A Note on the Radius of Convergence for the Logarithmic Signature

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Abstract.

In this note, we prove that if the winding angle of a C^2 path falls in a certain range, its logarithmic signature must have finite radius of convergence (i.e. decaying at most geometrically fast). Our technique provides an alternative approach to a conjecture of T. Lyons and N. Sidorova [10], which cannot be handled by the more general technique developed in the recent work of [3].

§1. Introduction and statement of the main result

The *signature transform* of a continuous path $\gamma : [0, T] \to \mathbb{R}^d$ with bounded total variation (BV) is the formal tensor series defined by

$$S(\gamma) := \sum_{n=0}^{\infty} \int_{0<t_1<\cdots<t_n<T} d\gamma_{t_1} \otimes \cdots \otimes d\gamma_{t_n}.$$

This transformation was initially introduced by K.T. Chen [6] to study the topology of loop spaces over manifolds. It also appeared as a basic tool in the study of problems related to stochastic and controlled differential equations ([1, 7, 9]).

It was a fundamental result of Hambly-Lyons [8] in 2010, which was later extended to the rough path context in [4], that every rough path is uniquely determined by its signature transform up to treelike equivalence. The essential mechanism behind such a uniqueness theorem is that the signature transform satisfies an algebraic property which is known as the shuffle product formula, or equivalently, *the logarithmic signature is always a Lie series* (Chen's theorem; see [5]). The *logarithmic signature of a path* γ is defined by the formal Taylor series

$$\log S(\gamma) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \big(S(\gamma) - \mathbf{1}\big)^{\otimes n},$$

where $\otimes$ denotes the tensor product on the tensor algebra $T((\mathbb{R}^d)) := \prod_{n=0}^{\infty}(\mathbb{R}^d)^{\otimes n}$ and $\mathbf{1} := (1, 0, 0, \cdots)$. An important reason of considering the logarithmic signature is that there are no algebraic dependencies among different components of the logarithmic signature. Another important application of the logarithmic signature is the log-ODE method for solving controlled differention equation (the Chen-Strichartz formula; see [1]).

From an analytic viewpoint, the signature transform (of a continuous BV path) always has factorial decay, namely,

$$\left\|\pi_n S(\gamma)\right\|_\pi \leqslant \frac{\|\gamma\|_{1\text{-var}}^n}{n!} \tag{1}$$

for all $n \geqslant 1$, where

$$\pi_n : T((\mathbb{R}^d)) := \prod_{n=0}^{\infty}(\mathbb{R}^d)^{\otimes n} \to (\mathbb{R}^d)^{\otimes n}$$

is the canonical projection and $\|\cdot\|_\pi$ denotes the projective tensor norm on $(\mathbb{R}^d)^{\otimes n}$. The inequality (1) is an immediate consequence of the triangle inequality. Let us define the power series

$$P(z) := \sum_{n=0}^{\infty} \|\pi_n S(\gamma)\|_\pi z^n, \quad z \in \mathbb{C}.$$

The factorial decay property (1) implies that $\pi_n S(\gamma)$ decays faster than any geometric rate, and as a consequence, $P(z)$ is an entire function on $\mathbb{C}$. In other words, the signature $S(\gamma)$ always has *infinite radius of convergence* (*ROC*).

On the other hand, whether the logarithmic signature has infinite or finite ROC is a highly nontrivial problem. It was conjectured by Lyons-Sidorova [10] that straight lines are the only (tree-reduced) BV paths whose logarithmic signature can have infinite ROC. In other words, the logarithmic signature of any nonlinear BV path can decay at most geometrically fast. This decay property is drastically different from the signature decay. In some sense, it is an infinite dimensional analogue of the fact that e^z is an entire function while $\log(1 + z)$ only possesses a finite ROC. The above conjecture was confirmed in the same paper for two special classes of paths which the authors called 1-monotone paths and unpaired piecewise linear paths.

A substantial progress towards the Lyons-Sidorova conjecture was made in a recent paper [3]. First of all, it was pointed out that the conjecture cannot hold in its originally stated form. This is simply because any path that is conjugate to a straight line always has infinite

ROC. As a result, the conjecture should be revised into a conjugacy class property: the logarithmic signature of a tree-reduced BV path has infinite ROC if and only if it is conjugate to a straight line. By using the technique of Cartan's path developments into complex semisimple Lie algebras, it was shown in [3] that if the logarithmic signature of a non-closed geometric rough path has infnite ROC, its signature coefficients must satisfy an infinite system of rigid algebraic relations, which in turn imposes strong geometric constraint on the original path. As an application, the authors proved a weak version of the conjecture which asserts that a BV path whose logarithmic signature has infinite ROC on any subinterval of time must be a straight line.

One limitation of [3] is that their technique breaks down if the underlying path is a loop, namely, if $\gamma_0 = \gamma_T$. The purpose of this article is to provide a different attempt which allows the case of loops. Our main result is stated as follows.

Theorem 1. *Let $\hat{\theta} : [0, T] \to \mathbb{R}$ be a piecewise C^1 function. Suppose that $\hat{\theta}_{T-} - \hat{\theta}_{0+} \in (4k\pi + \pi, 4k\pi + 3\pi)$ for some integer k. Then the logarithmic signature of the planar path*

$$\gamma_t := \Big(\int_0^t \cos\hat{\theta}_s ds, \int_0^t \sin\hat{\theta}_s ds \Big)$$

has finite radius of convergence.

§2. $\mathrm{SL}_2(\mathbb{R})$-development

2.1. Escape from exponential image

Our proof of Theorem 1 relies on a key insight of Lyons-Sidorova [10] which is recaptured as follows.

An Euclidean path $\gamma : [0, T] \to \mathbb{R}^d$ can be *lifted* onto a path in an arbitrary Lie group by solving a linear controlled differential equation. Let G be a (matrix) Lie group with Lie algebra $\mathfrak{g}$. Let $F : \mathbb{R}^d \to \mathfrak{g}$ be a given linear map. The solution to the ODE

$$d\Gamma_t = \Gamma_t F(d\gamma_t), \quad \Gamma_0 = I \tag{2}$$

defines a path in G (here I denotes the identity). Note that the equation (2) is well-posed if γ_t is a weakly geometric rough path. The path Γ_t is known as *Cartan's development* of γ_t with respect to $(G, \mathfrak{g}, F)$.

The key insight of Lyons-Sidorova [10] is that if the logarithmic signature of the path γ_t has infinite ROC, the end point of its Cartan's development must lie inside the image of the exponential map for any

choice of $(G, , F)$. In other words,

$$\log S(\gamma) \text{ has infinite ROC } \implies \Gamma_T \in \exp(\mathfrak{g}) \quad \forall (G, \mathfrak{g}, F).$$

As a consequence, in order to show that $\log S(\gamma)$ has finite ROC, it is enough to construct *one specific development* $(G, \mathfrak{g}, F)$, under which $\Gamma_T \notin \exp(\mathfrak{g})$.

2.2. Image of the exponential map for $\mathrm{SL}_2(\mathbb{R})$

The simplest Lie group with a "sufficiently non-surjective" exponential map is

$$\mathrm{SL}_2(\mathbb{R}) := \big\{A \in \mathrm{Mat}(2;\mathbb{R}) : \det A = 1\big\}.$$

Its Lie algebra is

$$\mathfrak{sl}_2(\mathbb{R}) := \big\{A \in \mathrm{Mat}(2;\mathbb{R}) : \mathrm{tr}(A) = 0\big\}.$$

According to [11, Section 2.1],

$$A \notin \exp(\mathfrak{sl}_2(\mathbb{R})) \iff \mathrm{tr}(A) \leqslant -2 \text{ and } A \neq -I. \tag{3}$$

for any $A \in \mathrm{SL}_2(\mathbb{R})$. We will rely on the following alternative characterisation of $\exp(\mathfrak{sl}_2(\mathbb{R}))$.

Lemma 1. *Let $A \in SL_2(R)$. Then $A \notin \exp(\mathfrak{sl}_2(\mathbb{R}))$ if and only if*

$$\int_l^{l+\pi} u(\zeta)^T A u(\zeta) d\zeta \leqslant -\pi \quad \textit{and} \quad G \neq -I$$

for some $l \in \mathbb{R}$, where $u(\zeta) := (\cos(\zeta), \sin(\zeta))^T$.

Proof. Write $G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Simple calculation shows that

$$u(\zeta)^T A u(\zeta) = \frac{1}{2}\big((a+d) + (a-d)\cos(2\zeta) + (b+c)\sin(2\zeta)\big).$$

Therefore, one has

$$\int_l^{l+\pi} u(\zeta)^T A u(\zeta) d\zeta = \frac{\pi}{2}(a+d) = \frac{\pi}{2}\mathrm{tr}(A).$$

The desired claim follows from the above relation and (3).

Q.E.D.

2.3. Dynamics of $\mathrm{SL}_2(\mathbb{R})$-development

We now construct a specific $\mathrm{SL}_2(\mathbb{R})$-development that will be used for proving Theorem 1. First of all, there is a Lorentzian metric on $\mathfrak{sl}_2(\mathbb{R})$ which is defined by

$$\langle A, B\rangle := \frac{1}{2}\mathrm{tr}(AB).$$

An orthonormal basis under this metric is given by

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is readily checked that

$$\langle E_1, E_1\rangle = \langle E_2, E_2\rangle = -\langle E_3, E_3\rangle = 1$$

and $\langle E_i, E_j\rangle = 0$ for all $i \neq j$.

We consider the development $F_\lambda : \mathbb{R}^2 \to \mathfrak{sl}_2(\mathbb{R})$ which is induced (as a linear map) by the relations

$$F_\lambda(e_1) := \lambda E_1, \quad F_\lambda(e_2) := \lambda E_2.$$

Here $\lambda > 0$ is a scale parameter and $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. Let

$$\gamma_t := \Big(\int_0^t \cos\hat{\theta}_s ds, \int_0^t \sin\hat{\theta}_s ds\Big), \quad 0 \leqslant t \leqslant T \tag{4}$$

be a planar BV path which is parametrised at unit speed. The Cartan development of the path $t \mapsto \gamma_{s,t}$ $(t \geqslant s)$ onto $\mathrm{SL}_2(\mathbb{R})$ under F_λ is then described by the ODE

$$\frac{d\Gamma^\lambda_{s,t}}{dt} = \lambda\Gamma^\lambda_{s,t} \cdot \begin{pmatrix} \cos\hat{\theta}_t & \sin\hat{\theta}_t \\ \sin\hat{\theta}_t & -\cos\hat{\theta}_t \end{pmatrix}, \quad \Gamma^\lambda_{s,s} = I. \tag{5}$$

Let $u = (\cos(\zeta), \sin(\zeta))^T \in \mathbb{R}^2$. Define $q^\lambda_t := \Gamma^\lambda_{T-t,T}u$. According to (5), it is easily seen that q^λ_t solves the ODE

$$\frac{dq^\lambda_t}{dt} = \lambda \begin{pmatrix} \cos\theta_t & \sin\theta_t \\ \sin\theta_t & -\cos\theta_t \end{pmatrix} \cdot q^\lambda_t, \quad q^\lambda_0 = u. \tag{6}$$

where $\theta_t := \hat{\theta}_{T-t}$. Let us write $q^\lambda_t = \rho^\lambda_t e^{i\varphi^\lambda_t}$, $t \in [0, T]$ in polar coordinates. The initial condition is that $u = \rho^\lambda_0 e^{i\varphi^\lambda_0}$, where $\rho^\lambda_0 = 1$ and $\varphi^\lambda_0 = \zeta$. Note that both ρ^λ_t and φ^λ_t depend on ζ (which we omit for the ease of notation). The dynamics of $(\rho^\lambda_t, \varphi^\lambda_t)$ is described by the following ODE system (cf. [2]).

Lemma 2. *The pair $(\rho_t^\lambda, \varphi_t^\lambda)$ satisfies the following ODE system:*

$$\begin{cases} \dot{\rho}_s^\lambda = \lambda \rho_s^\lambda \cos(\theta_s - 2\varphi_s^\lambda), \\ \dot{\varphi}_s^\lambda = \lambda \sin(\theta_s - 2\varphi_s^\lambda), \\ \varphi_0^\lambda = \zeta. \end{cases} \tag{7}$$

Proof. Plugging $q_t^\lambda = \rho_t^\lambda e^{i\varphi_t^\lambda}$ into (6) gives

$$dq_t^\lambda = \lambda \rho_t^\lambda dt \begin{pmatrix} \cos\theta_t & \sin\theta_t \\ \sin\theta_t & -\cos\theta_t \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi_t^\lambda \\ \sin\varphi_t^\lambda \end{pmatrix}.$$

On the other hand, one also has by definition that

$$\begin{aligned} dq_t^\lambda =& d\rho_t^\lambda \cdot \begin{pmatrix} \cos\varphi_t^\lambda \\ \sin\varphi_t^\lambda \end{pmatrix} + \rho_t^\lambda d\varphi_t^\lambda \cdot \begin{pmatrix} -\sin\varphi_t^\lambda \\ \cos\varphi_t^\lambda \end{pmatrix} \\ =& \begin{pmatrix} \cos\varphi_t^\lambda & -\sin\varphi_t^\lambda \\ \sin\varphi_t^\lambda & \cos\varphi_t^\lambda \end{pmatrix} \cdot \begin{pmatrix} d\rho_t^\lambda \\ \rho_t^\lambda d\varphi_t^\lambda \end{pmatrix}. \end{aligned}$$

Comparing the above equations, one obtains that

$$\begin{pmatrix} d\rho_t^\lambda \\ \rho_t^\lambda d\varphi_t^\lambda \end{pmatrix} = \lambda \rho_t^\lambda dt \begin{pmatrix} \cos(\theta_t - 2\varphi_t^\lambda) \\ \sin(\theta_t - 2\varphi_t^\lambda) \end{pmatrix},$$

which is exactly the ODE system (7). Q.E.D.

§3. Proof of Theorem 1

In this section, we develop the proof of Theorem 1. Let γ be a planar BV path defined by (4). Let $\Gamma_{s,t}^\lambda$ be the development of γ under F_λ (see (5)). We define the radial and angular components $\rho_t^\lambda, \varphi_t^\lambda$ as in Section 2.3. Recall that they are both functions of ζ.

3.1. Strategy and heuristics

The following observation is a direct corollary of Lemma 1.

Lemma 3. *Suppose that*

$$\int_a^{a+\pi} \rho_T^\lambda \cos(\varphi_T^\lambda - \zeta) d\zeta < -\pi$$

for some $\lambda > 0$ and $a \in \mathbb{R}$. Then $\log S(\gamma)$ has finite ROC.

Proof. By definition, one has

$$u^T \Gamma_{0,T}^\lambda u = \rho_T^\lambda \big(\cos(\zeta)\cos(\varphi_T^\lambda) + \sin(\zeta)\sin(\varphi_T^\lambda) \big) = \rho_T^\lambda \cos(\varphi_T^\lambda - \zeta).$$

The claim follows immediately from Lemma 1. Q.E.D.

Now the strategy of proving Theorem 1 becomes quite clear. Our aim is to show that

$$\int_a^{a+\pi} \rho_T^\lambda \cos(\varphi_T^\lambda - \zeta) d\zeta \to -\infty, \quad \text{as } \lambda \to \infty. \tag{8}$$

Let us explain heuristically why the relation (8) is expected to hold. Let $\alpha_s := 2\varphi_s^\lambda - \theta_s$. Recall that $(\rho_t^\lambda, \varphi_t^\lambda)$ satisfy the ODE system (7). It will be clear from Lemma 6 that

$$\rho_T^\lambda = \exp\left(\lambda \int_0^T \cos(\alpha_s) ds\right) \approx e^{\lambda T} \cos(\frac{\alpha_0}{2}), \quad \lambda \to +\infty.$$

At the same time, one also has $\alpha_T \to 0$ as $\lambda \to +\infty$ which implies that $\varphi_T^\lambda \to \theta_T/2$ (see [2, Lemma A.1]). As a result,

$$\begin{aligned} \int_a^{a+\pi} \rho_T^\lambda \cos\left(\varphi_T^\lambda - \zeta\right) d\zeta &\approx e^{\lambda T} \int_a^{a+\pi} \cos\Big(\zeta - \frac{\theta_0}{2}\Big) \cos\Big(\zeta - \frac{\theta_T}{2}\Big) d\zeta \\ &= \frac{e^{\lambda T}\pi}{2} \cos\Big(\frac{\theta_T - \theta_0}{2}\Big). \end{aligned} \tag{9}$$

Hence, a sufficient condition for (8) to hold is $\cos(\frac{\theta_T - \theta_0}{2}) < 0$, or equivalently, $\hat{\theta}_T - \hat{\theta}_0 \in (4k\pi + \pi, 4k\pi + 3\pi)$ for some integer k.

Remark 1. *A technical difficulty here is that the convergence*

$$e^{-\lambda T} \rho_T^\lambda \cos(\varphi_T^\lambda - \zeta) \to \cos\Big(\zeta - \frac{\theta_0}{2}\Big) \cos\Big(\zeta - \frac{\theta_T}{2}\Big)$$

cannot hold uniformly for all $\zeta \in (a, a+\pi)$. But thanks to Lemma 7, one can select some suitable a and some compact subset K_a on which the uniform convergence can be justified. In addition, the integral over $(a, a+\pi) \setminus K_a$ will be seen to be negligible.

3.2. The C^1 case

We first treat the C^1 case. The main result of this subsection is stated as follows.

Proposition 1. *Let γ be a planar path given by (4). Suppose that $\hat{\theta} \in C^1([0,T])$ and $\hat{\theta}_T - \hat{\theta}_0 \in (4k\pi + \pi, 4k\pi + 3\pi)$ for some integer k. Then $\log S(\gamma)$ has finite ROC.*

Remark 2. *As a special example, the logarithmic signature of any C^1 simple closed curve has finite ROC.*

Before proving Proposition 1, one needs several technical lemmas as preparation. First of all, one has the following comparison estimate.

Lemma 4. *Let $\theta \in C^1([0,T])$ and let $\varepsilon \in (0,\pi/2)$. Let (α_s,β_s) be the solution to the following ODE system:*

$$\begin{cases} \dot{\alpha}_s = -2\lambda \sin\alpha_s - \dot{\theta}_s, \\ \dot{\beta}_s = -2\lambda \sin\beta_s, \\ \alpha_0 = \beta_0 \in [-\pi/2+\varepsilon, \pi/2-\varepsilon]. \end{cases} \tag{10}$$

Then there exists a positive constant $\lambda_0 = \lambda_0(\varepsilon)$, such that the following estimate

$$\sup_{s\in[0,t]} |\alpha_s - \beta_s| \leqslant c_{\theta,t} \frac{1-e^{-2\lambda \sin(\varepsilon t)}}{2\lambda \sin\varepsilon}$$

holds uniformly for all $\lambda > \lambda_0$ and $t \in [0,T]$, where $c_{\theta,t} := \sup_{s\in[0,t]} |\dot{\theta}_s|$.

Proof. By solving the equation for β_t explicitly, one finds that

$$\beta_t = 2\arctan\Big(e^{-2\lambda t}\tan\Big(\frac{\beta_0}{2}\Big)\Big).$$

Due to monotonicity, it is easily seen that

$$|\beta_t| \leqslant |\beta_0| \leqslant \Big|\frac{\pi}{2}-\varepsilon\Big|.$$

In particular, β_t never leaves the interval $[-\pi/2+\varepsilon, \pi/2-\varepsilon]$.

Next, we consider the α_t dynamics. Set

$$\lambda_0 := \frac{\sup_{s\in[0,T]} |\dot{\theta}_s|}{2\sin(\pi/2-\varepsilon)}.$$

For any $\lambda > \lambda_0$ and any s satisfying $\alpha_s = \pi/2-\varepsilon$, it holds that

$$\dot{\alpha}_s = -2\lambda\sin(\alpha_s) - \dot{\theta}_s < -\sup_{s\in[0,T]} |\dot{\theta}_s| - \dot{\theta}_s < 0.$$

Similar estimate also holds if $\alpha_s = -\pi/2+\varepsilon$. As a consequence, the path α_t never leaves the interval $[-\pi/2+\varepsilon, \pi/2-\varepsilon]$ either.

Now let $\lambda > \lambda_0$ and $s \in [0,T]$. Observe that

$$\begin{aligned} \dot{\alpha}_s - \dot{\beta}_s &\leqslant -2\lambda\cos\Big(\frac{\pi}{2}-\varepsilon\Big)(\alpha_s-\beta_s) - \dot{\theta}_s \\ \dot{\alpha}_s - \dot{\beta}_s &\geqslant -2\lambda(\alpha_s-\beta_s) - \dot{\theta}_s. \end{aligned}$$

The above inequalities imply that

$$-\int_0^s \dot{\theta}_u\, e^{-2\lambda(s-u)}du \leqslant \alpha_s - \beta_s \leqslant -\int_0^s \dot{\theta}_u\, e^{-2\lambda(s-u)\sin\varepsilon}du.$$

Therefore, one has

$$|\alpha_s - \beta_s| \leqslant \frac{1 - e^{-2\lambda \sin(\varepsilon s)}}{2\lambda \sin \varepsilon} \sup_{u \in [0,s]} |\dot{\theta}_u|$$

for all $s \in [0,T]$. Taking $c_{\theta,t} := \sup_{s\in[0,t]} |\dot{\theta}_s|$ gives the desired result. Q.E.D.

Remark 3. *The angular function $\alpha_s := 2\varphi_s^\lambda - \theta_s$ satisfies the first ODE of* (10).

The next lemma shows how fast the dynamics enters the central region $[-\pi/2 + \varepsilon, \pi/2 - \varepsilon]$ from outside.

Lemma 5. *Let $\varepsilon \in (0, \pi/2)$ and $k \in (\pi/2 - \varepsilon, \pi)$ be given fixed. Let α_t satisfy the first ODE of* (10). *Define*

$$\tau := \inf\Big\{ t > 0 : |\alpha_t| = \frac{\pi}{2} - \varepsilon \Big\}.$$

(i) Suppose that $\alpha_0 \in [-k, k] \backslash [-\pi/2 + \varepsilon, \pi/2 - \varepsilon]$. Then one has

$$\tau \leqslant \frac{\log(\tan(\alpha_0/2)) - \log(\tan(\pi/4 - \varepsilon/2))}{2\lambda - \frac{\sup_{t\in[0,T]} |\dot{\theta}_t|}{\cos\varepsilon \wedge \sin k}}. \tag{11}$$

for all $\lambda \geqslant \frac{\sup_{t\in[0,T]} |\dot{\theta}_t|}{2(\cos\varepsilon \wedge \sin k)}$.

(ii) Suppose that $\alpha_0 \in (-\pi, \pi) \backslash (-k, k)$. Then one has

$$\tau \geqslant \frac{\log(\tan(k/2)) - \log(\tan(\pi/4 - \varepsilon/2))}{2\lambda + \frac{\sup_{t\in[0,T]} |\dot{\theta}_t|}{\cos\varepsilon \wedge \sin k}} + \frac{\alpha_0 - k}{2\lambda \sin k + \sup_{t\in[0,T]} |\dot{\theta}_t|} \tag{12}$$

for all $\lambda \geqslant \frac{\sup_{t\in[0,T]} |\dot{\theta}_t|}{2(\cos\varepsilon \wedge \sin k)}$.

Proof. Let $\lambda \geqslant \frac{\sup_{t\in[0,T]} |\dot{\theta}_t|}{2(\cos\varepsilon \wedge \sin k)}$ be given fixed. Assume without loss of generality that $\alpha_0 > 0$. By using the equation for α_t, one finds that

$$\log\Big(\tan\Big(\frac{\pi}{4} - \frac{\varepsilon}{2}\Big)\Big) - \log\Big(\tan\Big(\frac{\alpha_0}{2}\Big)\Big) = -2\lambda\tau - \int_0^\tau \frac{\dot{\theta_u}}{\sin(\alpha_u)} du.$$

(i) Suppose that $\pi/2 - \varepsilon < \alpha_0 \leqslant k$. Similar to the proof of Lemma 4, one can show that $\alpha_u \in [\pi/2 - \varepsilon, k]$ for all $u \in [0, \tau]$. As a result,

$$-2\lambda\tau - \int_0^\tau \frac{\dot{\theta_u}}{\sin(\alpha_u)} du \begin{cases} \leqslant \Big(-2\lambda + \frac{\sup_{t\in[0,T]} |\dot{\theta}_t|}{\cos\varepsilon \wedge \sin k} \Big)\tau \\ \\ \geqslant \Big(-2\lambda - \frac{\sup_{t\in[0,T]} |\dot{\theta}_t|}{\cos\varepsilon \wedge \sin k} \Big)\tau \end{cases}$$

It follows that

$$\tau \leqslant \frac{\log(\tan(\alpha_0/2)) - \log(\tan(\pi/4 - \varepsilon/2))}{2\lambda - \frac{\sup_{t\in[0,T]} |\dot{\theta}_t|}{\cos\varepsilon \wedge \sin k}} \tag{13}$$

and

$$\tau \geqslant \frac{\log(\tan(\alpha_0/2)) - \log(\tan(\pi/4 - \varepsilon/2))}{2\lambda + \frac{\sup_{t\in[0,T]} |\dot{\theta}_t|}{\cos\varepsilon \wedge \sin k}}. \tag{14}$$

This proves the estimate (11).

(ii) Suppose that $k < \alpha_0 < \pi$. In this case, the dynamics of α_t has two stages: (a) α_t travels from α_0 to k and (b) α_t travels from k to $\pi/2 - \varepsilon$. One only needs to lower bound the travel time in Stage (a) (the Stage (b) case is handled by (14)). Let $\tau_k := \inf t : \alpha_t = k$. For $s \in [0, \tau_k]$, one has

$$\dot{\alpha}_s \geqslant -2\lambda \sin k - \sup_{t\in[0,T]} |\dot{\theta}_t|.$$

Integrating both sides on $[0, \tau_k]$ yields that

$$k - \alpha_0 \geqslant -\Big(2\lambda \sin k + \sup_{t\in[0,T]} |\dot{\theta}_t|\Big)\tau_k \iff \tau_k \geqslant \frac{\alpha_0 - k}{2\lambda \sin k + \sup_{t\in[0,T]} |\dot{\theta}_t|}. \tag{15}$$

The desired estimate (12) follows from (15) and (14). Q.E.D.

The next two lemmas provide the key analysis for the large-λ behaviour of the radial dynamics. We continue to use the same assumptions and notation as in Lemma 5.

Lemma 6. *Suppose that $\frac{\pi}{2} - \varepsilon < |\alpha_0| \leqslant k < \pi$. Then one has*

$$\lambda \int_0^\tau \cos\alpha_s ds = \frac{1}{2} \log\Big[\frac{\sin|\alpha_0|}{\sin(\pi/2 - \varepsilon)}\Big] + o(1/\lambda) \tag{16}$$

as $\lambda \to +\infty$, where the $o(1/\lambda)$ is uniform for all $|\alpha_0| \in [\pi/2 - \varepsilon, k]$.

Proof. We assume without loss of generality that $\alpha_0 > 0$. By integrating the equation of α on $[0, t]$, one finds that

$$\alpha_t = 2\arctan\Big[\exp\Big(-2\lambda t - \int_0^t \frac{\dot{\theta}_s}{\sin\alpha_s} ds\Big) \tan\Big(\frac{\alpha_0}{2}\Big)\Big].$$

Similar to the proof of Lemma 4, one can show that $\alpha_t \in [\pi/2-\varepsilon, k]$ for all $\lambda \geqslant \frac{\sup_{s\in[0,T]}|\dot{\theta_s}|}{2\cos\varepsilon\wedge\sin(\pi/2-\varepsilon)}$. It follows that

(17)
$$\alpha_t \begin{cases} \geqslant 2\arctan\Big[\exp\Big(-2\lambda t - \int_0^t \frac{c}{\cos\varepsilon\wedge\sin k}ds\Big)\tan(\alpha_0/2)\Big] =: \alpha_t^1, \\ \leqslant 2\arctan\Big[\exp\Big(-2\lambda t + \int_0^t \frac{c}{\cos\varepsilon\wedge\sin k}ds\Big)\tan(\alpha_0/2)\Big] =: \alpha_t^2, \end{cases}$$

where $c := \sup_{s\in[0,T]}|\dot{\theta}_s|$. For the entrance time τ, one also recall from (11) and (12) that

$$\begin{aligned} \tau_1 := \frac{\log(\tan(\alpha_0/2)) - \log(\tan(\pi/4-\varepsilon/2))}{2\lambda + \frac{c}{\cos\varepsilon\wedge\sin k}} \leqslant \tau \\ \leqslant \frac{\log(\tan(\alpha_0/2)) - \log(\tan(\pi/4-\varepsilon/2))}{2\lambda - \frac{c}{\cos\varepsilon\wedge\sin k}} =: \tau_2 \end{aligned} \tag{18}$$

We are going to use (17) and (18) to obtain upper and lower bounds on $\int_0^\tau \cos\alpha_s ds$.

For the upper bound, note that $0 < \alpha_t^1 < \alpha_t^2 \leqslant k < \pi$. By monotonicity, one has

$$\int_0^\tau \cos\alpha_s ds \leqslant \int_0^\tau \cos\alpha_s^1 ds.$$

In addition, once α_s hits $\pi/2-\varepsilon$ it will stay within $[-\pi/2+\varepsilon, \pi/2-\varepsilon]$ afterwards (because $\lambda \geqslant \frac{c}{2\cos\varepsilon\wedge\sin(\pi/2-\varepsilon)}$). In particular, the integrand is positive on $[\tau, \tau_2]$. Therefore,

$$\begin{aligned} \int_0^\tau \cos\alpha_s^1 ds \leqslant \int_0^{\tau_2} \cos\alpha_s^1 ds = \int_0^{\tau_2} \cos^2\Big(\frac{\alpha_s^1}{2}\Big)ds - \int_0^{\tau_2}\sin^2\Big(\frac{\alpha_s^1}{2}\Big)ds \\ := U_1 + U_2. \end{aligned}$$

We will estimate U_1, U_2 separately.

For the term U_1, by using the definition (17) of α_s^1, one has

$$\begin{aligned} U_1 &= \int_0^{\tau_2} \frac{\cos^2(\alpha_0/2)dt}{\cos^2(\alpha_0/2) + \sin^2(\alpha_0/2)\exp(-2t(2\lambda + \frac{c}{\cos\varepsilon\wedge\sin k}))} \\ &= \frac{\log\big[\cos^2(\alpha_0/2)\exp\big(2\tau_2(2\lambda + \frac{c}{\cos\varepsilon\wedge\sin k})\big) + \sin^2(\alpha_0/2)\big]}{4\lambda + \frac{2c}{\cos\varepsilon\wedge\sin k}} \end{aligned} \tag{19}$$

By the definition (18) of τ_2,

$$\begin{aligned}
&\exp\Big[2\tau_2\Big(2\lambda+\frac{c}{\cos\varepsilon\wedge\sin k}\Big)\Big]\\
&=\frac{\tan^2(\alpha_0/2)}{\tan^2(\pi/4-\varepsilon/2)}\exp\Big[4\frac{\frac{c}{\cos\varepsilon\wedge\sin k}}{\lambda-\frac{c}{\cos\varepsilon\wedge\sin k}}\frac{\tan^4(\alpha_0/2)}{\tan^4(\pi/4-\varepsilon/2)}\Big]\\
&=\frac{\tan^2(\alpha_0/2)}{\tan^2(\pi/4-\varepsilon/2)}+o(1/\lambda).
\end{aligned}\tag{20}$$

It follows that

$$\begin{aligned}
&\log\Big[\cos^2(\alpha_0/2)\exp\Big(2\tau_2\Big(2\lambda+\frac{c}{\cos\varepsilon\wedge\sin k}\Big)\Big)+\sin^2(\alpha_0/2)\Big]\\
&=\log\Big[\cos^2(\alpha_0/2)\frac{\tan^2(\alpha_0/2)}{\tan^2(\pi/4-\varepsilon/2)}+\sin^2(\alpha_0/2)\Big]+o(1/\lambda)\\
&=2\log\Big[\sin\Big(\frac{\alpha_0}{2}\Big)\Big]-2\log\Big[\sin\Big(\frac{\pi}{4}-\frac{\varepsilon}{2}\Big)\Big]+o(1/\lambda).
\end{aligned}$$

By substituting this relation into (19), one obtains that

$$\begin{aligned}
\lambda U_1=&\lambda\frac{2\log[\sin(\alpha_0/2)]-2\log[\sin(\pi/4-\varepsilon/2)]}{4\lambda+\frac{2c}{\cos\varepsilon\wedge\sin k}}+o(1/\lambda)\\
=&\frac{1}{2}\log\Big[\frac{\sin(\alpha_0/2)}{\sin(\pi/4-\varepsilon/2)}\Big]+o(1/\lambda).
\end{aligned}\tag{21}$$

The term U_2 is estimated in a similar way and one finds that

$$\lambda U_2=\frac{1}{2}\log\Big[\frac{\cos(\alpha_0/2)}{\cos(\pi/4-\varepsilon/2)}\Big]+o(1/\lambda).\tag{22}$$

Combining (21) and (22), it follows that

$$\lambda\int_0^\tau\cos\alpha_s ds\leqslant\lambda U_1+\lambda U_2=\frac{1}{2}\log\Big[\frac{\sin\alpha_0}{\sin(\pi/2-\varepsilon)}\Big]+o(1/\lambda).\tag{23}$$

This proves the upper bound in (16).

The lower bound of $\int_0^\tau\cos\alpha_s ds$ is proved in essentially the same way. Since $\frac{\pi}{2}-\varepsilon<\alpha_t<\alpha_t^2\leqslant k<\pi$, one has

$$\int_0^\tau\cos\alpha_s ds\geqslant\int_0^\tau\cos\alpha_s^2 ds.\tag{24}$$

By using the estimate (18) for τ, it is seen that

$$2\tau\Big(2\lambda-\frac{c}{\cos\varepsilon\wedge\sin k}\Big)=\log\Big[\frac{\tan^2(\alpha_0/2)}{\tan^2(\pi/4-\varepsilon/2)}\Big]+o(1/\lambda).$$

The same analysis as in the upper bound argument leads to

$$\lambda \int_0^\tau \cos \alpha_s^2 ds \geqslant \frac{1}{2} \log \Big[\frac{\sin \alpha_0}{\sin(\pi/2 - \varepsilon)} \Big]) + o(1/\lambda). \tag{25}$$

Combining with (24), one obtains the matching lower bound in (16). Q.E.D.

Lemma 7. *Suppose that* $|\alpha_0| \leqslant k < \pi$. *Then one has*

$$\lambda \int_0^T \cos \alpha_s ds = \lambda T + \log \Big[\cos \Big(\frac{\alpha_0}{2} \Big) \Big] + o(1/\sqrt{\lambda}) \tag{26}$$

as $\lambda \to \infty$, *where the error term* $o(1/\sqrt{\lambda})$ *is uniform with respect to all* $\alpha_0 \in [-k, k]$.

Proof. As before, assume that $\alpha_0 > 0$. Let β_s satisfy the second equation of (10) with $\beta_\tau = \pi/2 - \varepsilon$. One can write

$$\begin{aligned} \lambda \int_\tau^T \cos(\alpha_s) ds = & \lambda \int_\tau^T \cos(\beta_s) \cos(\alpha_s - \beta_s) ds \\ & - \lambda \int_\tau^T \sin(\beta_s) \sin(\alpha_s - \beta_s) ds. \end{aligned}$$

We now estimate the above two integrals separately.

First of all, by solving the equation for β_s explicitly one finds that

$$\beta_s = 2 \arctan \Big[e^{-2\lambda(s-\tau)} \tan \Big(\frac{\pi}{4} - \frac{\varepsilon}{2} \Big) \Big].$$

It follows that

$$\begin{aligned} \int_\tau^T \cos \beta_s ds &= \int_\tau^T \cos^2 \Big(\frac{\beta_s}{2} \Big) ds - \int_\tau^T \sin^2 \Big(\frac{\beta_s}{2} \Big) ds \\ &= \int_\tau^T \frac{\cos^2(\pi/4 - \varepsilon/2) - \sin^2(\pi/4 - \varepsilon/2) e^{-4\lambda(s-\tau)}}{\cos^2(\pi/4 - \varepsilon/2) + \sin^2(\pi/4 - \varepsilon/2) e^{-4\lambda(s-\tau)}} ds \\ &= \frac{1}{2\lambda} \log \Big[\cos^2 \Big(\frac{\pi}{4} - \frac{\varepsilon}{2} \Big) \, e^{2\lambda s} + \sin^2 \Big(\frac{\pi}{4} - \frac{\varepsilon}{2} \Big) \, e^{-2\lambda s} \Big) \Big|_{s=0}^{T-\tau} \\ &= T - \tau + \frac{1}{2\lambda} \log \Big[1 + \tan^2 \Big(\frac{\pi}{4} - \frac{\varepsilon}{2} \Big) \, e^{-4\lambda s} \Big] \Big|_{s=0}^{T-\tau}. \end{aligned}$$

Note that $\beta_s \in [-\pi/2+\varepsilon, \pi/2-\varepsilon]$ for all $s \in [\tau, T]$, in particular, $\cos\beta_s \geqslant 0$. By applying the estimate (18), one obtains that

$$\begin{aligned}
&\lambda \int_\tau^T \cos\beta_s \cos(\alpha_s - \beta_s) ds \\
&\leqslant \lambda \int_\tau^T \cos\beta_s ds \leqslant \lambda(T-\tau_1) + \frac{1}{2}\log\Big[1+\tan^2\Big(\frac{\pi}{4}-\frac{\varepsilon}{2}\Big)\, e^{-4\lambda s}\Big]\Big|_{s=0}^{T-\tau_2} \\
&\leqslant \lambda T - \frac{1}{2}\log\Big[\tan\Big(\frac{\alpha_0}{2}\Big)\Big] + \frac{1}{2}\log\Big[\tan\Big(\frac{\pi}{4}-\frac{\varepsilon}{2}\Big)\Big] \\
&\qquad + \log\Big[\cos\Big(\frac{\pi}{4}-\frac{\varepsilon}{2}\Big)\Big] + o(1/\lambda).
\end{aligned} \tag{27}$$

The reversed inequality of (27) also holds (with a different $o(1\lambda)$) which is seen as follows. According to Lemma 4, one has

$$\sup_{s\in[\tau,T]} |\alpha_s - \beta_s| \leqslant \frac{c_{\theta,T}}{2\lambda \sin\varepsilon}. \tag{28}$$

It follows from the monotonicity of the cosine function that

$$\begin{aligned}
&\lambda \int_\tau^T \cos\beta_s \cos(\alpha_s-\beta_s) ds \geqslant \lambda \int_\tau^T \cos\beta_s ds \ \cos\Big(\frac{c_{\theta,T}}{2\lambda\sin\varepsilon}\Big) \\
&= \ \lambda\Big(\int_\tau^T \cos\beta_s ds\Big)\ \big(1+o(1/\lambda^{3/2})\big) = \ \lambda\int_\tau^T \cos\beta_s ds \ + o(1/\sqrt{\lambda}).
\end{aligned}$$

Similar to the derivation of (27) with the roles of τ_1, τ_2 being interchanged, one can obtain a lower bound of $\lambda\int_\tau^T \cos\beta_s ds$ given by the right hand side of (27) (of course, with a different $o(1/\lambda)$). Combining with the upper bound, one thus concludes that

$$\begin{aligned}
&\lambda \int_\tau^T \cos\beta_s \cos(\alpha_s-\beta_s) ds \\
&= \lambda T - \frac{1}{2}\log\Big[\tan\Big(\frac{\alpha_0}{2}\Big)\Big] + \frac{1}{2}\log\Big[\tan\Big(\frac{\pi}{4}-\frac{\varepsilon}{2}\Big)\Big] \\
&\qquad + \log\Big[\cos\Big(\frac{\pi}{4}-\frac{\varepsilon}{2}\Big)\Big] + o(1/\sqrt{\lambda}).
\end{aligned} \tag{29}$$

To estimate the sine integral, one notes that

$$\Big|\lambda\int_\tau^T \sin\beta_s \sin(\alpha_s-\beta_s) ds\Big| \leqslant \frac{c_{\theta,T}}{2\sin\varepsilon}\int_\tau^T \sin\beta_s ds.$$

In addition,

$$\begin{aligned}
&\int_\tau^T \sin\beta_s ds = \int_\tau^T 2\sin\Big(\frac{\beta_s}{2}\Big)\cos\Big(\frac{\beta_s}{2}\Big)ds \\
&= \int_\tau^T \frac{2\cos(\pi/4-\varepsilon/2)\sin(\pi/4-\varepsilon/2)e^{-2\lambda(s-\tau)}}{\cos^2(\pi/4-\varepsilon/2)+\sin^2(\pi/4-\varepsilon/2)e^{-4\lambda(s-\tau)}}ds \\
&= -\frac{1}{\lambda}\arctan\Big[\tan\Big(\frac{\pi}{4}-\frac{\varepsilon}{2}\Big)\, e^{-2\lambda s}\Big]\Big|_{s=0}^{T-\tau} \\
&\leqslant \frac{1}{\lambda}\Big(\frac{\pi}{4}-\frac{\varepsilon}{2}\Big) = o(1/\lambda).
\end{aligned}$$

Therefore,

$$\Big|\lambda\int_\tau^T \sin(\beta_s)\sin(\alpha_s-\beta_s)ds\Big| \leqslant o(1/\lambda). \tag{30}$$

Combining the estimates (16), 6, (29) and (30), one obtains that

$$\begin{aligned}
&\lambda\int_0^T \cos\alpha_s ds \\
&= \lambda\int_0^\tau \cos\alpha_s ds + \lambda\int_\tau^T \cos\beta_s\cos(\alpha_s-\beta_s)ds \\
&\quad - \lambda\int_\tau^T \sin\beta_s\sin(\alpha_s-\beta_s)ds \\
&= \frac{1}{2}\log\Big[\frac{\sin\alpha_0}{\sin(\pi/2-\varepsilon)}\Big] + \lambda T - \frac{1}{2}\log\Big[\tan\Big(\frac{\alpha_0}{2}\Big)\Big] \\
&\quad + \frac{1}{2}\log\Big[\tan\Big(\frac{\pi}{4}-\frac{\varepsilon}{2}\Big)\Big] + \log\Big[\cos\Big(\frac{\pi}{4}-\frac{\varepsilon}{2}\Big)\Big] + o(1/\sqrt{\lambda}) \\
&= \lambda T + \log\Big[\cos\Big(\frac{\alpha_0}{2}\Big)\Big] + o(1/\sqrt{\lambda}).
\end{aligned}$$

The desired estimate (26) thus follows. Q.E.D.

We are now in a position to prove Proposition 1.

Proof of Proposition 1. It suffices to establish (8). To this end, let $\delta, \eta, \varepsilon$ be small positive numbers to be chosen later on. According to Lemma 7, $e^{-\lambda T}\rho_T^\lambda$ converges to $\cos(\alpha_0/2)$ uniformly for $\alpha_0 \in [-\pi+\delta, \pi-\delta]$ or equivalently, for $\zeta \in [\frac{-\pi+\theta_0}{2}+\frac{\delta}{2}, \frac{\pi+\theta_0}{2}-\frac{\delta}{2}]$ as $\lambda\to\infty$. More quantitatively, there exists $\lambda_{\theta,\delta,\eta} > 0$ such that

$$\Big|\lambda\Big(\int_0^T \cos\alpha_s ds - T\Big) - \log\Big[\cos\Big(\frac{\alpha_0}{2}\Big)\Big]\Big| \leqslant \eta \tag{31}$$

for all $\lambda > \lambda_{\theta,\delta,\eta}$ and $\zeta \in [\frac{-\pi+\theta_0}{2} + \frac{\delta}{2}, \frac{\pi+\theta_0}{2} - \frac{\delta}{2}]$.

The next step is to show that $\varphi_T^\lambda \to \theta_T/2$ (equivalently, $\alpha_T \to 0$) as $\lambda \to \infty$ uniformly for $\zeta \in [\frac{-\pi+\theta_0}{2} + \frac{\delta}{2}, \frac{\pi+\theta_0}{2} - \frac{\delta}{2}]$. Recall from (13) that

$$\tau \leqslant \tau_2(\alpha_0) = \frac{\log(\tan(\alpha_0/2)) - \log(\tan(\pi/4 - \varepsilon/2))}{2\lambda - \frac{\sup_{t\in[0,T]} |\dot{\theta}_t|}{\cos\varepsilon \wedge \sin(\pi-\delta)}}.$$

We choose $\lambda_{\theta,\delta,\eta,\varepsilon} := \frac{\sup_{t\in[0,T]} |\dot{\theta}_t|}{2\cos(\varepsilon)\wedge\sin(\pi-\delta)} \vee \lambda_{\delta,\eta}$ so that

$$\tau_2(\alpha_0) \leqslant \frac{\log(\tan(\pi/2 - \delta/2)) - \log(\tan(\pi/4 - \varepsilon/2))}{2\lambda_{\theta,\varepsilon,\delta,\eta} - \frac{\sup_{t\in[0,T]} |\dot{\theta}_t|}{\cos\varepsilon\wedge\sin(\pi-\delta)}} < T.$$

holds uniformly for $\alpha_0 \in [-\pi + \delta, \pi - \delta]$. It follows that

$$\alpha_s \in \Big[-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon \Big] \tag{32}$$

for all $s \in [\tau, T]$ and $\lambda > \lambda_{\theta,\delta,\eta,\varepsilon}$. In particular, $|\alpha_T| \leqslant \pi/2 - \varepsilon$.

Let us set $a = \frac{-\pi+\theta_0}{2}$. According to (31) and (32), one has

$$\begin{aligned}
&\int_a^{a+\pi} \rho_T^\lambda \cos(\varphi_T^\lambda - \zeta) d\zeta \leqslant \int_{a+\delta/2}^{a+\pi-\delta/2} \rho_T^\lambda \cos(\varphi_T^\lambda - \zeta) d\zeta + e^{\lambda T}\delta \\
&\leqslant \int_{a+\delta/2}^{a+\pi-\delta/2} e^{\lambda T + \eta} \cos\Big(\zeta - \frac{\theta_0}{2}\Big) \cos\Big[\Big(\frac{\theta_T}{2} - \zeta\Big) - \Big(\frac{\pi}{4} - \frac{\varepsilon}{2}\Big)\Big] d\zeta + e^{\lambda T}\delta \\
&:= e^{\lambda T} A(\eta, \delta, \varepsilon),
\end{aligned} \tag{33}$$

for all $\lambda > \lambda_{\theta,\delta,\eta,\varepsilon}$, where

$$A(\eta, \delta, \varepsilon) := \int_{a+\delta/2}^{a+\pi-\delta/2} e^{\eta} \cos\Big(\zeta - \frac{\theta_0}{2}\Big) \cos\Big[\Big(\frac{\theta_T}{2} - \zeta\Big) - \Big(\frac{\pi}{4} - \frac{\varepsilon}{2}\Big)\Big] d\zeta + \delta.$$

Note that

$$A(\eta, \delta, \varepsilon) \to \int_a^{a+\pi} \cos\Big(\zeta - \frac{\theta_0}{2}\Big) \cos\Big(\frac{\theta_T}{2} - \zeta\Big) d\zeta = \frac{\pi}{2} \cos\Big(\frac{\theta_T - \theta_0}{2}\Big) < 0$$

if one takes $\eta \to 0+$, $\varepsilon \to \frac{\pi}{2}-$ and $\delta \to 0+$ in order. In particular, one can choose the parameters $\eta, \varepsilon, \delta$ so that $A(\eta, \delta, \varepsilon) < 0$. It follows from (33) that

$$\int_a^{a+\pi} \rho_T^\lambda \cos(\varphi_T^\lambda - \zeta) d\zeta \longrightarrow -\infty \quad \text{as} \quad \lambda \to \infty.$$

This completes the proof of Proposition 1. Q.E.D.

3.3. The piecewise C^1 case

Now we are in a position to complete the proof of Theorem 1. The argument is a simple adaptation of the previous analysis to the piecewise C^1 case.

Proof of Theorem 1. Suppose that $\gamma|_{[0,T]} = \gamma|_{[0,T_1]} \sqcup \gamma|_{[T_1,T_2]} \sqcup \cdots \sqcup \gamma|_{[T_{n-1},T_n]}$ with $T_n = T$, where each piece $\gamma|_{[T_{k-1},T_k]}$ is given by the form (4) with some $\hat{\theta} \in C^1$. Let us fix $\varphi^\lambda_{0-} := \varphi^\lambda_0 = \zeta$. On each piece $\gamma|_{[T_k,T_{k+1}]}$, we consider the ODE

$$\begin{cases} \dot{\alpha}_s = -2\lambda \sin \alpha_s - \dot{\theta}_s \\ \alpha_s = 2\varphi^\lambda_s - \theta_s \\ \alpha_{T_k+} = 2\varphi^\lambda_{T_k-} - \theta_{T_k+} \\ \rho^\lambda_s = \exp\left(\lambda \int_0^s \cos\alpha_r dr\right). \end{cases}$$

for $s \in (T_k, T_{k+1})$.

According to Lemma 7, one has

$$\lambda \int_{T_k}^{T_{k+1}} \cos\alpha_s ds = \lambda(T_{k+1} - T_k) + \log\left[\cos\left(\frac{\alpha_{T_k+}}{2}\right)\right] + o(1/\sqrt{\lambda}), \tag{34}$$

where the term $o(1/\sqrt{\lambda})$ is uniform for all $\alpha_{T_k+} = 2\varphi^\lambda_{T_k-} - \theta_{T_k+}$ belonging to a compact subset of $(-\pi, \pi)$. In addition, based on the argument around (32), one knows that $\varphi^\lambda_{T_k-}$ converges to $\theta_{T_k-}/2$ uniformly for all $\alpha_{T_{k-1}+} = 2\varphi^\lambda_{T_{k-1}-} - \theta_{T_{k-1}+}$ belonging to a compact subset of $(-\pi, \pi)$. By arguing recursively, one concludes that $\alpha_{T_k+} \to \theta_{T_k-} - \theta_{T_k+}$ and the term $o(1/\sqrt{\lambda})$ appearing in (34) is uniform for all $k = 0, \cdots, n$ and all $2\zeta - \theta_{0+}$ belonging to a compact subset of $(-\pi, \pi)$ as $\lambda \to \infty$.

Let $\delta > 0$ be a given fixed small number. As a consequence, one concludes that

$$\exp\left[\lambda \int_0^T \cos\alpha_s ds - \lambda T\right] \longrightarrow \cos\left[\zeta - \frac{\theta_{0+}}{2}\right] \prod_{k=1}^{n-1} \cos\left[\frac{\theta_{T_k+} - \theta_{T_k-}}{2}\right]$$

$$\varphi^\lambda_T \longrightarrow \frac{\theta_{T-}}{2}$$

as $\lambda \to \infty$ uniformly for $2\zeta - \theta_0 \in [-\pi + \delta, \pi - \delta]$. A similar argument as in the proof of Proposition 1 to shows that the integral

$$\int_a^{a+\pi} \rho^\lambda_T \cos(\varphi^\lambda_T - \zeta) d\zeta$$

has the same sign as

$$\prod_{k=1}^{n-1}\cos\Big(\frac{\theta_{T_k+}-\theta_{T_k-}}{2}\Big)\int_a^{a+\pi}\cos\Big(\zeta-\frac{\theta_{0+}}{2}\Big)\cos\Big(\zeta-\frac{\theta_{T-}}{2}\Big)d\zeta$$
$$=\prod_{k=1}^{n-1}\cos\Big(\frac{\theta_{T_k+}-\theta_{T_k-}}{2}\Big)\cos\Big(\frac{\theta_{T-}-\theta_{0+}}{2}\Big).$$

for all large λ. Since $\theta_{T_k\pm}\in(-\pi,\pi)$, one always has

$$\prod_{k=1}^{n-1}\cos\Big(\frac{\theta_{T_k+}-\theta_{T_k-}}{2}\Big)>0.$$

In addition, the assumption of the theorem ensures that

$$\cos\Big(\frac{\theta_{T-}-\theta_{0+}}{2}\Big)<0.$$

Therefore,

$$\lim_{\lambda\to\infty}\int_a^{a+\pi}\rho_T^\lambda\cos(\varphi_T^\lambda-\zeta)d\zeta=-\infty.$$

According to Lemma 3, this completes the proof of Theorem 1.

Q.E.D.

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