When Is the Conway-Maxwell-Poisson Distribution Infinitely Divisible?

Xi Geng[*1] and Aihua Xia[†1]

[1]School of Mathematics and Statistics, the University of Melbourne, Parkville VIC 3010, Australia

Abstract

An essential characteristic for a distribution to play a central role in limit theory is infinite divisibility. In this note, we prove that the Conway-Maxwell-Poisson (CMP) distribution is infinitely divisible iff it is the Poisson or geometric distribution. This explains that, despite its applications in a wide range of fields, there is no theoretical foundation for the CMP distribution to be a natural candidate for the law of small numbers.

Key words and phrases: Conway-Maxwell-Poisson distribution, infinite divisibility, entire function.

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1   Introduction and the main result

A fundamental driving force for the success of modelling count data is the law of small numbers and the Poisson distribution is undoubtedly the cornerstone of the approximation theory in capturing the distribution of counts of rare events [Barbour, Holst and Janson (1992)]. The main disadvantage of the Poisson distribution

is that its mean and variance are identical, so when the count data is over-dispersed (resp. under-dispersed), i.e. its variance is bigger (resp. less) than its mean, the Poisson distribution is inadequate in fitting the data. There are many alternatives aiming to overcome the deficiency, e.g., compound Poisson (including negative binomial), translated Poisson [Barbour and Xia (1999), Röllin (2005)] and convolutions of infinitely divisible distributions [Presman (1983), Kruopis (1986)]. By introducing an extra parameter to the Poisson distribution for looking after the dispersion behaviour of the count data, the *Conway-Maxwell-Poisson distribution* $\text{CMP}(\lambda, \nu)$ [Conway and Maxwell (1962)] assumes the probability mass function

$$\mathbb{P}(X = k) = \frac{1}{Z(\lambda, \nu)} \cdot \frac{\lambda^k}{(k!)^\nu}, \ \ k = 0, 1, 2, \cdots,$$

where the parameters (λ, ν) satisfy $\lambda, \nu > 0$ or $\nu = 0, \lambda \in (0, 1)$, and $Z(\lambda, \nu)$ denotes the normalising constant. When $\nu = 1$, it reduces to the Poisson distribution with parameter λ, denoted as $\text{Pn}(\lambda)$, and when $\nu = 0$, it becomes the geometric distribution. The CMP distribution is over-dispersed if $\nu < 1$ and under-dispersed if $\nu > 1$, see [Kokonendji, Mizère and Balakrishnan]. Due to its smooth transition between over-dispersion and under-dispersion, it plays a significant role in modelling count data [Hilbe (2014)] and has an extraordinarily diverse range of applications, see [Sellers, Borle and Shmueli (2012)] for a brief survey. However, despite some initiatives [Daly and Gaunt (2016), Li, Zhang and He (2020)], there is disproportionately little advance in its analytical properties and an approximation theory based on this popular distribution. This note aims to explain the fundamental reason behind the unbalanced development.

An essential characteristic for a distribution to play a central role in the limit theory is infinite divisibility. For the CMP distribution, we have the following result.

Theorem 1. *The distribution* $\text{CMP}(\lambda, \nu)$ *is infinitely divisible if and only if* $\nu = 0$ *or* $\nu = 1$.

In other words, the CMP distribution is infinitely divisible iff it is the Poisson or geometric distribution. This explains that, despite its applications in a wide range of fields, the only member in the CMP family that can capture the distribution of the count of weakly dependent rare events is the Poisson distribution and there is no theoretical foundation for the CMP distribution to be a natural candidate for the law of small numbers. In particular, Theorem 1 implies that the CMP process possessing independent stationary increments with $\nu > 0$ in [Zhu *et al.* (2017)] does not exist except when it is a Poisson process.

2 The proof

The sufficiency part requires no comment. In addition, it is classical that a distribution on nonnegative integers with positive mass at zero is infinitely divisible if and only if it is compound Poisson [Feller (1968), p. 290]. Since compound Poisson distributions cannot be under-dispersed (see subsection 2.1), we know that $\mathrm{CMP}(\lambda, \nu)$ is not infinitely divisible when $\nu > 1$. By comparing with the probability mass function of a compound Poisson distribution, [Mao (2020)] concludes that $\mathrm{CMP}(\lambda, \nu)$ is not infinitely divisible when $\nu \in (0.33, 1)$. However, as ν edges towards 0, the elementary approach of comparing the probability mass functions offers little hope in determining infinite divisibility of the CMP distribution, and we need an entirely different approach to tackle the problem. Our strategy here is based on complex analysis.

Throughout the rest, let $\nu \in (0, 1)$ and $\lambda > 0$ be given. We assume on the contrary that $X \sim \mathrm{CMP}(\lambda, \nu)$ is infinitely divisible. The main steps towards reaching a contradiction are summarised as follows.

1. Under the assumption, X must be a compound Poisson random variable, say

$$X = \sum_{n=1}^{N} Y_n,$$

   where N is Poisson distributed and $\{Y_1, Y_2, \cdots\}$ are i.i.d. positive integer-valued random variables and independent of N.

2. The probability generating function $G(z)$ of Y_1 must be an entire function, i.e. holomorphic on the entire complex plane.

3. The exponential-type growth property of the probability generating function of X forces $G(z)$ to have no more than polynomial growth.

4. $G(z)$ must be a polynomial as a consequence of complex analysis.

5. The precise growth estimate for the probability generating function of X obtained in Step 3 further forces $G(z)$ to be a monomial, which then leads to a contradiction easily.

In the rest of this note, we develop the above steps carefully.

2.1 The compound Poisson decomposition and related probability generating functions

Since X has positive mass at the origin, under the assumption of infinite divisibility we know from [Feller (1968), p. 290] that X is compound Poisson, i.e.

$$X \stackrel{d}{=} \sum_{n=1}^{N} Y_n,$$

where $N \sim \mathrm{Pn}(\mu)$ with some $\mu > 0$, $\{Y_1, Y_2, \cdots\}$ is an i.i.d. sequence of $\mathbb{N} \triangleq \{0, 1, 2, \dots\}$-valued random variables that is independent of N. By absorbing the mass of Y_1 at the origin into the parameter of N if necessary, we may assume that Y_1 takes values in the positive integers $\mathbb{N}_+ \triangleq \{1, 2, \dots\}$. Note that in this case we have

$$\mathbb{V}[X] - \mathbb{E}[X] = \mathbb{E}[N]\big(\mathbb{E}[Y_1^2] - \mathbb{E}[Y_1]\big) \geqslant 0.$$

This already implies that ν cannot be greater than 1 if $\mathrm{CMP}(\lambda, \nu)$ were infinitely divisible.

We now write

$$q_k \triangleq \mathbb{P}(Y_1 = k), \quad k \in \mathbb{N}_+,$$

and let $G(z)$ denote the probability generating function of Y_1, i.e.

$$G(z) = \sum_{r=1}^{\infty} q_r z^r. \tag{2.1}$$

We treat z as a complex variable and note that $G(z)$ is holomorphic at least in the unit disk. Since X is a compound Poisson random variable, from elementary probability theory, $e^{\mu(G(z)-1)}$ is the probability generating function of X. By using the CMP-distribution of X, we have

$$e^{\mu G(z)} = \frac{e^{\mu}}{Z(\lambda, \nu)} \sum_{k=0}^{\infty} \frac{(\lambda z)^k}{(k!)^{\nu}} \tag{2.2}$$

for those z's within the radius of convergence of $G(z)$. By taking $z = 0$, we get $e^{\mu} = Z(\lambda, \nu)$ and thus

$$e^{\mu G(z)} = \sum_{k=0}^{\infty} \frac{(\lambda z)^k}{(k!)^{\nu}}.$$

Call this $F(z)$ and note that it is an entire function.

2.2 Holomorphicity of $G(z)$ on $\mathbb{C}$

Our next step is to show that $G(z)$ must be an entire function. Before doing so, we first recall the construction of the logarithm of a holomorphic function (cf. [Lang (1999), p. 123]). Suppose that $f(z)$ is a holomorphic function on a simply connected domain Ω and is everywhere non-zero, then there exists a holomorphic function $L(z)$ on Ω such that

$$e^{L(z)} = f(z). \tag{2.3}$$

Indeed, the equation (2.3) suggests that $L'(z) = \frac{f'(z)}{f(z)}$, which leads us to defining

$$L(z) \triangleq \int_{z_0}^{z} \frac{f'(w)}{f(w)} dw, \quad z \in \Omega,$$

where z_0 is a fixed based point in Ω and the integral is performed along an arbitrary path joining z_0 to z. The well-definedness of $L(z)$ is a simple consequence of the simply connectedness of Ω and Cauchy's theorem.

Lemma 2. *The function $G(z)$ defined by the power series (2.1) is an entire function.*

Proof. Let R be the radius of convergence of $G(z)$. Suppose on the contrary that $R < \infty$. We know that

$$e^{\mu G(z)} = F(z), \quad z \in B_R \triangleq \{z : |z| < R\}.$$

In particular,

$$e^{\mu G(\rho)} = F(\rho), \quad \rho \in (0, R).$$

Since the coefficients of $G(z)$ are non-negative, by the monotone convergence theorem we have

$$e^{\mu G(R)} = F(R) < \infty.$$

It follows that $G(z)$ is convergent on the entire boundary of B_R. The dominated convergence theorem further implies that $G(z)$ is continuous on the closed ball $\overline{B_R}$ and we thus have

$$e^{\mu G(z)} = F(z), \quad z \in \overline{B_R}. \tag{2.4}$$

Now let $U \triangleq \{z \in \mathbb{C} : F(z) \neq 0\}$. Then U is an open subset of $\mathbb{C}$. Since the exponential function is everywhere non-vanishing, it follows from (2.4) that $\overline{B_R} \subseteq U$, and hence $B_{R+\varepsilon} \subseteq U$ for some $\varepsilon > 0$. As $B_{R+\varepsilon}$ is simply connected, there

is a well-defined logarithm of $F(z)$ on $B_{R+\varepsilon}$, namely a holomorphic function $L(z)$ such that

$$e^{L(z)} = F(z), \quad z \in B_{R+\varepsilon}.$$

Combining with (2.4), we obtain

$$L(z) = \mu G(z) + 2\pi i k(z), \quad z \in B_R$$

for some function $k : B_R \to \mathbb{Z} \triangleq \{0, \pm 1, \pm 2, \dots\}$. Since both $L(z)$ and $G(z)$ are continuous, the function $k(z)$ must be constant (say $k(z) \equiv k^*$) and we arrive at

$$L(z) = 2\pi k^* i + \sum_{r=1}^{\infty} \mu q_r z^r.$$

The power series on the right hand side gives the Taylor expansion of $L(z)$. Since $L(z)$ is holomorphic on $B_{R+\varepsilon}$, its radius of convergence must be at least $R + \varepsilon$. This contradicts the assumption that R is the radius of convergence for $G(z)$. Therefore, the series (2.1) is convergent on the entire complex plane and $G(z)$ is thus an entire function. $\qquad\square$

2.3 Precise growth-type estimate of $F(z)$

In this part, we investigate the precise growth of $F(z)$. This is the core step of the argument.

We first prepare a simple analytical lemma.

Lemma 3. *There exist positive constants C_1, C_2 and K depending only on ν, such that*

$$C_1 k^{\frac{1-\nu}{2}} \nu^{\nu k} \leqslant \frac{(k\nu)!}{(k!)^\nu} \leqslant C_2 k^{\frac{1-\nu}{2}} \nu^{\nu k} \tag{2.5}$$

for all $k \geqslant K$.

Proof. We recall the following Stirling's approximation for the Gamma function for all positive real x:

$$x! \triangleq \Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \quad \text{as } x \to \infty.$$

As a result, we have

$$\frac{(k\nu)!}{(k!)^\nu} \sim \sqrt{\nu}(2\pi k)^{\frac{1-\nu}{2}} \nu^{\nu k} \quad \text{as } k \to \infty,$$

and the claim thus follows. $\qquad\square$

The main result for this part is stated below. It quantifies the precise growth rate of $F(z)$ as z approaches infinity along the positive axis.

Lemma 4. *Let $M \triangleq \lambda \nu^\nu$. There exists a constant $C > 0$ as well as two polynomials p_1, p_2 with positive coefficients such that*

$$\frac{C}{R^{1/\nu}} e^{(MR)^{1/\nu}} - p_1(R) \leqslant F(R) \leqslant p_2(R) e^{(MR)^{1/\nu}} \tag{2.6}$$

for all R with $MR > 1$.

Proof. We first establish the upper bound. Let K be as in Lemma 3. Enlarging K if necessary, we assume $K\nu \geq 2$. For $k \geq K$, the right hand side of (2.5) ensures

$$\frac{(\lambda R)^k}{(k!)^\nu} \leq C_2 \frac{(MR)^k}{(k\nu)!} k^{\frac{1-\nu}{2}} \leq C_3 \frac{(MR)^k}{(k\nu - 1)!},$$

where $C_3 \triangleq C_2/\nu$. Hence,

$$F(R) = \sum_{k=0}^{K-1} \frac{(\lambda R)^k}{(k!)^\nu} + \sum_{k=K}^{\infty} \frac{(\lambda R)^k}{(k!)^\nu} \leqslant \sum_{k=0}^{K-1} \frac{(\lambda R)^k}{(k!)^\nu} + C_3 \sum_{k=K}^{\infty} \frac{(MR)^k}{(k\nu - 1)!}. \tag{2.7}$$

We now examine the last summation in the above inequality by introducing the division $k = mp + r$ where $p \triangleq 1/\nu$, $m \in \mathbb{N}$ and $r \in [0, p)$. To be more precise, for each $m \in \mathbb{N}$ we set

$$R_m \triangleq \{r \in [0, p) : mp + r \in \mathbb{Z}\}.$$

Note that $R_m \neq \emptyset$ (since $p > 1$) and contains at most $[p] + 1$ elements, where $[p]$ denotes the integer part of p. It is clear that each $k \in \mathbb{N}$ can be written as $k = mp + r$ with some $m \in \mathbb{N}$ and $r \in R_m$. To see the uniqueness of such decomposition, suppose that $mp + r = m'p + r'$. Then $(m - m')p = r' - r \in (-p, p)$. As a result, $m = m'$ and $r = r'$. It follows that $k \leftrightarrow (m, r)$ is a one-to-one correspondence. By using this decomposition, we have

$$\sum_{k=K}^{\infty} \frac{(MR)^k}{(k\nu - 1)!} \leq \sum_{m=1}^{\infty} \sum_{r \in R_m} \frac{(MR)^{mp+r}}{(m + r\nu - 1)!}$$

$$\leqslant \sum_{m=1}^{\infty} \frac{(MR)^{mp}}{(m-1)!} \sum_{r \in R_m} (MR)^r$$

$$\leqslant ([p] + 1)(MR)^{2p} e^{(MR)^p},$$

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provided that $MR > 1$. Since the second last summation in (2.7) is polynomial, the desired upper bound follows.

The idea of establishing the lower bound is similar. By using Lemma 3, we have

$$
\begin{aligned}
F(R) &\geqslant C_1 \sum_{k=K}^{\infty} \frac{k^{\frac{1-\nu}{2}}(MR)^k}{(k\nu)!} \geqslant C_1 \sum_{k=K}^{\infty} \frac{(MR)^k}{(k\nu)!} \\
&= C_1 \left(\sum_{k=0}^{\infty} \frac{(MR)^k}{(k\nu)!} - \sum_{k=0}^{K-1} \frac{(MR)^k}{(k\nu)!} \right) \\
&= C_1 \sum_{m=0}^{\infty} \sum_{r \in R_m} \frac{(MR)^{mp+r}}{(m+r\nu)!} - p_1(R),
\end{aligned}
$$

where

$$
p_1(R) \triangleq C_1 \sum_{k=0}^{K-1} \frac{(MR)^k}{(k\nu)!}.
$$

To estimate the double summation, we observe that (assuming $MR > 1$)

$$
\sum_{r \in R_m} \frac{(MR)^{mp+r}}{(m+r\nu)!} \geqslant \frac{(MR)^{mp}}{(m+1)!} \sum_{r \in R_m} (MR)^r \geqslant \frac{(MR)^{mp}}{(m+1)!}
$$

where the last inequality follows from the fact that $R_m \neq \emptyset$. Therefore,

$$
\sum_{m=0}^{\infty} \sum_{r \in R_m} \frac{(MR)^{mp+r}}{(m+r\nu)!} \geqslant \sum_{m=0}^{\infty} \frac{(MR)^{mp}}{(m+1)!} = (MR)^{-p}\left(e^{(MR)^p} - 1\right).
$$

The desired lower bound thus follows.

$\square$

2.4 The function $G(z)$ is a polynomial

Let $d \geqslant 2$ be the unique integer such that $1/\nu \in (d-1, d]$. It follows from Lemma 4 that

$$
F(R) \leqslant e^{M'R^d} \quad \text{for } R \text{ sufficiently large}
$$

with some $M' > 0$ which is independent of R. On the other hand, since $G(z)$ has non-negative coefficients, we have

$$
e^{\mu|G(z)|} \leqslant e^{\mu G(|z|)} = F(|z|).
$$

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As a result, we have

$$|G(z)| \leqslant \frac{1}{\mu} \log F(|z|) \leqslant \frac{M'}{\mu}|z|^d \quad \text{for } z \text{ large.}$$

Since $G(z)$ is an entire function, the following complex analysis lemma implies that $G(z)$ has to be a polynomial.

Lemma 5. *Let $g(z)$ be an entire function such that $|g(z)| \leqslant C|z|^d$ for all large z. Then $g(z)$ is a polynomial of degree at most d.*

Proof. We write

$$g(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in \mathbb{C}.$$

From Cauchy's integral formula, the Taylor coefficients are given by

$$a_k = \frac{1}{2\pi i} \int_{\partial B_R} \frac{g(z)}{z^{k+1}} dz = \frac{1}{2\pi} \int_0^{2\pi} \frac{g(Re^{i\theta})}{(Re^{i\theta})^k} d\theta, \quad k = 0, 1, 2, \cdots.$$

Note that the above formula is true for all $R > 0$. By the assumption, we have

$$|a_k| \leqslant \frac{C}{2\pi} \int_0^{2\pi} \frac{R^d}{R^k} d\theta = CR^{d-k}$$

for all large R. For each $k > d$, by taking $R \to \infty$ we conclude that $a_k = 0$. $\qquad \square$

Corollary 6. *The function $G(z)$ is a polynomial of degree at most d, i.e. $G(z) = q_1 z + \cdots + q_d z^d$. In particular, Y_1 is supported on $\{1, \cdots, d\}$.*

Remark 7. Under the conclusion of Corollary 6, for instance X is Poisson if $d = 1$ and X is Hermite if $d = 2$.

2.5   Reaching the contradiction

We are now in a position to complete the proof of Theorem 1. The main point is that the growth estimate (2.6) of $F(R)$ forces the polynomial $G(z)$ to consist of the single monomial $q_d z^d$ only, which then trivially leads to a contradiction since X achieves all possible values in $\mathbb{N}$. Recall that $d \geqslant 2$ is the unique integer such that $1/\nu \in (d-1, d]$.

Proof of Theorem 1. According to Lemma 4 and Corollary 6, we have

$$\frac{C}{R^{1/\nu}} e^{(MR)^{1/\nu}} - p_1(R) \leqslant e^{\mu(q_1 R + \cdots + q_d R^d)} \leqslant p_2(R) e^{(MR)^{1/\nu}} \tag{2.8}$$

for all sufficiently large R. The lower bound forces q_d to be non-zero. In the case when $\nu \neq 1/d$, the upper bound in (2.8) cannot hold true when $R \to \infty$, giving a contradiction. Now it remains to consider the case when $\nu = 1/d$. In this case, the upper bound implies $\mu q_d \leqslant M^{1/\nu}$ while the lower bound implies that $\mu q_d \geqslant M^{1/\nu}$. Therefore, $\mu q_d = M^{1/\nu}$. If $q_1, \cdots, q_{d-1}$ were not all zero, the upper bound cannot hold true. As a result, we have $q_1 = \cdots = q_{d-1} = 0$. In particular, $q_d = 1$ and $G(z) = z^d$. Since $d \geqslant 2$, this clearly contradicts the fact that $\mathbb{P}(X = 1) > 0$.

The proof of Theorem 1 is now complete. $\qquad\square$

3   Closing Remarks

It was known by Steutel [Steutel (1972), Theorem 5.11'] that the tail of an infinitely divisible non-Gaussian random variable cannot decay faster than a certain rate. More specifically, let $H(x)$ denote the cumulative distribution function of a non-Gaussian random variable ξ. For ξ to be infinitely divisible, it is necessary that

$$-\log(1 - H(x)) \leqslant ax \log x \tag{3.1}$$

for some $a > 0$ and all large x.

We claim that the CMP distribution satisfies Steutel's condition (3.1) for all parameters $\lambda, \nu > 0$. Indeed, let

$$T(K) \triangleq \frac{1}{Z(\lambda, \nu)} \sum_{k=K}^{\infty} \frac{\lambda^k}{(k!)^\nu}, \quad K \in \mathbb{N}$$

be the tail probability function of the $\mathrm{CMP}(\lambda, \nu)$ distribution. Since multiplying the tail probability function by a constant does not affect the conclusion of satisfying the condition (3.1), we say that the tail of the distribution of a random variable η_1 is larger than that of a random variable η_2 if $\mathbb{P}(\eta_2 \geq x)/\mathbb{P}(\eta_1 \geq x)$ is bounded above by a positive constant for all large x. If $\nu < 1$, it is trivial that $T(K)$ is larger than the tail of a Poisson distribution with parameter λ. If $\nu > 1$, according to Lemma 3 we have

$$T(K) \geqslant C_1 \sum_{k=K}^{\infty} \frac{k^{\frac{1-\nu}{2}} (\lambda \nu^\nu)^k}{(k\nu)!} \geqslant C_2 \sum_{k=K}^{\infty} \frac{C_3^k}{(k\nu)!} \quad \text{for all large } K, \tag{3.2}$$

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where the C_i's are suitable constants depending only on λ, ν. Let us assume for simplicity that ν is a positive integer. In this case, we have

$$\sum_{m=K\nu}^{\infty} \frac{C_3^{m/\nu}}{m!} = \sum_{r=0}^{\nu-1} \sum_{k=K}^{\infty} \frac{1}{(k\nu+r)!} C_3^{\frac{k\nu+r}{\nu}} \leqslant C_4 \sum_{k=K}^{\infty} \frac{1}{(k\nu)!} C_3^k. \tag{3.3}$$

It follows from (3.2) and (3.3) that the tail $T(K)$ is larger than the tail of a Poisson distribution with parameter $C_3^{1/\nu}$. Since the Poisson distribution is infinitely divisible, it satisfies the condition (3.1). As a result, the CMP distribution also satisfies (3.1). Indeed, one can further show that the tail of a CMP distribution (for all $\lambda, \nu > 0$) is bounded above and below by suitable Poisson tails.

As an interesting consequence, our main theorem suggests that Steutel's condition (3.1) is necessary but not sufficient for infinite divisibility, and the CMP distribution with $\nu \notin \{0, 1\}$ provides an explicit counterexample. In other words, the decay rate of the tail probability cannot be used to detect infinite divisibility. This also indicates that Theorem 1 cannot be proved by simply invalidating the condition (3.1).

On the other hand, in [Chakraborty and Imoto (2016)], the authors introduced an extended family of CMP-type distributions whose p.m.f. is defined by

$$\mathbb{P}(X = k) = \frac{1}{Z(\alpha, \lambda, \nu, \beta)} \cdot \frac{\Gamma(\alpha+k)^\beta \lambda^k}{\Gamma(\alpha)^\beta (k!)^\nu}, \quad k \in \mathbb{N},$$

where $Z(\alpha, \lambda, \nu, \beta)$ is the normalising constant. The distribution is defined in the parameter space $\{\alpha \geq 0, \lambda > 0, \nu > \beta\} \cup \{\alpha > 0, 0 < \lambda < 1, \nu = \beta\}$ and it reduces to the CMP distribution when $\beta = 0$. It was shown in [Zhang (2015)] that this distribution is log-convex and is thus infinitely divisible provided that

$$\frac{1}{2^\nu}\Big(1 + \frac{1}{\alpha}\Big)^\beta \geqslant 1. \tag{3.4}$$

[Zhang, Tan and Li (2018)] also studied the infinite divisibility of the distribution for the subclass $\nu = \beta$. The classical CMP distribution satisfies the condition (3.4) in the case $\nu = 0$, but not for any other values of ν, including $\nu = 1$ for which the distribution is known to be infinitely divisible. It will be interesting to see if the method in the present note can be used to investigate the sharpness of the condition (3.4) for the extended CMP distribution.

Acknowledgement

References

[Barbour, Holst and Janson (1992)] A. D. Barbour, L. Holst and S. Janson, Poisson approximation, Oxford Univ. Press, 1992.

[Barbour and Xia (1999)] A. D. Barbour and A. Xia, *Poisson perturbations*, European Series in Applied and Industrial Mathematics: Probability and Statistics **3** (1999), 131–150.

[Chakraborty and Imoto (2016)] S. Chakraborty and T. Imoto, *Extended Conway-Maxwell-Poisson distribution and its properties and applications*, Journal of Statistical Distributions and Applications **3** (1) (2016), 1–19.

[Conway and Maxwell (1962)] R. W. Conway and W. L. Maxwell, *A queuing model with state dependent service rates*, Journal of Industrial Engineering **12** (1962), 132–136.

[Daly and Gaunt (2016)] F. Daly and R. E. Gaunt, *The Conway-Maxwell-Poisson distribution: distributional theory and approximation*, ALEA Latin American Journal of Probabability and Mathematical Statistics **13** (2016), 635–658.

[Feller (1968)] W. Feller, *An introduction to probability theory and its applications I*, Third Edition, John Wiley & Sons, 1968.

[Hilbe (2014)] J. M. Hilbe, *Modeling count data*, Cambridge University Press, 2014.

[Kokonendji, Mizère and Balakrishnan] C. C. Kokonendji, D. Mizère and N. Balakrishnan, *Connections of the Poisson weight function to overdispersion and underdispersion*, Journal of Statistical Planning and Inference **138** (2008), 1287–1296.

[Kruopis (1986)] J. Kruopis, *Precision of approximations of the generalized Binomial distribution by convolutions of Poisson measures*, Lithuanian Math. J. **26** (1986), 37–49.

[Lang (1999)] S. Lang, Complex analysis, Graduate Texts in Mathematics, Springer-Verlag, 1999.

[Li, Zhang and He (2020)] B. Li, H. Zhang and J. He, *Some characterizations and properties of COM-Poisson random variables*, Communications in Statistics - Theory and Methods **49** (2020), 1311–1329.

[Mao (2020)] C. Mao, *The Conway-Maxwell Poisson distribution – from infinite divisibility to its applications*, MSc Thesis, University of Melbourne, 2020.

[Presman (1983)] E. L. Presman, *Approximation of binomial distributions by infinitely divisible ones*, Theory. Probab. Appl. 28 (1983), 393–403.

[Röllin (2005)] A. Röllin, *Approximation of sums of conditionally independent variables by the translated Poisson distribution*, Bernoulli **11** (2005), 1115–1128.

[Sellers, Borle and Shmueli (2012)] K. F. Sellers, S. Borle and G. Shmueli, *The COM-Poisson model for count data: a survey of methods and applications*, Applied Stochastic Models in Business and Industry **28** (2012), 104–116.

[Steutel (1972)] F. W. Steutel, *Some recent results in infinite divisibility*, Stochastic Process. Appl. **1** (1973), 125–143.

[Zhang (2015)] H. Zhang, *Characterizations and infinite divisibility of extended COM-Poisson distribution*, International Journal of Statistical Distributions and Applications **1** (2015), 1–5.

[Zhang, Tan and Li (2018)] H. Zhang, K. Tan and B. Li, *COM-negative binomial distribution: modeling overdispersion and ultrahigh zero-inflated count data*, Frontiers of Mathematics in China **13** (2018), 967–998.

[Zhu *et al.* (2017)] L. Zhu, K. F. Sellers, D. S. Morris and G. Shmueli, *Bridging the gap: a generalized stochastic process for count data*, Amer. Statist. **71** (2017), 71–80.