand ϕ_i:

$$\partial_j\phi_i\otimes\varphi\otimes\phi_k\otimes\phi_l\times\Big(-\frac{1}{9}\delta^{il}\delta^{jk}-\frac{1}{12}\delta^{ik}\delta^{jl}\Big).$$

Freeze both and expand ϕ_k:

$$\phi_i\otimes\varphi\otimes\partial_j\phi_k\otimes\phi_l\times\Big(-\frac{1}{6}\delta^{ij}\delta^{kl}-\frac{1}{12}\delta^{ik}\delta^{jl}\Big).$$

B.2.15 The $(JK; KI)$ term

Target:

$$-t^2\int_0^1\Big(\int_0^\rho \varphi(X_{tr})dr\Big)\otimes\phi_i(X_{t\rho})(\sigma dB^t)^i_\rho\otimes\int_0^1\Big(\int_0^\theta \phi_k(X_{t\delta})(\sigma dB^t)^k_\delta\Big)\otimes\phi_l(X_{t\theta})\frac{X^l_{t\theta}}{1-\theta}d\theta.$$

Expand ϕ_l:

$$\varphi\otimes\phi_i\otimes\phi_k\otimes\partial_j\phi_l\times\Big(-\frac{1}{9}\delta^{il}\delta^{jk}-\frac{1}{8}\delta^{ik}\delta^{jl}-\frac{1}{9}\delta^{ij}\delta^{kl}\Big).$$

Freeze ϕ_l and expand ϕ_i:

$$\varphi\otimes\partial_j\phi_i\otimes\phi_k\otimes\phi_l\times\Big(-\frac{5}{36}\delta^{il}\delta^{jk}-\frac{1}{12}\delta^{ik}\delta^{jl}\Big)$$

Freeze ϕ_l and expand ϕ_k:

$$\varphi\otimes\phi_i\otimes\partial_j\phi_k\otimes\phi_l\times\Big(-\frac{1}{12}\delta^{ij}\delta^{kl}-\frac{1}{12}\delta^{ik}\delta^{jl}\Big)$$

Freeze ϕ_l and expand φ:

$$\partial_j\varphi\otimes\phi_i\otimes\phi_k\otimes\phi_l\times\Big(-\frac{1}{9}\delta^{il}\delta^{jk}-\frac{1}{24}\delta^{ik}\delta^{jl}\Big)$$

B.2.16 The $(IJ;KK)$ term

Target:

$$-t^2\int_0^1\Big(\int_0^\rho \phi_i(X_{tr})\frac{X^i_{tr}}{1-r}dr\Big)\otimes\varphi(X_{t\rho})d\rho\otimes\int_0^1\Big(\int_0^\theta \phi_k(X_{t\delta})(\sigma dB^t)^k_\delta\Big)\otimes\phi_l(X_{t\theta})(\sigma dB^t)^l_\theta.$$

Expand ϕ_i:

$$\partial_j\phi_i\otimes\varphi\otimes\phi_k\otimes\phi_l\times\Big(-\frac{1}{27}\delta^{il}\delta^{jk}-\frac{1}{27}\delta^{ik}\delta^{jl}\Big).$$

Expand φ:

$$\phi_i\otimes\partial_j\varphi\otimes\phi_k\otimes\phi_l\times\Big(-\frac{1}{54}\delta^{il}\delta^{jk}-\frac{5}{108}\delta^{ik}\delta^{jl}\Big).$$

Freeze both and expand ϕ_k: the result is 0.
Freeze both and expand ϕ_l:

$$\phi_i\otimes\varphi\otimes\phi_k\otimes\partial_j\phi_l\times\Big(-\frac{1}{18}\delta^{il}\delta^{jk}\Big).$$

B.2.17 The $(JI;KK)$ term

Target:

$$-t^2\int_0^1\Big(\int_0^\rho \varphi(X_{tr})dr\Big)\otimes\phi_i(X_{t\rho})\frac{X^i_{t\rho}}{1-\rho}d\rho\otimes\int_0^1\Big(\int_0^\theta \phi_k(X_{t\delta})(\sigma dB^t)^k_\delta\Big)\otimes\phi_l(X_{t\theta})(\sigma dB^t)^l_\theta.$$

Expand φ:

$$\partial_j\varphi\otimes\phi_i\otimes\phi_k\otimes\phi_l\times\Big(-\frac{4}{27}\delta^{il}\delta^{jk}-\frac{1}{27}\delta^{ik}\delta^{jl}\Big).$$

Expand ϕ_i:

$$\varphi\otimes\partial_j\phi_i\otimes\phi_k\otimes\phi_l\times\Big(-\frac{19}{216}\delta^{il}\delta^{jk}-\frac{19}{216}\delta^{ik}\delta^{jl}\Big).$$

Freeze both and expand ϕ_k: the result is 0.
Freeze both and expand ϕ_l:

$$\varphi\otimes\phi_i\otimes\phi_k\otimes\partial_j\phi_l\times\Big(-\frac{7}{36}\delta^{il}\delta^{jk}\Big).$$

B.2.18 The $(JK;KK)$ term

Target:

$$t^{5/2}\int_0^1\big(\int_0^\rho \varphi(X_{tr})dr\big)\otimes\phi_i(X_{t\rho})(\sigma dB^t)^i_\rho\otimes\int_0^1\big(\int_0^\theta \phi_k(X_{t\delta})(\sigma dB^t)^k_\delta\big)\otimes\phi_l(X_{t\theta})(\sigma dB^t)^l_\theta.$$

Expand φ:

$$\partial_j\varphi\otimes\phi_i\otimes\phi_k\otimes\phi_l\times\big(\frac{1}{9}\delta^{il}\delta^{jk}\big)$$

Expand ϕ_i:

$$\varphi\otimes\partial_j\phi_i\otimes\phi_k\otimes\phi_l\times\big(\frac{5}{36}\delta^{il}\delta^{jk}\big)$$

Expand ϕ_k: the result is 0.
Expand ϕ_l:

$$\varphi\otimes\phi_i\otimes\phi_k\otimes\partial_j\phi_l\times\big(\frac{5}{36}\delta^{il}\delta^{jk}\big)$$

B.2.19 The $(KJ;KK)$ term

Target:

$$t^{5/2}\int_0^1\big(\int_0^\rho \phi_i(X_{tr})(\sigma dB^t)^i_r\big)\otimes\varphi(X_{t\rho})d\rho\otimes\int_0^1\big(\int_0^\theta \phi_k(X_{t\delta})(\sigma dB^t)^k_\delta\big)\otimes\phi_l(X_{t\rho})(\sigma dB^t)^l_\rho.$$

Expand φ:

$$\phi_i\otimes\partial_j\varphi\otimes\phi_k\otimes\phi_l\times\big(\frac{1}{18}\delta^{il}\delta^{jk}+\frac{1}{12}\delta^{ik}\delta^{jl}\big).$$

Freeze φ and expand ϕ_i:

$$\partial_j\phi_i\otimes\varphi\otimes\phi_k\otimes\phi_l\times\big(\frac{1}{9}\delta^{il}\delta^{jk}\big).$$

Freeze φ and expand ϕ_k: the result is 0.
Freeze φ and expand ϕ_l:

$$\phi_i\otimes\varphi\otimes\phi_k\otimes\partial_j\phi_l\times\big(\frac{1}{9}\delta^{il}\delta^{jk}\big).$$

B.3 Total degree $= 2$

Here we present the results for all cases listed in (5.33). The calculation for this part is also computer-assisted by Wolfram Mathematica. We only present the final expressions; all source codes and documentation are provided at the link [2]. Again, we only display the results for half of the list because the corresponding results for the permuted cases (i.e. interchanging the $(1,2)$ and $(3,4)$ tensor slots) are obtained directly by tensor permutation.

We first introduce some notation. We fix the following list of products of Kronecker deltas that goes through all possible different combinations of the indices i,j,k,l,p,q in a specific order:

$$\Delta \triangleq (\delta^{ij}\delta^{kl}\delta^{pq}, \delta^{ij}\delta^{kp}\delta^{lq}, \delta^{ij}\delta^{kq}\delta^{lp}, \delta^{ik}\delta^{jl}\delta^{pq}, \delta^{ik}\delta^{jp}\delta^{lq}, \delta^{ik}\delta^{jq}\delta^{lp}, \delta^{il}\delta^{jk}\delta^{pq}, \delta^{il}\delta^{jp}\delta^{kq}, \delta^{il}\delta^{jq}\delta^{kp}, \delta^{ip}\delta^{jk}\delta^{lq}, \delta^{ip}\delta^{jl}\delta^{kq}, \delta^{ip}\delta^{jq}\delta^{kl}, \delta^{iq}\delta^{jk}\delta^{lp}, \delta^{iq}\delta^{jl}\delta^{kp}, \delta^{iq}\delta^{jp}\delta^{kl}).$$

This list of product Kronecker deltas is regarded as a basis, so that one only needs to record the list of coefficients to represent the final results instead of writing down full expressions. For instance, the vector

$$(2,2,2,0,0,0,0,0,0,0,0,0,0,0,0),$$

represents the expression

$$\Delta \cdot (2,2,2,0,0,0,0,0,0,0,0,0,0,0,0) = 2\delta^{ij}\delta^{kl}\delta^{pq} + 2\delta^{ij}\delta^{kp}\delta^{lq} + 2\delta^{ij}\delta^{kq}\delta^{lp}.$$

We will also use Notation 5.12 exclusively. Recall that $\bar{b}$ is the function defined by (5.11).

B.3.1 The $(II;II)$ term

Target:

$$\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})\frac{X^i_{tr}}{1-r}dr\Big)\otimes\phi_j(X_{t\rho})\frac{X^j_{t\rho}}{1-\rho}d\rho\otimes\int_0^1\Big(\int_0^\theta \phi_k(X_{t\delta})\frac{X^k_{t\delta}}{1-\delta}d\delta\Big)\otimes\phi_l(X_{t\theta})\frac{X^l_{t\theta}}{1-\theta}d\theta.$$

Expand ϕ_i to order t:

$$\phi_{i,pq|jkl}\times\Big(\frac{1}{48},\frac{37}{3456},\frac{37}{3456},\frac{533}{17280},\frac{37}{3456},\frac{37}{3456},\frac{257}{17280},\frac{37}{3456},\frac{37}{3456},\frac{257}{17280},\frac{533}{17280},\frac{1}{48},\frac{257}{17280},\frac{533}{17280},\frac{1}{48}\Big).$$

Expand ϕ_j to order t:

$$\phi_{i|j,pq|kl}\times\Big(\frac{1}{72},\frac{1}{128},\frac{1}{128},\frac{253}{17280},\frac{253}{17280},\frac{253}{17280},\frac{157}{17280},\frac{157}{17280},\frac{157}{17280},\frac{1}{128},\frac{1}{128},\frac{1}{72},\frac{1}{128},\frac{1}{128},\frac{1}{72}\Big).$$

[2] https://github.com/DeepIntoStreams/ESig_BM_on_Manifold

Expand ϕ_k to order t:

$$\phi_{ij|k,pq|l} \times (\frac{1}{48}, \frac{1}{48}, \frac{1}{48}, \frac{533}{17280}, \frac{37}{3456}, \frac{37}{3456}, \frac{257}{17280}, \frac{257}{17280}, \frac{37}{3456}, \frac{533}{17280}, \frac{37}{3456}, \frac{37}{3456}, \frac{37}{3456}, \frac{533}{17280}, \frac{37}{3456}).$$

Expand ϕ_l to order t:

$$\phi_{ijk|l,pq} \times (\frac{1}{72}, \frac{1}{72}, \frac{1}{72}, \frac{253}{17280}, \frac{253}{17280}, \frac{253}{17280}, \frac{157}{17280}, \frac{1}{128}, \frac{1}{128}, \frac{157}{17280}, \frac{1}{128}, \frac{1}{128}, \frac{157}{17280}, \frac{1}{128}, \frac{1}{128}).$$

Expand both ϕ_i and ϕ_j to order $\sqrt{t}$:

$$\phi_{i,p|j,q|kl} \times (\frac{1}{48}, \frac{347}{17280}, \frac{233}{17280}, \frac{347}{17280}, \frac{347}{17280}, \frac{101}{3456}, \frac{233}{17280}, \frac{233}{17280}, \frac{101}{3456}, \frac{115}{3456}, \frac{115}{3456}, \frac{1}{16}, \frac{233}{17280}, \frac{347}{17280}, \frac{1}{48}).$$

Expand both ϕ_i and ϕ_k to order $\sqrt{t}$:

$$\phi_{i,p|j|k,q|l} \times (\frac{1}{45}, \frac{1}{45}, \frac{3}{80}, \frac{11}{270}, \frac{1}{45}, \frac{1}{54}, \frac{1}{54}, \frac{3}{80}, \frac{1}{54}, \frac{3}{80}, \frac{37}{360}, \frac{3}{80}, \frac{1}{54}, \frac{11}{270}, \frac{1}{45}).$$

Expand both ϕ_i and ϕ_l to order $\sqrt{t}$:

$$\phi_{i,p|jk|l,q} \times (\frac{23}{1440}, \frac{23}{1440}, \frac{23}{1440}, \frac{11}{576}, \frac{23}{720}, \frac{11}{576}, \frac{1}{80}, \frac{23}{1440}, \frac{11}{576}, \frac{1}{20}, \frac{97}{2880}, \frac{97}{2880}, \frac{1}{80}, \frac{11}{576}, \frac{23}{1440}).$$

Expand both ϕ_j and ϕ_k to order $\sqrt{t}$:

$$\phi_{i|j,p|k,q|l} \times (\frac{23}{1440}, \frac{23}{1440}, \frac{97}{2880}, \frac{11}{576}, \frac{23}{720}, \frac{11}{576}, \frac{1}{80}, \frac{1}{20}, \frac{1}{80}, \frac{23}{1440}, \frac{97}{2880}, \frac{23}{1440}, \frac{11}{576}, \frac{11}{576}, \frac{23}{720}).$$

Expand both ϕ_j and ϕ_l to order $\sqrt{t}$:

$$\phi_{i|j,p|k|l,q} \times (\frac{7}{540}, \frac{7}{320}, \frac{7}{540}, \frac{19}{1080}, \frac{17}{360}, \frac{19}{1080}, \frac{1}{96}, \frac{7}{320}, \frac{1}{96}, \frac{7}{320}, \frac{7}{540}, \frac{7}{540}, \frac{1}{96}, \frac{1}{96}, \frac{7}{320}).$$

Expand both ϕ_k and ϕ_l to order $\sqrt{t}$:

$$\phi_{ij|k,p|l,q} \times (\frac{1}{48}, \frac{1}{16}, \frac{1}{48}, \frac{347}{17280}, \frac{101}{3456}, \frac{347}{17280}, \frac{233}{17280}, \frac{233}{17280}, \frac{115}{3456}, \frac{101}{3456}, \frac{347}{17280}, \frac{347}{17280}, \frac{233}{17280}, \frac{115}{3456}, \frac{233}{17280}).$$

Expand $\frac{X^i_{tr}}{1-r}$ or $\frac{X^j_{t\rho}}{1-\rho}$ or $\frac{X^k_{t\delta}}{1-\delta}$ or $\frac{X^l_{t\theta}}{1-\theta}$ to order $t^{1.5}$:

$$\phi_{ijkl} \times \Big[\partial_p \bar{b}^i (\frac{1}{24}\delta^{pj}\delta^{kl} + \frac{19}{216}\delta^{jl}\delta^{kp} + \frac{7}{216}\delta^{jk}\delta^{lp}) + \partial_p \bar{b}^j (\frac{1}{12}\delta^{ip}\delta^{kl} + \frac{7}{108}\delta^{il}\delta^{kp} + \frac{7}{108}\delta^{ik}\delta^{lp})$$
$$+\partial_p \bar{b}^k (\frac{19}{216}\delta^{ip}\delta^{jl} + \frac{7}{216}\delta^{il}\delta^{jp} + \frac{1}{24}\delta^{ij}\delta^{lp}) + \partial_p \bar{b}^l (\frac{7}{108}\delta^{ip}\delta^{jk} + \frac{7}{108}\delta^{ik}\delta^{jp} + \frac{1}{12}\delta^{ij}\delta^{kp})\Big].$$

Expand a to order t:

$$\phi_{ijkl} \times \Big[\partial^2_{pq} a^{ij} (\frac{1}{108}\delta^{kq}\delta^{lp} + \frac{1}{108}\delta^{kp}\delta^{lq} + \frac{1}{96}\delta^{kl}\delta^{pq}) + \partial^2_{pq} a^{kl} \frac{1}{96}\delta^{ij}\delta^{pq}$$
$$+ \partial^2_{pq} a^{ik} (\frac{1}{240}\delta^{jq}\delta^{lp} + \frac{1}{240}\delta^{jp}\delta^{lq} + \frac{1}{240}\delta^{jl}\delta^{pq}) + \partial^2_{pq} a^{jl} \frac{1}{288}\delta^{ik}\delta^{pq}$$
$$+ \partial^2_{pq} a^{il} (\frac{23}{2880}\delta^{jq}\delta^{kp} + \frac{23}{2880}\delta^{jp}\delta^{kq} + \frac{23}{2880}\delta^{jk}\delta^{pq}) + \partial^2_{pq} a^{jk} \frac{1}{144}\delta^{il}\delta^{pq}$$
$$+ \partial^2_{pq} a^{jk} (\frac{23}{2880}\delta^{iq}\delta^{lp} + \frac{23}{2880}\delta^{ip}\delta^{lq} + \frac{23}{2880}\delta il\delta^{pq}) + \partial^2_{pq} a^{il} \frac{1}{144}\delta^{jk}\delta^{pq}$$
$$+ \partial^2_{pq} a^{jl} (\frac{73}{4320}\delta^{iq}\delta^{kp} + \frac{73}{4320}\delta^{ik}\delta^{kq} + \frac{31}{1440}\delta^{ik}\delta^{pq}) + \partial^2_{pq} a^{ik} \frac{7}{288}\delta^{jl}\delta^{pq}$$
$$+ \partial^2_{pq} a^{kl} (\frac{1}{108}\delta^{iq}\delta^{jp} + \frac{1}{108}\delta^{ip}\delta^{jq} + \frac{1}{96}\delta^{ij}\delta^{pq}) + \partial^2_{pq} a^{ij} \frac{1}{96}\delta^{kl}\delta^{pq}\Big].$$

B.3.2 The $(II;IK)$ term

Target:

$$-\sqrt{t}\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})\frac{X^i_{tr}}{1-r}dr\Big)\otimes\phi_j(X_{t\rho})\frac{X^j_{t\rho}}{1-\rho}d\rho\otimes\int_0^1 \Big(\int_0^\theta \phi_k(X_{t\delta})\frac{X^k_{t\delta}}{1-\delta}d\delta\Big)\otimes\phi_l(X_\theta)(\sigma dB^t)^l_\theta.$$

Expand ϕ_i to order t:

$$\phi_{i,pq|jkl}\times(0,-\frac{53}{17280},-\frac{53}{17280},-\frac{329}{17280},-\frac{53}{17280},-\frac{53}{17280},-\frac{53}{17280},-\frac{53}{17280},-\frac{53}{17280},-\frac{53}{17280},-\frac{329}{17280},0,-\frac{53}{17280},-\frac{329}{17280},0).$$

Expand ϕ_j to order t:

$$\phi_{i|j,pq|kl}\times(0,-\frac{47}{17280},-\frac{47}{17280},-\frac{221}{51840},-\frac{221}{51840},-\frac{221}{51840},-\frac{53}{51480},-\frac{53}{51840},-\frac{53}{51840},-\frac{47}{17280},-\frac{47}{17280},0,-\frac{47}{17280},-\frac{47}{17280},0).$$

Expand ϕ_k to order t:

$$\phi_{ij|k,pq|l}\times(0,0,0,-\frac{29}{1440},0,0,-\frac{1}{240},-\frac{1}{240},-\frac{1}{240},0,-\frac{29}{1440},0,0,-\frac{29}{1440},0).$$

Expand ϕ_l to order t:

$$\phi_{ijk|l,pq}\times(0,0,0,-\frac{13}{960},0,0,-\frac{11}{2880},-\frac{11}{2880},-\frac{11}{2880},0,-\frac{77}{8640},0,0,-\frac{77}{8640},0).$$

Expand both ϕ_i and ϕ_j to order $\sqrt{t}$:

$$\phi_{i,p|j,q|kl}\times(0,-\frac{167}{17280},-\frac{53}{17280},-\frac{167}{17280},-\frac{167}{17280},-\frac{53}{17280},-\frac{53}{17280},-\frac{53}{17280},-\frac{53}{17280},-\frac{227}{17280},-\frac{227}{17280},0,-\frac{53}{17280},-\frac{167}{17280},0).$$

Expand both ϕ_i and ϕ_k to order $\sqrt{t}$:

$$\phi_{i,p|j|k,q|l}\times(0,0,-\frac{11}{720},-\frac{7}{270},0,-\frac{1}{270},-\frac{1}{270},-\frac{11}{720},-\frac{1}{270},0,-\frac{47}{720},0,-\frac{1}{270},-\frac{7}{270},0).$$

Expand both ϕ_i and ϕ_l to order $\sqrt{t}$:

$$\phi_{i,p|jk|l,q}\times(0,0,-\frac{7}{720},-\frac{79}{4320},0,-\frac{11}{2160},-\frac{11}{2160},-\frac{7}{720},-\frac{11}{2160},0,-\frac{43}{1440},0,-\frac{11}{2160},-\frac{79}{4320},0).$$

Expand both ϕ_j and ϕ_k to order $\sqrt{t}$:

$$\phi_{i|j,p|k,q|l}\times(0,0,-\frac{17}{960},-\frac{73}{8640},0,-\frac{73}{8640},-\frac{1}{540},-\frac{13}{720},-\frac{1}{540},0,-\frac{17}{960},0,-\frac{73}{8640},-\frac{73}{8640},0).$$

Expand both ϕ_j and ϕ_l to order $\sqrt{t}$:

$$\phi_{i|j,p|k|l,q}\times(0,0,-\frac{43}{4320},-\frac{53}{4320},0,-\frac{53}{4320},-\frac{11}{4320},-\frac{17}{1440},-\frac{11}{4320},0,-\frac{43}{4320},0,-\frac{11}{1440},-\frac{11}{1440},0).$$

Expand both ϕ_k and ϕ_l to order $\sqrt{t}$:

$$\phi_{ij|k,p|l,q} \times \big(0,0,0,-\frac{3}{160},0,0,-\frac{1}{180},-\frac{1}{180},-\frac{1}{80},0,-\frac{3}{160},0,0,-\frac{47}{1440},0\big).$$

Expand $\frac{X^i_{tr}}{1-r}$ or $\frac{X^j_{t\rho}}{1-\rho}$ or $\frac{X^k_{t\delta}}{1-\delta}$ to order $t^{1.5}$:

$$\phi_{ijkl} \times \Big[\partial_p \bar{b}^i\big(-\frac{13}{216}\delta^{jl}\delta^{kp}-\frac{1}{216}\delta^{jk}\delta^{lp}\big)+\partial_p \bar{b}^j\big(-\frac{5}{216}\delta^{il}\delta^{kp}-\frac{5}{216}\delta^{ik}\delta^{lp}\big)$$
$$+\partial_p \bar{b}^k\big(-\frac{1}{144}\delta^{il}\delta^{kp}-\frac{7}{144}\delta^{ik}\delta^{jl}\big)\Big].$$

Expand a to order t:

$$\phi_{ijkl} \times \Big[\partial_{pq}a^{il}\big(-\frac{11}{2880}\delta^{jq}\delta^{kp}-\frac{11}{2880}\delta^{jp}\delta^{kq}-\frac{11}{2880}\delta^{jk}\delta^{pq}\big)+\partial_{pq}a^{jk}\big(\big(-\frac{1}{288}\delta^{il}\delta^{pq}\big)$$
$$+\partial_{pq}a^{jl}\big(-\frac{77}{8640}\delta^{iq}\delta^{kq}-\frac{77}{8640}\delta^{ip}\delta^{kq}-\frac{13}{960}\delta^{ik}\delta^{pq}\big)$$
$$+\partial_{pq}a^{ik}\big(-\frac{5}{288}\delta^{jl}\delta^{pq}\big)+\partial_{pq}a^{ij}\big(-\frac{1}{432}\delta^{kq}\delta^{lp}-\frac{1}{432}\delta^{kp}\delta^{lq}\big)\Big].$$

B.3.3 The $(II;KI)$ term

Target:

$$-\sqrt{t}\int_0^1\Big(\int_0^\rho \phi_i(X_{tr})\frac{X^i_{tr}}{1-r}dr\Big)\otimes\phi_j\big(X_{t\rho}\big)\frac{X^j_{t\rho}}{1-\rho}d\rho\otimes\int_0^1\Big(\int_0^\theta \phi_k(X_{t\delta})(\sigma dB^t)^k_\delta\Big)\otimes\phi_l(X_{t\theta})\frac{X^l_{t\theta}}{1-\theta}d\theta.$$

Expand ϕ_i to order t:

$$\phi_{i,pq|jkl}\times\big(-\frac{1}{24},-\frac{317}{17280},-\frac{317}{17280},-\frac{737}{17280},-\frac{317}{17280},-\frac{317}{17280},-\frac{461}{17280},-\frac{317}{17280},-\frac{317}{17280},-\frac{461}{17280},-\frac{737}{17280},-\frac{1}{24},-\frac{461}{17280},-\frac{737}{17280},-\frac{1}{24}\big).$$

Expand ϕ_j to order t:

$$\phi_{i|j,pq|kl}\times\big(-\frac{1}{36},-\frac{223}{17280},-\frac{223}{17280},-\frac{379}{17280},-\frac{379}{17280},-\frac{379}{17280},-\frac{283}{17280},-\frac{283}{17280},-\frac{283}{17280},-\frac{223}{17280},-\frac{223}{17280},-\frac{1}{36},-\frac{223}{17280},-\frac{223}{17280},-\frac{1}{36}\big).$$

Expand ϕ_k to order t:

$$\phi_{ij|k,pq|l}\times\big(-\frac{1}{24},0,0,-\frac{19}{480},-\frac{1}{120},-\frac{1}{120},-\frac{19}{720},0,0,-\frac{23}{1440},0,-\frac{1}{54},-\frac{23}{1440},0,-\frac{1}{54}\big).$$

Expand ϕ_l to order t:

$$\phi_{ijk|l,pq}\times\big(-\frac{5}{144},-\frac{5}{144},-\frac{5}{144},-\frac{73}{2880},-\frac{73}{2880},-\frac{73}{2880},-\frac{19}{960},-\frac{1}{54},-\frac{1}{54},-\frac{19}{960},-\frac{1}{54},-\frac{1}{54},-\frac{19}{960},-\frac{1}{54},-\frac{1}{54}\big).$$

Expand both ϕ_i and ϕ_j to order $\sqrt{t}$:

$$\phi_{i,p|j,q|kl}\times\big(-\frac{1}{24},-\frac{527}{17280},-\frac{413}{17280},-\frac{527}{17280},-\frac{527}{17280},-\frac{877}{17280},-\frac{413}{17280},-\frac{413}{17280},-\frac{877}{17280},-\frac{923}{17280},-\frac{923}{17280},-\frac{1}{8},-\frac{413}{17280},-\frac{527}{17280},-\frac{1}{24}\big).$$

Expand both ϕ_i and ϕ_k to order $\sqrt{t}$:

$$\phi_{i,p|j|k,q|l} \times \big(-\frac{1}{24}, -\frac{1}{40}, 0, -\frac{13}{270}, -\frac{1}{40}, -\frac{11}{540}, -\frac{17}{540}, 0, -\frac{11}{540}, -\frac{1}{20}, 0, -\frac{1}{16}, -\frac{17}{540}, -\frac{13}{270}, -\frac{1}{24}\big).$$

Expand both ϕ_i and ϕ_l to order $\sqrt{t}$:

$$\phi_{i,p|jk|l,q} \times \big(-\frac{11}{288}, -\frac{13}{240}, -\frac{11}{288}, -\frac{293}{8640}, -\frac{13}{240}, -\frac{293}{8640}, -\frac{59}{2160}, -\frac{11}{288}, -\frac{293}{8640}, -\frac{7}{80}, -\frac{41}{576}, -\frac{41}{576}, -\frac{59}{2160}, -\frac{293}{8640}, -\frac{11}{288}\big).$$

Expand both ϕ_j and ϕ_k to order $\sqrt{t}$:

$$\phi_{i|j,p|k,q|l} \times \big(-\frac{1}{36}, -\frac{29}{1440}, 0, -\frac{37}{2160}, -\frac{1}{30}, -\frac{37}{2160}, -\frac{77}{4320}, 0, -\frac{77}{4320}, -\frac{29}{1440}, 0, -\frac{1}{36}, -\frac{137}{4320}, -\frac{137}{4320}, -\frac{1}{16}\big).$$

Expand both ϕ_j and ϕ_l to order $\sqrt{t}$:

$$\phi_{i|j,p|k|l,q} \times \big(-\frac{25}{864}, -\frac{109}{2880}, -\frac{25}{864}, -\frac{61}{2160}, -\frac{19}{240}, -\frac{61}{2160}, -\frac{91}{4320}, -\frac{31}{576}, -\frac{91}{4320}, -\frac{109}{2880}, -\frac{25}{864}, -\frac{25}{864}, -\frac{91}{4320}, -\frac{91}{4320}, -\frac{31}{576}\big).$$

Expand both ϕ_k and ϕ_l to order $\sqrt{t}$:

$$\phi_{ij|k,p|l,q} \times \big(-\frac{1}{24}, 0, -\frac{1}{24}, -\frac{1}{45}, -\frac{7}{240}, -\frac{1}{45}, -\frac{11}{480}, -\frac{11}{432}, 0, -\frac{73}{1440}, -\frac{17}{432}, -\frac{17}{432}, -\frac{11}{480}, 0, -\frac{11}{432}\big).$$

Expand $\frac{X^i_{tr}}{1-r}$ or $\frac{X^j_{t\rho}}{1-\rho}$ or $\frac{X^l_{t\theta}}{1-\theta}$ to order $t^{1.5}$:

$$\begin{aligned}\phi_{ijkl} \times \Big[&\partial_p \bar{b}^i \big(\big(-\frac{1}{12}\delta^{jp}\delta^{kl} - \frac{25}{216}\delta^{jl}\delta^{kp} - \frac{13}{216}\delta^{jk}\delta^{lp}\big) + \partial_p \bar{b}^j \big(-\frac{1}{6}\delta^{ip}\delta^{kl} - \frac{23}{216}\delta^{il}\delta^{kp} - \frac{23}{216}\delta^{ik}\delta^{lp}\big)\\ &+ \partial_p \bar{b}^l \big(-\frac{5}{48}\delta^{ip}\delta^{jk} - \frac{13}{144}\delta^{ik}\delta^{jp} - \frac{1}{8}\delta^{ij}\delta^{kp}\big)\Big].\end{aligned}$$

Expand a to order t:

$$\begin{aligned}\phi_{ijkl} \times \Big[&\partial^2_{pq} a^{ik}\big(-\frac{1}{120}\delta^{jq}\delta^{lp} - \frac{1}{120}\delta^{jp}\delta^{lq} - \frac{1}{120}\delta^{jl}\delta^{pq}\big) + \partial^2_{pq} a^{jk}\big(-\frac{23}{1440}\delta^{iq}\delta^{lp} - -\frac{23}{1440}\delta^{ip}\delta^{lq} - \frac{23}{1440}\delta^{il}\delta^{pq}\big)\\ &+ \partial^2_{pq} a^{lk}\big(-\frac{1}{54}\delta^{iq}\delta^{jp} - -\frac{1}{54}\delta^{ip}\delta^{jq} - \frac{1}{48}\delta^{ij}\delta^{pq}\big) + \partial^2_{pq} a^{ij}\big(-\frac{7}{432}\delta^{kq}\delta^{lp} - -\frac{7}{432}\delta^{kp}\delta^{lq} - \frac{1}{48}\delta^{kl}\delta^{pq}\big)\\ &+ \partial^2_{pq} a^{il}\big(-\frac{7}{576}\delta^{jq}\delta^{kp} - -\frac{7}{576}\delta^{jp}\delta^{kq} - \frac{7}{576}\delta^{jk}\delta^{pq}\big) + \partial^2_{pq} a^{jl}\big(-\frac{43}{1728}\delta^{iq}\delta^{kp} - -\frac{43}{1728}\delta^{ip}\delta^{kq} - \frac{17}{576}\delta^{ik}\delta^{pq}\big)\\ &+ \partial^2_{pq} a^{jl}\big(-\frac{1}{144}\delta^{ik}\delta^{pq}\big) + \partial^2_{pq} a^{il} \cdot \big(-\frac{1}{72}\delta^{jk}\delta^{pq}\big) + \partial^2_{pq} a^{ij}\big(-\frac{1}{48}\delta^{lk}\delta^{pq}\big) + \partial^2_{pq} a^{lk}\big(-\frac{1}{48}\delta^{ij}\delta^{pq}\big)\\ &+ \partial^2_{pq} a^{jk}\big(-\frac{1}{96}\delta^{il}\delta^{pq}\big) + \cdot\partial^2_{pq} a^{ik}\big(-\frac{1}{32}\delta^{jl}\delta^{pq}\big)\Big].\end{aligned}$$

B.3.4 The $(II;KK)$ term

Target:

$$t\int_0^1 \big(\int_0^\rho \phi_i(X_{tr})\frac{X^i_{tr}}{1-r}dr\big)\otimes\phi_j(X_{t\rho})\frac{X^j_{t\rho}}{1-\rho}d\rho\otimes\int_0^1 \big(\int_0^\theta \phi_k(X_{t\delta})(\sigma dB^t)^k_\delta\big)\otimes\phi_l(X_\theta)(\sigma dB^t)^l_\theta.$$

Expand ϕ_i to order t:

$$\phi_{i,pq|jkl}\times\Big(0,\frac{37}{3456},\frac{37}{3456},\frac{121}{3456},\frac{37}{3456},\frac{37}{3456},\frac{37}{3456},\frac{37}{3456},\frac{37}{3456},\frac{37}{3456},\frac{121}{3456},0,\frac{37}{3456},\frac{121}{3456},0\Big).$$

Expand ϕ_j to order t:

$$\phi_{i|j,pq|kl}\times\Big(0,\frac{1}{128},\frac{1}{128},\frac{59}{3456},\frac{59}{3456},\frac{59}{3456},\frac{23}{3456},\frac{23}{3456},\frac{23}{3456},\frac{1}{128},\frac{1}{128},0,\frac{1}{128},\frac{1}{128},0\Big).$$

Expand ϕ_k to order t:

$$\phi_{ij|k,pq|l}\times\Big(0,0,0,\frac{1}{32},0,0,\frac{1}{96},0,0,0,0,0,0,0,0\Big).$$

Expand ϕ_l to order t:

$$\phi_{ijk|l,pq}\times\Big(0,0,0,\frac{17}{576},0,0,\frac{7}{576},0,0,\frac{7}{576},\frac{43}{1728},0,\frac{7}{576},\frac{43}{1728},0\Big).$$

Expand both ϕ_i and ϕ_j to order $\sqrt{t}$:

$$\phi_{i,p|j,q|kl}\times\Big(0,\frac{79}{3456},\frac{79}{3456},\frac{79}{3456},\frac{79}{3456},\frac{101}{3456},\frac{37}{3456},\frac{37}{3456},\frac{101}{3456},\frac{115}{128},\frac{115}{128},0,\frac{37}{3456},\frac{79}{3456},0\Big).$$

Expand both ϕ_i and ϕ_k to order $\sqrt{t}$:

$$\phi_{i,p|j|k,q|l}\times\Big(0,0,0,\frac{1}{27},0,\frac{1}{108},\frac{1}{108},0,\frac{1}{108},0,0,0,\frac{1}{108},\frac{1}{27},0\Big).$$

Expand both ϕ_i and ϕ_l to order $\sqrt{t}$:

$$\phi_{i,p|jk|l,q}\times\Big(0,0,\frac{5}{144},\frac{35}{864},0,\frac{7}{432},\frac{7}{432},\frac{5}{144},\frac{7}{432},0,\frac{23}{288},0,\frac{7}{432},\frac{35}{864},0\Big).$$

Expand both ϕ_j and ϕ_k to order $\sqrt{t}$:

$$\phi_{i|j,p|k,q|l}\times\Big(0,0,0,\frac{5}{432},0,\frac{5}{432},\frac{1}{216},0,\frac{1}{216},0,0,0,\frac{1}{54},\frac{1}{54},0\Big).$$

Expand both ϕ_j and ϕ_l to order $\sqrt{t}$:

$$\phi_{i|j,p|k|l,q}\times\Big(0,0,\frac{13}{432},\frac{11}{432},0,\frac{11}{432},\frac{7}{864},\frac{13}{288},\frac{7}{864},0,\frac{13}{432},0,\frac{1}{48},\frac{1}{48},0\Big).$$

Expand both ϕ_k and ϕ_l to order $\sqrt{t}$:

$$\phi_{ij|k,p|l,q}\times\Big(0,0,0,\frac{1}{36},0,0,\frac{1}{72},\frac{1}{72},0,0,\frac{1}{24},0,0,0,0\Big).$$

Expand $\frac{X^i_{tr}}{1-r}$ or $\frac{X^j_{t\rho}}{1-\rho}$ to order $t^{1.5}$:

$$\phi_{ijkl}\times\Big[\partial_p\bar{b}^i\big(\frac{11}{108}\delta^{jl}\delta^{kp}+\frac{1}{54}\delta^{jk}\delta^{lp}\big)+\partial_p\bar{b}^j\big(\frac{7}{108}\delta^{il}\delta^{kp}+\frac{7}{108}\delta^{ik}\delta^{ip}\big)\Big].$$

Expand a to order t:

$$\begin{aligned}\phi_{ijkl}\times\Big[&\partial_{pq}a^{il}\big(\frac{7}{576}\delta^{jq}\delta^{kp}+\frac{7}{576}\delta^{jp}\delta^{kq}+\frac{7}{576}\delta^{jk}\delta^{pq}\big)+\partial_{pq}a^{jk}\frac{1}{96}\delta^{il}\delta^{pq}\\&+\partial_{pq}a^{jl}\big(\frac{43}{1728}\delta^{iq}\delta^{kp}+\frac{43}{1728}\delta^{ip}\delta^{kq}+\frac{17}{576}\delta^{ik}\delta^{pq}\big)+\partial_{pq}a^{ik}\big(\frac{1}{32}\delta^{jl}\delta^{pq}\big)\\&+\partial_{pq}a^{ij}\big(\frac{1}{108}\delta^{kq}\delta^{lp}+\frac{1}{108}\delta^{kp}\delta^{lq}\big)\Big].\end{aligned}$$

B.3.5 The $(IK;IK)$ term

Target:

$$t\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})\frac{X_{tr}^i}{1-r}dr\Big)\otimes\phi_j(X_{t\rho})(\sigma dB^t)_\rho^j\otimes\int_0^1 \Big(\int_0^\theta \phi_k(X_{t\delta})\frac{X_{t\delta}^k}{1-\delta}d\delta\Big)\otimes\phi_l(X_{t\theta})(\sigma dB^t)_\theta^l.$$

Expand ϕ_i to order t:

$$\phi_{i,pq|jkl}\times\Big(0,0,0,\frac{23}{1440},0,0,0,0,0,0,\frac{23}{1440},0,0,\frac{23}{1440},0\Big).$$

Expand ϕ_j to order t:

$$\phi_{i|j,pq|kl}\times\Big(0,0,0,\frac{7}{720},0,0,0,0,0,0,\frac{11}{2160},0,0,\frac{11}{2160},0\Big).$$

Expand ϕ_k to order t:

$$\phi_{ij|k,pq|l}\times\Big(0,0,0,\frac{23}{1440},0,0,0,0,0,0,\frac{23}{1440},0,0,\frac{23}{1440},0\Big).$$

Expand ϕ_l to order t:

$$\phi_{ijk|l,pq}\times\Big(0,0,0,\frac{7}{720},0,0,0,0,0,0,\frac{11}{2160},0,0,\frac{11}{2160},0\Big).$$

Expand both ϕ_i and ϕ_j to order $\sqrt{t}$:

$$\phi_{i,p|j,q|kl}\times\Big(0,0,0,\frac{19}{1440},0,0,0,0,0,0,\frac{29}{1440},0,0,\frac{19}{1440},0\Big).$$

Expand both ϕ_i and ϕ_k to order $\sqrt{t}$:

$$\phi_{i,p|j|k,q|l}\times\Big(0,0,0,\frac{1}{45},0,0,0,0,0,0,\frac{1}{20},0,0,\frac{1}{45},0\Big).$$

Expand both ϕ_i and ϕ_l to order $\sqrt{t}$:

$$\phi_{i,p|jk|l,q}\times\Big(0,0,0,\frac{19}{1440},0,0,0,0,0,0,\frac{29}{1440},0,0,\frac{19}{1440},0\Big).$$

Expand both ϕ_j and ϕ_k to order $\sqrt{t}$:

$$\phi_{i|j,p|k,q|l}\times\Big(0,0,0,\frac{19}{1440},0,0,0,0,0,0,\frac{29}{1440},0,0,\frac{19}{1440},0\Big).$$

Expand both ϕ_j and ϕ_l to order $\sqrt{t}$:

$$\phi_{i|j,p|k|l,q}\times\Big(0,0,0,\frac{7}{360},0,0,0,0,0,0,\frac{11}{1080},0,0,\frac{11}{1080},0\Big).$$

Expand both ϕ_k and ϕ_l to order $\sqrt{t}$:

$$\phi_{ij|k,p|l,q} \times \big(0, 0, 0, \frac{19}{1440}, 0, 0, 0, 0, 0, 0, \frac{19}{1440}, 0, 0, \frac{29}{1440}, 0\big).$$

Expand $\frac{X^i_{tr}}{1-r}$ or $\frac{X^k_{t\delta}}{1-\delta}$ to order $t^{1.5}$:

$$\phi_{ijkl} \times \Big[\partial_p \bar{b}^i \frac{1}{24}\delta^{jl}\delta^{kp} + \partial_p \bar{b}^k \frac{1}{24}\delta^{ip}\delta^{jl}\Big].$$

Expand a to order t:

$$\phi_{ijkl} \times \Big[\partial^2_{pq} a^{jl}\big(\frac{11}{2160}\delta^{iq}\delta^{kp} + \frac{11}{2160}\delta^{ip}\delta^{kq} + \frac{7}{720}\delta^{ik}\delta^{pq}\big) + \partial^2_{pq} a^{ik}\frac{1}{72}\delta^{jl}\delta^{pq}\Big]$$

B.3.6 The $(IK;KI)$ term

Target:

$$t\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})\frac{X^i_{tr}}{1-r}du\Big)\otimes\phi_j(X_{t\rho})(\sigma dB^t)^j_\rho\otimes\int_0^1 \Big(\int_0^\theta \phi_k(X_{t\delta})(\sigma dB^t)^k_\delta\Big)\otimes\phi_l(X_{t\theta})\frac{X^l_{t\theta}}{1-\theta}d\theta.$$

Expand ϕ_i to order t:

$$\phi_{i,pq|jkl} \times \big(0, 0, 0, \frac{7}{288}, 0, 0, \frac{1}{120}, 0, 0, \frac{1}{120}, \frac{7}{288}, 0, \frac{1}{120}, \frac{7}{288}, 0\big).$$

Expand ϕ_j to order t:

$$\phi_{i|j,pq|kl} \times \big(0, 0, 0, \frac{5}{288}, 0, 0, \frac{11}{1440}, 0, 0, \frac{11}{1440}, \frac{11}{864}, 0, \frac{11}{1440}, \frac{11}{864}, 0\big).$$

Expand ϕ_k to order t:

$$\phi_{ij|k,pq|l} \times \big(0, 0, 0, \frac{1}{48}, 0, 0, \frac{11}{1440}, 0, 0, \frac{11}{1440}, 0, \frac{1}{216}, \frac{11}{1440}, 0, \frac{1}{216}\big).$$

Expand ϕ_l to order t:

$$\phi_{ijk|l,pq} \times \big(0, 0, 0, \frac{1}{96}, \frac{1}{96}, \frac{1}{96}, \frac{7}{1440}, \frac{5}{864}, \frac{5}{864}, \frac{7}{1440}, \frac{5}{864}, \frac{5}{864}, \frac{7}{1440}, \frac{5}{864}, \frac{5}{864}\big).$$

Expand both ϕ_i and ϕ_j to order $\sqrt{t}$:

$$\phi_{i,p|j,q|kl} \times \big(0, 0, 0, \frac{7}{288}, 0, 0, \frac{1}{90}, 0, 0, \frac{1}{40}, \frac{13}{288}, 0, \frac{1}{90}, \frac{7}{288}, 0\big).$$

Expand both ϕ_i and ϕ_k to order $\sqrt{t}$:

$$\phi_{i,p|j|k,q|l} \times \big(0, 0, 0, \frac{1}{36}, 0, 0, \frac{1}{90}, 0, 0, \frac{1}{40}, 0, \frac{1}{48}, \frac{1}{90}, \frac{1}{36}, 0\big).$$

Expand both ϕ_i and ϕ_l to order $\sqrt{t}$:

$$\phi_{i,p|jk|l,q} \times (0,0,0,\frac{7}{576}0,\frac{7}{576},\frac{1}{180},0,\frac{7}{576},\frac{1}{30},\frac{19}{576},\frac{19}{576},\frac{1}{180},\frac{7}{576},0).$$

Expand both ϕ_j and ϕ_k to order $\sqrt{t}$:

$$\phi_{i|j,p|k,q|l} \times (0,0,0,\frac{1}{72},0,0,\frac{11}{720},0,0,\frac{11}{720},0,\frac{1}{72},\frac{11}{720},\frac{1}{36},0).$$

Expand both ϕ_j and ϕ_l to order $\sqrt{t}$:

$$\phi_{i|j,p|k|l,q} \times (0,0,0,\frac{5}{288},0,\frac{5}{288},\frac{11}{1440},0,\frac{11}{864},\frac{31}{1440},\frac{11}{864},\frac{11}{864},\frac{11}{1440},\frac{11}{864},0).$$

Expand both ϕ_k and ϕ_l to order $\sqrt{t}$:

$$\phi_{ij|k,p|l,q} \times (0,0,0,\frac{1}{144},0,\frac{1}{144},\frac{11}{1440},\frac{1}{216},0,\frac{31}{1440},\frac{1}{54},\frac{1}{54},\frac{11}{1440},0,\frac{1}{216}).$$

Expand $\frac{X^i_{tr}}{1-r}$ or $\frac{X^l_{t\theta}}{1-\theta}$ to order $t^{1.5}$:

$$\phi_{ijkl} \times \Big[\partial_p \bar{b}^i (\frac{1}{72}\delta^{jk}\delta^{lp} + \frac{1}{18}\delta^{jl}\delta^{kp}) + \partial_p \bar{b}^l (\frac{1}{24}\delta^{ip}\delta^{jk} + \frac{1}{36}\delta^{ik}\delta^{jp})\Big].$$

Expand a to order t:

$$\begin{aligned}\phi_{ijkl} \times \Big[&\partial^2_{pq} a^{jk} (\frac{11}{1440}\delta^{iq}\delta^{lp} + \frac{11}{1440}\delta^{ip}\delta^{lq} + \frac{11}{1440}\delta^{il}\delta^{pq}) + \partial^2_{pq} a^{il} \frac{1}{144}\delta^{jk}\delta^{pq} \\ &+ \partial^2_{pq} a^{jl} (\frac{11}{864}\delta^{iq}\delta^{kp} + \frac{11}{864}\delta^{ip}\delta^{kq} + \frac{5}{288}\delta^{ik}\delta^{pq}) + \partial^2_{pq} a^{ik} \frac{1}{48}\delta^{jl}\delta^{pq} \\ &+ \partial^2_{pq} a^{kl} (\frac{1}{216}\delta^{iq}\delta^{jp} + \frac{1}{216}\delta^{ip}\delta^{jq})\Big].\end{aligned}$$

B.3.7 The $(IK; KK)$ term

Target:

$$-t^{3/2}\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})\frac{X^i_{tr}}{1-r}dr\Big)\otimes\phi_j(X_{t\rho})(\sigma dB^t)^j_\rho \otimes \int_0^1 \Big(\int_0^\theta \phi_k(X_{t\delta})(\sigma dB)^k_\delta\Big)\otimes\phi_l(X_{t\theta})(\sigma dB^t)^l_\theta.$$

Expand ϕ_i to order t:

$$\phi_{i,pq|jkl} \times (0,0,0,-\frac{7}{288},0,0,0,0,0,0,-\frac{7}{288},0,0,-\frac{7}{288},0).$$

Expand ϕ_j to order t:

$$\phi_{i|j,pq|kl} \times (0,0,0,-\frac{5}{288},0,0,0,0,0,0,-\frac{11}{864},0,0,-\frac{11}{864},0).$$

Expand ϕ_k to order t:

$$\phi_{ij|k,pq|l} \times \left(0,0,0,-\frac{1}{48},0,0,0,0,0,0,0,0,0,0,0\right).$$

Expand ϕ_l to order t:

$$\phi_{ijk|l,pq} \times \left(0,0,0,-\frac{5}{288},0,0,0,0,0,0,-\frac{11}{864},0,0,-\frac{11}{864},0\right).$$

Expand both ϕ_i and ϕ_j to order $\sqrt{t}$:

$$\phi_{i,p|j,q|kl} \times \left(0,0,0,-\frac{7}{288},0,0,0,0,0,0,-\frac{13}{288},0,0,-\frac{7}{288},0\right).$$

Expand both ϕ_i and ϕ_k to order $\sqrt{t}$:

$$\phi_{i,p|j|k,q|l} \times \left(0,0,0,-\frac{1}{36},0,0,0,0,0,0,0,0,0,-\frac{1}{36},0\right).$$

Expand both ϕ_i and ϕ_l to order $\sqrt{t}$:

$$\phi_{i,p|jk|l,q} \times \left(0,0,0,-\frac{7}{288},0,0,0,0,0,0,-\frac{13}{288},0,0,-\frac{7}{288},0\right).$$

Expand both ϕ_j and ϕ_k to order $\sqrt{t}$:

$$\phi_{i|j,p|k,q|l} \times \left(0,0,0,-\frac{1}{72},0,0,0,0,0,0,0,0,0,-\frac{1}{36},0\right).$$

Expand both ϕ_j and ϕ_l to order $\sqrt{t}$:

$$\phi_{i|j,p|k|l,q} \times \left(0,0,0,-\frac{5}{144},0,0,0,0,0,0,-\frac{11}{432},0,0,-\frac{11}{432},0\right).$$

Expand both ϕ_k and ϕ_l to order $\sqrt{t}$:

$$\phi_{ij|k,p|l,q} \times \left(0,0,0,-\frac{1}{72},0,0,0,0,0,0,-\frac{1}{36},0,0,0,0\right).$$

Expand $\frac{X^i_{tr}}{1-r}$ to order $t^{1.5}$:

$$\phi_{ijkl} \times \Big[\partial_p \bar{b}^i\big(-\frac{1}{18}\delta^{jl}\delta^{kp}\big)\Big].$$

Expand a to order t:

$$\phi_{ijkl} \times \Big[\partial^2_{pq}a^{jl}\big(-\frac{11}{864}\delta^{iq}\delta^{kp}-\frac{11}{864}\delta^{ip}\delta^{kq}-\frac{5}{288}\delta^{ik}\delta^{pq}\big)+\partial^2_{pq}a^{ik}\big(-\frac{1}{48}\delta^{jl}\delta^{pq}\big)\Big].$$

B.3.8 The $(KI;KI)$ term

Target:

$$t\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})(\sigma dB^t)^i_r\Big)\otimes\phi_j(X_{t\rho})\frac{X^j_{t\rho}}{1-\rho}dv\otimes\int_0^1 \Big(\int_0^\theta \phi_k(X_{t\delta})(\sigma dB^t)^k_\delta\Big)\otimes\phi_l(X_{t\theta})\frac{X^l_{t\theta}}{1-\theta}d\theta.$$

Expand ϕ_i to order t:

$$\phi_{i,pq|jkl}\times(\frac{1}{12},\frac{7}{216},\frac{7}{216},\frac{7}{120},\frac{1}{60},\frac{1}{60},\frac{13}{288},\frac{7}{288},\frac{7}{288},0,0,0,0,0,0).$$

Expand ϕ_j to order t:

$$\phi_{i|j,pq|kl}\times(\frac{5}{72},\frac{1}{32},\frac{1}{32},\frac{29}{720},\frac{29}{720},\frac{29}{720},\frac{5}{144},\frac{5}{144},\frac{5}{144},\frac{1}{32},\frac{1}{32},\frac{5}{72},\frac{1}{32},\frac{1}{32},\frac{5}{72}).$$

Expand ϕ_k to order t:

$$\phi_{ij|k,pq|l}\times(\frac{1}{12},0,0,\frac{7}{120},\frac{1}{60},\frac{1}{60},\frac{13}{288},0,0,\frac{7}{288},0,\frac{7}{216},\frac{7}{288},0,\frac{7}{216}).$$

Expand ϕ_l to order t:

$$\phi_{ijk|l,pq}\times(\frac{5}{72},\frac{5}{72},\frac{5}{72},\frac{29}{720},\frac{29}{720},\frac{29}{720},\frac{5}{144},\frac{1}{32},\frac{1}{32},\frac{5}{144},\frac{1}{32},\frac{1}{32},\frac{5}{144},\frac{1}{32},\frac{1}{32}).$$

Expand both ϕ_i and ϕ_j to order $\sqrt{t}$:

$$\phi_{i,p|j,q|kl}\times(\frac{1}{12},\frac{13}{216},\frac{5}{108},\frac{3}{80},\frac{3}{80},\frac{7}{120},\frac{11}{288},\frac{11}{288},\frac{23}{288},0,0,0,\frac{5}{108},\frac{13}{216},\frac{1}{12}).$$

Expand both ϕ_i and ϕ_k to order $\sqrt{t}$:

$$\phi_{i,p|j|k,q|l}\times(\frac{1}{12},\frac{1}{24},0,\frac{7}{60},\frac{1}{30},\frac{1}{30},\frac{1}{18},0,\frac{1}{36},0,0,0,\frac{1}{36},0,\frac{1}{24}).$$

Expand both ϕ_i and ϕ_l to order $\sqrt{t}$:

$$\phi_{i,p|jk|l,q}\times(\frac{5}{72},\frac{5}{48},\frac{5}{72},\frac{3}{80},\frac{7}{120},\frac{3}{80},\frac{11}{288},\frac{13}{288},\frac{5}{96},0,0,0,\frac{11}{288},\frac{5}{96},\frac{13}{288}).$$

Expand both ϕ_j and ϕ_k to order $\sqrt{t}$:

$$\phi_{i|j,p|k,q|l}\times(\frac{5}{72},\frac{13}{288},0,\frac{3}{80},\frac{7}{120},\frac{3}{80},\frac{11}{288},0,\frac{11}{288},\frac{13}{288},0,\frac{5}{72},\frac{5}{96},\frac{5}{96},\frac{5}{48}).$$

Expand both ϕ_j and ϕ_l to order $\sqrt{t}$:

$$\phi_{i|j,p|k|l,q}\times(\frac{29}{432},\frac{53}{576},\frac{29}{432},\frac{1}{20},\frac{2}{15},\frac{1}{20},\frac{37}{864},\frac{53}{576},\frac{37}{864},\frac{53}{576},\frac{29}{432},\frac{29}{432},\frac{37}{864},\frac{37}{864},\frac{53}{576}).$$

Expand both ϕ_k and ϕ_l to order $\sqrt{t}$:

$$\phi_{ij|k,p|l,q} \times \big(\frac{1}{12}, 0, \frac{1}{12}, \frac{3}{80}, \frac{7}{120}, \frac{3}{80}, \frac{11}{288}, \frac{5}{108}, 0, \frac{23}{288}, \frac{13}{216}, \frac{13}{216}, \frac{11}{288}, 0, \frac{5}{108}\big).$$

Expand $\frac{X^j_{t\rho}}{1-\rho}$ or $\frac{X^l_{t\theta}}{1-\theta}$ to order $t^{1.5}$:

$$\phi_{ijkl} \times \Big[\partial_p \bar{b}^j \big(\frac{1}{4}\delta^{ip}\delta^{kl} + \frac{1}{6}\delta^{ip}\delta^{kl} + \frac{11}{72}\delta^{ik}\delta^{lp}\big) + \partial_p \bar{b}^l \big(\frac{1}{6}\delta^{ip}\delta^{jk} + \frac{11}{72}\delta^{ik}\delta^{jp} + \frac{1}{4}\delta^{ij}\delta^{kp}\big)\Big].$$

Expand a to order t:

$$\begin{aligned}\phi_{ijkl} \times \Big[&\partial^2_{pq} a^{ij}\big(\frac{7}{216}\delta^{kq}\delta^{lp} + \frac{7}{216}\delta^{kp}\delta^{lq} + \frac{1}{24}\delta^{kl}\delta^{pq}\big) + \partial^2_{pq} a^{ik}\big(\frac{1}{60}\delta^{jq}\delta^{lp} + \frac{1}{60}\delta^{jp}\delta^{lq} + \frac{1}{60}\delta^{jl}\delta^{pq}\big) \\ &+ \partial^2_{pq} a^{il}\big(\frac{7}{288}\delta^{jq}\delta^{kp} + \frac{7}{288}\delta^{jp}\delta^{kq} + \frac{7}{288}\delta^{jk}\delta^{pq}\big) + \partial^2_{pq} a^{jk}\big(\frac{7}{288}\delta^{iq}\delta^{lp} + \frac{7}{288}\delta^{ji}\delta^{lq} + \frac{7}{288}\delta^{il}\delta^{pq}\big) \\ &+ \partial^2_{pq} a^{jl}\big(\frac{1}{27}\delta^{iq}\delta^{kp} + \frac{1}{27}\delta^{ip}\delta^{kq} + \frac{1}{24}\delta^{ik}\delta^{pq}\big) + \partial^2_{pq} a^{kl}\big(\frac{7}{216}\delta^{iq}\delta^{jp} + \frac{7}{216}\delta^{ip}\delta^{jq} + \frac{1}{24}\delta^{ij}\delta^{pq}\big) \\ &+ \partial^2_{pq} a^{kl}\big(\frac{1}{24}\delta^{ij}\delta^{pq}\big) + \partial^2_{pq} a^{jl}\big(\frac{1}{72}\delta^{ik}\delta^{pq}\big) + \partial^2_{pq} a^{jk}\big(\frac{1}{48}\delta^{il}\delta^{pq}\big) \\ &+ \partial^2_{pq} a^{il}\big(\frac{1}{48}\delta^{jk}\delta^{pq}\big) + \partial^2_{pq} a^{ik}\big(\frac{1}{24}\delta^{jl}\delta^{pq}\big) + \partial^2_{pq} a^{ij}\big(\frac{1}{24}\delta^{kl}\delta^{pq}\big)\Big].\end{aligned}$$

B.3.9 The $(KI;KK)$ term

Target:

$$-t^{3/2}\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})(\sigma dB^t)^i_r\Big) \otimes \phi_j(X_{t\rho}) \frac{X^j_{t\rho}}{1-\rho} d\rho \otimes \int_0^1 \Big(\int_0^\theta \phi_k(X_{t\delta})(\sigma dB^t)^k_\delta\Big) \otimes \phi_l(X_{t\theta})(\sigma dB^t)^l_\theta.$$

Expand ϕ_i to order t:

$$\phi_{i,pq|jkl} \times \big(0, -\frac{1}{54}, -\frac{1}{54}, -\frac{1}{24}, 0, 0, -\frac{7}{288}, -\frac{7}{288}, -\frac{7}{288}, 0, 0, 0, 0, 0, 0\big).$$

Expand ϕ_j to order t:

$$\phi_{i|j,pq|kl} \times \big(0, -\frac{1}{54}, -\frac{1}{54}, -\frac{1}{36}, -\frac{1}{36}, -\frac{1}{36}, -\frac{5}{288}, -\frac{5}{288}, -\frac{5}{288}, -\frac{1}{54}, -\frac{1}{54}, 0, -\frac{1}{54}, -\frac{1}{54}, 0\big).$$

Expand ϕ_k to order t:

$$\phi_{ij|k,pq|l} \times \big(0, 0, 0, -\frac{1}{24}, 0, 0, -\frac{1}{48}, 0, 0, 0, 0, 0, 0, 0, 0\big).$$

Expand ϕ_l to order t:

$$\phi_{ijk|l,pq} \times \big(0, 0, 0, -\frac{1}{24}, 0, 0, -\frac{7}{288}, -\frac{7}{288}, -\frac{7}{288}, 0, -\frac{1}{27}, 0, 0, -\frac{1}{27}, 0\big).$$

Expand both ϕ_i and ϕ_j to order $\sqrt{t}$:

$$\phi_{i,p|j,q|kl} \times \big(0, -\frac{1}{108}, -\frac{1}{54}, -\frac{1}{48}, -\frac{1}{48}, 0, -\frac{7}{288}, -\frac{7}{288}, -\frac{23}{288}, 0, 0, 0, -\frac{1}{54}, -\frac{5}{108}, 0\big).$$

Expand both ϕ_i and ϕ_k to order $\sqrt{t}$:

$$\phi_{i,p|j|k,q|l} \times \big(0, 0, 0, -\frac{1}{12}, 0, 0, -\frac{1}{36}, 0, -\frac{1}{36}, 0, 0, 0, 0, 0, 0\big).$$

Expand both ϕ_i and ϕ_l to order $\sqrt{t}$:

$$\phi_{i,p|jk|l,q} \times \big(0, 0, -\frac{1}{18}, -\frac{1}{24}, 0, 0, -\frac{7}{288}, -\frac{7}{288}, -\frac{7}{288}, 0, 0, 0, 0, -\frac{1}{18}, 0\big).$$

Expand both ϕ_j and ϕ_k to order $\sqrt{t}$:

$$\phi_{i|j,p|k,q|l} \times \big(0, 0, 0, -\frac{1}{48}, 0, -\frac{1}{48}, 0, 0, 0, 0, 0, 0, -\frac{1}{36}, -\frac{1}{36}, 0\big).$$

Expand both ϕ_j and ϕ_l to order $\sqrt{t}$:

$$\phi_{i|j,p|k|l,q} \times \big(0, 0, -\frac{7}{108}, -\frac{1}{24}, 0, -\frac{1}{24}, -\frac{7}{288}, -\frac{23}{288}, -\frac{7}{288}, 0, -\frac{7}{108}, 0, -\frac{1}{27}, -\frac{1}{27}, 0\big).$$

Expand both ϕ_k and ϕ_l to order $\sqrt{t}$:

$$\phi_{ij|k,p|l,q} \times \big(0, 0, 0, -\frac{1}{24}, 0, 0, -\frac{1}{36}, -\frac{1}{36}, 0, 0, -\frac{1}{18}, 0, 0, 0, 0\big).$$

Expand $\frac{X^j_{t\rho}}{1-\rho}$ to order $t^{1.5}$:

$$\phi_{ijkl} \times \Big[\partial_p \bar{b}^j \big(-\frac{1}{9}\delta^{il}\delta^{kp} - \frac{1}{12}\delta^{ik}\delta^{lp}\big)\Big].$$

Expand a to order t:

$$\phi_{ijkl} \times \Big[\partial^2_{pq} a^{ij} \big(-\frac{1}{54}\delta^{kq}\delta^{lp} - \frac{1}{54}\delta^{kp}\delta^{lq}\big) + \partial^2_{pq} a^{il} \big(-\frac{7}{288}\delta^{jq}\delta^{kp} - \frac{7}{288}\delta^{jp}\delta^{kq} - \frac{7}{288}\delta^{jk}\delta^{pq}\big)$$
$$+ \partial^2_{pq} a^{jl} \big(-\frac{1}{27}\delta^{iq}\delta^{kp} - \frac{1}{27}\delta^{ip}\delta^{kq} - \frac{1}{24}\delta^{ik}\delta^{pq}\big) + \partial^2_{pq} a^{jk} \big(-\frac{1}{24}\delta^{il}\delta^{pq}\big) + \partial^2_{pq} a^{ik} \big(-\frac{1}{12}\delta^{jl}\delta^{pq}\big)\Big].$$

B.3.10 The $(KK; KK)$ term

Target:

$$t^2 \int_0^1 \big(\int_0^\rho \phi_i(X_{tr})(\sigma dB^t)^i_r \big) \otimes \phi_j(X_{t\rho})(\sigma dB^t)^j_\rho \otimes \int_0^1 \big(\int_0^\theta \phi_k(X_{t\delta})(\sigma dB^t)^k_\delta \big) \otimes \phi_l(X_{t\theta})(\sigma dB^t)^l_\theta.$$

Expand ϕ_i to order t:

$$\phi_{i,pq|jkl} \times \big(0,0,0,\frac{1}{24},0,0,0,0,0,0,0,0,0,0,0\big).$$

Expand ϕ_j to order t:

$$\phi_{i|j,pq|kl} \times \big(0,0,0,\frac{1}{24},0,0,0,0,0,0,\frac{1}{27},0,0,\frac{1}{27},0\big).$$

Expand ϕ_k to order t:

$$\phi_{ij|k,pq|l} \times \big(0,0,0,\frac{1}{24},0,0,0,0,0,0,0,0,0,0,0,0,0\big).$$

Expand ϕ_l to order t:

$$\phi_{ijk|l,pq} \times \big(0,0,0,\frac{1}{24},0,0,0,0,0,0,\frac{1}{27},0,0,\frac{1}{27},0\big).$$

Expand both ϕ_i and ϕ_j to order $\sqrt{t}$:

$$\phi_{i,p|j,q|kl} \times \big(0,0,0,\frac{1}{24},0,0,0,0,0,0,0,0,0,\frac{1}{18},0\big).$$

Expand both ϕ_i and ϕ_k to order $\sqrt{t}$:

$$\phi_{i,p|j|k,q|l} \times \big(0,0,0,\frac{1}{12},0,0,0,0,0,0,0,0,0,0,0\big).$$

Expand both ϕ_i and ϕ_l to order $\sqrt{t}$:

$$\phi_{i,p|jk|l,q} \times \big(0,0,0,\frac{1}{24},0,0,0,0,0,0,0,0,0,\frac{1}{18},0\big).$$

Expand both ϕ_j and ϕ_k to order $\sqrt{t}$:

$$\phi_{i|j,p|k,q|l} \times \big(0,0,0,\frac{1}{24},0,0,0,0,0,0,\frac{1}{18},0,0,0,0\big).$$

Expand both ϕ_j and ϕ_l to order $\sqrt{t}$:

$$\phi_{i|j,p|k|l,q} \times \big(0,0,0,\frac{1}{12},0,0,0,0,0,0,\frac{2}{27},0,0,\frac{2}{27},0\big).$$

Expand both ϕ_k and ϕ_l to order $\sqrt{t}$:

$$\phi_{ij|k,p|l,q} \times \big(0,0,0,\frac{1}{24},0,0,0,0,0,0,\frac{1}{18},0,0,0,0\big).$$

Expand a to order t:

$$\phi_{ijkl} \times \Big[\partial^2_{pq} a^{jl}\big(\frac{1}{27}\delta^{iq}\delta^{kp} + \frac{1}{27}\delta^{ip}\delta^{kq} + \frac{1}{24}\delta^{ik}\delta^{pq}\big) + \partial^2_{pq} a^{ik}\frac{1}{24}\delta^{jl}\delta^{pq}\Big].$$

B.3.11 The $(II;P)$ term

Target:

$$t\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})\frac{X^i_{tr}}{1-r}dr\Big)\otimes\phi_j(X_{t\rho})\frac{X^j_{t\rho}}{1-\rho}d\rho\otimes\int_0^1\frac{1}{2}\phi_k\big(X_{t\theta}\big)\otimes\phi_l\big(X_{t\theta}\big)a^{kl}\big(X_{t\theta}\big)d\theta.$$

Expand ϕ_i to order t:

$$\phi_{i,pq|jkl}\times(\frac{1}{48},0,0,0,0,0,0,0,0,0,0,0,\frac{1}{48},0,0,\frac{1}{48}).$$

Expand ϕ_j to order t:

$$\phi_{i|j,pq|kl}\times(\frac{1}{72},0,0,0,0,0,0,0,0,0,0,0,\frac{1}{72},0,0,\frac{1}{72}).$$

Expand ϕ_k to order t:

$$\phi_{ij|k,pq|l}\times(\frac{1}{48},0,0,0,0,0,0,0,0,0,0,0,\frac{1}{108},0,0,\frac{1}{108}).$$

Expand ϕ_l to order t:

$$\phi_{ijk|l,pq}\times(\frac{1}{48},0,0,0,0,0,0,0,0,0,0,0,\frac{1}{108},0,0,\frac{1}{108}).$$

Expand both ϕ_i and ϕ_j to order $\sqrt{t}$:

$$\phi_{i,p|j,q|kl}\times(\frac{1}{48},0,0,0,0,0,0,0,0,0,0,0,\frac{1}{16},0,0,\frac{1}{48}).$$

Expand both ϕ_i and ϕ_k to order $\sqrt{t}$:

$$\phi_{i,p|j|k,q|l}\times(\frac{1}{48},0,0,0,0,0,0,0,0,0,0,0,\frac{1}{32},0,0,\frac{1}{48}).$$

Expand both ϕ_i and ϕ_l to order $\sqrt{t}$:

$$\phi_{i,p|jk|l,q}\times(\frac{1}{48},0,0,0,0,0,0,0,0,0,0,0,\frac{1}{32},0,0,\frac{1}{48}).$$

Expand both ϕ_j and ϕ_k to order $\sqrt{t}$:

$$\phi_{i|j,p|k,q|l}\times(\frac{1}{72},0,0,0,0,0,0,0,0,0,0,0,\frac{1}{72},0,0,\frac{1}{32}).$$

Expand both ϕ_j and ϕ_l to order $\sqrt{t}$:

$$\phi_{i|j,p|k|l,q}\times(\frac{1}{72},0,0,0,0,0,0,0,0,0,0,0,\frac{1}{72},0,0,\frac{1}{32}).$$

Expand both ϕ_k and ϕ_l to order $\sqrt{t}$:

$$\phi_{ij|k,p|l,q} \times (\frac{1}{24}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{54}, 0, 0, \frac{1}{54}).$$

Expand $\frac{X^i_{tr}}{1-r}$ or $\frac{X^j_{t\rho}}{1-\rho}$ to order $t^{1.5}$:

$$\phi_{ijkl} \times \left[\partial_p \bar{b}^i \frac{1}{24}\delta^{jp}\delta^{kl} + \partial_p \bar{b}^j \frac{1}{12}\delta^{ip}\delta^{kl}\right].$$

Expand a to order t:

$$\phi_{ijkl} \times \left[\partial^2_{pq} a^{kl}(\frac{1}{108}\delta^{iq}\delta^{jp} + \frac{1}{108}\delta^{ip}\delta^{jq} + \frac{1}{48}\delta^{ij}\delta^{pq}) + \partial^2_{pq} a^{ij}(\frac{1}{48}\delta^{kl}\delta^{pq})\right].$$

B.3.12 The $(IK;P)$ term

Target:

$$-t^{3/2}\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})\frac{X^i_{tr}}{1-r}dr\Big) \otimes \phi_j(X_{t\rho})(\sigma dB^t)^j_\rho \otimes \int_0^1 \frac{1}{2}\phi_k\big(X_{t\theta}\big) \otimes \phi_l\big(X_{t\theta}\big)a^{kl}\big(X_{t\theta}\big)d\theta.$$

Expand ϕ_k to order t:

$$\phi_{ij|k,pq|l} \times (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{432}, 0, 0, -\frac{1}{432}).$$

Expand ϕ_l to order t:

$$\phi_{ijk|l,pq} \times (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{432}, 0, 0, -\frac{1}{432}).$$

Expand both ϕ_i and ϕ_k to order $\sqrt{t}$:

$$\phi_{i,p|j|k,q|l} \times (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{96}, 0, 0, 0).$$

Expand both ϕ_i and ϕ_l to order $\sqrt{t}$:

$$\phi_{i,p|jk|l,q} \times (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{96}, 0, 0, 0).$$

Expand both ϕ_j and ϕ_k to order $\sqrt{t}$:

$$\phi_{i|j,p|k,q|l} \times (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{144}, 0, 0, 0).$$

Expand both ϕ_j and ϕ_l to order $\sqrt{t}$:

$$\phi_{i|j,p|k|l,q} \times (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{144}, 0, 0, 0).$$

Expand both ϕ_k and ϕ_l to order $\sqrt{t}$:

$$\phi_{ij|k,p|l,q} \times (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{216}, 0, 0, -\frac{1}{216}).$$

Expand a to order t:

$$\phi_{ijkl} \times \left[\partial^2_{pq} a^{kl}(-\frac{1}{432}\delta^{iq}\delta^{jp} - \frac{1}{432}\delta^{ip}\delta^{jq})\right].$$

B.3.13 The $(KI;P)$ term

Target:

$$-t^{3/2}\int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})(\sigma dB^t)_r^i\Big)\otimes\phi_j(X_{t\rho})\frac{X_{t\rho}^j}{1-\rho}d\rho\otimes\int_0^1\frac{1}{2}\phi_k\big(X_{t\theta}\big)\otimes\phi_l\big(X_{t\theta}\big)a^{kl}\big(X_{t\theta}\big)d\theta.$$

Expand ϕ_i to order t:

$$\phi_{i,pq|jkl}\times\big(-\frac{1}{24},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\big).$$

Expand ϕ_j to order t:

$$\phi_{i|j,pq|kl}\times\big(-\frac{5}{144},0,0,0,0,0,0,0,0,0,0,0,-\frac{5}{144},0,0,-\frac{5}{144}\big).$$

Expand ϕ_k to order t:

$$\phi_{ij|k,pq|l}\times\big(-\frac{1}{24},0,0,0,0,0,0,0,0,0,0,0,-\frac{7}{432},0,0,-\frac{7}{432}\big).$$

Expand ϕ_l to order t:

$$\phi_{ijk|l,pq}\times\big(-\frac{1}{24},0,0,0,0,0,0,0,0,0,0,0,-\frac{7}{432},0,0,-\frac{7}{432}\big).$$

Expand both ϕ_i and ϕ_j to order $\sqrt{t}$:

$$\phi_{i,p|j,q|kl}\times\big(-\frac{1}{24},0,0,0,0,0,0,0,0,0,0,0,0,0,0,-\frac{1}{24}\big).$$

Expand both ϕ_i and ϕ_k to order $\sqrt{t}$:

$$\phi_{i,p|j|k,q|l}\times\big(-\frac{1}{24},0,0,0,0,0,0,0,0,0,0,0,0,0,0,-\frac{1}{48}\big).$$

Expand both ϕ_i and ϕ_l to order $\sqrt{t}$:

$$\phi_{i,p|jk|l,q}\times\big(-\frac{1}{24},0,0,0,0,0,0,0,0,0,0,0,,0,0,-\frac{1}{48}\big).$$

Expand both ϕ_j and ϕ_k to order $\sqrt{t}$:

$$\phi_{i|j,p|k,q|l}\times\big(-\frac{5}{144},0,0,0,0,0,0,0,0,0,0,0,-\frac{5}{144},0,0,-\frac{5}{96}\big).$$

Expand both ϕ_j and ϕ_l to order $\sqrt{t}$:

$$\phi_{i|j,p|k|l,q}\times\big(-\frac{5}{144},0,0,0,0,0,0,0,0,0,0,0,-\frac{5}{144},0,0,-\frac{5}{96}\big).$$

Expand both ϕ_k and ϕ_l to order $\sqrt{t}$:

$$\phi_{ij|k,p|l,q} \times \big(- \frac{1}{12}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{7}{216}, 0, 0, -\frac{7}{216} \big).$$

Expand $\frac{X^j_{t\rho}}{1-\rho}$ to order $t^{1.5}$:

$$\phi_{ijkl} \times \Big[\partial_p \bar{b}^j \cdot \big(-\frac{1}{8} \delta^{ip} \delta^{kl} \big) \Big].$$

Expand a to order t:

$$\phi_{ijkl} \times \Big[\partial^2_{pq} a^{kl} \big(-\frac{7}{432} \delta^{iq} \delta^{jp} - \frac{7}{432} \delta^{ip} \delta^{jq} - \frac{1}{24} \delta^{ij} \delta^{pq} \big) + \partial^2_{pq} a^{ij} \big(-\frac{1}{24} \delta^{kl} \delta^{pq} \big) \Big].$$

B.3.14 The $(KK; P)$ term

Target:

$$t^2 \int_0^1 \Big(\int_0^\rho \phi_i(X_{tr})(\sigma dB^t)^i_r \Big) \otimes \phi_j(X_{t\rho})(\sigma dB^t)^j_\rho \otimes \int_0^1 \frac{1}{2} \phi_k\big(X_{t\theta}\big) \otimes \phi_l\big(X_{t\theta}\big) a^{kl}\big(X_{t\theta}\big) d\theta.$$

Expand ϕ_k to order t:

$$\phi_{ij|k,pq|l} \times \big(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{108}, 0, 0, \frac{1}{108} \big).$$

Expand ϕ_l to order t:

$$\phi_{ijk|l,pq} \times \big(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{108}, 0, 0, \frac{1}{108} \big).$$

Expand both ϕ_j and ϕ_k to order $\sqrt{t}$:

$$\phi_{i|j,p|k,q|l} \times \big(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{36}, 0, 0, 0 \big).$$

Expand both ϕ_j and ϕ_l to order $\sqrt{t}$:

$$\phi_{i|j,p|k|l,q} \times \big(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{36}, 0, 0, 0 \big).$$

Expand both ϕ_k and ϕ_l to order $\sqrt{t}$:

$$\phi_{ij|k,p|l,q} \times \big(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{54}, 0, 0, \frac{1}{54} \big).$$

Expand a to order t:

$$\phi_{ijkl} \times \Big[\partial^2_{pq} a^{kl} \big(\frac{1}{108} \delta^{iq} \delta^{jp} + \frac{1}{108} \delta^{ip} \delta^{jq} \big) \Big].$$

B.3.15 The $(P;P)$ term

Target:

$$t^2 \int_0^1 \frac{1}{2}\phi_i(X_{t\rho}) \otimes \phi_j(X_{t\rho}) a^{ij}(X_{t\rho}) d\rho \otimes \int_0^1 \frac{1}{2}\phi_k(X_{t\theta}) \otimes \phi_l(X_{t\theta}) a^{kl}(X_{t\theta}) d\theta.$$

Expand ϕ_i to order t:

$$\phi_{i,pq|jkl} \times (\frac{1}{48}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Expand ϕ_j to order t:

$$\phi_{i|j,pq|kl} \times (\frac{1}{48}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Expand ϕ_k to order t:

$$\phi_{ij|k,pq|l} \times (\frac{1}{48}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Expand ϕ_l to order t:

$$\phi_{ijk|l,pq} \times (\frac{1}{48}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Expand both ϕ_i and ϕ_j to order $\sqrt{t}$:

$$\phi_{i,p|j,q|kl} \times (\frac{1}{24}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Expand both ϕ_i and ϕ_k to order $\sqrt{t}$:

$$\phi_{i,p|j|k,q|l} \times (\frac{1}{48}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Expand both ϕ_i and ϕ_l to order $\sqrt{t}$:

$$\phi_{i,p|jk|l,q} \times (\frac{1}{48}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Expand both ϕ_j and ϕ_k to order $\sqrt{t}$:

$$\phi_{i|j,p|k,q|l} \times (\frac{1}{48}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Expand both ϕ_j and ϕ_l to order $\sqrt{t}$:

$$\phi_{i|j,p|k|l,q} \times (\frac{1}{48}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Expand both ϕ_k and ϕ_l to order $\sqrt{t}$:

$$\phi_{ij|k,p|l,q} \times (\frac{1}{24}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Expand a to order t:

$$\phi_{ijkl} \times \left[\partial^2_{pq} a^{ij} \frac{1}{48} \delta^{kl} \delta^{pq} + \partial^2_{pq} a^{kl} \frac{1}{48} \delta^{ij} \delta^{pq}\right].$$

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