Parabolic Anderson Model in the Hyperbolic Space. Part II: Quenched Asymptotics

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Abstract

We establish the exact quenched asymptotic growth of the solution to the parabolic Anderson model (PAM) in the hyperbolic space with a regular, stationary, time-independent Gaussian potential. More precisely, we show that with probability one, the solution u to PAM with constant initial data has pointwise growth asymptotics

$$u(t,x) \sim e^{L^* t^{5/3} + o(t^{5/3})}$$

as $t \to +\infty$. Both the power $t^{5/3}$ on the exponential and the exact value of L^* are different from their counterparts in the Euclidean situation. They are determined through an explicit optimisation procedure. Our proof relies on certain fine localisation techniques, which also reveals a stronger non-Euclidean localisation mechanism.

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1 The main theorem

This article is the second part of our investigation on the long time asymptotics of the parabolic Anderson model (PAM) on the hyperbolic space with a regular, stationary, time-independent Gaussian potential. We consider the solution u to

$$\partial_t u = \Delta u + \xi \cdot u, \qquad u(0,\cdot) \equiv 1 \tag{1.1}$$

on the standard hyperbolic space $\mathbb{H}^d$ with constant curvature $\kappa \equiv -1$. Here ξ is a Gaussian field satisfying the following assumption.

Assumption 1.1. *ξ is a real-valued, centered, time-independent, stationary Gaussian field on $\mathbb{H}^d$. Furthermore, its covariance function*[1]

$$C(x,y) = C\big(d(x,y)\big) = \mathbb{E}\big(\xi(x)\xi(y)\big)$$

[1] Stationarity means for every n, every $x_1,\dots,x_n \in \mathbb{H}^d$ and every $g \in SO^+(d,1)$ (the group of orientation preserving isometries over $\mathbb{H}^d$), one has

$$\big(\xi(g\cdot x_1),\dots,\xi(g\cdot x_n)\big) \overset{\text{law}}{=} \big(\xi(x_1),\dots,\xi(x_n)\big)\,.$$

has two continuous derivatives on $\mathbb{R}$, and there exists $R_0 > 0$ such that $C(\rho) = 0$ whenever $\rho > R_0$. Here d denotes the hyperbolic distance.

Except for the finiteness of correlation length, the setting here is identical to the one in the article [GX25]. A simple way to construct such a ξ is to convolute the white noise on the isometry group with a smooth and compactly supported mollifier.

In the first part [GX25], we established the second order annealed asymptotics and demonstrated the same moment intermittency property as in the Euclidean case. One interesting phenomena there is that the fluctuation exponent is determined by the geometry of the *Euclidean* (rather than hyperbolic) Laplacian. The purpose of the present article is to investigate the quenched behaviour of the PAM. Our main theorem is the following.

Theorem 1.1. *For almost every realisation of ξ, one has*

$$\lim_{t\to\infty} \frac{1}{t^{5/3}} \log u(t,o) = \frac{3 \cdot 2^{4/3}}{5^{5/3}} (\sigma^2(d-1))^{2/3}, \tag{1.2}$$

where $o \in \mathbb{H}^d$ is a given fixed base point and $\sigma^2 \triangleq \mathrm{Var}[\xi(o)]$.

Remark 1.1. The solution $u(t,o)$ is the same as the total mass (whole-space integral) of the PAM with δ_o-initial condition.

Unlike the moment asymptotics, the almost sure behaviour revealed by Theorem 1.1 is drastically different from the Euclidean situation (under the same assumptions on the Gaussian field) and as its proof reveals, the PAM also undergoes a stronger non-Euclidean localisation mechanism through its Feynman-Kac representation.

1.1 Introduction

Background and motivation. Motivated from problems in physics, the study of the asymptotic behaviour of directed polymers in random environments have attracted much interest over the past decades in both discrete and continuous settings (cf. [Com17, Kön16] and the references therein). Mathematically, a random polymer is described by a random walk or a Brownian motion propagating through space where random rewards and penalties are distributed. One important question is to understand how the global behaviour of the polymer (i.e. the random walk or Brownian motion) is affected by the impurities of the random environment. A typical phenomenon is that the polymer would exhibit certain localisation property (e.g. favouring particular locations or trajectories) under the polymer measure (Gibbs measure). Such behaviour is intimately related to the (quenched) long time

By stationarity, the covariance function of ξ must be a function of the distance only (see [GX25, Lemma 2.1]).

asymptotics of the so-called partition function, which is the solution to the parabolic Anderson model with potential function describing the random environment.

The long time behaviour of PAM in Euclidean settings (over $\mathbb{Z}^d$ or $\mathbb{R}^d$ with various potentials) have been extensively studied in the literature. The monograph [Kön16] contains an excellent exposition on various asymptotic results in the discrete setting. The annealed and quenched asymptotics for the continuous PAM with a regular Gaussian potential were studied in the seminal works [CM95, GK00, GKM00]. The quenched asymptotics was extended to singular Gaussian potentials in [Che14] and the white noise potential in [KPZ22].

A basic motivation of the current work is to investigate a similar model in a different geometric setting and look for new non-Euclidean asymptotic behaviour and localisation mechanism. As an initial attempt, we choose to work on the standard hyperbolic space where the global geometry and large scale behaviour of Brownian motion differ from the Euclidean situation in several fundamental ways. The underlying random environment is simply described by a time-independent, regular Gaussian potential whose distribution is invariant under isometries. Mathematically, the PAM under consideration is described by the parabolic equation (1.1). In the first part of our investigation [GX25], we established the annealed asymptotics for the PAM (1.1). It turns out that the moment asymptotics is identical to the Euclidean result established in [GK00] and the PAM exhibits the same moment intermittency property as in the Euclidean case. A surprising point over there is that the fluctuation exponent is described by a variational problem induced by the *Euclidean* (rather than hyperbolic) Laplacian.

Main result and novelty. The goal of the present article is to investigate the quenched asymptotics of the PAM (1.1). Our main finding is that the almost sure growth of the solution, as revealed by (1.2), is much faster than the Euclidean situation where the growth rate was found to be (cf. [CM95, GKM00])

$$u(t,o) = \exp\left(\sqrt{2d\sigma^2}t\sqrt{\log t} + o(t\sqrt{\log t})\right) \quad \text{as } t \to \infty. \tag{1.3}$$

The $t^{5/3}$-scale and the exact growth constant in (1.2) are determined through an explicit optimisation procedure that is related to the competition between the reward from peak values of the Gaussian field and the cost of travelling to those regions. There is a new and interesting localisation mechanism for the Brownian motion in this situation: the Brownian motion over $[0,t]$ intends to reach the peak region of the Gaussian field *over a definite ball* (precisely, with radius $K^*t^{4/3}$ where K^* is an explicit number) *at a definite proportion of time* (precisely, at $s = t/5$) and stay there for the rest of time. This localisation mechanism is different from the Euclidean situation (cf. Section 1.2.3 below for a more detailed comparison). We will discuss our methodology for proving this result as well as the main difficulties in Section 1.2 below.

Related works. There are several related works on the behaviour of PAM in non-Euclidean spaces. As a closely related setting (discrete analogue of hyperbolic space), [HKS21] established the quenched asymptotics on a Galton-Watson tree and [HW23] established the annealed asymptotics on a regular tree. In both works, the random potential is assumed to have a double-exponential tail and as a result, the asymptotic theorems obtained in these works are very different in nature from the ones presented in [GX25] and here.

In the continuous setting, the PAM with white noise potential and the analysis of the singular Anderson operator on Riemann surfaces (heat kernel estimates, spectral properties etc.) were studied in depth in a series of works [DDD18, Mou22, BDM25] (see also the references therein). The general framework for singular SPDEs in manifolds was established recently by [HS23].

On the other hand, [BCO24] considered the PAM with a time-dependent (white in time and coloured in space) Gaussian potential on a Cartan-Hadamard manfold, where the authors obtained well-posedness results as well as moment estimates for the solution. The role of non-positive curvature (and global geometry) also plays an essential role in the recent work [CO25] where similar questions as in [BCO24] were investigated. Other works related to the asymptotic behaviour of PAM with a time-dependent Gaussian potential in a geometric setting include e.g. [TV02, BOT23, BCH24, COV24]. We should point out that the case of time-dependent potential is different from the time-independent case in several fundamental ways (the asymptotic results, methodology and underlying mechanism have very different nature in these two settings). The current work focus on the time-independent case and we will not comment much on the other situation.

1.2 Heuristics: methodology and localisation mechanism

In this section, we discuss how Theorem 1.1 is proved at a heuristic but rather non-rigorous level. We will also describe the localisation mechanism for achieving the optimal growth rate $e^{L^* t^{5/3}}$ (L^* is the exact constant given in (1.2)) and compare such mechanism to the Euclidean situation. The proof of Theorem 1.1 contains two parts:

$$\varliminf_{t\to\infty} \frac{1}{t^{5/3}} \log u(t,o) \geqslant L^* \text{ a.s.} \quad \text{(lower asymptotics)} \tag{1.4}$$

and

$$\varlimsup_{t\to\infty} \frac{1}{t^{5/3}} \log u(t,o) \leqslant L^* \text{ a.s.} \quad \text{(upper asymptotics)}. \tag{1.5}$$

The starting points are the following two basic facts.

(i) The solution admits the Feynman-Kac representation:

$$u(t,o) = \mathbb{E}[e^{\int_0^t \xi(W_s)ds}], \tag{1.6}$$

where W is a Brownian motion starting from o and the expectation is taken with respect to W.

(ii) Sample functions of the Gaussian field ξ admit the following almost-sure growth property:

$$\max_{B(o,R)} \xi(x) \approx \sqrt{2\sigma^2(d-1)R} \quad \text{as } R \to \infty. \tag{1.7}$$

According to (1.6) and (1.7), it is natural to expect that the growth of $u(t,o)$ is intimately related to the quantitative competition between peak values of ξ (reward) and probabilities of reaching and staying at those peak regions by the Brownian motion W (cost). This viewpoint is robust for studying this kind of models in various general settings (Euclidean / discrete / other potentials).

1.2.1 Formal derivation of optimal Brownian scenario and growth exponent

In order to prove the lower asymptotics (1.4), one needs to identify a particular Brownian scenario under which the growth rate $e^{L^* t^{5/3}}$ is essentially achieved.

Let $K(t)$ and $\varepsilon(t)$ be parameters to be determined later on ($K(t) \to \infty$ as $t \to \infty$ and $\varepsilon(t) \in [0,t]$). On the ball $B(o,K(t))$, the Gaussian field ξ achieves and maintains its maximal value $\approx \sqrt{2\sigma^2(d-1)K(t)}$ on some (random) island B^t near $\partial B(o,K(t))$. Inspired by the Euclidean situation, we consider the following ansatz (localisation) for the Brownian motion which potentially yields the main contribution of $\mathbb{E}[e^{\int_0^t \xi(W_s)ds}]$.

Ansatz. The Brownian motion W enters the island B^t at time $\varepsilon(t)$ and stays inside it over $[\varepsilon(t),t]$ to pick up the peak values of ξ. In addition, the integral $\int_0^{\varepsilon(t)} \xi(W_s)ds$ and the probability of staying inside B^t during $[\varepsilon(t),t]$ do not contribute to the growth of $u(t,o)$.

By localising at the above scenario, one can formally write down a lower bound for $u(t,o)$:

$$u(t,o) \gtrsim e^{(t-\varepsilon(t))\cdot\sqrt{2\sigma^2(d-1)K(t)}} \times \mathbb{P}(W_{\varepsilon(t)} \in B^t),$$

where the exponential term comes from (1.7). Since B^t is near the boundary $\partial B(o,K(t))$, the hyperbolic heat kernel estimate shows that

$$\mathbb{P}(W_{\varepsilon(t)} \in B^t) \approx e^{-\frac{K(t)^2}{4\varepsilon(t)}}.$$

As a consequence, one has

$$u(t,o) \gtrsim \exp\big((t-\varepsilon(t))\cdot\sqrt{2\sigma^2(d-1)K(t)} - \frac{K(t)^2}{4\varepsilon(t)}\big). \tag{1.8}$$

It is elementary to see that the right hand side of (1.8) can only be large *when* $\varepsilon(t)$ *is proportional to* t *and* $K(t)$ *is of scale* $t^{4/3}$. Therefore, one is led to taking

$$\varepsilon(t) = \varepsilon t,\ \ K(t) = Kt^{4/3}$$

where $\varepsilon \in (0,1)$ and $K > 0$ are given parameters. With these choices, the right hand side of (1.8) becomes $e^{f(\varepsilon,K)t^{5/3}}$ where

$$f(\varepsilon,K) \triangleq (1-\varepsilon)\sqrt{2\sigma^2(d-1)K} - \frac{K^2}{4\varepsilon}. \tag{1.9}$$

Since ε, K can be arbitrary, one obtains that

$$\varliminf_{t\to\infty} \frac{1}{t^{5/3}} \log u(t,o) \geqslant \max\big\{f(\varepsilon,K) : \varepsilon \in (0,1), K > 0\big\}.$$

A standard optimisation procedure yields that

$$\max f(\varepsilon,K) = \frac{3\cdot 2^{4/3}}{5^{5/3}}(\sigma^2(d-1))^{2/3}, \tag{1.10}$$

which then gives the optimal growth constant L^*. The maximum of f is attained at

$$\varepsilon^* = \frac{1}{5},\ \ K^* = \frac{2^{5/3}}{5^{4/3}}(\sigma^2(d-1))^{1/3}. \tag{1.11}$$

To summarise, we have formally obtained a potentially optimal Brownian scenario which is given by the following event:

$$\begin{aligned} \mathcal{O}_t^* = \text{“}W \text{ enters the peak region of } \xi \text{ on } B(o,K^*t^{4/3}) \text{ at time } \varepsilon^* t \\ \text{and stays there over } [\varepsilon^* t, t].\text{”} \end{aligned} \tag{1.12}$$

Under this scenario, one expects the following lower bound to hold true:

$$u(t,o) \geqslant \mathbb{E}\big[e^{\int_0^t \xi(W_s)ds}; \mathcal{O}_t^*\big] \approx e^{L^* t^{5/3}}$$

for all sufficiently large t.

We should point out that the negligibility of the integral $\int_0^{\varepsilon^* t} \xi(W_s)ds$ (which is part of the ansatz) is non-trivial. Intuitively, by requiring the Brownian motion W to reach the peak region B^t of ξ at time $\varepsilon^* t$ which is near $\partial B(o,K^*t^{4/3})$, the Brownian motion is forced to travel at a speed much faster than its normal speed. As a result, the Brownian motion intends to travel like a geodesic with high probability. This makes it reasonable to replace $W|_{[0,\varepsilon^* t]}$ by the corresponding geodesic γ^t connecting o and the center of B^t. Once this is possible, it is then not hard to see that the integral $\int_0^{\varepsilon^* t} \xi(\gamma_s^t)ds$ is negligible since the geodesic γ^t rarely passes through very negative values of ξ.

1.2.2 The upper asymptotics

The proof of the matching upper asymptotics (1.5) is more challenging. Essentially, one needs to analyse every Brownian scenario (i.e. all possible ways of propagating through ξ-peak values) and show that none of these scenarios (nor their total contribution) produces a growth rate that is larger than $e^{L^* t^{5/3}}$. Here we give an intuitive explanation about why this is expected to be true and also point out a major difficulty in the argument.

Optimality of the Brownian scenario (1.12). Let $\varepsilon_1, \varepsilon_2, \eta_1, \eta_2 \in (0,1)$ be such that $\varepsilon_1 + \eta_1 + \varepsilon_2 + \eta_2 = 1$ and let $0 < K_1 < K_2$. For $i = 1, 2$, there is a peak region (say, B_i^t) of ξ near the boundary of $B(o, K_i t^{4/3})$ with ξ-values $\approx \mu_0 \sqrt{K_i t^{4/3}}$ ($\mu_0 \triangleq \sqrt{2\sigma^2(d-1)}$). Consider the following Brownian scenario: *the Brownian motion W enters B_1^t at time $\varepsilon_1 t$, stays there for a period of $\eta_1 t$, then jumps to B_2^t using $\varepsilon_2 t$ amount of time and stays there for the rest of time.* It turns out that the contribution produced by this scenario is given by

$$\exp\big[\big(\sqrt{\mu_0}(\eta_1\sqrt{K_1} + \eta_2\sqrt{K_2})t^{5/3}\big] \times \exp\big[\big(-\frac{K_1^2}{4\varepsilon_1^2} - \frac{(K_2-K_1)^2}{4\varepsilon_2^2}\big)t^{5/3}\big]. \tag{1.13}$$

The first exponential comes from the periods where W stays inside B_i^t ($i = 1, 2$) to pick up the maximal values of ξ. The second exponential comes from the jumping probabilities. Here we have implicitly presumed that neither the ξ-integral outside the staying periods nor the staying probabilities would contribute to the leading growth (which is of course not obvious).

The crucial observation is the following elementary inequality:

$$\begin{aligned}\sqrt{\mu_0}(\eta_1\sqrt{K_1} + \eta_2\sqrt{K_2}) - \frac{K_1^2}{4\varepsilon_1^2} - \frac{(K_2-K_1)^2}{4\varepsilon_2^2}& \\ \leqslant \sqrt{\mu_0}(1 - \varepsilon_1 - \varepsilon_2)\sqrt{K}_2 - \frac{K_2^2}{4(\varepsilon_1+\varepsilon_2)}&. \end{aligned}\tag{1.14}$$

This inequality suggests that the current scenario is not as good as just directly jumping to B_2^t in $(\varepsilon_1 + \varepsilon_2)t$ amount of time and then staying there for the rest of time. An extension of this argument shows that the same conclusion holds even when there are n different ξ-peak regions and n steps of hoppings / stayings among these regions in any order. Therefore, hopping and staying among intermediate peak regions of ξ (in any arbitrary way) is not as good as directly travelling to the furthest peak region and staying there for the rest of time. The latter is defined by the following event:

$$\mathcal{O}_t(\varepsilon, K) = \text{“entering a } \xi\text{-peak region at time } \varepsilon t \text{ of distance } Kt^{4/3} \\ \text{and staying there over } [\varepsilon t, t]\text{"},$$

whose corresponding contribution (at the logarithmic $t^{5/3}$-scale) is just $f(\varepsilon, K)$ defined by (1.9). The optimisation (1.10) then leads to the determination of the parameters (ε^*, K^*) given by (1.11).

A major difficulty. The above argument is only formal and there is a major challenge to make it work. The elementary inequality (1.14), or its extension to n hoppings, is good enough if one pretends that the ξ-peak regions had no size (i.e. they are just degenerate points). This is unfortunately not true in our situation. It turns out that a natural size scale of ξ-peak regions one shall consider is of order $\eta t^{4/3}$ where η is an arbitrarily small number (for otherwise, one would lose control on the length of hoppings and also cannot make the jumping probabilities in the expression (1.13) accurate). In other words, the ξ-peak regions (more precisely, clusters of these regions) now have size of order $\eta t^{4/3}$. If one considers a scenario with N hoppings among the clusters (N needs to be of order η^{-1} to make scenarios with $> N$ hoppings negligible), the use of the inequality (1.13) (more precisely, its N-step extension) would introduce an error of order $N\eta t^{4/3}$. This is due to the fact that the entrance and exit locations in these clusters are generically different; one has to correct the triangle inequality with the cluster size multiplied by the length of steps. The issue is that this error term *cannot* be made negligible (i.e. $\ll e^{L^* t^{5/3}}$) because $N\eta \gtrsim 1$. The resolution of this difficulty relies crucially on finer analysis of the geometry of the Gaussian field ξ as well as the consideration of the Brownian discrete route through clusters at a finer scale.

1.2.3 Comparison with the Euclidean mechanism

Apart from the $t^{5/3}$-growth rate, the localisation mechanism in the hyperbolic space is also quite different from the Euclidean situation. In the Euclidean case, the almost sure growth of the Gaussian field is given by

$$\max_{x \in B(o,R)} \xi(x) \approx \sqrt{2d\sigma^2 \log R} \quad \text{as } R \to \infty.$$

If one applies the same ansatz (localisation) leading to (1.8), one is led to optimising the following inequality instead:

$$u(t,o) \gtrsim \exp\Big((t - \varepsilon(t)) \cdot \sqrt{2\sigma^2 d \log K(t)} - \frac{K(t)^2}{4\varepsilon(t)}\Big).$$

The maximum of the right hand side (under suitable choices of $\varepsilon(t)$ and $K(t)$) is given by

$$\sqrt{2\sigma^2 d} t \sqrt{\log t} + o(t\sqrt{\log t}), \tag{1.15}$$

which explains the growth rate (1.3). The interpretation of the quantity

$$\sqrt{2\sigma^2 d} t \sqrt{\log t} \approx t \times \max_{B(o,O(t))} \xi$$

is very simple: the Brownian motion picks up the maximal values of ξ on a ball with $O(t)$-radius for essentially all time during $[0,t]$. However, there are a few key differences between the hyperbolic and Euclidean situations.

(i) The maximum (1.15) is attained e.g. by taking

$$\varepsilon(t) = \frac{t}{(\log t)^{\alpha}},\ K(t) = \frac{Kt}{(\log t)^{\beta}} \tag{1.16}$$

for any $\alpha, \beta > 0$ with $\alpha - 2\beta < 1/2$ and any $K > 0$. In other words, the Euclidean asymptotics (1.3) is insensitive to a unique localisation scale (there is not a uniquely favorable scale K^*t on which the Brownian motion is localised). In the hyperbolic case, there is such a uniquely defined localisation scale $K^*t^{4/3}$ which is determined by the optimisation problem (1.10).

(ii) In the Euclidean case, the Brownian motion takes essentially no time ($\varepsilon(t)/t \to 0$) to enter the ξ-island; neither the location of the island nor the cost (probability) of reaching there in time $\varepsilon(t)$ would contribute to the growth of $u(t,o)$. In other words, the growth constant $\sqrt{2\sigma^2 d}$ in the asymptotics (1.3) only reflects the maximum of ξ. In the hyperbolic case, the Brownian motion needs to spend a non-trivial proportion of time (i.e. $\varepsilon^* t = t/5$) to enter the ξ-island which is located near the boundary of $B(o, K^*t)$. In addition, there is a competition between the ξ-maximum (reward) and the probability of travelling to the island (cost); the exact growth constant L^* is determined from the compromise between the two quantities.

(iii) As a consequence of (ii), in our modest opinion the upper bound techniques developed in [CM95], [GKM00] may not apply in the hyperbolic case. In fact, both arguments over there (which are very different indeed) are quite global and do not reflect the Brownian localisation at ξ-islands since its cost is negligigle anyway. In the hyperbolic case, such a localisation has to enter the upper bound argument in an essential way. In other words, one needs to compare every generic Brownian scenario to the optimal one (defined by (1.12)) and show that none of them produces a larger growth.

We conclude this section with a few remarks on our results.

Remark 1.2. One can consider the hyperbolic space with curvature $-\kappa$ (instead of just having $\kappa = 1$). The curvature constant κ will be reflected in the Gaussian field growth as well as the heat kernel estimate which is used to compute probability costs. The aforementioned constants L^*, K^* will both depend on κ. An interesting point is that the constant ε^* is always $1/5$ regardless of what κ is. It is also interesting to note that L^* is non-linear with respect to the strength of the Gaussian field (i.e. its standard variation σ) in contrast to the Euclidean case.

Remark 1.3. To some extent, our proof of Theorem 1.1 implicitly reveals a localisation phenomenon for the Brownian motion under the Gibbs measure: with high

(Gibbs) probability, W reaches a particular location near $\partial B(o, K^* t^{4/3})$ at time $\varepsilon^* t$ and stays there for the rest of time. On the other hand, it is not clear whether the PAM is *completely localised*, i.e. does there exist a ξ-measurable site $Z(t)$ such that $v(t, Z_t)/u(t, o) \to 1$ in probability? Here $v(t, x)$ is the solution to the PAM with δ_o-initial condition and recall from Remark 1.1 that $u(t, o) = \int_{\mathbb{H}^d} v(t, x)dx$.

Remark 1.4. Although we work on the standard hyperbolic space, we do not use fine geometric properties in a substantial way (only the exponential volume growth plays an essential role). Therefore, we expect that the methods developed in the current work can be utilised and extended to establish (sharp) asymptotic estimates for similar kind of models in more general geometric settings.

1.3 Further questions

The current article (along with the first part [GX25]) provides a simple initial attempt towards a deeper understanding on the fine asymptotic behaviour of PAM in non-Euclidean settings. We mention a few natural questions for future investigation.

1. Can one derive the quenched fluctuation asymptotics? Since the precise localisation regime is identified in our methodology, the fluctuation behaviour might be described in a suitably rescaled picture with respect to the current localisation. It could be the case that the fluctuation exponent is again determined by the Euclidean Laplacian in a similar fashion as in the annealed asymptotics.

2. How does the asymptotic behaviour of the PAM change when one considers the Gaussian potential with a more singular covariance structure or more intrinsically, the spatial white noise (on the 2D/3D hyperbolic space in this case)? For the white noise case, one needs to apply techniques from singular SPDEs and the picture is not clear at this stage.

3. What would the picture look like if one considers a time-dependent Gaussian potential (e.g. white in time and coloured in space)? In general, questions related to annealed and quenced asymptotics, fluctuation asymptotics, description of localisation vs delocalisation mechanism (in low vs high temperature) and the large scale behaviour of Brownian motion under Gibbs measure are widely open in continuous geometric settings.

2 Some preliminary tools

We list a few preliminary tools that will be used frequently in the sequel. The reader is referred to [GX25, Section 2] for a more general review. Throughout the rest of this article, we fix a base point $o \in \mathbb{H}^d$. We will use Q_R to denote the geodesic ball of radius R centered at o.

Volume form and heat kernel estimates

The exponential volume growth of Q_R as $R \to \infty$ plays an essential role in our study. The volume form on $\mathbb{H}^d$ under geodesic polar coordinates is given by

$$\mathrm{vol}(d\rho, d\sigma) = \sinh^{d-1}\rho d\rho d\sigma.$$

The following two-sided uniform estimate for the hyperbolic heat kernel will also be important to us (cf. [DM98, Theorem 3.1]).

Lemma 2.1. *Let $p(t,x,y)$ be the heat kernel on $\mathbb{H}^d$. Then one has*

$$\begin{aligned} p(t,x,y) \asymp t^{-d/2} \exp\big[-\frac{(d-1)^2 t}{4} - \frac{\rho^2}{4t} - \frac{(d-1)\rho}{2} \big] \\ \times (1+\rho+t)^{\frac{d-3}{2}}(1+\rho), \end{aligned} \tag{2.1}$$

where $\rho \triangleq d(x,y)$. Here the notation $f(\cdot) \asymp g(\cdot)$ means that there are positive constants C_1, C_2 such that

$$C_1 g(\cdot) \leqslant f(\cdot) \leqslant C_2 g(\cdot)$$

uniformly for all variables in the arguments of f, g.

Hyperbolic Brownian motion

The Brownian motion on $\mathbb{H}^d$ is the Markov family $\{W_t^x : t \geqslant 0\}$ ($x \in M$ denotes its starting point) generated by Δ. Its behaviour differs from the Euclidean Brownian motion in several fundamental ways. For instance, it travels a distance of order t when t is large (more precisely, $\frac{d(W_t^x, x)}{(d-1)t} \to 1$ a.s. as $t \to \infty$) and its angular component with respect to the starting point x converges to a definite limiting angle (cf. [Sul83]). In other words, the Brownian motion converges a.s. to a (random) limiting point on the boundary at infinity (which is homeomorphic to the sphere S^{d-1}).

The following lemma gives an SDE description for the radial process $R_t \triangleq d(W_t^o, o)$. It is a direct consequence of the explicit expression of Δ under geodesic polar coordinates with respect to o.

Lemma 2.2. *The process R_t satisfies an SDE*

$$dR_t = \sqrt{2} d\beta_t + (d-1)\coth R_t dt, \tag{2.2}$$

where β_t is a one-dimensional Euclidean Brownian motion.

We also recall an exit time estimate which was proved in [GX25, Lemma 4.1].

Lemma 2.3. *Let τ_R be the exit time of Brownian motion (starting at o) for the ball Q_R. There exist constants $C_1, C_2 > 0$ such that*

$$\mathbb{P}(\tau_R \leqslant t) \leqslant C_1 e^{-C_2 R^2/t}$$

for all $R \gg t > 1$.

Ball packing

The following elementary observation will be used for several times in the article. Let E be a subset of a metric space (X,d). An r-ball packing of E is a maximal way of putting balls of radius r inside E such that they are all disjoint to each other. Suppose that $\mathcal{P} = \{B(x_i,r) : i \in \mathcal{I}\}$ is an r-ball packing of Q_R $(r < R)$. Then $\mathcal{P}' = \{B(x_i,2r) : i \in \mathcal{I}\}$ is a covering of $B(x,R-r)$. Similarly, if $\mathcal{P}$ is an r-ball packing of the annulus $A_{a,b} \triangleq \{x : d(x,o) \in [a,b]\}$, then $\mathcal{P}'$ is a covering of $A_{a+r,b-r}$.

Feynman-Kac representation

It is standard that (cf. [KS88]) the solution $u(t,o)$ to the PAM (1.1) admits the following stochastic representation:

$$u(t,o) = \mathbb{E}\big[e^{\int_0^t \xi(W_s)ds}\big], \quad t \geqslant 0. \tag{2.3}$$

Here $W = \{W_s : s \geqslant 0\}$ is a Brownian motion on $\mathbb{H}^d$ starting from o and the expectation $\mathbb{E}$ is taken with respect to W. Our entire analysis will be based on this representation as a starting point.

3 The lower asymptotics

In this section, we establish the precise lower asymptotics of $u(t,o)$. Recall that

$$L^* \triangleq \frac{3 \cdot 2^{4/3}}{5^{5/3}} (\sigma^2(d-1))^{2/3}$$

is the growth constant which comes from optimising the function $f(\varepsilon,K)$ (cf. (1.10)).

Theorem 3.1. *With probability one,*

$$\varliminf_{t\to\infty} \frac{1}{t^{5/3}} \log u(t,o) \geqslant L^*. \tag{3.1}$$

The proof of (3.1) is conceptually easier than the corresponding (matching) upper asymptotics, which will be the main task in Section 4 below. Essentially, one only needs to identify a particular (optimal) Brownian scenario under which the growth $e^{L^* t^{5/3}}$ is achieved. In the following subsections, we develop the major steps towards proving Theorem 3.1 in a precise mathematical way.

3.1 Sample function growth of Gaussian field

We begin by establishing a simple but important lemma about the growth of the Gaussian field ξ. This plays a prominent role in the proof of the main theorem.

Lemma 3.1. *Let Q_R denote the geodesic ball of radius R centered at the origin o. Then with probability one,*

$$\overline{\lim_{R\to\infty}} \frac{\max_{x\in Q_R}|\xi(x)|}{\sqrt{2\sigma^2(d-1)R}} \leqslant 1.$$

In particular, for each fixed $\mu > \mu_0 \triangleq \sqrt{2\sigma^2(d-1)}$, with probability one

$$\max_{x\in Q_R}|\xi(x)| \leqslant \mu\sqrt{R} \tag{3.2}$$

for all sufficiently large R.

Lemma 3.1 is a simple consequence of the following well-known result (cf. [Adl90, Theorem 2.1]).

Borell's Inequality. Let $\{X_t : t \in T\}$ be a centered Gaussian process with a.s. bounded sample paths. Then one has

$$\mathbb{P}\big(\sup_{t\in T}|X_t| > \lambda\big) \leqslant 4e^{-\frac{1}{2}(\lambda-\mathbb{E}[\sup_{t\in T}X_t])^2/\sigma_T^2} \tag{3.3}$$

for all $\lambda > \mathbb{E}[\sup_{t\in T}X_t]$, where $\sigma_T^2 \triangleq \sup_{t\in T}\mathbb{E}[X_t^2]$.

Proof of Lemma 3.1. Let $r > 0$ be a fixed number. For each $R \gg r$, let $\mathcal{P}_r^R$ be an r-ball packing of Q_{R+r} so that $\{B_i \triangleq B(z_i, 2r) : i \in \mathcal{P}_r^R\}$ is a covering of Q_R. Then one has

$$\mathbb{P}\big(\max_{x\in Q_R}|\xi(x)| > \lambda\big) \leqslant \sum_{i\in\mathcal{P}_r^R}\mathbb{P}\big(\max_{x\in B_i}|\xi(x)| > \lambda\big) = |\mathcal{P}_r^R| \times \mathbb{P}\big(\max_{x\in Q_{2r}}|\xi(x)| > \lambda\big),$$

where $|\mathcal{P}_r^R|$ denotes the cardinality of $\mathcal{P}_r^R$ and the last equality follows from the stationarity of ξ. Since $\{B(z_i, r) : i \in \mathcal{P}_r^R\}$ is a disjoint collection of balls inside Q_{R+r}, one knows that

$$|\mathcal{P}_r^R| \leqslant \frac{\mathrm{vol}(Q_{R+r})}{\mathrm{vol}(Q_r)} \leqslant C_{d,r}e^{(d-1)R}.$$

In addition, according to Borell's inequality (3.3) one has

$$\mathbb{P}\big(\max_{x\in Q_{2r}}|\xi(x)| > \lambda\big) \leqslant 4e^{-\frac{(\lambda-K_r)^2}{2\sigma^2}} \quad \forall\lambda > K_r \triangleq \mathbb{E}\big[\max_{x\in Q_{2r}}\xi(x)\big].$$

Therefore, one obtains that

$$\mathbb{P}\big(\max_{x\in Q_R}|\xi(x)| > \lambda\big) \leqslant 4C_{d,r}e^{(d-1)R-\frac{1}{2\sigma^2}(\lambda-K_r)^2} \tag{3.4}$$

for all such λ.

Given fixed $\varepsilon > 0$, we choose $\lambda \triangleq \sqrt{2\sigma^2(d-1)R} \cdot \sqrt{1+\varepsilon}$. The exponent on the right hand side of (3.4) becomes

$$(d-1)R - \frac{1}{2\sigma^2}(\lambda - K_r)^2 = -\varepsilon(d-1)R + O(\sqrt{R}).$$

As a result, (3.4) is summable over R. By the first Borel-Cantelli lemma, the following estimate

$$\max_{x \in Q_R} |\xi(x)| \leqslant \sqrt{2\sigma^2(d-1)R} \cdot \sqrt{1+\varepsilon} \quad \forall \text{ large } R$$

holds with probability one. The lemma follows since ε is arbitrary.

□

Remark 3.1. Technically, one needs to discretise R in order to apply the Borel-Cantelli lemma. This causes no issue due to the monotonicity of $R \mapsto \max_{x \in Q_R} |\xi(x)|$.

The following lemma is also a direct consequence of Borell's inequality (3.3). This will be useful to us later on.

Lemma 3.2. *For each $R > 0$, there exists a positive constant C_R such that*

$$\mathbb{P}\big(\sup_{x \in Q_R} \xi(x) > \lambda\big) \leqslant e^{-C_R \lambda^2}$$

for all $\lambda > 0$.

3.2 Peak of Gaussian field near boundary

Next, we show that Lemma 3.1 is indeed sharp. More precisely, there is a (random) location z_* near the boundary of Q_R where ξ maintains its peak value $\sqrt{2\sigma^2(d-1)R} + o(\sqrt{R})$ near z_*.

We need to introduce some notation in order to state the main technical lemma here. Throughout the rest, we will set $K(t) \triangleq K^* t^{4/3}$ where K^* is the optimal scale factor defined by (1.11) (cf. Section 1.2.1 for the reason behind this particular choice). We will also fix another parameter δ ($0 < \delta \ll K^*$) as well as a scale function $\delta(t)$. The parameter δ will be sent to zero eventually and $\delta(t)$ will be specified in (3.10) later on (for now, one only needs to keep in mind that $\delta(t) \to 0$ as $t \to \infty$).

Consider the annulus (with width $\delta t^{4/3}$) defined by

$$A_\delta^t \triangleq \{x \in \mathbb{H}^d : (K^* - \delta)t^{4/3} < d(x, o) < K^* t^{4/3}\}. \tag{3.5}$$

Recall that R_0 is the correlation length of ξ. Let $\mathcal{P}_{R_0}^t$ be a given fixed R_0-ball packing of A_δ^t. Elements of $\mathcal{P}_{R_0}^t$ are disjoint balls $B(z_i, R_0)$ ($i \in \mathcal{P}_{R_0}^t$ by abuse of notation). Let

$$h_R \triangleq \sqrt{2\sigma^2(d-1)R} \quad (R > 0). \tag{3.6}$$

The main result of this subsection is stated as follows.

Lemma 3.3. *The following statement holds true with probability one: for all sufficiently large t there exists some $i^* \in \mathcal{P}^t_{R_0}$, such that*

$$\xi(x) \geqslant h_{K(t)} - 2\sqrt{h_{K(t)}} \tag{3.7}$$

for all $x \in B(z_{i^*}, 2\delta(t))$.

We need a basic fluctuation estimate in order to prove Lemma 3.3. Here we introduce one more scale function $\zeta(h) \triangleq h^{-\beta}$ where $\beta \in (1/4, 1/2)$ is given fixed (e.g. just choose $\beta = 1/3$). The choice of $\delta(t)$ will be closely related to the function $\zeta(h)$. We also set

$$H(t) \triangleq \frac{1}{2}\sigma^2 t^2,\ L(h) \triangleq \frac{h^2}{2\sigma^2},\ \rho(h) \triangleq \frac{h}{\sigma^2}.$$

Basically, $H(t)$ is the cumulant generating function of $\xi(o)$, the function

$$L(h) = \sup_{\rho>0}\{\rho h - H(\rho)\} \tag{3.8}$$

is its Legendre transform and $\rho(h)$ is the maximiser in (3.8) (explicitly, $L(h) = \rho(h)h - H(\rho(h))$).

Lemma 3.4. *For any given $\eta \in (0,1)$, one has*

$$\mathbb{P}\big(\max_{x\in Q_{2\zeta(h)}} \big|\xi(x) - h\big| < \eta\sqrt{h}\big) \geqslant \frac{1}{2}e^{-L(h)-\eta\rho(h)\sqrt{h}}$$

for all sufficiently large h.

Proof. This is essentially [GKM00, Lemma 3.3]. Given fixed $h > 0$, consider the change of measure

$$\frac{d\mathbb{P}_h}{d\mathbb{P}} \triangleq e^{\rho(h)\xi(o)-H(\rho(h))}.$$

Let

$$G_\eta \triangleq \big\{\max_{x\in Q_{2\zeta(h)}} \big|\xi(x) - h\big| < \eta\sqrt{h}\big\}.$$

Then one has

$$\begin{aligned}\mathbb{P}(G_\eta) &= \langle \mathbf{1}_{G_\eta} e^{H(\rho)-\rho(h)\xi(o)}\rangle_h \\ &= \langle \mathbf{1}_{G_\eta} e^{-\rho(h)(\xi(o)-h)-L(h)}\rangle_h \geqslant e^{-L(h)-\delta\rho(h)\sqrt{h}}\mathbb{P}_h\big(G_\eta\big).\end{aligned} \tag{3.9}$$

Now a basic observation is that under $\mathbb{P}_h$, the field ξ is a Gaussian with mean function $x \mapsto \rho(h)C(d(x,o))$ and the same covariance function $(x,y) \mapsto C(d(x,y))$. This can be seen e.g. by explicit calculations based on the characteristic function.

Since $C'(0)=0$, there exists a constant $L>0$ such that

$$\Big|\frac{C(d(x,o))}{\sigma^2}-1\Big|\leqslant Ld(x,o)^2$$

for all $x\in Q_1$. It follows that

$$\big|\rho(h)C(d(x,o))-h\big|\leqslant 4Lh\zeta(h)^2=4Lh^{1-2\beta}\quad \forall x\in Q_{2\zeta(h)}.$$

Since $\beta>1/4$, one has

$$\big|\rho(h)C(d(x,o))-h\big|<\frac{\eta}{2}\sqrt{h}\quad \forall x\in Q_{2\zeta(h)}$$

provided that h is large. As a result,

$$\begin{aligned}\mathbb{P}_h(G_\eta)&\geqslant\mathbb{P}_h\big(\max_{x\in Q_{2\zeta(h)}}\big|\xi(x)-\rho(h)C(d(x,o))\big|<\frac{\eta\sqrt{h}}{2}\big)\\&=\mathbb{P}\big(\max_{x\in Q_{2\zeta(h)}}|\xi(x)|<\frac{\eta\sqrt{h}}{2}\big).\end{aligned}$$

Note that the last probability converges to $\mathbb{P}(|\xi(o)|<\infty)=1$ as $h\to\infty$. As a consequence, the right hand side of (3.9) is bounded below by $e^{-L(h)-\eta\rho(h)\sqrt{h}}/2$ for all large h. The lemma thus follows. □

We now proceed to prove Lemma 3.3. We define

$$\delta(t)\triangleq\zeta(\tilde{h}(t))=\tilde{h}(t)^{-\beta},\tag{3.10}$$

where $\tilde{h}(t)\triangleq h_{K(t)}-\sqrt{h_{K(t)}}$. Note that $\delta(t)=O(t^{-2\beta/3})$ as $t\to\infty$.

Proof of Lemma 3.3. Let $\eta\in(0,1)$ be a given fixed number. Recall that $\mathcal{P}^t_{R_0}$ is an R_0-ball packing of the annulus A^t_δ defined by (3.5) and we also set $\hat{B}^t_i\triangleq B(z_i,2\delta(t))$ $(i\in\mathcal{P}^t_{R_0})$. For each $t>0$, consider the event

$$E(t)\triangleq\bigcup_{i\in\mathcal{P}^t_{R_0}}\big\{\xi(x)>\tilde{h}(t)-\eta\sqrt{\tilde{h}(t)}\ \forall x\in\hat{B}^t_i\big\}.$$

By definition, one has

$$E(t)^c\subseteq\bigcap_{i\in\mathcal{P}^t_{R_0}}\big\{\max_{x\in\hat{B}^t_i}|\xi(x)-\tilde{h}(t)|\geqslant\eta\sqrt{\tilde{h}(t)}\big\}.$$

Since the correlation length of ξ is R_0 and $\delta(t) \to 0$ as $t \to \infty$, the Gaussian families $\mathcal{G}_i^t \triangleq \{\xi(x) : x \in \hat{B}_i^t\}$ $(i \in \mathcal{P}_{R_0}^t)$ are independent from each other when t is large. By the stationarity of ξ, one has

$$\mathbb{P}(E(t)^c) \leqslant \mathbb{P}\big(\max_{x \in Q_{2\zeta(t)}} |\xi(x) - \tilde{h}(t)| \geqslant \eta\sqrt{\tilde{h}(t)}\big)^{|\mathcal{P}_{R_0}^t|}, \tag{3.11}$$

where $|\mathcal{P}_{R_0}^t|$ denotes the cardinality of $\mathcal{P}_{R_0}^t$. Let us set

$$p \triangleq \mathbb{P}\big(\max_{x \in Q_{2\zeta(t)}} |\xi(x) - \tilde{h}(t)| < \eta\sqrt{\tilde{h}(t)}\big).$$

The right hand side of (3.11) is further estimated as

$$(1-p)^{|\mathcal{P}_{R_0}^t|} \leqslant e^{-p|\mathcal{P}_{R_0}^t|}.$$

On the one hand, by using Lemma 3.4 as well as the explicit expressions of $L(h), \rho(h)$, one finds that

$$\begin{aligned} p &\geqslant \frac{1}{2} e^{-L(\tilde{h}(t)) - \eta\rho(\tilde{h}(t))\sqrt{\tilde{h}(t)}} \\ &= \frac{1}{2}\exp\big[-\frac{1}{2\sigma^2}(h_{K(t)}^2 - 2h_{K(t)}^{3/2} + h_{K(t)}) - \frac{\eta}{\sigma^2}(h_{K(t)} - \sqrt{h_{K(t)}})^{3/2}\big] \\ &= \frac{1}{2}\exp\big[-(d-1)K(t) + \frac{1}{\sigma^2}h_{K(t)}^{3/2} - \frac{\eta}{\sigma^2}h_{K(t)}^{3/2}(1 - h_{K(t)}^{-1/2})^{3/2} - \frac{1}{2\sigma^2}h_{K(t)}\big]. \end{aligned}$$

On the other hand, note that $\{B(z_i, 2R_0) : i \in \mathcal{P}_{R_0}^t\}$ is a covering of the R_0-reduced annulus

$$\hat{A}_\delta^t \triangleq \{x \in \mathbb{H}^d : (K^* - \delta)t^{4/3} + R_0 < d(x, o) < K^* t^{4/3} - R_0\}.$$

In particular, one has

$$|\mathcal{P}_{R_0}^t| \geqslant \frac{\mathrm{vol}\hat{A}_\delta^t}{\mathrm{vol}Q_{2R_0}} = \frac{\int_{K(t)-\delta t^{4/3}+R_0}^{K(t)-R_0} \sinh^{d-1}\rho d\rho}{\int_0^{2R_0} \sinh^{d-1}\rho d\rho}. \tag{3.12}$$

It is easily seen that when t is sufficiently large,

$$|\mathcal{P}_{R_0}^t| \geqslant C_{d,R_0}\big(e^{(d-1)(K(t)-R_0)} - e^{(d-1)(K(t)-\delta t^{4/3}+R_0)}\big) \geqslant C'_{d,R_0}e^{(d-1)K(t)}.$$

Therefore, with a suitable constant $C = C''_{d,R}$ one concludes that

$$p|\mathcal{P}_{R_0}^t| \geqslant C\exp\big[\frac{1}{\sigma^2}h_{K(t)}^{3/2} - \frac{\eta}{\sigma^2}h_{K(t)}^{3/2}(1 - h_{K(t)}^{-1/2})^{3/2} - \frac{1}{2\sigma^2}h_{K(t)}\big].$$

By fixing $\eta \in (0, 1/2)$, it follows that

$$e^{-p|\mathcal{P}^t_{R_0}|} \leqslant \exp\big[-C\exp\big(\frac{1}{2\sigma^2}h^{3/2}_{K(t)} + o(h^{3/2}_{K(t)})\big)\big] \tag{3.13}$$

for all sufficiently large t.

It is now clear that (3.13) is summable in t. According to the first Borel-Cantelli lemma,

$$\mathbb{P}(E(t)^c \text{ for infinitely many } t) = 0.$$

In other words, with probability one $E(t)$ occurs for all sufficiently large t. By further reducing η if necessary, one can ensure that

$$\tilde{h}(t) - \eta\sqrt{\tilde{h}(t)} > h_{K(t)} - 2\sqrt{h_{K(t)}}$$

for all large t. In particular,

$$E(t) \subseteq \bigcup_{i \in \mathcal{P}^t_{R_0}} \big\{\xi(x) > h_{K(t)} - 2\sqrt{h_{K(t)}}\ \forall x \in \hat{B}^t_i\big\}.$$

The result of the lemma thus follows.

□

3.3 The main localisation

We are now able to define an optimal Brownian scenario under which the lower asymptotics for $u(t, o)$ can be obtained precisely. Let ε^*, K^* be the optimal parameters given by (1.11). We choose $K(t) \triangleq K^* t^{4/3}$ and the parameters δ, $\delta(t)$ as in Section 3.2. In what follows, we fix a realisation of the Gaussian field ξ (outside a suitable null set where the relevant almost sure properties of ξ hold true). Recall that $B(z^t, \delta(t))$ is the (ξ-dependent) ball contained in the annulus A^t_δ such that the estimate (3.7) holds on $B(z^t, 2\delta(t))$. We let $s \mapsto \gamma^t_s$ (parametrised on $[0, \varepsilon^* t]$) be the geodesic connecting o and z^t.

Our optimal Brownian scenario is defined by

$$\mathcal{M}(t) \triangleq C(t) \cap E(t) \cap S(t),$$

where the three events

$$\begin{aligned} C(t) &\triangleq \big\{W_s \in B(\gamma^t_s, \delta) \cap Q_{K(t)}\ \forall s \in [0, \varepsilon^* t]\big\}, \\ E(t) &\triangleq \{W_{\varepsilon^* t} \in B(z^t, \delta(t))\}, \\ S(t) &\triangleq \{W([\varepsilon^* t, t]) \subseteq B(z^t, 2\delta(t))\} \end{aligned}$$

correspond to "concentrating near the geodesic γ^t", "entering $B(z^t,\delta(t))$" and "staying inside $B(z^t,2\delta(t))$" respectively. The role of $C(t)$ is to allow one to replace the Brownian trajectory by the geodesic γ^t, along which the integral of ξ over $[0,\varepsilon^* t]$ is easier to estimate. The role of $S(t)$, as we explained in Section 1.2.1, is to let the Brownian motion stay at the peak region of ξ to pick up its maximal values.

By localising on $\mathcal{M}(t)$ and conditioning on $\mathcal{F}_{\varepsilon^* t}$, one has

$$\begin{aligned} u(t,o) &\geqslant \mathbb{E}_o\big[e^{\int_0^t \xi(W_s)ds}; C(t)\cap E(t)\cap S(t)\big] \\ &= \mathbb{E}_o\big[e^{\int_0^{\varepsilon^* t}\xi(W_s)ds}\mathbf{1}_{C(t)\cap E(t)}\mathbb{E}[e^{\int_{\varepsilon^* t}^t \xi(W_s)ds}\mathbf{1}_{S(t)}|\mathcal{F}_{\varepsilon^* t}]\big]. \end{aligned}$$

According to Lemma 3.3,

$$\int_{\varepsilon^* t}^t \xi(W_s)ds \geqslant (1-\varepsilon^*)t\big(h_{K(t)} - 2\sqrt{h_{K(t)}}\big) \quad \text{on } S(t).$$

It follows from the Markov property that

$$\begin{aligned} u(t,o) \geqslant & e^{(1-\varepsilon^*)t(h_{K(t)}-2\sqrt{h_{K(t)}})}\mathbb{E}_o\big[e^{\int_0^{\varepsilon^* t}\xi(W_s)ds}\mathbf{1}_{C(t)\cap E(t)} \\ & \times \mathbb{P}_{W_{\varepsilon^* t}}\big(\tilde{W}([0,(1-\varepsilon^*)t] \subseteq B(z^t,2\delta(t))\big)\big] \\ \geqslant & e^{(1-\varepsilon^*)t(h_{K(t)}-2\sqrt{h_{K(t)}})} \times \inf_{x\in B(z^t,\delta(t))}\mathbb{P}_x\big(\tilde{W}([0,(1-\varepsilon^*)t]) \subseteq B(z^t,2\delta(t))\big) \\ & \times \int_{B(z^t,\delta(t))}\mathbb{E}_o\big[e^{\int_0^{\varepsilon^* t}\xi(W_s)ds}\mathbf{1}_{C(t)}|W_{\varepsilon^* t}=q\big]p(\varepsilon^* t,o,q)\mathrm{vol}(dq). \end{aligned} \tag{3.14}$$

Here $\tilde{W}$ denotes a new hyperbolic Brownian motion and $p(s,x,y)$ is the heat kernel on $\mathbb{H}^d$. Our main task is now reduced to estimating the quantities appearing on the right hand side of (3.14). For this purpose, we divide the discussion into several individual steps.

3.3.1 Step 1: Staying probability after time $\varepsilon^* t$

We first estimate the "staying probability":

$$\inf_{x\in B(z^t,\delta(t))}\mathbb{P}_x\big(\tilde{W}([0,(1-\varepsilon^*)t]) \subseteq B(z^t,2\delta(t))\big).$$

This is achieved by the lemma below.

Lemma 3.5. *There exist universal constants $\Lambda_1,\Lambda_2>0$, such that*

$$\inf_{x\in B(z^t,\delta(t))}\mathbb{P}_x\big(\tilde{W}([0,(1-\varepsilon^*)t]) \subseteq B(z^t,2\delta(t))\big) \geqslant \Lambda_1 e^{-\Lambda_2\frac{(1-\varepsilon^*)t}{\delta(t)^2}} \tag{3.15}$$

for all sufficiently large t.

Proof. First of all, for any $x \in B(z^t, \delta(t))$ one has

$$\mathbb{P}_x\big(\tilde{W}([0,(1-\varepsilon^*)t]) \subseteq B(z^t, 2\delta(t))\big) \geqslant \mathbb{P}_x\big(d(\tilde{W}_s, x) < \delta(t)\ \forall s \in [0,(1-\varepsilon^*)t]\big).$$

To estimate the latter probability, let $\tilde{R}_s$ be the process

$$d\tilde{R}_s = \sqrt{2}d\beta_s + (d-1)F(\tilde{R}_s)ds,$$

where $F(r) = \coth r$ for $r \leqslant \delta(t)/4$ and $F(r) = 2/\delta(t)$ for $r \geqslant \delta(t)/4$. It follows that

$$\mathbb{P}\big(\sup_{s\in[0,(1-\varepsilon^*)t]} R_s < \delta(t)\big) \geq \mathbb{P}\big(\sup_{s\in[0,(1-\varepsilon^*)t]} \tilde{R}_s < \delta(t)\big)\ .$$

The latter event contains the event that $\tilde{R}$ reaches $\delta(t)/2$ at some $\tau \in [0,(1-\varepsilon^*)t]$ and then remains in the interval $(\delta(t)/4, 3\delta(t)/4)$. Hence, by strong Markov property, one has

$$\mathbb{P}\big(\sup_{s\in[0,(1-\varepsilon^*)t]} \tilde{R}_s < \delta(t)\big) \geq \mathbb{P}\big(\sup_{s\in[0,(1-\varepsilon^*)t]} \big|\beta_s + \frac{Cs}{\delta(t)}\big| < \frac{\delta(t)}{8}\big)$$

for some $C > 0$ independent of t. By scaling, one has

$$\mathbb{P}\big(\sup_{s\in[0,(1-\varepsilon^*)t]} \big|\beta_s + \frac{Cs}{\delta(t)}\big| < \frac{\delta(t)}{8}\big) = \mathbb{P}\Big(\sup_{s\in[0,64(1-\varepsilon^*)t/\delta^2(t)}]|\beta_s - C's| < 1\Big)$$

for some possibly different C'. The claim then follows from standard large deviation estimates for Brownian motion.

□

3.3.2 Step 2: Analysis of the ξ-integral before time $\varepsilon^* t$

Next, we show that the integral $\int_0^{\varepsilon^* t} \xi(W_s)ds$ is essentially negligible on the event $C(t)$ (with respect to the desired growth $e^{L^* t^{5/3}}$). The main estimate is summarised as follows.

Lemma 3.6. *There exist deterministic constants $\Lambda_3, \Lambda_{4,\delta} > 0$ depending on d and the distribution of ξ ($\Lambda_{4,\delta}$ depends additionally on δ), such that with (ξ-)probability one, the following estimate*

$$\int_0^{\varepsilon^* t} \xi(W_s)ds \geqslant -\Lambda_3 \delta t^{5/3} - \Lambda_{4,\delta} t^{1/3} \tag{3.16}$$

holds true on the (W-)event $C(t)$ for all sufficiently large t.

Our strategy for proving Lemma 3.6 is to first analyse the integral where W is replaced by the geodesic γ^t and then to estimate the error. We discuss these two steps separately, each being achieved by a key lemma.

I. Estimation of the integral $\int_0^{\varepsilon^* t} \xi(\gamma_s^t)ds$

Lemma 3.7. *There exists a deterministic constant $C_{d,\xi,\delta} > 0$, such that with probability one, the following estimate*

$$\int_0^{\varepsilon^* t} \xi(\gamma_s^t)ds \geqslant -\frac{\varepsilon^*}{(K^* - \delta)}\left[C_{d,\xi,\delta}t^{1/3} + \delta K^* t^{5/3}\right] \tag{3.17}$$

holds true for all large t.

To prove Lemma 3.7, we will borrow another lemma from Section 4.2 below which will be used in the upper bound argument in an essential way.

Lemma 3.8. *Let*

$$\bar{I}_\delta^t \triangleq \{x \in Q_{K(t)} : \xi(x) < -\delta t^{2/3}\}.$$

Then there exist deterministic constants C_1, C_2 depending only on d and the distribution of ξ, such that by setting

$$L_\delta \triangleq \frac{C_1}{\delta^2},\ \eta_\delta \triangleq C_2\delta^4$$

the following statement holds true:

"With probability one, for all large t no balls of radius $\sqrt{\eta_\delta}t^{4/3}$ inside $Q_{K(t)}$ can contain $\geqslant L_\delta$ points in $\bar{I}_\delta^t$ that are at least $9R_0$-apart from each other".

Proof. This is just Lemma 4.2 (i) below with K_0 replaced by K^* and ξ replaced by $-\xi$. □

Proof of Lemma 3.7. First of all, a simple change of variables shows that

$$\int_0^{\varepsilon^* t} \xi(\gamma_s^t)ds = \frac{\varepsilon^* t}{d(o, z^t)} \int_0^{d(o,z^t)} \xi(\hat{\gamma}_s^t)ds, \tag{3.18}$$

where $\hat{\gamma}^t$ denotes the unit speed reparametrisation of γ^t. In order to apply Lemma 3.8, let us partition the geodesic arc $\hat{\gamma}^t$ into equal sub-arcs $\{\Gamma_i\}$ of length $2\sqrt{\eta_\delta}t^{4/3}$; the total number of sub-arcs is estimated as

$$N(t) = \frac{d(o, z^t)}{2\sqrt{\eta_\delta}t^{4/3}} \leqslant \frac{K^*}{2\sqrt{\eta_\delta}}. \tag{3.19}$$

Each sub-arc Γ_i is further divided into equal pieces $\{\Gamma_{i,j} : 1 \leqslant j \leqslant 2\sqrt{\eta_\delta}t^{4/3}/9R_0\}$ of length $9R_0$. Note that Γ_i is contained in the ball B_i of radius $\sqrt{\eta_\delta}t^{4/3}$ centered at the midpoint of Γ_i. According to Lemma 3.8, at most $2L_\delta$ number of $\Gamma_{i,j}$'s (with i fixed) can have non-empty intersection with $\bar{I}_\delta^t$ (for all large t). Indeed, suppose that $\Gamma_{i,j_1}, \cdots, \Gamma_{i,j_{2L_\delta}}$ all contain elements in $\bar{I}_\delta^t$. One would obtain L_δ points

$$x_1 \in \Gamma_{i,j_2} \cap \bar{I}_\delta^t,\ x_2 \in \Gamma_{i,j_4} \cap \bar{I}_\delta^t, \cdots,\ x_{L_\delta} \in \Gamma_{i,j_{2L_\delta}} \cap \bar{I}_\delta^t$$

inside B_i which are all $9R_0$-apart from each other. This leads to a contradiction to Lemma 3.8.

Let ζ_i denote the union of the sub-arcs $\Gamma_{i,j}$ inside Γ_i which have non-empty intersection with $\bar{I}^t_\delta$. One can write

$$\int_0^{d(o,z^t)} \xi(\hat{\gamma}^t_s)ds = \sum_i \big(\int_{\zeta_i} \xi(\hat{\gamma}^t_s)ds + \int_{\Gamma_i\backslash\zeta_i} \xi(\hat{\gamma}^t_s)ds \big).$$

For the integral over the "bad" region ζ_i, since $\hat{\gamma}^t \subseteq Q_{K(t)}$ one simply applies the crude lower bound (cf. Lemma 3.1)

$$\xi(\hat{\gamma}^t_s) \geqslant -\mu\sqrt{K^*}t^{2/3} \quad \forall s\in\zeta_i,$$

where $\mu > \sqrt{2\sigma^2(d-1)}$ is a given fixed number. It follows from (3.19) that

$$\sum_i \int_{\zeta_i} \xi(\hat{\gamma}^t_s)ds \geqslant -\mu\sqrt{K^*}t^{2/3} \times 2L_\delta \cdot 9R_0 \times \frac{K^*}{2\sqrt{\eta_\delta}} = -\frac{9\mu R_0 L_\delta K^{*3/2}}{\sqrt{\eta_\delta}}t^{2/3}. \quad (3.20)$$

For the "good" region $\Gamma_i\backslash\zeta_i$, one has

$$\sum_i \int_{\Gamma_i\backslash\zeta_i} \xi(\hat{\gamma}^t_s)ds \geqslant -\delta t^{2/3} \times d(o,z^t) \geqslant -\delta K^* t^{6/3}. \quad (3.21)$$

By substituting (3.20, 3.21) into (3.18), one obtains that

$$\int_0^{\varepsilon^* t} \xi(\gamma^t_s)ds \geqslant -\frac{\varepsilon^*}{(K^*-\delta)}\big[\frac{9\mu R_0 L_\delta K^{*3/2}}{\sqrt{\eta_\delta}}t^{1/3} + \delta K^* t^{5/3}\big]$$

for all sufficiently large t. The estimate (3.17) thus follows.

□

II. Estimation of the difference $\int_0^{\varepsilon^* t}\xi(W_s)ds - \int_0^{\varepsilon^* t}\xi(\gamma^t_s)ds$

Lemma 3.9. *There exists a deterministic constant $D_{d,\xi} > 0$, such that with (ξ-)probability one, the following estimate*

$$\big|\int_0^{\varepsilon^* t}\xi(W_s)ds - \int_0^{\varepsilon^* t}\xi(\gamma^t_s)ds\big| \leqslant D_{d,\xi}\varepsilon^*\sqrt{K^*}\delta t^{5/3} \quad (3.22)$$

holds true on the (W-)event $C(t)$ for all sufficiently large t.

Proof. On the event $C(t)$, the Brownian motion W is within the δ-tubular neighbourhood of the geodesic γ^t. In particular, one knows that

$$\big|\int_0^{\varepsilon^* t}\xi(W_s)ds - \int_0^{\varepsilon^* t}\xi(\gamma^t_s)ds\big| \leqslant \|\nabla\xi\|_{\infty;Q_{K(t)}}\delta\varepsilon^* t. \quad (3.23)$$

It remains to estimate the growth of the field $\nabla\xi$.

We claim that there exists a deterministic constant $D_{d,\xi} > 0$, such that with probability one,

$$\sup_{x\in Q_R} |\nabla\xi(x)| \leqslant D_{d,\xi}\sqrt{R} \quad \forall \text{ sufficiently large } R. \tag{3.24}$$

The proof of this claim is essentially the same as the one for Lemma 3.1. Indeed, on the geodesic ball Q_r (under some coordinate system), one has

$$|\nabla\xi|^2 \leqslant C_r \max_{1\leqslant i\leqslant d} \sup_{x\in Q_r} |\partial_i\xi(x)|^2. \tag{3.25}$$

Since $\{\partial_i\xi(x) : x \in Q_r\}$ is also a Gaussian field, one concludes from Borell's inequality (3.3) and (3.25) that

$$\mathbb{P}\big(\max_{x\in Q_r} |\nabla\xi| > \lambda\big) \leqslant e^{-C_{r,\xi}\lambda^2}$$

for all large λ. The same estimate holds for any ball of radius r due to the stationarity of the field $|\nabla\xi|$. The rest of the argument is identical to the proof of Lemma 3.1.

By substituting (3.24) into (3.23), the desired estimate (3.22) thus follows. □

It is now clear that Lemma 3.6 follows directly from Lemma 3.7 and Lemma 3.9.

3.3.3 Step 3: Localisation around geodesic before time $\varepsilon^* t$

We need one more estimate to complete the proof of Theorem 3.1: the conditional probability

$$\mathbb{P}_o(C(t)|W_{\varepsilon^* t} = q) = \mathbb{P}_o\big(W_s \in B(\gamma_s^t, \delta) \cap Q_{K(t)} \ \forall s \in [0, \varepsilon^* t] | W_{\varepsilon^* t} = q\big) \tag{3.26}$$

where $q \in B(z^t, \delta(t))$. This step is technically more involved than the previous ingredients; some geometric consideration (nonpositive curvature) is needed along with large deviation estimates for Brownian bridge. The main estimate for this part is contained in the lemma below.

Lemma 3.10. *Let δ, η be fixed parameters such that*

$$\delta < K^*, \ \ \eta < \min\Big\{\frac{\delta}{24}, \frac{\delta^2}{2560K^*}\Big\}. \tag{3.27}$$

There exist deterministic constants $\Lambda_5, \Lambda_{6,\eta} > 0$ depending on d and the distribution of ξ ($\Lambda_{6,\eta}$ depends additionally on η), such that

$$\mathbb{P}_o(C(t)|W_{\varepsilon^* t} = q) \geqslant \exp\big(-\Lambda_5\eta t^{5/3} - \Lambda_{6,\eta} t^{4/3} \log t\big) \tag{3.28}$$

for all sufficiently large t and $q \in B(z^t, \delta(t))$.

The probability $\mathbb{P}_o(C(t)|W_{\varepsilon^* t} = q)$ is essentially concerned with a Brownian bridge. To summarise our main strategy, we partition the geodesic γ^t into normal scale so that one can apply large deviation estimates for Brownian bridges on each piece. This allows one to easily get rid of the supremum in the event $C(t)$, leaving a suitable event that is only concerned with the bridge at the partition points. The probability of the latter event is expressed in terms of the heat kernel, which is then easily estimated by using heat kernel bounds.

In what follows , we develop the main steps towards proving Lemma 3.10. Given $u < v$ and $x, y \in M$, we will use $\mathbb{P}^{u,x;v,y}(\cdot)$ to denote the law of the Brownian bridge from x (at time u) to y (at time v). This is the Brownian motion starting from x at time u conditioned on reaching y at time v.

Remark 3.2. It is not surprising to expect a reasonable lower bound for the probability $\mathbb{P}_o(C(t)|W_{\varepsilon^* t} = q)$. Indeed, the condition "$W_{\varepsilon^* t} = q$" forces the Brownian motion to travel a distance of order $t^{4/3}$ in time scale t, which is much faster than its normal speed. As a result, it should travel like a geodesic (in order to reach q fast). We do not expect the estimate (3.28) to be sharp but it is enough for our purpose.

I. Geodesic partition

We partition the geodesic γ^t into $N \triangleq d(o,q)/K^*$ equal pieces. This means partitioning $[0, \varepsilon^* t]$ into

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = \varepsilon^* t$$

so that $d(\gamma^t_{t_{i-1}}, \gamma^t_{t_i}) = K^*$ and

$$t_i - t_{i-1} = \frac{\varepsilon^* t}{N} \in \big[\frac{\varepsilon^* t^{-1/3}}{K^*}, \frac{2\varepsilon^* t^{-1/3}}{K^*}\big].$$

One also notes that the intersection with $Q_{K(t)}$ in (3.26) is not needed; since $B(z^t, R_0) \subseteq Q_{K(t)}$ one has

$$\begin{aligned} d(W_s, o) &\leqslant d(W_s, \gamma^t_s) + d(\gamma^t_s, o) \leqslant d(W_s, \gamma^t_s) + d(z^t, o) \\ &\leqslant \delta + K(t) - R_0 < K(t) \end{aligned}$$

provided that $W_s \in B(\gamma^t_s, \delta)$ and $\delta < R_0$. Under above partition, one can further write

$$\begin{aligned} \mathbb{P}_o(C(t)|W_{\varepsilon^* t} = q) &= \mathbb{P}^{0,o;t_N,q}\big(\sup_{0 \leqslant s \leqslant t_N} d(W_s, \gamma^t_s) < \delta\big) \\ &\geqslant \mathbb{P}^{0,o;t_N,q}\big(\bigcap_{i=1}^{N}\big\{\sup_{t_{i-1} \leqslant s \leqslant t_i} d(W_s, \gamma^t_s) < \delta,\ d(W_{t_i}, \gamma^t_{t_i}) < \eta\big\}\big). \end{aligned}$$

Here $\eta \ll \delta$ is another independent parameter which is to be specified later on. We will denote the right hand side by $\phi_{\delta,\eta}(t)$.

Remark 3.3. Technically, one needs to take integer part of $d(o,q)/K^*$ to define the number of partition points. This will not affect the main discussion and we decide not to bother with this small technicality.

II. A uniform LDP estimate for Brownian bridge

We will make use of a large deviation estimate to analyse the probability $\phi_{\delta,\eta}(t)$. The key technical lemma is summarised as follows.

Lemma 3.11. *Let δ,η be given fixed which satisfy the relations in (3.27). Then there exist positive constants C,κ,s_* depending on d,K^*,δ,η, such that the following estimate*

$$\mathbb{P}^{0,x;s,y}\big(\sup_{0\leqslant u\leqslant s} d(W_u,\gamma_u^{x,y;s})>\frac{\delta}{2}\big)\leqslant Ce^{-\kappa/s} \tag{3.29}$$

holds uniformly for all $s\in(0,s_*)$, $x\in B(\gamma_{t_{i-1}}^t,\eta)$, $y\in B(\gamma_{t_i}^t,\eta)$ and all i. Here $\gamma^{x,y;s}$ denotes the geodesic parametrised on $[0,s]$ joining x to y.

Proof. The detailed proof of this lemma is deferred to Appendix A. The exponential decay is a consequence of Hsu's LDP for Riemannian Brownian bridge (cf. [Hsu90]); the extra technical effort here is to obtain uniform estimates with respect to the endpoints x,y, which is not immediate from Hsu's original work. The positivity of κ follows from the positivity of energy excess for paths deviating from the geodesic (cf. Lemma A.2). This part relies on the geometry of nonpositive curvature and is not true in general. □

III. Small ball estimate at partition points

The LDP estimate (3.29) is used for getting rid of the supremum inside $\phi_{\delta,\eta}(t)$, leaving the small ball events at the partition points. We now summarise another lemma which is used for estimating such events.

Lemma 3.12. *There are positive constants C_η,D depending on d,ε^*,K^* (C_η depends additionally on η) and a universal power function $P(t)$ of t, such that*

$$\mathbb{P}^{(t_{i-1},x;t_N,q)}\big(d(W_{t_i},\gamma_{t_i}^t)<\eta\big)\geqslant C_\eta P(t)\exp\big(-D\eta t^{1/3}\big) \tag{3.30}$$

for all $i\leqslant N-1$, $x\in B(\gamma_{t_{i-1}}^t,\eta)$, $q\in B(z^t,\delta(t))$, $\eta\in(0,1)$ and $t>1$.

Proof. The distribution of the Brownian bridge is given by

$$\mathbb{P}^{(t_{i-1},x;t_N,q)}(W_{t_i}\in dy)=\frac{p(t_i-t_{i-1},x,y)p(t_N-t_i,y,q)}{p(t_N-t_{i-1},x,q)}\mathrm{vol}(dy).$$

Using this formula and the two-sided heat kernel estimate (2.1), one finds that

$$\begin{aligned}
&\mathbb{P}^{(t_{i-1},x;t_N,q)}\big(d(W_{t_i},\gamma^t_{t_i})<\eta\big)\\
&=\int_{B(\gamma^t_{t_i},\eta)}\frac{p(t_i-t_{i-1},x,y)p(t_N-t_i,y,q)}{p(t_N-t_{i-1},x,q)}\mathrm{vol}(dy)\\
&\geqslant C_{d,\varepsilon^*,K^*}P(t)\int_{B(\gamma^t_{t_i},\eta)}\exp\big[-\frac{1}{4}\big(\frac{d(x,y)^2}{t_i-t_{i-1}}+\frac{d(y,q)^2}{t_N-t_i}\\
&\qquad-\frac{d(x,q)^2}{t_N-t_{i-1}}\big)-\frac{d-1}{2}\big(d(x,y)+d(y,q)-d(x,q)\big)\big]\mathrm{vol}(dy),
\end{aligned}\tag{3.31}$$

where $P(t)$ denotes some power function of t (of the form t^{-r} for some $r>0$). By the assumptions on x,y,q and noting that $\delta(t)<\eta$ for t large, one has

$$\begin{aligned}
&d(x,y)\leqslant d(x,\gamma^t_{t_{i-1}})+d(\gamma^t_{t_{i-1}},\gamma^t_{t_i})+d(\gamma^t_{t_i},y)<2\eta+\frac{t_i-t_{i-1}}{t_N}d(o,z^t),\\
&d(y,q)\leqslant d(y,\gamma^t_{t_i})+d(\gamma^t_{t_i},z^t)+d(z^t,q)<2\eta+\frac{t_N-t_i}{t_N}d(o,z^t),\\
&d(x,q)\geqslant d(\gamma^t_{t_{i-1}},z^t)-d(x,\gamma^t_{t_{i-1}})-d(z^t,q)>\frac{t_N-t_{i-1}}{t_N}d(o,z^t)-2\eta.
\end{aligned}$$

After substituting the above three estimates into (3.31) and simplification, one arrives at the following estimate:

$$\mathbb{P}^{(t_{i-1},x;t_N,q)}\big(d(W_{t_i},\gamma^t_{t_i})<\eta\big)\geqslant C_{d,\varepsilon^*,K^*,\eta}P(t)\exp\big[-\big(\frac{3\eta K^*}{\varepsilon^*}+\frac{\eta^2}{\varepsilon^*}\big)t^{1/3}-3(d-1)\eta\big].$$

The desired inequality (3.30) follows by absorbing and renaming constants. □

IV. Proof of Lemma 3.10

We are now in a position to prove Lemma 3.10. Let δ,η be given fixed parameters as in Lemma 3.11. To ease notation, we write

$$\psi(t)\triangleq\mathbb{P}^{0,o;t_N,q}\big(\bigcap_{i=1}^N(S_i\cap E_i)\big)$$

where

$$S_i\triangleq\big\{\sup_{t_{i-1}\leqslant s\leqslant t_i}d(W_s,\gamma^t_s)<\delta\big\},\ E_i\triangleq\big\{d(W_{t_i},\gamma^t_{t_i})<\eta\big\}.$$

By conditioning on $\mathcal{F}_{t_{N-1}}$ and noting that E_N is trivially satisfied (since $\delta(t)<\eta$ for large t), one has

$$\psi(t)=\mathbb{E}^{0,o;t_N,q}\big[\mathbb{P}^{t_{N-1},W_{t_{N-1}};t_N,q}\big(\sup_{t_{N-1}\leqslant s\leqslant t_N}d(W_s,\gamma^t_s)<\delta\big);\bigcap_{i=1}^{N-1}(S_i\cap E_i)\big].$$

Let α be the geodesic parametrised on $[t_{N-1}, t_N]$ joining $W_{t_{N-1}}$ to q. Since $\eta \vee \delta(t) < \delta/2$ (for t large), it is easily seen from the geodesic convexity property in hyperbolic space (cf. (A.6) below) that

$$\begin{aligned}&\mathbb{P}^{t_{N-1},W_{t_{N-1}};t_N,q}\big(\sup_{t_{N-1}\leqslant s\leqslant t_N} d(W_s,\gamma_s^t)<\delta\big)\\ &\geqslant \mathbb{P}^{t_{N-1},W_{t_{N-1}};t_N,q}\big(\sup_{t_{N-1}\leqslant s\leqslant t_N} d(W_s,\alpha_s)<\frac{\delta}{2}\big).\end{aligned} \tag{3.32}$$

In addition, note that

$$t_N - t_{N-1} = \frac{\varepsilon^* t}{N} \leqslant 2\varepsilon^* t^{-1/3}.$$

According to Lemma 3.11, the right hand side of (3.32) is bounded below by $1 - Ce^{-\frac{\kappa}{2\varepsilon^*}t^{1/3}}$ provided that $t > (2\varepsilon^*/s_*)^3$. Here C, κ, s_* are constants depending on d, K^*, δ, η. We further assume that $t > T \triangleq (2\varepsilon^*\kappa^{-1}\log 2C)^3$ so that $1 - Ce^{-\frac{\kappa}{2\varepsilon^*}t^{1/3}} > 1/2$. As a consequence, one obtains that

$$\psi(t) \geqslant \frac{1}{2}\mathbb{P}^{0,o;t_N,q}\big(\bigcap_{i=1}^{N-1}(S_i\cap E_i)\big) \tag{3.33}$$

for all $t > T$.

The next conditioning involves the use of Lemma 3.12. One has from (3.33) that

$$\psi(t) \geqslant \frac{1}{2}\mathbb{E}^{0,o;t_N,q}\big[\mathbb{P}^{t_{N-2},W_{t_{N-2}};t_N,q}(S_{N-1}\cap E_{N-1});\bigcap_{i=1}^{N-2}(S_i\cap E_i)\big].$$

The inner probability is estimated as ($x \triangleq W_{t_{N-2}}$)

$$\begin{aligned}&\mathbb{P}^{t_{N-2},x;t_N,q}(S_{N-1}\cap E_{N-1})\\ &= \mathbb{E}^{t_{N-2},x;t_N,q}\big(\mathbf{1}_{B(\gamma^t_{t_{N-1}},\eta)}(W_{t_{N-1}})\mathbb{P}^{t_{N-2},x;t_{N-1},W_{t_{N-1}}}(S_{N-1})\big)\\ &\geqslant \frac{1}{2}\mathbb{P}^{t_{N-2},x;t_N,q}\big(d(W_{t_{N-1}},\gamma^t_{t_{N-1}})<\eta\big)\\ &\geqslant \frac{1}{2}C_\eta P(t)\exp\big(-D\eta t^{1/3}\big).\end{aligned}$$

The first inequality above follows from the same argument leading to (3.33) based on the use of Lemma 3.11. The second inequality is a consequence of Lemma 3.12. Therefore, one obtains that

$$\psi(t) \geqslant 2^{-2}C_\eta P(t)\exp\big(-D\eta t^{1/3}\big)\times\mathbb{P}^{0,o;t_N,q}\big(\bigcap_{i=1}^{N-2}(S_i\cap E_i)\big).$$

One can now proceed by conditioning backward recursively to obtain that (recall $N \leqslant t^{4/3}$)

$$\psi(t) \geqslant (2^{-1}C_\eta)^{t^{4/3}} e^{t^{4/3}\log P(t)} e^{-D\eta t^{5/3}}.$$

Since $P(t)$ is a power function, the estimate (3.16) thus follows by renaming and merging constants.

3.3.4 Step 4: Completing the proof of Theorem 3.1

We are now in a position to complete the proof of the main lower asymptotics (3.1).

Proof of Theorem 3.1. We shall apply Lemma 3.5, Lemma 3.6, Lemma 3.10 as well as the heat kernel estimate (2.1) to the initial lower bound (3.14). First of all, one has from (2.1) that

$$p(\varepsilon^* t, o, q) \geqslant C_d P(t) \exp\Big(-\frac{K^{*2}}{4\varepsilon^*} t^{5/3} - \frac{(d-1)K^* t^{4/3}}{2} - \frac{(d-1)^2 \varepsilon^* t}{4}\Big)$$

where C_d is a positive constant depending only on d and $P(t)$ is some universal power function of t. After all these substitutions, one arrives at the following lower estimate:

$$\begin{aligned} u(t,o) \geqslant & \Lambda_1 C_d \delta(t)^d P(t) \exp\Big[(1-\varepsilon^*)t(h_{K(t)} - \sqrt{h_{K(t)}}) - \frac{(1-\varepsilon^*)\Lambda_2 t}{\delta(t)^2} \\ & - \Lambda_3 \delta t^{5/3} - \Lambda_{4,\delta} t^{1/3} - \Lambda_5 \eta t^{5/3} - \Lambda_{6,\eta} t^{4/3} \log t \\ & - \frac{K^{*2}}{4\varepsilon^*} t^{5/3} - \frac{(d-1)K^* t^{4/3}}{2} - \frac{(d-1)^2\varepsilon^* t}{4}\Big] \end{aligned} \tag{3.34}$$

for all δ, η satisfying (3.27) and all large t.

Recall from Section 1.2.1 that the optimal parameters (ε^*, K^*) are defined so that the optimal growth exponent

$$L^* = (1-\varepsilon^*)\sqrt{2\sigma^2(d-1)K^*} - \frac{K^{*2}}{4\varepsilon^*}. \tag{3.35}$$

Also recall that $h_{K(t)} = \sqrt{2\sigma^2(d-1)K^*} t^{2/3}$ and $\delta(t) = O(t^{-2\beta/3})$ with $\beta \in (1/4, 1/2)$. By substituting (3.35) into (3.34), one finally concludes that

$$u(t,o) \geqslant \exp\big(L^* t^{5/3} - \Lambda_3 \delta t^{5/3} - \Lambda_5 \eta t^{5/3} - o(t^{5/3})\big)$$

where the $o(t^{5/3})$-term can depend on δ, η. The desired lower estimate (3.1) thus follows by first taking $t \to \infty$ and then sending $\eta, \delta \to 0^+$.

The proof of Theorem 3.1 is now complete.

□

4 The matching upper asymptotics

In this section, we prove the matching upper bound for (3.1). The main theorem is stated as follows.

Theorem 4.1. *Let L^* be the number defined by (1.10). Then with probability one,*

$$\overline{\lim_{t\to\infty}}\frac{1}{t^{5/3}}\log u(t,o)\leqslant L^*. \tag{4.1}$$

In the lower bound proof, we essentially identified a particular scenario of Brownian motion under which the solution $u(t,o)$ picks up the growth of $e^{L^*t^{4/3}}$. To prove the matching upper bound (4.1), one has to take into account all possible Brownian trajectories and show that *no Brownian scenarios (nor their total contribution) could produce a growth exponent that is larger than* L^*. This is much harder than the lower bound. In the following subsections, we develop the major steps for the proof of Theorem 4.1 in a precise mathematical way.

4.1 An initial localisation

As an initial step, we first perform a simple localisation of Brownian motion on a large fixed ball (in the scale of $t^{4/3}$).

Lemma 4.1. *There exist deterministic numbers $K_0, C_{K_0} > 0$, such that for almost every realisation of ξ, one has*

$$\mathbb{E}\big[e^{\int_0^t \xi(W_s)ds}; \sup_{0\leqslant s\leqslant t} d(W_s,o) > K_0 t^{4/3}\big] \leqslant C_{K_0}$$

for all large t.

Proof. Given $t, K > 0$, let us denote

$$\mathcal{N}_K^t \triangleq \big\{\sup_{0\leqslant s\leqslant t} d(W_s,o) > Kt^{4/3}\big\}.$$

One further writes

$$\mathbb{E}\big[e^{\int_0^t \xi(W_s)ds};\mathcal{N}_K^t\big] = \sum_{n=1}^{\infty}\mathbb{E}\big[e^{\int_0^t \xi(W_s)ds};\mathcal{N}_n\big]$$

where

$$\mathcal{N}_n \triangleq \big\{\sup_{0\leqslant s\leqslant t} d(W_s,o) \in [K_n t^{4/3}, K_{n+1}t^{4/3}]\big\} \quad (K_n \triangleq nK).$$

According to Lemma 3.1, on each event $\mathcal{N}_n$ one has

$$\sup_{0\leqslant s\leqslant t}\xi(W_s)\leqslant 2\mu_0\sqrt{K_{n+1}t^{4/3}}$$

for all sufficiently large t (for almost every realisation of ξ). It follows from Lemma 2.3 that

$$\mathbb{E}\big[e^{\int_0^t \xi(W_s)ds}; \mathcal{N}_n\big] \leqslant e^{2\mu_0 t\sqrt{K_{n+1}t^{4/3}}}\mathbb{P}\big(\sup_{0\leqslant s\leqslant t} d(W_s,o) \geqslant K_n t^{4/3}\big)$$

$$\leqslant e^{2\mu_0\sqrt{K_{n+1}}t^{5/3}}e^{-CK_n^2 t^{5/3}} = e^{\left(2\mu_0\sqrt{(n+1)K}-CK^2n^2\right)t^{5/3}}. \tag{4.2}$$

By choosing a sufficiently large $K = K_0$, one can ensure that

$$2\mu_0\sqrt{K_0(n+1)} - CK_0^2 n^2 < 0 \quad \forall n \geqslant 1.$$

For this choice of K_0, with probability one

$$\mathbb{E}\big[e^{\int_0^t \xi(W_s)ds}; \mathcal{N}_{K_0}^t\big] \leqslant \sum_{n=1}^{\infty} e^{2\mu_0\sqrt{(n+1)K}-CK^2n^2} =: C_{K_0}. \tag{4.3}$$

The result of the lemma thus follows. □

From now on, we fix the parameter K_0 as in Lemma 4.1. Our task thus reduces to proving the upper bound (4.1) with the unconditional expectation replaced by

$$\mathbb{E}\big[e^{\int_0^t \xi(W_s)ds}; M^t\big], \tag{4.4}$$

where $M^t \triangleq \{d(W_s,o) \leqslant K_0 t^{4/3}\ \forall s \in [0,t]\}$.

4.2 Islands and clusters of the Gaussian field

Next, we discuss some aspects about the geometry of the background Gaussian environment through which the Brownian motion propagates. In particular, we will introduce the notion of islands and clusters for the Gaussian field ξ and prove a few properties that will be important to us later on.

In what follows, we fix two parameters $\delta, \eta > 0$ ($\eta \ll \delta$). Let $t > 0$ be given.

Definition 4.1. (i) A δ-*island* of ξ (inside $Q_{K_0 t^{4/3}}$) is a connected component of the set

$$I_\delta^t \triangleq \{x \in Q_{K_0 t^{4/3}} : \xi(x) > \delta t^{2/3}\}. \tag{4.5}$$

The collection of δ-islands is denoted as $\mathcal{I}_\delta^t$.

(ii) Two δ-islands $E, E' \in \mathcal{I}_\delta^t$ are said to be η-*connected* if there exists a finite sequence of islands

$$E_0 = E, E_1, \cdots, E_{n-1}, E_n \triangleq E'$$

such that the distance between E_{i-1} and E_i is not greater than $\eta t^{4/3}$ ($i = 1, \cdots, n$) An η-*cluster* (of δ-islands) is an equivalence class of δ-islands with respect to η-connectedness. The collection of η-clusters of δ-islands is denoted as $\mathcal{J}_{\eta,\delta}^t$.

By definition, different η-clusters are at least $\eta t^{4/3}$-apart from each other. Note that the islands and clusters are random sets since they are defined for each realisation of ξ. The following lemma summarises the essential properties of η-clusters that are needed for our purpose. Recall that R_0 is the correlation length of the Gaussian field ξ and C_{R_0} is the constant appearing in Lemma 3.2.

Lemma 4.2. *Let $\delta > 0$ be given fixed. Define*

$$L_\delta \triangleq 1.01 \times \frac{2(d-1)K_0}{C_{R_0}\delta^2}. \tag{4.6}$$

Let η be another parameter such that

$$0 < \eta < \eta_\delta \triangleq 0.99 \times \min\left\{\frac{1}{4L_\delta^2}, \frac{C_{R_0}^2\delta^4}{36(d-1)^2}\right\}. \tag{4.7}$$

Then with probability one, for all large t the following properties hold true:

(i) No balls of radius $\sqrt{\eta}t^{4/3}$ inside $Q_{K_0t^{4/3}}$ can contain $\geqslant L_\delta$ points in I_δ^t (cf. (4.5) for its definition) that are at least $9R_0$-apart from each other;

(ii) No η-clusters inside $Q_{K_0t^{4/3}}$ can have diameter $> \sqrt{\eta}t^{4/3}$;

(iii) No η-clusters inside $Q_{K_0t^{4/3}}$ can contain $\geqslant L_\delta$ points in I_δ^t that are at least $9R_0$-apart from each other.

Proof. We just write $L = L_\delta$ for simplicity. It is obvious that (iii) is an immediate consequence of (i) and (ii), since any η-cluster has diameter $\leqslant \sqrt{\eta}t^{4/3}$ and is thus contained in a ball of radius $\sqrt{\eta}t^{4/3}$. Next, we show that (ii) is a consequence of (i). Suppose that (i) is true and assume on the contrary that $\mathfrak{c}$ is some η-cluster with diameter $> \sqrt{\eta}t^{4/3}$. Let $x, y \in \mathfrak{c} \subseteq I_\delta^t$ be two points achieving this diameter so that $d(x,y) > \sqrt{\eta}t^{4/3}$. According to the definition of η-cluster, for each $k = 1, \cdots, \sqrt{\eta}^{-1}$ there is at least one point (say x_k) in $\mathfrak{c}$ that is contained in the annulus

$$A_k \triangleq \{z : (k-1)\eta t^{4/3} \leqslant d(z,x) \leqslant k\eta t^{4/3}\}.$$

In particular, the points $x_2, x_4, \cdots, x_{1/\sqrt{\eta}}$ are contained in the ball $B(x, \sqrt{\eta}t^{4/3})$ but are all at least $\eta t^{4/3}$-apart from each other. Since η satisfies (4.7), the total number of such points is

$$N = \frac{1}{2\sqrt{\eta}} > L.$$

This contradicts Property (i) when t is large enough so that $\eta t^{4/3} > 9R_0$. Therefore, all η-clusters must have diameter $\leqslant \sqrt{\eta}t^{4/3}$. This proves Property (ii).

It remains to prove Property (i). We will use C_i to denote constants depending only on d and the distribution of ξ. First of all, let B be a given fixed ball of radius

$3\sqrt{\eta}t^{4/3}$ inside $Q_{K_0 t^{4/3}}$. Let $\mathcal{U}_{R_0}$ be an R_0-ball packing of the R_0-enlargement of B, so that $\mathcal{U}_{2R_0}$ (defined by enlarging the radius of each member in $\mathcal{U}_{R_0}$ to $2R_0$) is a covering of B. Let E denote the event that $B \cap I_\delta^t$ contains $\geqslant L$ points that are at least $9R_0$-apart from each other. Respectively, let F denote the event that there exist L members in $\mathcal{U}_{2R_0}$ such that each of them has non-empty intersection with I_δ^t and they are all at least R_0-apart from each other. We claim that $E \subseteq F$. Indeed, let $\{y_1, \cdots, y_L\} \subseteq B \cap I_\delta^t$ be such that

$$d(y_i, y_j) > 9R_0 \quad \forall i \neq j.$$

Each y_i is contained in some ball $B(x_i, 2R_0) \in \mathcal{U}_{2R_0}$. As a result, one has

$$d(x_i, x_j) \geqslant d(y_i, y_j) - d(x_i, y_i) - d(x_j, y_j) > 9R_0 - 4R_0 = 5R_0.$$

In particular, the balls $B(x_i, 2R_0)$ $(i = 1, \cdots, L)$ are at least R_0-apart from each other and $B(x_i, 2R_0) \cap I_\delta^t \neq \emptyset$ (because it contains y_i). Therefore, one has $E \subseteq F$.

To estimate the probability of F, for each $B_i \in \mathcal{U}_{2R_0}$ we define

$$\Xi_i \triangleq \sup_{B_i} \xi.$$

Suppose that $B_{i_1}, \cdots, B_{i_L} \in \mathcal{U}_{2R_0}$ are at least R_0-apart from each other. Since the correlation length of ξ is R_0, one knows that $\{\Xi_{i_1}, \cdots, \Xi_{i_L}\}$ are independent from each other. It follows from Lemma 3.2 that

$$\mathbb{P}\big(\Xi_{i_l} > \delta t^{2/3}\ \forall l = 1, \cdots, L\big) \leqslant e^{-LC_{R_0}\delta^2 t^{4/3}}.$$

On the other hand, the cardinality of $\mathcal{U}_{2R_0}$ is estimated as

$$\big|\mathcal{U}_{2R_0}\big| \leqslant \frac{\mathrm{Vol}(Q_{3\sqrt{\eta}t^{4/3}+R_0})}{\mathrm{Vol}(Q_{R_0})} \leqslant C_2 e^{3(d-1)\sqrt{\eta}t^{4/3}}. \tag{4.8}$$

It follows that

$$\begin{aligned}\mathbb{P}(E) \leqslant \mathbb{P}(F) &\leqslant \binom{|\mathcal{U}_{2R_0}|}{L} e^{-LC_{R_0}\delta^2 t^{4/3}} \\ &\leqslant \frac{|\mathcal{U}_{2R_0}|^L}{L!} e^{-LC_{R_0}\delta^2 t^{4/3}} \leqslant \frac{C_2^L}{L!} e^{-L(C_{R_0}\delta^2 - 3(d-1)\sqrt{\eta})t^{4/3}}.\end{aligned}$$

Now suppose that $\mathcal{U}_{\sqrt{\eta}t^{4/3}}$ is a $\sqrt{\eta}t^{4/3}$-ball packing of $Q_{(K_0+\sqrt{\eta})t^{4/3}}$ (so $\mathcal{U}_{2\sqrt{\eta}t^{4/3}}$ covers $Q_{K_0 t^{4/3}}$). Similar to (4.8), the cardinality of $\mathcal{U}_{\sqrt{\eta}t^{4/3}}$ is estimated as

$$\big|\mathcal{U}_{\sqrt{\eta}t^{4/3}}\big| \leqslant \frac{\mathrm{Vol}(Q_{(K_0+\sqrt{\eta})t^{4/3}})}{\mathrm{Vol}(Q_{\sqrt{\eta}t^{4/3}})} \leqslant C_3 e^{(d-1)K_0 t^{4/3}}.$$

Let G^t denote the event that there exists a member $B \in \mathcal{U}_{3\sqrt{\eta}t^{4/3}}$ which contains $\geqslant L$ points in I_δ^t that are at least $9R_0$-apart. Then one has

$$\begin{aligned}\mathbb{P}(G^t) \leqslant \left|\mathcal{U}_{\sqrt{\eta}t^{4/3}}\right| \times \mathbb{P}(E) &\leqslant \frac{C_3 C_2^L}{L!} e^{(d-1)K_0 t^{4/3}} e^{-L(C_{R_0}\delta^2 - 3(d-1)\sqrt{\eta})t^{4/3}} \\ &\leqslant \frac{C_3 C_2^L}{L!} e^{((d-1)K_0 - LC_{R_0}\delta^2/2)t^{4/3}}, \end{aligned} \tag{4.9}$$

where the last inequality follows from the condition (4.7) of η. The exponent on the right hand side of (4.9) is negative due to the definition (4.6) of L. According to the first Borel-Cantelli lemma, one concludes that with probability one, for all large t no members in $\mathcal{U}_{3\sqrt{\eta}t^{4/3}}$ can contain $\geqslant L$ points in I_δ^t that are at least $9R_0$-apart. To demonstrate the desired Property (i), one only needs to observe that any ball of radius $\sqrt{\eta}t^{4/3}$ inside $Q_{K_0 t^{4/3}}$ is contained in some member of $\mathcal{U}_{3\sqrt{\eta}t^{4/3}}$ because $\mathcal{U}_{2\sqrt{\eta}t^{4/3}}$ is a covering of $Q_{K_0 t^{4/3}}$. □

Remark 4.1. When one applies the Borel-Cantelli lemma, technically one should discretise t (e.g. by looking at $\lfloor t \rfloor$ and $I_\delta^{\lfloor t \rfloor}$ instead). This will not cause any essential difference to the above argument (e.g. by a monotonicity consideration).

The following corollary of Lemma 4.2 provides an estimate on the size of a generic η-cluster.

Corollary 4.1. *Let L_δ, η_δ be defined by (4.6, 4.7) respectively. Let $\eta \in (0, \eta_\delta)$ be given fixed. With probability one, for all large t no η-clusters can have diameter $\geqslant 2L_\delta \eta t^{4/3}$.*

Proof. Suppose that such a cluster (say $\mathfrak{c}$) exists and let x, y be two points on $\mathfrak{c}$ that are at least $2L_\delta \eta t^{4/3}$-apart. Consider the annulus

$$A_k \triangleq \{z \in Q_{K_0 t^{4/3}} : (k-1)\eta t^{4/3} \leqslant d(z,x) \leqslant k\eta t^{4/3}\}, \quad k = 1, \cdots, 2L_\delta.$$

By the definition of η-cluster, it follows that each A_k contains at least one point (say x_k) from $\mathfrak{c}$. In particular, the η-cluster $\mathfrak{c}$ contains L points $\{x_{2k}\}_{k=1}^{L_\delta}$ that are all at least $\eta t^{4/3}$-apart. This contradicts Lemma 4.2 (iii) since $\eta t^{4/3} > 9R_0$ when t is large. □

Remark 4.2. In a similar way, it can be shown that with probability one, for all large t no islands in $\mathcal{I}_\delta^t$ can have diameter $> 18R_0 L_\delta$.

4.3 Discrete routes for Brownian motion and a further localisation

In order to upper bound the expectation (4.4) for each fixed realisation of ξ, one needs to carefully analyse the contributions from *all possible ways* of propagating through islands and clusters by the Brownian motion (discrete routes).

4.3.1 Discrete and reduced routes

From now on, we assume that $\lambda < \eta < \delta$ are given fixed parameters where $\eta < \eta_\delta$ (η_δ is defined by (4.7)). Recall that $\mathcal{J}^t_{\eta,\delta}$ is the collection of η-clusters (of δ-islands) inside $Q_{K_0 t^{4/3}}$. We shall also consider the collection $\mathcal{J}^t_{\lambda,\delta}$ of λ-clusters. Since $\lambda < \eta$, it is clear that every λ-cluster is part of an η-cluster and every η-cluster is a disjoint union of certain λ-clusters. As the following lemma shows, the number of λ-clusters constituting any η-cluster is at most L_δ (L_δ is defined by (4.6)).

Lemma 4.3. *With probability one, for all large t every η-cluster inside $Q_{K_0 t^{4/3}}$ contains at most an L_δ number of λ-clusters.*

Proof. Note that different λ-clusters are at least $\lambda t^{4/3}$-apart. When t is large, one has $\lambda t^{4/3} > 9R_0$. The claim thus follows immediately from Lemma 4.2 (iii). □

Now we introduce the notions of discrete routes that will be used in our proof. Let W be a Brownian motion starting from o.

Definition 4.2. The *η-(discrete) route* of W over $[0,t]$, denoted as D^t_η, is the (finite) sequence of η-clusters in $\mathcal{J}^t_{\eta,\delta}$ visited by $W|_{[0,t]}$ in order.

It is possible that D^t_η has repeated letters (revisiting the same cluster after visiting other clusters) but they cannot be adjacent to each other. In our proof, we need to analyse the discrete route through the finer λ-clusters and we need a slightly different definition. By abuse of notation, a cluster can either refer to a label or the actual point set of the cluster depending on its use in the specific context. Given $\mathfrak{c} \in \mathcal{J}^t_{\lambda,\delta}$, we denote $\hat{\mathfrak{c}}$ as the $\lambda t^{4/3}/2$-neighbourhood of $\mathfrak{c}$.

Definition 4.3. Let W be the Brownian motion starting from o. We first set $\tau_0 \triangleq 0$ and

$$\sigma_1 \triangleq \inf\{s \in [0,t] : W_s \in \mathcal{J}^t_{\lambda,\delta}\},\ \ \tau_1 \triangleq \inf\{s \in [\sigma_1,t] : W_s \in \hat{\mathfrak{c}}_1^c\}$$

where $\mathfrak{c}_1$ denotes the λ-cluster to which W_{σ_1} belongs. Then we set

$$\sigma_2 \triangleq \inf\{s \in [\tau_1,t] : W_s \in \mathcal{J}^t_{\lambda,\delta}\},\ \ \tau_2 \triangleq \inf\{s \in [\sigma_2,t] : W_s \in \hat{\mathfrak{c}}_2^c\}$$

and inductively

$$\sigma_i \triangleq \inf\{s \in [\tau_{i-1},t] : W_s \in \mathcal{J}^t_{\lambda,\delta}\},\ \ \tau_i \triangleq \inf\{s \in [\sigma_i,t] : W_s \in \hat{\mathfrak{c}}_i^c\}$$

where $\mathfrak{c}_i$ denotes the λ-cluster to which W_{σ_i} belongs. Let m be the largest integer such that $\sigma_m < t$. The *extended λ-(discrete) route* of W on $[0,t]$ is the word over $\mathcal{J}^t_{\lambda,\delta}$ defined by $\hat{D}^t_\lambda \triangleq \mathfrak{c}_1 \cdots \mathfrak{c}_m$.

The extended λ-route allows consecutive letters to be identical. It is easily seen from the definition that one can read out the λ- and η-routes from the extended λ-route, provided that the configurations of λ- and η-clusters are known. For our purpose, we need to introduce one more simple operation: word reduction.

Definition 4.4. Let $\mathbf{w} = \mathfrak{c}_1 \cdots \mathfrak{c}_m$ be a given word. The *reduced word* of $\mathbf{w}$, denoted as $\bar{\mathbf{w}}$, is defined in the following way. Let i_1 be the last location in $\mathbf{w}$ where $\mathfrak{c}_{i_1} = \mathfrak{c}_1$. Let i_2 be the last location where $\mathfrak{c}_{i_2} = \mathfrak{c}_{i_1+1}$ and so forth. This reduction procedure must terminate at finitely many steps, say $\bar{m}$. The reduced word $\bar{\mathbf{w}}$ is then defined by $\bar{\mathbf{w}} \triangleq \mathfrak{c}_{i_1} \cdots \mathfrak{c}_{i_{\bar{m}}}$.

Example 4.1. Suppose that $\mathbf{w} = abcabbacbccb$. Then $\bar{\mathbf{w}} = acb$.

Remark 4.3. The reason for considering the extended route (rather than the usual one) is just technical; it allows one to apply Markovian argument to analysis probabilities related to Brownian excursions outside the λ-clusters which themselves have a non-Markovian nature. The consideration of reduced λ-route will play a critical role in the proof; it allows one to effectively control the error arising from the ambiguity of entrance locations for the λ-clusters (cf. Lemma 4.7 below).

4.3.2 Reduction to bounded discrete routes

We now return to the main course of proving Theorem 4.1. Before studying a generic Brownian scenario we shall first perform an addition step of localisation, i.e. on the event that discrete routes are bounded. As before, $\lambda < \eta < \delta$ ($\eta < \eta_\delta$) are given fixed.

Lemma 4.4. *Let $N \geqslant 1$ be given fixed. Then with probability one, for all large t one has*

$$\mathbb{E}\big[e^{\int_0^t \xi(W_s)ds}; M^t \cap \{|D^t_\eta| > N\}\big] \leqslant F(t;\eta,N)\cdot \exp\big[-\big(\frac{(\eta N)^2}{32} - 2\mu_0\sqrt{K_0}\big)t^{5/3}\big].$$

Here $F(t;\eta,N)$ is an explicit (deterministic) function which tends to zero as $t \to \infty$ for every fixed η, N.

Proof. Since we are working on the event M^t, we will just trivially bound the exponential by

$$e^{\int_0^t \xi(W_s)ds} \leqslant e^{2\mu_0\sqrt{K_0}t^{5/3}} \tag{4.10}$$

for a.e. realisation of ξ and all large t (cf. (3.2) with $\mu = 2\mu_0$). Our task is thus reduced to estimating the probability $\mathbb{P}(M^t \cap \{|D^t_\eta| > N\})$.

To this end, let $\sigma_1, \sigma_2, \cdots$ denote the successive visit times of the η-clusters by the Brownian motion (cf. Definition 4.3 with λ replaced by η). By conditioning on $\mathcal{F}_{\sigma_{N-1}}$, one has

$$\mathbb{P}\big(M^t \cap \{|D^t_\eta| > N\}\big) \leqslant \mathbb{P}(\sigma_N < t) = \mathbb{E}\big[\mathbf{1}_{\{\sigma_{N-1}<t\}}\mathbb{E}[\mathbf{1}_{\{\sigma_N<t\}}|\mathcal{F}_{\sigma_{N-1}}]\big]. \tag{4.11}$$

According to the Markov property, one can write

$$\mathbb{E}\big[\mathbf{1}_{\{\sigma_N<t\}}|\mathcal{F}_{\sigma_{N-1}}\big] = h^t(W_{\sigma_{N-1}}, t-\sigma_{N-1}), \tag{4.12}$$

where

$$h^t(x,s) \triangleq \mathbb{P}\big(W^x \text{ will visit a new } \eta\text{-cluster before time } s\big). \tag{4.13}$$

Here W^x denotes a Brownian motion starting at x. We assume that x is a point on some η-cluster and $s \in [0,t]$.

To estimate the function $h^t(x,s)$, one first observes by the definition of η-cluster that

$$\{W^x \text{ will visit a new } \eta\text{-cluster before time } s\} \subseteq \big\{\sup_{0\leqslant u\leqslant s} R_u > \eta t^{4/3}\big\},$$

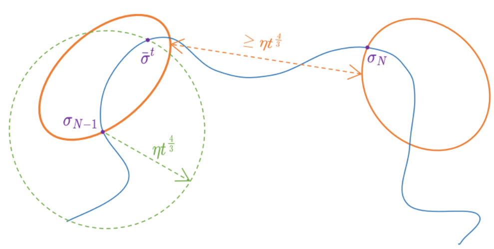
where $R_s \triangleq d(W^x_s, x)$. Next, recall from Lemma 2.2 that R_s satisfies the SDE (2.2) with some Euclidean Brownian motion β_s. Let $f : [0,\infty) \to \mathbb{R}$ be a non-decreasing, smooth function such that

$$f(x) = \frac{1}{2} \text{ on } [0, \frac{1}{2}]; \quad f(x) = x \text{ on } [1,\infty); \quad f'(x) \leqslant 1 \ \forall x.$$

By applying Itô's formula to $f(R_s)$, one finds that

$$\begin{aligned} f(R_s) &= \frac{1}{2} + \int_0^s f'(R_u)\sqrt{2}d\beta_u + \int_0^s \big((d-1)f'(R_u)\coth R_u + f''(R_u)\big)du \\ &=: \frac{1}{2} + M_s + Q_s. \end{aligned}$$

On a possibly enlarged probability space, one can write $M_s = \sqrt{2}\gamma_{\langle M\rangle_s/2}$ with γ being a one-dimensional Euclidean Brownian motion. Let $\bar{\sigma}^t$ denote the hitting time of the level $\eta t^{4/3}$ by the process R. By the construction of f, this is also the hitting time by the process $f(R)$ if t is large enough (so that $\eta t^{4/3} > 1$).



Let $\hat{\sigma}^t$ denote the hitting time of the level $\eta t^{4/3}/2\sqrt{2}$ by the Euclidean Brownian motion γ. Note that

$$\langle M\rangle_s = 2\int_0^s f'(R_u)^2 du \leqslant 2s; \ Q_s \leqslant C_Q s$$

where

$$C_Q \triangleq \sup_{x\geqslant 0} \big((d-1)f'(x)\coth x + f''(x)\big) < \infty. \tag{4.14}$$

Since $s \leqslant t$, it is clear that $\hat{\sigma}^t \leqslant \bar{\sigma}^t$ when t is large. As a result, one finds that

$$h^t(x,s) \leqslant \mathbb{P}(\bar{\sigma}^t < s) \leqslant \mathbb{P}(\hat{\sigma}^t < s). \tag{4.15}$$

On the other hand, it is well-known that (cf. [RW94, Section I.9]) the hitting time $\hat{\sigma}^t$ has an explicit density function

$$\mathbb{P}(\hat{\sigma}^t \in ds) = \frac{\eta t^{4/3}/2\sqrt{2}}{\sqrt{2\pi}s^{3/2}} e^{-\eta^2 t^{4/3}/16s} ds. \tag{4.16}$$

By substituting this into the estimate (4.15), it follows that

$$h^t(x,s) \leqslant \frac{\eta t^{4/3}}{4\sqrt{\pi}} \int_0^s u^{-3/2} e^{-\eta^2 t^{4/3}/16u} du =: \phi_{N-1}(s).$$

Note that ϕ_{N-1} is an increasing function. In view of (4.11) and (4.12), one thus obtains that

$$\mathbb{P}\big(M^t \cap \{|D^t_\eta| > N\}\big) \leqslant \mathbb{E}\big[\mathbf{1}_{\{\sigma_{N-1}<t\}} \phi_{N-1}(t-\sigma_{N-1})\big] \tag{4.17}$$

for all large t.

To estimate the right hand side of (4.17) we further condition on $\mathcal{F}_{\sigma_{N-2}}$. This gives

$$\begin{aligned} &\mathbb{E}\big[\mathbf{1}_{\{\sigma_{N-1}<t\}} \phi_{N-1}(t-\sigma_{N-1})\big] \\ &= \mathbb{E}\big[\mathbf{1}_{\{\sigma_{N-2}<t\}} \mathbb{E}[\mathbf{1}_{\{\sigma_{N-1}<t\}} \phi_{N-1}(t-\sigma_{N-1})|\mathcal{F}_{\sigma_{N-2}}]\big] \\ &= \mathbb{E}\big[\mathbf{1}_{\{\sigma_{N-2}<t\}} \tilde{h}^t(W_{\sigma_{N-2}}, t-\sigma_{N-2})\big], \end{aligned} \tag{4.18}$$

where

$$\tilde{h}^t(x,s) \triangleq \mathbb{E}\big[\mathbf{1}_{\{\sigma^t<s\}} \phi_{N-1}(s-\sigma^t)\big]$$

and σ^t is the hitting time of a new η-cluster by W^x. Let us define the hitting times $\bar{\sigma}^t, \hat{\sigma}^t$ in exactly the same way as before (on a possibly enlarged probability space). Note that $\hat{\sigma}^t \leqslant \bar{\sigma}^t \leqslant \sigma^t$ (for all large t). By the monotonicity of ϕ_{N-1} and the density formula (4.16), in a similar fashion one finds that

$$\begin{aligned} \tilde{h}^t(x,s) &\leqslant \mathbb{E}\big[\mathbf{1}_{\{\hat{\sigma}^t<s\}} \phi_{N-1}(s-\hat{\sigma}^t)\big] \\ &= \frac{\eta t^{4/3}}{4\sqrt{\pi}} \int_0^s u^{-3/2} e^{-\eta^2 t^{4/3}/16u} \phi_{N-1}(s-u) du =: \phi_{N-2}(s). \end{aligned}$$

We again remark that ϕ_{N-2} is an increasing function. By substituting this estimate into (4.18), it follows from (4.17) that

$$\mathbb{P}\big(M^t \cap \{|D^t_\eta| > N\}\big) \leqslant \mathbb{E}\big[\mathbf{1}_{\{\sigma_{N-2}<t\}} \phi_{N-2}(t-\sigma_{N-2})\big]$$

for all large t.

One can now argue recursively (by further conditioning backward) to conclude that

$$\begin{aligned}\mathbb{P}\big(M^t \cap \{|D_\eta^t| > N\}\big) &\leqslant \mathbb{E}\big[\mathbf{1}_{\{\sigma_{N-3}<t\}}\phi_{N-3}(t-\sigma_{N-3})\big] \\ &\cdots \\ &\leqslant \mathbb{E}\big[\mathbf{1}_{\{\sigma_1<t\}}\phi_1(t-\sigma_1)\big] \leqslant \phi_0(t) \qquad (4.19)\end{aligned}$$

for all large t. Here the increasing functions $\phi_j : [0,t] \to [0,\infty)$ are defined recursively by

$$\phi_{j-1}(s) \triangleq \frac{\eta t^{4/3}}{4\sqrt{\pi}} \int_0^s u^{-3/2} e^{-\eta^2 t^{4/3}/16u} \phi_j(s-u)du.$$

Explicitly, one has

$$\begin{aligned}\phi_0(t) &= \int_{0<u_1+\cdots+u_N<t} \big(\frac{\eta t^{4/3}}{4\sqrt{\pi}}\big)^N (u_1\cdots u_N)^{-3/2} \exp\big(-\frac{1}{16}\eta^2 t^{8/3}\sum_{i=1}^N \frac{1}{u_i}\big)d\mathbf{u} \\ &= \int_{0<v_1+\cdots+v_N<t^{-5/3}} \big(\frac{\eta}{4\sqrt{\pi}}\big)^N (v_1\cdots v_N)^{-3/2} \exp\big(-\frac{1}{16}\eta^2 \sum_{i=1}^N \frac{1}{v_i}\big)d\mathbf{v}. \qquad (4.20)\end{aligned}$$

To finish the proof, the key observation is that

$$\frac{N}{1/v_1+\cdots+1/v_N} \leqslant \frac{v_1+\cdots+v_N}{N} \leqslant \frac{t^{-5/3}}{N}, \qquad (4.21)$$

which is a consequence of the harmonic-arithmetic mean inequality. By applying this elementary estimate to (4.20), it follows from (4.19) that

$$\mathbb{P}\big(M^t \cap \{|D_\eta^t| > N\}\big) \leqslant F(t;\eta,N) \times e^{-(\eta N)^2 t^{5/3}/32}, \qquad (4.22)$$

where we define

$$F(t;\eta,N) \triangleq \int_{0<v_1+\cdots+v_N<t^{-5/3}} \big(\frac{\eta}{4\sqrt{\pi}}\big)^N (v_1\cdots v_N)^{-3/2} \exp\big(-\frac{1}{32}\eta^2 \sum_{i=1}^N \frac{1}{v_i}\big)d\mathbf{v}.$$

It is apparent that $F(t;\eta,N)$ vanishes as $t\to\infty$ for each fixed η, N. The result of the lemma thus follows by combining (4.10) and (4.22). □

Remark 4.4. In the above proof, we should point out that (technically) the visit of the first η-cluster (say $\mathfrak{c}_1$) is a bit different from the rest (since one does not know if $d(o,\mathfrak{c}_1) > \eta t^{4/3}$). However, since $\xi(x) > \delta t^{2/3}$ for any $x \in \mathfrak{c}_1$, one knows from (3.2) (with $\mu = 2\mu_0$) that $d(o,\mathfrak{c}_1) > (\delta/2\mu_0)^2 t^{4/3}$. To make the argument for the first visit consistent with the rest, one could further impose that $\eta < (\delta/2\mu_0)^2$.

In exactly the same way, one can also show that

$$\mathbb{E}\big[e^{\int_0^t \xi(W_s)ds}; M^t \cap \{|\hat{D}_\lambda^t| > N\}\big]$$
$$\leqslant F(t; \frac{\lambda}{2}, 2N-1) \cdot \exp\big[-\big(\frac{((\lambda/2)(2N-1))^2}{32} - 2\mu_0\sqrt{K_0}\big)t^{5/3}\big]. \tag{4.23}$$

In fact, by the definition of the extended λ-route, if $|\hat{D}_\lambda^t| > N$ the Brownian motion needs to jump a $\lambda t^{4/3}/2$-distance for at least $2N-1$ times. One just replace η be $\lambda/2$ and N by $2N-1$ in the above proof to conclude (4.23) directly.

Let us define

$$N_\eta \triangleq \eta^{-1}\sqrt{64\mu_0\sqrt{K_0}},\ \hat{N}_\lambda \triangleq \frac{1}{2}\big(1 + (\lambda/2)^{-1}\sqrt{64\mu_0\sqrt{K_0}}\big). \tag{4.24}$$

It follows from Lemma 4.4 and (4.23) that

$$\mathbb{E}\big[e^{\int_0^t \xi(W_s)ds}; M^t \cap \big(\{|D_\eta^t| > N_\eta\} \cup \{|\hat{D}_\lambda^t| > \hat{N}_\lambda\}\big)\big] \leqslant F(t;\eta,N_\eta) + F(t;\lambda/2, 2\hat{N}_\lambda - 1).$$

This is negligible as $t \to \infty$. As a result, in order to prove Theorem 4.1 one can further localise the expectation (4.4) on the event $\{|D_\eta^t| \leqslant N_\eta,\ |\hat{D}_\lambda^t| \leqslant \hat{N}_\lambda\}$. A benefit of this is that the total number of such words is of order $e^{O(t^{4/3})}$, which is also negligible with respect to the leading $e^{L^* t^{5/3}}$-growth. As we will see, the intermediate localisation $\{|D_\eta^t| \leqslant N_\eta\}$ will also play a crucial role in an error estimate.

4.4 The main estimate on generic routes

In this subsection, we estimate the contribution from a generic discrete route. The main result is stated as follows. Let $\mathcal{A}_{\lambda,\eta,\delta}^t$ denote the set of all words $\mathbf{w}$ over $\mathcal{J}_{\lambda,\delta}^t$ such that $|D_\eta^t(\mathbf{w})| \leqslant N_\eta$ and $|\hat{D}_\lambda^t(\mathbf{w})| \leqslant \hat{N}_\lambda$.

Proposition 4.1. *Let $0 < \lambda < \eta < \eta_\delta$ be given where η_δ is defined by (4.7). Suppose further that λ, η satisfy the following constraint:*

$$\frac{1-\alpha}{16}\big(\frac{\delta}{\mu}\big)^4 > \frac{\lambda K_0}{2} \times \big[N_\eta L_\delta(6L_\delta + 1/2) + 2L_\delta\big], \tag{4.25}$$

where K_0 is the constant in Lemma 4.1, L_δ is defined by (4.6) and N_η is defined by (4.24). Then with probability one, the following estimate

$$\mathbb{E}\big[e^{\int_0^t \xi(W_s)ds}; \hat{D}_\lambda^t = \mathbf{w}\big] \leqslant G(t; m, \lambda, \eta, \delta, \alpha, \mu)e^{(\delta + L^*(\alpha,\mu))t^{5/3}} \tag{4.26}$$

holds for all large t uniformly $\mathbf{w} \in \mathcal{A}_{\lambda,\eta,\delta}^t$. Here $L^*(\alpha,\mu)$ is defined by

$$L^*(\alpha,\mu) \triangleq \max_{K>0, v\in(0,1)} \big\{\mu\sqrt{K}(1-v) - \frac{\alpha K^2}{4v}\big\}.$$

The function $G(t; m, \lambda, \eta, \delta, \alpha, \mu)$ is an explicit (deterministic) function which vanishes as $t \to \infty$ uniformly with respect to $m \leqslant \hat{N}_\lambda$ when all other parameters are fixed.

Remark 4.5. It is elementary to figure out the exact value of $L^*(\alpha,\mu)$ (which will not be done here). The point is that $L^*(\alpha,\mu)\to L^*$ (the optimal exponent defined by (1.10)) when $\alpha\uparrow 1$ and $\mu\downarrow\mu_0$, as a simple consequence of continuity.

Remark 4.6. Since the right hand side of (4.26) precisely corresponds to the scenario defined by (1.12), the estimate (4.26) thus shows that no other Brownian scenarios can produce a larger growth than the optimal scenario (1.12) does.

In what follows, we develop the main steps towards proving Proposition 4.1.

4.4.1 Step 1: Decomposition of $\int_0^t \xi(W_s)ds$ into "excursion" and "staying" parts

Let $\mathbf{w}=\mathfrak{c}_1\cdots\mathfrak{c}_m$ be given fixed where $\mathfrak{c}_i\in\mathcal{J}^t_{\lambda,\delta}$. We define $\tau_0\triangleq 0$ and for $1\leqslant i\leqslant m$,

$$\sigma_i\triangleq\inf\{s\in[\tau_{i-1},t]:W_s\in\mathfrak{c}_i\},\ \ \tau_i\triangleq\inf\{s\in[\sigma_i,t]:W_s\in\hat{\mathfrak{c}}_i^c\},$$

where we recall that $\hat{\mathfrak{c}}_i$ is the $\lambda t^{4/3}/2$-neighbourhood of $\mathfrak{c}_i$. Note that the stopping times $\{\tau_0,\sigma_1,\tau_1,\cdots,\sigma_{m-1},\tau_{m-1},\sigma_m\}$ are different from the ones in Definition 4.3; here they are defined with respect to the given fixed λ-clusters $\mathfrak{c}_1,\cdots,\mathfrak{c}_m$. However, on the event $\{\hat{D}^\lambda_t=\mathbf{w}\}$ the two definitions clearly coincide. In what follows, our analysis will be performed on this event.

In order to estimate the integral $\int_0^t\xi(W_s)ds$, we decompose time into two parts:

$$\mathcal{S}^t\triangleq\Big(\bigcup_{i=1}^{m-1}[\sigma_i,\tau_i]\Big)\bigcup[\sigma_m,t],\ \ \mathcal{E}^t\triangleq\bigcup_{i=0}^{m-1}[\tau_i,\sigma_{i+1}].$$

Correspondingly, one can write

$$\int_0^t\xi(W_s)ds=\int_{\mathcal{E}^t}\xi(W_s)ds+\int_{\mathcal{S}^t}\xi(W_s)ds.$$

During the excursion period $\mathcal{E}^t$ (which means ξ is small), one just trivially bounds ξ from above by $\delta t^{2/3}$ because the Brownian motion is outside the δ-islands by definition. We refer to $\mathcal{S}^t$ as the "staying period". We pretend that W stays inside λ-clusters during this period (although this is not true!), in the sense that over this period one bounds ξ from above by its possible maximal value over the λ-clusters $\mathfrak{c}_i^j$.

More precisely, let us set

$$K^*\triangleq t^{-4/3}\sup\{d(x,o):x\in\cup_{i=1}^m\mathfrak{c}_i\}.\tag{4.27}$$

Note that K^* also depends on t but it satisfies

$$\Big(\frac{\delta}{\mu}\Big)^2\leqslant K^*\leqslant K_0\tag{4.28}$$

for all large t (cf. (3.2)). On the event $\{\hat{D}^t_\lambda = \mathbf{w}\}$, at a generic time s there are precisely two possibilities: either the Brownian motion W_s is outside I^t_δ in which case $\xi(W_s) \leqslant \delta t^{2/3}$ or W_s belongs to one of the λ-clusters $\mathfrak{c}^j_i$ in which case $\xi(W_s) \leqslant \mu\sqrt{K^* t^{4/3}}$ (for all large t). As a result, one obtains that

$$\begin{aligned}\int_0^t \xi(W_s)ds &\leqslant \delta t^{2/3}\sum_{i=0}^{m-1}(\sigma_{i+1}-\tau_i) + \mu\sqrt{K^* t^{4/3}}\big(t - \sum_{i=0}^{m-1}(\sigma_{i+1}-\tau_i)\big) \\ &\leqslant \delta t^{5/3} + \mu\sqrt{K^*}t^{2/3}\big(t - \sum_{i=0}^{m-1}(\sigma_{i+1}-\tau_i)\big).\end{aligned}$$

It follows that

$$\mathbb{E}\big[e^{\int_0^t \xi(W_s)ds}; M^t \cap \{\hat{D}^t_\lambda = \mathbf{w}\}\big] \leqslant e^{\delta t^{5/3}} \times \mathbb{E}\big[e^{\Xi(t)}; \hat{D}^t_\lambda = \mathbf{w}\big]. \tag{4.29}$$

Here we set

$$\Xi(t) \triangleq \mu\sqrt{K^*}t^{2/3}\big(t - \sum_{i=0}^{m-1}(\sigma_{i+1}-\tau_i)\big).$$

which corresponds to the "staying" part.

4.4.2 Step 2: Estimation of the "staying" part

Now our aim is to estimate the expectation $\mathbb{E}\big[e^{\Xi(t)}; \hat{D}^t_\lambda = \mathbf{w}\big]$. The strategy is quite similar to the proof of Lemma 4.4 (i.e. applying Markov property recursively and reducing to explicit calculations to Euclidean hitting times). The main difference is that the use of the harmonic-arithmetic mean inequality (4.21) will be replaced by the elementary inequality (4.34) in combination with a simple optimisation procedure.

To start with, one first observes that

$$\{\hat{D}^t_\lambda = \mathbf{w}\} \subseteq \mathbf{1}_{\{\sigma_1<\tau_1<\cdots<\sigma_{m-1}<\tau_{m-1}<\sigma_m<t\}} (= \mathbf{1}_{\{\sigma_m<t\}}).$$

By conditioning on $\mathcal{F}_{\tau_{m-1}}$, one has

$$\begin{aligned}&\mathbb{E}\big[e^{\Xi(t)}\mathbf{1}_{\{\hat{D}^t_\lambda=\mathbf{w}\}}|\mathcal{F}_{\tau_{m-1}}\big] \\ &\leqslant \exp\big(\mu\sqrt{K^*}t^{2/3}\big(t - \sum_{i=0}^{m-2}(\sigma_{i+1}-\tau_i)\big)\big)\mathbf{1}_{\{\sigma_{m-1}<t\}} \\ &\qquad \times \mathbb{E}\big[\exp\big(-\mu\sqrt{K^*}t^{2/3}(\sigma_m - \tau_{m-1})\big)\mathbf{1}_{\{\tau_{m-1}<\sigma_m<t\}}|\mathcal{F}_{\tau_{m-1}}\big] \\ &= \exp\big(\mu\sqrt{K^*}t^{2/3}\big(t - \sum_{i=0}^{m-2}(\sigma_{i+1}-\tau_i)\big)\big)\mathbf{1}_{\{\sigma_{m-1}<t\}} \times g^t\big(W_{\tau_{m-1}}, t-\tau_{m-1}\big).\end{aligned} \tag{4.30}$$

Here $g^t : \partial\hat{\mathfrak{c}}_{m-1} \times [0,t] \to \mathbb{R}$ is the function defined by

$$g^t(x,s) \triangleq \mathbb{E}\big[e^{-\mu\sqrt{K^*}t^{2/3}\sigma^t}\mathbf{1}_{\{\sigma^t<s\}}\big],$$

where σ^t is the hitting time of $\mathfrak{c}_m$ by W^x (Brownian motion starting at $x \in \partial\hat{\mathfrak{c}}_{m-1}$). Let us set

$$D_{m-1} \triangleq \mathrm{dist}(\partial\hat{\mathfrak{c}}_{m-1}, \mathfrak{c}_m).$$

Note that $D_{m-1} \geqslant \lambda t^{4/3}/2$. In the same way as the proof of Lemma 4.4, one sees that

$$g^t(x,s) \leqslant \mathbb{E}\big[e^{-\mu\sqrt{K^*}t^{2/3}\hat{\sigma}^t}\mathbf{1}_{\{\hat{\sigma}^t<s\}}\big].$$

Here $\hat{\sigma}^t$ denotes the hitting time of the level $\hat{D}_{m-1}/\sqrt{2}$ by a one-dimensional Euclidean Brownian motion γ, where

$$\hat{D}_{m-1} \triangleq D_{m-1} - \big(\frac{1}{2} + C_Q t\big)$$

and C_Q is the constant defined by (4.14). By using the explicit density formula (4.16) for $\hat{\sigma}^t$, one finds that

$$g^t(x,s) \leqslant \int_0^s e^{-\mu\sqrt{K^*}t^{2/3}u}\frac{\hat{D}_{m-1}}{2\sqrt{\pi}u^{3/2}}e^{-\hat{D}_{m-1}^2/4u}du =: \psi_{m-1}(s).$$

By substituting this back into (4.30), it follows that

$$\begin{aligned}\mathbb{E}\big[e^{\Xi(t)};\hat{D}_\lambda^t = \mathbf{w}\big] &= \mathbb{E}\big[\mathbf{1}_{\{\sigma_{m-1}<t\}}\mathbb{E}\big[e^{\Xi(t)}\mathbf{1}_{\{\hat{D}_\lambda^t=\mathbf{w}\}}|\mathcal{F}_{\tau_{m-1}}\big]\big] \\ &\leqslant \mathbb{E}\big[\exp\big(\mu\sqrt{K^*}t^{2/3}\big(t - \sum_{i=0}^{m-2}(\sigma_{i+1}-\tau_i)\big)\big) \\ &\qquad \times \psi_{m-1}(t-\sigma_{m-1})\mathbf{1}_{\{\sigma_{m-1}<t\}}\big],\end{aligned} \tag{4.31}$$

where we also used the fact that

$$\psi_{m-1}(t-\tau_{m-1}) \leqslant \psi_{m-1}(t-\sigma_{m-1}),$$

which holds since $\sigma_{m-1} < \tau_{m-1}$ and ψ_{m-1} is increasing.

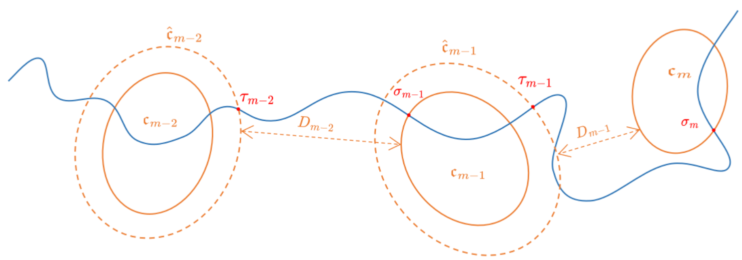
To estimate the right hand side of (4.31), we further condition on $\mathcal{F}_{\tau_{m-1}}$. By the same argument, one finds that

$$\begin{aligned}\mathbb{E}\big[e^{\Xi(t)};\hat{D}_\lambda^t = \mathbf{w}\big] &\leqslant \mathbb{E}\big[\exp\big(\mu\sqrt{K^*}t^{2/3}\big(t - \sum_{i=0}^{m-3}(\sigma_{i+1}-\tau_i)\big)\big) \\ &\qquad \times \psi_{m-2}(t-\sigma_{m-2})\mathbf{1}_{\{\sigma_{m-2}<t\}}\big].\end{aligned}$$

Here

$$\psi_{m-2}(s) \triangleq \int_0^s e^{-\mu\sqrt{K^*}t^{2/3}u}\psi_{m-1}(s-u)\frac{\hat{D}_{m-2}}{2\sqrt{\pi}u^{3/2}}e^{-\hat{D}_{m-2}^2/4u}du,$$
$$\hat{D}_{m-2} \triangleq D_{m-2} - \big(\frac{1}{2}+C_Q t\big),\ D_{m-2} \triangleq \mathrm{dist}(\partial\hat{\mathfrak{c}}_{m-2}, \mathfrak{c}_{m-1}).$$

By construction, one has $D_{m-2} \geqslant \lambda t^{4/3}/2$.



Now one can continue the above argument recursively to obtain the following estimate:

$$\mathbb{E}\big[e^{\Xi(t)}; \hat{D}_\lambda^t = \mathbf{w}\big] \leqslant \cdots \leqslant \mathbb{E}\big[e^{\mu\sqrt{K^*}t^{2/3}(t-\sigma_1)}\psi_1(t-\sigma_1)\mathbf{1}_{\{\sigma_1<t\}}\big]$$
$$\leqslant \int_0^t e^{\mu\sqrt{K^*}t^{2/3}(t-u)}\psi_1(t-s)\frac{\hat{D}_0}{2\sqrt{\pi}u^{3/2}}e^{-\hat{D}_0^2/4u}du,$$

where one defines (with $j=2,\cdots,m$)

$$\psi_{j-1}(s) \triangleq \int_0^s e^{-\mu\sqrt{K^*}t^{2/3}u}\psi_j(s-u)\frac{\hat{D}_{j-1}}{2\sqrt{\pi}u^{3/2}}e^{-\hat{D}_{j-1}^2/4u}du$$

and (with $j=0,\cdots,m-1$)

$$\hat{D}_j \triangleq D_j - \big(\frac{1}{2}+C_Q t\big),\ D_j \triangleq \mathrm{dist}(\partial\hat{\mathfrak{c}}_j, \mathfrak{c}_{j+1}) > \frac{\lambda t^{4/3}}{2}.$$

By fully expanding the function ψ_1, one arrives at the following inequality:

$$\mathbb{E}\big[e^{\Xi(t)}; \hat{D}_\lambda^t = \mathbf{w}\big] \leqslant \int_{0<u_0+\cdots+u_{m-1}<t} \prod_{i=0}^{m-1}\frac{\hat{D}_i}{2\sqrt{\pi}u_i^{3/2}}\exp\big(\mu\sqrt{K^*}t^{2/3}(t-\sum_{i=0}^{m-1}u_i)\big)$$
$$\times\exp\big(-\sum_{i=0}^{m-1}\frac{\hat{D}_i^2}{4u_i}\big)d\mathbf{u}. \tag{4.32}$$

We denote the integral on the right hand side as $I(t;\lambda,\eta,\delta)$.

4.4.3 Step 3: An elementary inequality and reduced λ-routes

Recall that $\alpha \in (0,1)$ is a given fixed parameter. In order to analyse the integral $I(t;\lambda,\eta,\delta)$, one can write

$$\exp\big(-\sum_{i=0}^{m-1}\frac{\hat{D}_i^2}{4u_i}\big)=\exp\big(-\frac{\alpha}{4}\sum_{i=0}^{m-1}\frac{\hat{D}_i^2}{u_i}\big)\times\exp\big(-\frac{(1-\alpha)}{4}\sum_{i=0}^{m-1}\frac{\hat{D}_i^2}{u_i}\big). \tag{4.33}$$

To analyse the former exponential, we first present a rather elementary inequality which plays a critical role in deducing the optimality of the scenario (1.12).

Lemma 4.5. *Let $D_1,\cdots,D_r$ and $t_1,\cdots,t_r$ be positive numbers. Then*

$$\sum_{i=1}^{r}\frac{D_i^2}{t_i}\geqslant\frac{(D_1+\cdots+D_r)^2}{t_1+\cdots+t_r}. \tag{4.34}$$

Proof. It suffices to consider the case when $r=2$, i.e.

$$\frac{D_1^2}{t_1}+\frac{D_2^2}{t_2}\geqslant\frac{(D_1+D_2)^2}{t_1+t_2}. \tag{4.35}$$

Setting $\lambda\triangleq\frac{D_1}{D_1+D_2}$ and $\rho\triangleq\frac{t_1}{t_1+t_2}$, the inequality (4.35) is equivalent to that

$$\frac{\lambda^2}{\rho}+\frac{(1-\lambda)^2}{(1-\rho)}\geqslant 1 \iff (\lambda-\rho)^2\geqslant 0.$$

The result thus follows. □

The next crucial point is to consider the reduced λ-route from $\mathbf{w}$. Recall that K^* is the maximal distance from the $\mathfrak{c}_i$'s to o (cf. (4.27)). Let m' be such that $\mathfrak{c}_{m'}$ contains the furthest location. Define $\bar{\mathbf{w}}=\mathfrak{c}_{i_1}\cdots\mathfrak{c}_{i_{\bar{m}}}$ to be the reduced word obtained from $\mathfrak{c}_1\cdots\mathfrak{c}_{m'}$ in the sense of Definition 4.4. It is not hard to see from the definition that $\mathfrak{c}_{i_l+1}=\mathfrak{c}_{i_{l+1}}$ and thus

$$\mathrm{dist}(\partial\hat{\mathfrak{c}}_{i_l},\mathfrak{c}_{i_{l+1}})=D_{i_l}$$

for all $l=0,\cdots,\bar{m}-1$ ($i_0\triangleq 0$). It follows from the above discussion and Lemma 4.5 that

$$\begin{aligned}\exp\big(-\frac{\alpha}{4}\sum_{i=0}^{m-1}\frac{\hat{D}_i^2}{u_i}\big)&\leqslant\exp\big(-\frac{\alpha}{4}\sum_{l=0}^{\bar{m}-1}\frac{\hat{D}_{i_l}^2}{u_{i_l}}\big)\\&\leqslant\exp\big(-\frac{\alpha}{4}\frac{(\hat{D}_{i_0}+\cdots+\hat{D}_{i_{\bar{m}-1}})^2}{u_{i_0}+\cdots+u_{i_{\bar{m}-1}}}\big).\end{aligned} \tag{4.36}$$

To further upper bound the right hand side, we need two important observations.

Lemma 4.6. *One has $\bar{m} \leqslant N_\eta L_\delta$.*

Proof. Note that every the λ-clusters $\mathfrak{c}_i$ is contained in some η-cluster along the η-discrete route $D^t_\eta(\mathbf{w})$. In addition, the letters $\mathfrak{c}_{i_l}$ are all distinct by the definition of reduced word. Since $D^t_\eta(\mathbf{w}) \leqslant N_\eta$ by assumption and every η-cluster can contain at most L_δ λ-clusters (cf. Lemma 4.3), one immediately knows that $\bar{m} \leqslant N_\eta L_\delta$. □

Lemma 4.7. *One has*

$$K^* t^{4/3} \leqslant \sum_{l=0}^{\bar{m}-1} D_{i_l} + \big(N_\eta L_\delta (6L_\delta + 1/2) + 2L_\delta\big)\lambda t^{4/3}.$$

Proof. Recall from Corollary 4.1 that when t is large, the diameter of any λ-cluster inside $Q_{K_0 t^{4/3}}$ is $\leqslant 2L_\delta \lambda t^{4/3}$. Let $x^* \in \mathfrak{c}_{i_{\bar{m}}}$ be such that $d(x^*, o) = K^* t^{4/3}$. Then one has

$$K^* t^{4/3} \leqslant \mathrm{dist}(o, \mathfrak{c}_{i_{\bar{m}}}) + 2L_\delta \lambda t^{4/3}. \tag{4.37}$$

In addition,

$$\begin{aligned}
\mathrm{dist}(o, \mathfrak{c}_{i_{\bar{m}}}) &\leqslant \mathrm{dist}(o, \mathfrak{c}_{i_1}) + \mathrm{dist}(\mathfrak{c}_{i_1}, \mathfrak{c}_{i_2}) + \cdots + \mathrm{dist}(\mathfrak{c}_{i_{\bar{m}-1}}, \mathfrak{c}_{i_{\bar{m}}}) + \sum_{l=1}^{\bar{m}} \mathrm{diam}(\mathfrak{c}_{i_l}) \\
&\leqslant \sum_{l=1}^{\bar{m}} D_{i_l} + \sum_{l=1}^{\bar{m}-1} \big(\frac{\lambda}{2} t^{4/3} + \mathrm{diam}(\mathfrak{c}_{i_l}) + \mathrm{diam}(\mathfrak{c}_{i_{l+1}})\big) + \sum_{l=1}^{\bar{m}} \mathrm{diam}(\mathfrak{c}_{i_l}) \\
&\leqslant \sum_{l=0}^{\bar{m}-1} D_{i_l} + \bar{m} \times \big(6L_\delta + \frac{1}{2}\big)\lambda t^{4/3} \\
&\leqslant \sum_{l=0}^{\bar{m}-1} D_{i_l} + N_\eta L_\delta \big(6L_\delta + \frac{1}{2}\big)\lambda t^{4/3},
\end{aligned} \tag{4.38}$$

where the last inequality comes from Lemma 4.6. The result follows by substituting (4.38) into (4.37). □

It follows from Lemma 4.7 that

$$\sum_{l=0}^{\bar{m}-1} \hat{D}_{i_l} \geqslant \sum_{l=0}^{\bar{m}-1} D_{i_l} - N_\eta L_\delta \big(\frac{1}{2} + C_Q t\big) \geqslant (K^* - \mathcal{R}^t(\lambda, \eta, \delta)) t^{4/3}.$$

Here the error term $\mathcal{R}^t$ is defined by

$$\mathcal{R}^t(\lambda, \eta, \delta) \triangleq \lambda\big(N_\eta L_\delta (6L_\delta + 1/2) + 2L_\delta\big) + N_\eta L_\delta \big(\frac{1}{2} + C_Q t\big) t^{-4/3}. \tag{4.39}$$

In particular, one has

$$(\hat{D}_{i_0} + \cdots + \hat{D}_{i_{\bar{m}-1}})^2 \geqslant (K^* t^{4/3})^2 - 2K^* \mathcal{R}^t(\lambda, \eta, \delta) t^{8/3}.$$

By substituting this back into (4.36), one finds that

$$\begin{aligned}\exp\big(-\frac{\alpha}{4}\sum_{l=0}^{\bar{m}-1}\frac{\hat{D}_{i_l}^2}{u_{i_l}}\big) &\leqslant \exp\big(-\frac{\alpha K^{*2}t^{8/3}}{4(u_{i_0}+\cdots+u_{i_{\bar{m}-1}})}\big)\times\exp\big(\frac{\alpha K^*\mathcal{R}^t(\lambda,\eta,\delta)t^{8/3}}{2(u_{i_0}+\cdots+u_{i_{\bar{m}-1}})}\big)\\ &\leqslant \exp\big(-\frac{\alpha K^{*2}t^{8/3}}{4(u_{i_0}+\cdots+u_{i_{\bar{m}-1}})}\big)\times\exp\big(\frac{K^*\mathcal{R}^t(\lambda,\eta,\delta)t^{8/3}}{2u_0}\big).\end{aligned}$$

In view of the decomposition (4.33), the integral $I(t;\lambda,\eta,\delta)$ can thus be estimated as

$$\begin{aligned}I(t;\lambda,\eta,\delta) \leqslant \int_{0<u_0+\cdots+u_{m-1}<t}&\prod_{i=0}^{m-1}\frac{\hat{D}_i}{2\sqrt{\pi}u_i^{3/2}}\exp\big(\mu\sqrt{K^*}t^{2/3}(t-\sum_{l=0}^{\bar{m}-1}u_{i_l})\big)\\ &\times\exp\big(-\frac{\alpha K^{*2}t^{8/3}}{4(u_{i_0}+\cdots+u_{i_{\bar{m}-1}})}\big)\times\exp\big(\frac{K^*\mathcal{R}^t(\lambda,\eta,\delta)t^{8/3}}{2u_0}\big)\\ &\times\exp\big(-\frac{(1-\alpha)}{4}\sum_{i=0}^{m-1}\frac{\hat{D}_i^2}{u_i}\big)d\mathbf{u}\\ \leqslant J(t;\lambda,\eta,\delta)&\times L(t;\alpha,\mu).\end{aligned}$$

Here the main term $L(t;\alpha,\mu)$ is defined by

$$\begin{aligned}L(t;\alpha,\mu) &\triangleq \sup_{K>0,\ \rho\in(0,t)}\exp\big(\mu\sqrt{K}t^{2/3}(t-\rho)-\frac{\alpha K^2t^{8/3}}{4\rho}\big)\\ &= \exp\big(t^{5/3}\times\sup_{K>0,v\in(0,1)}(\mu\sqrt{K}(1-v)-\frac{\alpha K^2}{4v})\big)\\ &=: \exp(L^*(\alpha,\mu)t^{5/3}).\end{aligned}$$

The error term $J(t;\lambda,\eta,\delta)$ is defined by

$$\begin{aligned}J(t;\lambda,\eta,\delta) \triangleq \int_{0<u_0+\cdots+u_{m-1}<t}&\prod_{i=0}^{m-1}\frac{\hat{D}_i}{2\sqrt{\pi}u_i^{3/2}}\exp\big(-\frac{(1-\alpha)}{4}\sum_{i=0}^{m-1}\frac{\hat{D}_i^2}{u_i}\big)\\ &\times\exp\big(\frac{K^*\mathcal{R}^t(\lambda,\eta,\delta)t^{8/3}}{2u_0}\big)d\mathbf{u}.\end{aligned}$$

The final step of the proof is to estimate this function.

4.4.4 Step 4: Estimation of the error term $J(t;\lambda,\eta,\delta)$

First of all, by applying the change of variables $u_i = v_i t^{8/3}$ one can rewrite

$$\begin{aligned} J(t;\lambda,\eta,\delta) = & \int_{0<v_0+\cdots+v_{m-1}<t^{-5/3}} \prod_{i=0}^{m-1} \frac{\hat{D}_i t^{-4/3}}{2\sqrt{\pi} v_i^{3/2}} \\ & \times \exp\Big(-\frac{1-\alpha}{4}\sum_{i=0}^{m-1}\frac{\hat{D}_i^2 t^{-8/3}}{v_i} + \frac{K^* \mathcal{R}^t(\lambda,\eta,\delta)}{2v_0}\Big) d\mathbf{v}. \end{aligned} \tag{4.40}$$

Next, one observes that

$$K^* \leqslant K_0,\ \hat{D}_i \leqslant D_i \leqslant 2K_0 t^{4/3},$$

$$\hat{D}_i \geqslant \frac{\lambda t^{4/3}}{4} \text{ (for large } t),\ \hat{D}_0 \geqslant \frac{1}{2}\Big(\frac{\delta}{\mu}\Big)^2 t^{4/3}.$$

By applying these relations to (4.40), one finds that (for large t)

$$\begin{aligned} J(t;\lambda,\eta,\delta) \leqslant & \int_{0<v_0+\cdots+v_{m-1}<t^{-5/3}} \prod_{i=0}^{m-1} \frac{2K_0}{2\sqrt{\pi} v_i^{3/2}} \times \exp\Big(-\frac{1-\alpha}{4}\sum_{i=1}^{m-1}\frac{\lambda^2}{16 v_i}\Big) \\ & \times \exp\Big(-\Big(\frac{1-\alpha}{4}\frac{(\delta/\mu)^4}{4} - \frac{K_0\mathcal{R}^t(\lambda,\eta,\delta)}{2}\Big)v_0^{-1}\Big) d\mathbf{v} \\ =: & F(t;\lambda,\eta,\delta,\alpha,\mu) \end{aligned}$$

To complete the proof, we now fix a choice of λ and η ($\lambda < \eta$) such that the constraint (4.25) holds. According to the definition (4.39) of $\mathcal{R}^t(\lambda,\eta,\delta)$, this will imply that the exponent

$$\frac{1-\alpha}{4}\frac{(\delta/\mu)^4}{4} - \frac{K_0\mathcal{R}^t(\lambda,\eta,\delta)}{2}$$

is uniformly positive for all large t. In particular, the integral $F(t;\lambda,\eta,\delta,\alpha,\mu)$ is always finite. It is easily seen from its expression that

$$\lim_{t\to\infty} F(t;\lambda,\eta,\delta,\alpha,\mu) = 0.$$

Such a convergence is clearly uniform with respect to all $m \leqslant \hat{N}_\lambda$ (with λ fixed). This completes the proof of Proposition 4.1.

4.5 Completing the proof of Theorem 4.1

Now we are in a position to complete the proof of the main upper bound.

Proof of Theorem 4.1. Given fixed (δ, α, μ), let (λ, η) be chosen (and fixed) to satisfy the constraint (4.25). Because of Lemma 4.1 and Lemma 4.4, one only needs to consider localised expectation

$$\begin{aligned} Q_t &\triangleq \mathbb{E}\big[e^{\int_0^t \xi(W_s)ds}; M^t \cap \big(\{|D_\eta^t| \leqslant N_\eta\} \cap \{|\hat{D}_\lambda^t| \leqslant \hat{N}_\lambda\}\big)\big] \\ &= \sum_{\mathbf{w}\in\mathcal{A}^t_{\lambda,\eta,\delta}} \mathbb{E}\big[e^{\int_0^t \xi(W_s)ds}; M^t \cap \{\hat{D}_\lambda^t = \mathbf{w}\}\big], \end{aligned} \tag{4.41}$$

where we recall that $\mathcal{A}^t_{\lambda,\eta,\delta}$ is the set of all extended λ-routes such that $|D_\eta^t(\mathbf{w})| \leqslant N_\eta$ and $|\hat{D}_\lambda^t(\mathbf{w})| \leqslant \hat{N}_\lambda$. For a.e. fixed realisation of ξ and any large t, one has Note that the total number P_λ of λ-clusters inside $Q_{K_0 t^{4/3}}$ is trivially estimated as

$$P_\lambda \leqslant \frac{\mathrm{Vol}(Q_{K_0 t^{4/3}})}{\mathrm{Vol}(Q_{\lambda t^{4/3}/2})} \leqslant C_d e^{(d-1)K_0 t^{4/3}}.$$

As a result,

$$\begin{aligned} \sharp\{\mathbf{w} \in \mathcal{A}^t_{\lambda,\eta,\delta}\} &\leqslant P_\lambda + P_\lambda^2 + \cdots + P_\lambda^{\hat{N}_\lambda} \\ &\leqslant \hat{N}_\lambda P_\lambda^{\hat{N}_\lambda} \leqslant \hat{N}_\lambda C_d^{\hat{N}_\lambda} e^{\hat{N}_\lambda (d-1)K_0 t^{4/3}}. \end{aligned} \tag{4.42}$$

It follows from Proposition 4.1 that

$$Q_t \leqslant \hat{N}_\lambda C_d^{\hat{N}_\lambda} e^{\hat{N}_\lambda (d-1)K_0 t^{4/3}} \cdot G(t; m, \lambda, \eta, \delta, \alpha, \mu) \cdot e^{(\delta + L^*(\alpha,\mu))t^{5/3}}.$$

As a consequence, one has

$$\overline{\lim_{t\to\infty}} \frac{1}{t^{5/3}} \log Q_t \leqslant \delta + L^*(\alpha, \mu).$$

The desired estimate (4.1) thus follows by taking $\delta \downarrow 0, \alpha \uparrow 1$ and $\mu \downarrow \mu_0$.

□

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Appendix A Uniform LDP estimate for Brownian bridge

In this appendix, we develop the proof of Lemma 3.11. Since the argument is technically quite involved, we divide it into several major steps.

Step 1: Decomposition

First of all, we need to fix an additional parameter $\zeta \ll \eta$. The purpose for this is to consider the following decomposition:

$$\begin{aligned}
&\mathbb{P}^{0,x;s,y}\big(\sup_{0\leqslant u\leqslant s} d(W_u,\gamma_u^{x,y;s})>\frac{\delta}{2}\big)\\
&\leqslant \mathbb{Q}^{s,x,y}\big(\sup_{0\leqslant v\leqslant 1-\zeta} d(\omega_v,\hat{\gamma}_v^{x,y})>\frac{\delta}{2},\ d(\omega_{1-\zeta},\hat{\gamma}_{1-\zeta}^{x,y})\leqslant\eta\big)+\\
&\qquad+\mathbb{Q}^{s,x,y}\big(\sup_{1-\zeta\leqslant v\leqslant 1} d(\omega_v,\hat{\gamma}_v^{x,y})>\eta\big)=:A_s^{x,y}+B_s^{x,y}.
\end{aligned}\tag{A.1}$$

Here $\mathbb{Q}^{s,x,y}$ denotes the law of the Brownian bridge reparametrised on $[0,1]$; more precisely, if $\{W_u^{0,x;s,y}:0\leqslant u\leqslant s\}$ is the underlying Brownian bridge, then $\mathbb{Q}^{s,x,y}$ is the law of $[0,1]\ni v\mapsto W_{sv}^{0,x;s,y}$. Note that $\mathbb{Q}^{s,x,y}$ is a probability measure on the path space

$$\Omega_{x,y}=\{\omega:[0,1]\to M\mid \omega\text{ continuous},\,\omega_0=x,\ \omega_1=y\}.$$

We always use $(\omega_v)_{0\leqslant v\leqslant 1}$ to denote a generic element in $\Omega_{x,y}$. The path $[0,1]\ni v\mapsto \hat{\gamma}_v^{x,y}$ denotes the reparametrisation of the geodesic $\gamma^{x,y;s}$ on $[0,1]$. The decomposition (A.1) holds since $\eta<\delta/2$. In general, we will use $\hat{\gamma}^{p,q}$ to denote the geodesic parametrised on $[0,1]$ joining p to q.

Step 2: Reduction to a Brownian LDP event

This and the next steps are devoted to estimating $A_s^{x,y}$ (the main estimate for $A_s^{x,y}$ is provided by (A.9) below). We begin by recalling a simple relation between Brownian bridge and Brownian motion (cf. [Hsu90, Equation (2.1)]). Let $\mathbb{Q}^{s,x}$ denote the law of $[0,1]\ni v\mapsto W_{sv}^x$ where W^x is the Brownian motion starting at x. Then

$$\frac{d\mathbb{Q}^{s,x,y}}{d\mathbb{Q}^{s,x}}\Big|_{\mathcal{F}_v}=\frac{p(s(1-v),\omega_v,y)}{p(s,x,y)}\quad(0\leqslant v<1)\tag{A.2}$$

where $\{\mathcal{F}_v\}$ is the natural filtration on path space. By using the relation (A.2) with $v=1-\zeta$, one has

$$A_s^{x,y}=\mathbb{E}^{\mathbb{Q}^{s,x}}\big[\frac{p(s\zeta,\omega_{1-\zeta},y)}{p(s,x,y)};\sup_{0\leqslant v\leqslant 1-\zeta} d(\omega_v,\hat{\gamma}_v^{x,y})>\frac{\delta}{2},\ d(\omega_{1-\zeta},\hat{\gamma}_{1-\zeta}^{x,y})\leqslant\eta\big].$$

It follows from the two-sided heat kernel estimate (2.1) and a further reparametrisation that

$$A_s^{x,y} \leqslant C^{(1)}_{d,K^*,\zeta} e^{\frac{d(x,y)^2}{4s}} \mathbb{Q}^{s(1-\zeta),x}\big(\sup_{0\leqslant v\leqslant 1} d(\omega_v, \hat{\gamma}_v^{x,y_\zeta}) > \frac{\delta}{2},\ d(\omega_1, \hat{\gamma}_1^{x,y_\zeta}) \leqslant \eta\big), \tag{A.3}$$

where $y_\zeta \triangleq \hat{\gamma}^{x,y}_{1-\zeta}$.

Our task is thus reduced to estimating

$$Q^{s,x,y}_{\delta,\eta,\zeta} \triangleq \mathbb{Q}^{s(1-\zeta),x}\big(\sup_{0\leqslant v\leqslant 1} d(\omega_v, \hat{\gamma}_v^{x,y_\zeta}) > \frac{\delta}{2},\ d(\omega_1, \hat{\gamma}_1^{x,y_\zeta}) \leqslant \eta\big). \tag{A.4}$$

To this end, let us put $Q^{s,x,y}_{\delta,\eta,\zeta}$ into a more standard form of an LDP event for the Brownian motion. Recall from the assumption of the lemma that $x \in B(\gamma^t_{t_{i-1}}, \eta)$, $y \in B(\gamma^t_{t_i}, \eta)$. Therefore,

$$d(\gamma^t_{t_{i-1}}, \gamma^t_{t_i}) = K^* \implies d(x,y) \in [K^* - 2\eta, K^* + 2\eta] \text{ (and } d(x,y) \leqslant 2K^*).$$

Let y_{K^*} denote the point on the geodesic $\hat{\gamma}^{x,y}$ that is of distance K^* to x. Then for any ω in the event of (A.4), one has

$$d(\omega_1, \hat{\gamma}_1^{x,y_{K^*}}) \leqslant d(\omega_1, \hat{\gamma}_1^{x,y_\zeta}) + d(\hat{\gamma}_1^{x,y_\zeta}, y) + d(y, \hat{\gamma}_1^{x,y_{K^*}}) \leqslant 3\eta + 2K^*\zeta.$$

In addition, suppose that $d(\omega_v, \hat{\gamma}_v^{x,y_\zeta}) > \delta/2$ at some v. Then one also has

$$\begin{aligned}\sup_{0\leqslant v\leqslant 1} d(\omega_v, \hat{\gamma}_v^{x,y_{K^*}}) &\geqslant \sup_{0\leqslant v\leqslant 1} d(\omega_v, \hat{\gamma}_v^{x,y_\zeta}) - d(\hat{\gamma}_1^{x,y_{K^*}}, \hat{\gamma}_1^{x,y_\zeta}) \\ &\geqslant \frac{\delta}{2} - (3\eta + 2K^*\zeta) > \frac{\delta}{4},\end{aligned}$$

where the last inequality follows since $\eta < \delta/12$ by assumption and we further assume that ζ is small enough to make the inequality hold.

As a consequence, one obtains the following relation:

$$Q^{s,x,y}_{\delta,\eta,\zeta} \leqslant \mathbb{Q}^{s(1-\zeta),x}\big(\sup_{0\leqslant v\leqslant 1} d(\omega_v, \hat{\gamma}_v^{x,y_{K^*}}) \geqslant \frac{\delta}{4},\ d(\omega_1, \hat{\gamma}_1^{x,y_{K^*}}) \leqslant 3\eta + 2K^*\zeta\big).$$

Due to the rotational symmetry of Brownian motion, the probability on the right hand side depends only on the distance $d(x, y_{K^*}) = K^*$ and the parameters s, δ, η, ζ. In particular, it is independent of x, y. We denote this probability as $\phi(s; K^*, \delta, \eta, \zeta)$.

Step 3: Estimation of $\phi(s; K^*, \delta, \eta, \zeta)$

The estimation of $\phi(s; K^*, \delta, \eta, \zeta)$ relies on the classical LDP for Brownian motion which we now recall (cf. [FW98]).

Lemma A.1. *Define the energy functional*

$$E(\omega) \triangleq \begin{cases} \int_0^1 |\dot{\omega}_v|^2 dv, & \text{if } |\dot{\omega}| \in L^2([0,1]); \\ +\infty, & \text{otherwise.} \end{cases}$$

Let F be a closed subset of the path space

$$\Omega_x \triangleq \{\omega : [0,1] \to M \mid \omega \text{ continuous}, \omega_0 = x\}$$

with respect to the uniform topology. Then one has

$$\overline{\lim_{s \to 0^+}} s \log \mathbb{Q}^{s,x}(F) \leqslant -\frac{1}{4} \inf_{\omega \in F} E(\omega).$$

In our case, we choose F to be the closed subset defined by

$$F \triangleq \big\{\omega \in \Omega_x : \sup_{0 \leqslant v \leqslant 1} d(\omega_v, \hat{\gamma}_v^{x,y_{K^*}}) \geqslant \frac{\delta}{4},\ d(\omega_1, \hat{\gamma}_1^{x,y_{K^*}}) \leqslant 3\eta + 2K^*\zeta\big\}.$$

One needs to demonstrate an effective lower bound for the energy functional $E(\omega)$ on F. This is given by the lemma below whose proof relies on the assumption of nonpositive curvature.

Lemma A.2. *One has*

$$E(\omega) \geqslant d(x,y)^2 + \frac{\delta^2}{128} - 4K^*(5\eta + 2K^*\zeta) \tag{A.5}$$

for all $\omega \in F$.

Proof. We first recall the following convexity property of geodesics in the hyperbolic space (cf. [Jos05, Theorem 4.8.2]). Let $\alpha, \beta : [u,v] \to \mathbb{H}^d$ be geodesics. Then one has

$$d(\alpha_s, \beta_s) \leqslant \max\big\{d(\alpha_u, \beta_u), d(\alpha_v, \beta_v)\big\} \tag{A.6}$$

for all $s \in [u,v]$. Now fix $\omega \in F$. Let $\alpha : [0,1] \to \mathbb{H}^d$ be the geodesic from x to ω_1. By applying (A.6) to the geodesics $\alpha, \hat{\gamma}^{x,y_{K^*}}$, one easily finds that

$$\sup_{0 \leqslant v \leqslant 1} d(\omega_v, \alpha_v) \geqslant \frac{\delta}{4} - (3\eta + 2K^*\zeta) \geqslant \frac{\delta}{8}$$

where the second inequality follows again from the assumption (3.27) on η and that ζ is small enough to make the inequality hold.

Let $v \in (0,1)$ be such that $d(\omega_v, \alpha_v) \geqslant \delta/8$. Let $\bar{\omega}$ be the path defined by

$$\bar{\omega}_r \triangleq \begin{cases} \text{geodesic from } o \text{ to } \omega_v, & 0 \leqslant r \leqslant v; \\ \text{geodesic from } \omega_v \text{ to } \omega_1, & v \leqslant r \leqslant 1. \end{cases}$$

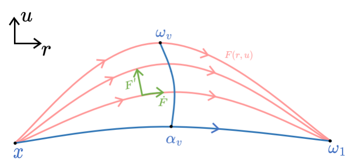
Since (minimising) geodesics minimise energy, one has

$$E(\omega) = \int_0^v |\dot{\omega}_r|^2 dr + \int_v^1 |\dot{\omega}_r|^2 dr \geqslant \int_0^v |\dot{\bar{\omega}}_r|^2 dr + \int_v^1 |\dot{\bar{\omega}}_r|^2 dr = E(\bar{\omega}).$$

It remains to lower bound $E(\bar{\omega})$. We will use the second variation of energy formula to achieve this.

Let $F(r,u) : [0,1] \times [0,1] \to \mathbb{H}^d$ be the variation of $\bar{\omega}$ defined in the following way. We first denote $[0,1] \ni u \mapsto \varphi_v(u)$ as the geodesic joining α_v to $\omega_v = \bar{\omega}_v$. For each fixed $u \in [0,1]$, we then define

$$F(r,u) \triangleq \begin{cases} \text{geodesic from } x \text{ to } \varphi_v(u), & 0 \leqslant r \leqslant v; \\ \text{geodesic from } \varphi_v(u) \text{ to } \omega_1 = \bar{\omega}_1, & v \leqslant r \leqslant 1. \end{cases}$$



We set $\dot{F} \triangleq \partial_r F$ and $F' \triangleq \partial_u F$ to be the tangential and variational fields respectively. Let $E(u) \triangleq \int_0^1 |\dot{F}(r,u)|^2 dr$ denote the energy of $F(\cdot, u)$. According to the second variation of energy formula (cf. [Jos05, Theorem 4.1.1]), one has

$$E''(u) = \int_0^1 |\nabla_{\partial_r} F'|^2 dr - \int_0^1 \langle R(\dot{F}, F')F', \dot{F} \rangle dr$$

where $R(\cdot,\cdot)\cdot$ is the Riemannian curvature tensor. Since $\mathbb{H}^d$ has negative curvature, one obtains that

$$E''(u) \geqslant \int_0^1 |\nabla_{\partial_r} F'|^2 dr. \tag{A.7}$$

To lower bound the integral on the right hand side, let $\{e_1, \cdots, e_d\}$ be an orthonormal basis of $T_x\mathbb{H}^d$ and parallel translate it along the geodesics $[0,v] \ni r \mapsto F(r,u)$ to obtain an orthonormal frame field $\{E_1, \cdots, E_d\}$. We write

$$F'(r,u) = \sum_{i=1}^d W^i(r,u) E_i, \quad r \in [0,v].$$

Since $W^i(0,u) \equiv 0$, one has

$$W^i(r,u)^2 = \Big(\int_0^r \partial_\rho W^i(\rho,u) d\rho \Big)^2 \leqslant v \int_0^v (\partial_\rho W^i(\rho,u))^2 d\rho$$

and thus

$$|F'(r,u)|^2 = \sum_{i=1}^{d} W^i(r,u)^2 \leqslant v \sum_{i=1}^{d} \int_0^v (\partial_\rho W^i(\rho,u))^2 d\rho \leqslant \int_0^1 |\nabla_{\partial_r} F'|^2 dr \qquad \text{(A.8)}$$

for all $r \in [0,v]$. The same inequality also holds for $r \in [v,1]$ (by the same argument). As a result, one concludes from (A.7, A.8) that

$$E''(u) \geqslant \sup_{r,u\in[0,1]} |F'(r,u)|^2 \geqslant \frac{\delta^2}{64},$$

where the last inequality follows from the fact that $|F'(v,0)| = d(\omega_v, \alpha_v) \geqslant \delta/8$. Since $E'(0) = 0$, it follows that

$$E(\bar{\omega}) = E(1) = E(0) + \int_0^1 \int_0^s E''(u)du \geqslant d(x,\omega_1)^2 + \frac{\delta^2}{128}.$$

By a further application of the triangle inequality

$$d(x,\omega_1)^2 \geqslant (d(x,y) - d(y,\omega_1))^2 \geqslant d(x,y)^2 - 4K^*(5\eta + 2K^*\zeta),$$

one obtains the desired inequality (A.5). □

Remark A.1. Lemma A.2 may fail in positively curved manifolds. For instance, all great semicircles connecting the north and south poles on the sphere are geodesics and they have the same energy. The lemma holds on general Riemannian manifolds when the geodesic does not contain conjugate points and the deviation is small.

One can now apply Lemma A.1 and Lemma A.2 to conclude that

$$\phi(s;K^*,\delta,\eta,\zeta) \leqslant \exp\big[-\frac{1}{4s}\big(d(x,y)^2 + \frac{\delta^2}{128} - 4K^*(5\eta+2K^*\zeta)\big)\big]$$

for all $s < s_1$, where s_1 is a constant depending only on K^*, δ, η, ζ. By substituting this estimate into (A.3), one thus concludes that

$$A_s^{x,y} \leqslant C^{(1)}_{d,K^*,\zeta} \exp\big[-\frac{1}{4s}\big(\frac{\delta^2}{128} - 4K^*(5\eta+2K^*\zeta)\big)\big]. \qquad \text{(A.9)}$$

This completes the estimate of $A_s^{x,y}$.

Step 4: Estimation of $B_s^{x,y}$

Recall that

$$B_s^{x,y} = \mathbb{Q}^{s,x,y}\big(\sup_{1-\zeta\leqslant v\leqslant 1} d(\omega_v, \hat{\gamma}_v^{x,y}) > \eta\big).$$

The strategy of estimating this term is quite similar to the previous step. First of all, by the time reversal symmetry of Brownian bridge (as a consequence of the symmetry of the heat kernel) one has

$$\begin{aligned} B_s^{x,y} &= \mathbb{Q}^{s,y,x}\big(\sup_{0\leqslant v\leqslant\zeta} d(\omega_v,\hat{\gamma}_v^{y,x})>\eta\big) \leqslant \mathbb{Q}^{s,y,x}\big(\sup_{0\leqslant v\leqslant\zeta} d(\omega_v,y)\geqslant \eta-2K^*\zeta\big) \\ &= \mathbb{E}^{\mathbb{Q}^{s,y}}\big[\frac{p(s(1-\zeta),\omega_\zeta,x)}{p(s,y,x)};\sup_{0\leqslant v\leqslant\zeta} d(\omega_v,y)\geqslant\eta-2K^*\zeta\big] \\ &\leqslant C^{(2)}_{d,K^*,\zeta}e^{\frac{K^{*2}}{s}}\mathbb{Q}^{s,y}\big(\sup_{0\leqslant v\leqslant\zeta} d(\omega_v,y)\geqslant\eta-2K^*\zeta\big), \end{aligned}$$

where the last inequality follows from the heat kernel estimate (2.1). We assume that ζ is small enough so that $\eta-2K^*\zeta>0$.

Let ω be a path such that $d(\omega_v,y)\geqslant\eta-2K^*\zeta$ for some $v\in[0,\zeta]$. Let $\beta:[0,v]\to\mathbb{H}^d$ be the geodesic joining y to ω_v. Then one has

$$E(\omega)\geqslant\int_0^v|\dot{\beta}_r|^2dr=\frac{d(\omega_v,y)^2}{v}\geqslant\frac{(\eta-2K^*\zeta)^2}{\zeta}.$$

It follows again from Lemma A.1 that

$$\mathbb{Q}^{s,y}\big(\sup_{0\leqslant v\leqslant\zeta} d(\omega_v,y)\geqslant\eta-2K^*\zeta\big)\leqslant e^{-\frac{(\eta-2K^*\zeta)^2}{4\zeta s}}$$

for all $s\in(0,s_2)$ where s_2 depends on K^*,η,ζ. As a consequence,

$$B_s^{x,y}\leqslant C^{(2)}_{d,K^*,\zeta}e^{-\frac{1}{4s}\big(\frac{(\eta-2K^*\zeta)^2}{\zeta}-4K^{*2}\big)} \tag{A.10}$$

for all $s\in(0,s_2)$.

Step 5: Conclusion

Now one can further reduce ζ to make sure that

$$\frac{\delta^2}{128}-4K^*(5\eta+2K^*\zeta)>0,\ \ \frac{(\eta-2K^*\zeta)^2}{\zeta}-4K^{*2}>0$$

With such fixed ζ (depending on K^*,δ,η), one then set

$$\begin{aligned} &C\triangleq C^{(1)}_{d,K^*,\zeta}+C^{(2)}_{d,K^*,\zeta},\ \ s_*\triangleq s_1\wedge s_2 \\ &\kappa\triangleq\frac{1}{4}\min\big\{\frac{\delta^2}{128}-4K^*(5\eta+2K^*\zeta),\frac{(\eta-2K^*\zeta)^2}{\zeta}-4K^{*2}\big\}. \end{aligned}$$

The desired estimate (3.29) with the above constants thus follows by substituting (A.9, A.10) into (3.29). This completes the proof of Lemma 3.11.

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