

Optimal Tail Estimates for Differential Equations Driven by Fractional Brownian Motion

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Abstract

The goal of the present paper is to investigate the exact decay rate of the tail probability $\mathbb{P}(|X_1 - x_0| > R)$ for large R , where X_t is the solution to a multidimensional stochastic differential equation driven by a fractional Brownian motion with initial condition x_0 .

In the first place, under the assumption of uniform ellipticity, we establish a general $(2H + 1)$ -Weibull lower tail estimate in Young's regime of $H \in (1/2, 1)$ and a general Gaussian lower tail estimate in the rough regime of $H \in (1/4, 1/2)$. These two estimates are seen to be sharp in their respective regimes.

In the second place, we prove a striking fact by constructing explicit examples that a uniformly elliptic system with C_b^∞ -coefficients could *fail* to have Gaussian lower tail in Young's regime. As a consequence, in our modest opinion, the multidimensional Gaussian lower estimate of [4] might not hold in its stated form of generality without further assumptions on the vector fields.

In the third place, we provide a simple nondegeneracy condition on the vector fields, under which a Gaussian lower tail can be established in Young's regime. In addition, given the existence of the aforementioned counterexamples, we prove another striking fact that any 2×2 periodic, uniformly elliptic system always has Gaussian lower tail in Young's regime.

Lastly, in the rough regime we establish a general $(2H + 1)$ -Weibull lower tail for a rich class of systems that satisfy a noncommutativity condition on the vector fields. This result provides a partially affirmative answer to a conjecture raised in [7] on the genericity of the well-known Cass-Litterer-Lyons estimate for noncommutative systems.

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1 Introduction and summary of main results

We consider a multidimensional stochastic differential equation (SDE)

$$dX_t = V_0(X_t)dt + \sum_{i=1}^d V_i(X_t)dB_t^i, \quad X_0 = x_0 \in \mathbb{R}^N \quad (1.1)$$

on the unit interval $[0, 1]$. Here $B_t = (B_t^1, \dots, B_t^d)$ is a d -dimensional fractional Brownian motion (fBM) with Hurst parameter $H \in (1/4, 1)$ and the vector fields $V_0, \dots, V_d : \mathbb{R}^N \rightarrow \mathbb{R}^N$ are assumed to be of class C_b^∞ (smooth, bounded and with bounded derivatives of all orders). Under the framework of Lyons' rough path theory, the SDE (1.1) has a unique pathwise solution (see [12, 26]).

Our study in the present paper is deeply driven by the following basic question.

Question. How fast does the tail probability

$$\mathbb{P}(|X_1 - x_0| > R) \quad (1.2)$$

decay as R goes to infinity?

We are mainly interested in *lower estimates* of the tail probability (1.2) under certain nondegeneracy condition on the vector fields (for instance, uniform ellipticity). We first survey a few related existing results from the literature and then discuss the main findings of the present paper.

1.1 Existing results in the literature

In the Brownian motion case (i.e. $H = 1/2$), it is classical that

$$\mathbb{P}(|X_1 - x_0| > R) \leq C_1 e^{-C_2 R^2} \quad (1.3)$$

for all large R with some positive constants C_1, C_2 depending only d, N and the vector fields. This Gaussian upper estimate can be established easily e.g. by using a martingale approach. On the other hand, suppose that $V_1, \dots, V_d$ are *uniformly elliptic*, in the sense that there are positive constants λ, Λ , such that

$$\lambda \cdot \text{id}_{\mathbb{R}^N} \leq V(x)V(x)^T \leq \Lambda \cdot \text{id}_{\mathbb{R}^N} \quad \text{for all } x \in \mathbb{R}^N. \quad (1.4)$$

Then it can be shown that the tail probability (1.2) admits a matching Gaussian lower bound:

$$\mathbb{P}(|X_1 - x_0| > R) \geq C_3 e^{-C_4 R^2} \quad (1.5)$$

for all large R . We refer the reader to [21, 22] for an in-depth quantitative study of these and related questions.

The situation becomes quite different when $H \neq 1/2$. In this case, one can no longer apply semimartingale or Markovian techniques as in the Brownian case. We divide our discussion into two cases: *Young's regime* $H > 1/2$ and the *rough regime* $H < 1/2$.

Young's regime

Suppose that $H \in (1/2, 1)$. By using Gaussian concentration method, it was proved by Baudoin, Ouyang and Tindel [3] that the solution X_1 also has a Gaussian upper tail, namely, the upper estimate (1.3) continues to hold.

On the other hand, under the assumption of uniform ellipticity, an argument was proposed by Besalú, Kohatsu-Higa and Tindel [4] to show that the tail probability of X_1 admits a matching *Gaussian* lower bound (1.5).

Rough regime

Suppose that $H \in (1/4, 1/2)$. It was proved by Cass, Litterer and Lyons [11] that

$$\mathbb{P}(|X_1 - x_0| > R) \leq C_1 e^{-C_2 R^\gamma} \quad (1.6)$$

for all large R , where γ is any exponent that is smaller than $2H + 1$. As pointed out in [2], a more careful application of Cameron-Martin embedding allows one to take $\gamma = 2H + 1$. In particular, one has the following $(2H + 1)$ -Weibull upper estimate:

$$\mathbb{P}(|X_1 - x_0| > R) \leq C_1 e^{-C_2 R^{2H+1}} \quad (1.7)$$

for all large R .

In the more recent work of Boedihardjo and Geng [7], the authors constructed a specific SDE example whose solution admits essentially a matching lower tail estimate. In fact, the equation considered in [7] was a rather special stochastic line integral example. It was shown that

$$\mathbb{P}\left(\left|\int_0^1 \phi(B_t^1) dB_t^2\right|\right) \geq C_1 e^{-C_2 R^\gamma} \quad (1.8)$$

with γ being any exponent strictly larger than $2H + 1$. Here $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a given function whose derivative satisfies a mild nondegeneracy condition. The result of (1.8) shows that the Cass-Litterer-Lyons estimate is essentially sharp.

To our best knowledge, no lower tail estimates are available for general SDEs in the rough regime.

Remark 1.1. The results of [2, 3, 4] are obtained in the stronger form of density estimates by using the Malliavin calculus. The corresponding tail probability estimates follow easily by integrating the density estimates.

1.2 Main results

We now summarise our main results in the present paper. Some of these findings are quite striking and counterintuitive.

General lower estimates for elliptic systems

Our first main result is a general lower estimate for uniformly elliptic systems (see Theorem 5.3 and Theorem 3.15 respectively).

Theorem A. *Suppose that the vector fields are of class C_b^∞ and satisfy the uniform ellipticity condition (1.4).*

(i) *Let $H \in (1/2, 1)$. There exist positive constants C_1, C_2 such that*

$$\mathbb{P}(|X_1 - x_0| > R) \geq C_1 e^{-C_2 R^{2H+1}}$$

for all large R .

(ii) *Let $H \in (1/4, 1/2)$. There exist positive constants C_1, C_2 such that*

$$\mathbb{P}(|X_1 - x_0| > R) \geq C_1 e^{-C_2 R^2} \quad (1.9)$$

for all large R .

In particular, the estimate (1.9) generalises the main result of [4] to the rough regime of $H \in (1/4, 1/2)$.

To summarise, one now has the following two-sided estimates for uniformly elliptic systems:

$$C_1 e^{-C_2 R^{2H+1}} \leq \mathbb{P}(|X_1 - x_0| > R) \leq C_3 e^{-C_4 R^2} \quad (1.10)$$

in Young's regime and

$$C_1 e^{-C_2 R^2} \leq \mathbb{P}(|X_1 - x_0| > R) \leq C_3 e^{-C_4 R^{2H+1}} \quad (1.11)$$

in the rough regime. The estimate (1.11) is clearly optimal, which can be seen by the trivial example $X_t = B_t$ (i.e. $d = N$, $V_0 = 0$, $(V_1, \dots, V_d) = \text{id}$) and the line integral example (1.8). The upper estimate of (1.10) is also optimal by the same example $X_t = B_t$.

Optimality of the lower estimate in Young's regime

The first striking result of the present paper is that the lower estimate of (1.10) in Young's regime is also optimal.

Theorem B. *There are examples of uniformly elliptic systems X_t with C_b^∞ -coefficients driven by fBM with Hurst parameter $H \in (1/2, 1)$, such that*

$$\mathbb{P}(|X_1 - x_0| > R) \leq C_1 e^{-C_2 R^{2H+1}}$$

for all large R .

As a consequence of Theorem B, a uniformly elliptic system with C_b^∞ -coefficients could fail to have a Gaussian lower tail. In contrast, note from Theorem A (ii) that the Gaussian lower tail is always valid in the rough regime.

In the present paper, we construct two such counterexamples (see Theorem 4.1 and Theorem 4.10 respectively). In our modest opinion, these two results demonstrate that the Gaussian lower estimate of [4, Theorem 1.2] in the multidimensional setting might not hold in its stated form of generality without further

assumptions on the vector fields. It will be clear that what governs a Gaussian lower tail is a delicate dynamical mechanism in which uniform ellipticity does not play an essential role. Nonetheless, the main theorem of [4] remains valid when $d = N = 1$.

Our third main result provides a simple sufficient condition on the vector fields, under which a Gaussian lower estimate indeed holds (see Theorem 5.2).

Theorem C. *Consider the SDE (1.1) with $H \in (1/2, 1)$. Suppose that Condition (H) in Section 5 holds. Then there exist positive constants C_1, C_2 , such that*

$$\mathbb{P}(|X_1 - x_0| > R) \geq C_1 e^{-C_2 R^2}$$

for all large R .

The counterexamples we constructed in Theorem B have a radially periodic nature: the underlying vector fields are periodic along radial directions. Given the existence of such exotic examples, our second striking result is that the Gaussian lower estimate is indeed always valid for *coordinate-periodic*, 2×2 elliptic systems (see Theorem 6.1).

Theorem D. *Let $\{V_1, V_2\}$ be smooth, uniformly elliptic vector fields on the torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$. Let X_t be the solution to the SDE*

$$dX_t = \tilde{V}_1(X_t)dB_t^1 + \tilde{V}_2(X_t)dB_t^2, \quad X_0 = x_0 \in \mathbb{R}^2,$$

where (B_t^1, B_t^2) is a 2D fBM with Hurst parameter $H \in (1/2, 1)$ and $\tilde{V}_i$ is the periodic extension of V_i to $\mathbb{R}^2$ ($i = 1, 2$). Then there exist positive constants C_1, C_2 , such that

$$\mathbb{P}(|X_1 - x_0| > R) \geq C_1 e^{-C_2 R^2}$$

for all large R .

Due to the existence of the aforementioned counterexamples, there is no hope for proving a Gaussian lower estimate by using a general and robust argument that is solely based on ellipticity. In fact, the underlying mechanism for Theorem C and Theorem D has a strong flavour of dynamical systems, while ellipticity only plays a nonessential role in the argument. It is plausible that Theorem D fails in dimension three and higher.

The $(2H + 1)$ -Weibull lower tail for noncommutative systems in the rough Regime

It was conjectured in [7] that the $(2H + 1)$ -Weibull lower tail (1.8) should be a generic phenomenon for noncommutative systems (noncommutativity means nonvanishing Lie brackets in a certain sense). In our last main result, we give a partially affirmative solution to this conjecture (see Theorem 3.1).

Theorem E. *Consider the SDE (1.1) with $H \in (1/4, 1/2)$. Suppose that Condition (NC) in Section 3 holds. Then there exist positive constants C_1, C_2 , such that*

$$\mathbb{P}(|X_1 - x_0| > R) \geq C_1 e^{-C_2 R^{2H+1}}$$

for all large R .

Condition (NC) is exactly a type of noncommutativity condition on the vector fields (while ellipticity is irrelevant here). It is robust and is expected to be generic. As a result, Theorem E provides a partially affirmative answer to the main question raised in [6]. As an application of the new method developed in Section 3, we are also able to strengthen the result of [7] to an exact $(2H+1)$ -Weibull lower tail under a weaker nondegeneracy assumption (see Proposition 3.13).

1.3 The main dynamical mechanism

We briefly summarise the main dynamical mechanism that governs the underlying tail estimates. Stated in vague terms, what determines the precise tail estimate is the large-scale behaviour of the *control distance function*

$$d_H(x, y) := \inf \{ \|h\|_{\bar{\mathcal{H}}} : h \in \bar{\mathcal{H}}, x \overset{h}{\rightsquigarrow} y \},$$

where the notation $x \overset{h}{\rightsquigarrow} y$ means that the ODE controlled by h drives the starting location x to the terminal location y at time one.

Let x_0 be the initial condition of the SDE (1.1) and let $B_R(x_0) := \{x \in \mathbb{R}^N : |x - x_0| \leq R\}$. The moral is that

$$d_H(x_0, B_R(x_0)^c) \gtrsim R^\alpha \text{ “} \implies \text{” upper } (2\alpha)\text{-Weibull tail,} \quad (1.12)$$

while

$$d_H(x_0, B_R(x_0)^c) \lesssim R^\alpha \text{ “} \implies \text{” lower } (2\alpha)\text{-Weibull tail.} \quad (1.13)$$

Of course, none of the above two statements are mathematically valid nor are they proved in any precise form in the current paper. In fact, the control distance estimates are deterministic, while one must establish robust *perturbative* estimates to turn the deterministic properties into probabilistic tail estimates.

General lower estimates in both regimes

For a uniformly elliptic system, a typical control that drives the dynamics to a distance of order R should have Cameron-Martin norm $\lesssim R^{H+1/2}$ in Young’s regime and $\lesssim R$ in the rough regime. This naturally leads to a lower $(2H+1)$ -Weibull tail for the SDE solution in Young’s regime and a lower Gaussian tail in the rough regime (Theorem A). This is essentially the mechanism of (1.13).

Young’s regime: Gaussian lower tail vs $(2H+1)$ -Weibull upper tail

The mechanism behind the Gaussian lower tail (Theorem C) is simple to describe. Geometrically, Condition (H) creates a ballistic (i.e. nearly linear) tunnel around the integral curve controlled by one particular direction (say u). The dynamics is able to propagate a distance of order R under any driving path

that is close enough to the reference control $h_t^R := Rtu$ under p -variation. A crucial point here is that the reference control h^R has an exceptionally small Cameron-Martin norm ($\|h^R\|_{\mathcal{H}} = O(R) \ll R^{H+1/2}$). The existence of a ballistic tunnel enforced by a control with small Cameron-Martin norm leads to a heavy tail (Gaussian tail in this case). This is again the mechanism of (1.13). In the periodic setting of Theorem D, the identification of such a ballistic tunnel turns out to be much subtler. One has to carefully examine the deterministic dynamics of torus flows associated with the vector fields. See Section 6.1 for a more detailed discussion on the proof strategy for Theorem D.

On the other hand, in the counterexamples given by Theorem B, the above mechanism fails completely. What we will prove is that any control that drives the dynamics to a distance of order R must have Cameron-Martin norm at least of order $R^{H+1/2}$. This phenomenon is caused by the radially periodic nature of the underlying vector fields. Such a property forces the SDE solution to have an exceptionally light tail ($(2H+1)$ -Weibull in this case). This is essentially the mechanism of (1.12).

Rough regime: $(2H+1)$ -Weibull lower tail

Under the noncommutativity condition (NC), one can construct a control leading to $O(R)$ -dynamical displacement, whose Cameron-Martin norm is only of order $R^{H+1/2}$ (note that $R^{H+1/2} \ll R$ in this case). The existence of such a control with small Cameron-Martin norm results in the heavier $(2H+1)$ -Weibull tail. This is again the mechanism of (1.13).

1.4 Organisation of the paper

In Section 2, we recall the basic definition of fBM and its Cameron-Martin space.

In Section 3, we prove the optimal $(2H+1)$ -Weibull lower tail in the rough regime under a natural noncommutativity condition on the vector fields (Theorem 3.1). As a straightforward byproduct of the technique developed in that section, we also prove a general Gaussian lower tail under the assumption of uniform ellipticity (Theorem 3.15).

In Section 4, we discuss two explicit examples of uniformly elliptic equations which possess $(2H)+1$ -Weibull upper tail in Young's regime. The main interesting result is the genuine elliptic system provided by Theorem 4.10. Nonetheless, the simpler example of Theorem 4.1 is helpful for understanding the essential mechanism behind Theorem 4.10.

In Section 5, we establish the optimal Gaussian lower tail in Young's regime under a ballistic-type nondegeneracy condition (Theorem 5.2). We also prove a general $(2H)+1$ -Weibull lower tail for uniformly elliptic systems (Theorem 5.3).

In Section 6, we prove that every periodic, uniformly elliptic 2×2 system has Gaussian lower tail in Young's regime (Theorem 6.1). This is of course only striking in Young's regime, in view of the aforementioned counterexamples as well as the general Theorem 3.15 in the rough regime.

2 Fractional Brownian motion and its Cameron-Martin space

In this section, we recall the definition of a fractional Brownian motion and several basic properties of its Cameron-Martin space that are needed in our analysis. The main reference for this part is [13, 20].

Definition 2.1. A d -dimensional *fractional Brownian motion* (fBM) with Hurst parameter $H \in (0, 1)$ is an $\mathbb{R}^d$ -valued stochastic process $B_t = (B_t^1, \dots, B_t^d)$ ($t \geq 0$) which satisfies the following properties.

- (i) $B_0 = 0$ a.s.
- (ii) B_t is a centered Gaussian process whose covariance structure is given by

$$\mathbb{E}[B_s^i B_t^j] = R(s, t) \delta_{ij}, \quad s, t \in [0, 1], \quad i, j = 1, \dots, d,$$

where

$$R(s, t) = \frac{1}{2}(s^{2H} + t^{2H} - |s - t|^{2H}).$$

- (iii) B_t has continuous sample paths a.s.

Throughout the paper, the time horizon is fixed to be $[0, 1]$. Unless otherwise stated, the fBM is always realised on the canonical path space $(\mathcal{W}, \mathcal{B}(\mathcal{W}), \mu_H)$. Here $\mathcal{W} = C_0([0, 1]; \mathbb{R}^d)$ denotes the space of continuous paths on $[0, 1]$ that vanish at $t = 0$. The space $\mathcal{W}$ is equipped with the uniform topology and $\mathcal{B}(\mathcal{W})$ denotes its Borel σ -algebra. The probability measure μ_H is the law of the fBM. Under μ_H , the coordinate process

$$B_t(w) := w_t, \quad w \in \mathcal{W}$$

is an fBM. When $H = 1/2$, the process B_t is a Brownian motion and $\mu_{1/2}$ is the Wiener measure.

Next, we define the Cameron-Martin space of an fBM. Let $\mathcal{C}_1$ denote the L^2 -closure of

$$\text{Span}\{B_t^i : 0 \leq t \leq 1, \quad i = 1, \dots, d\}$$

in $L^2(\mathcal{W}, \mu)$.

Definition 2.2. Let $\bar{\mathcal{H}}$ be the space of elements $h \in \mathcal{W}$ which can be expressed as

$$h_t = \mathbb{E}[B_t Z], \quad 0 \leq t \leq 1 \tag{2.1}$$

for some $Z \in \mathcal{C}_1$. The space $\bar{\mathcal{H}}$ is equipped with an inner product structure which is defined by

$$\langle h_1, h_2 \rangle_{\bar{\mathcal{H}}} := \mathbb{E}[Z_1 Z_2], \quad h_1, h_2 \in \bar{\mathcal{H}},$$

where Z_i is the element of $\mathcal{C}_1$ representing h_i through (2.1) ($i = 1, 2$). The Hilbert space $(\bar{\mathcal{H}}, \langle \cdot, \cdot \rangle_{\bar{\mathcal{H}}})$ is called the (intrinsic) *Cameron-Martin space* of the fBM.

A useful analytic tool for the computation of Cameron-Martin norm is fractional calculus which we now recall.

Definition 2.3. Let f be a function defined on $[0, 1]$ and let $\alpha > 0$. Its *left fractional integral* of order α is defined by

$$(I_{0+}^{\alpha}f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t f(s)(t-s)^{\alpha-1} ds, \quad 0 \leq t \leq 1.$$

Its *left fractional derivative* of order α is defined by

$$(D_{0+}^{\alpha}f)(t) := \left(\frac{d}{dt}\right)^{[\alpha]+1} (I_{0+}^{1-\{\alpha\}}f)(t), \quad 0 \leq t \leq 1,$$

where $[\alpha]$ is the integer part of α and $\{\alpha\} := \alpha - [\alpha]$.

Remark 2.4. If $\alpha \in (0, 1)$ and f is absolutely continuous, the left fractional derivative can also be computed as

$$(D_{0+}^{\alpha}f)(t) := \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(0)}{t^{\alpha}} + \int_0^t (t-s)^{-\alpha} \dot{f}(s) ds \right]. \quad (2.2)$$

The following result of [13, Theorem 2.1 and Theorem 3.3] is of fundamental importance.

Theorem 2.5. Let $H \in (0, 1)$. There is a linear isomorphism

$$\mathcal{K}_H : L^2([0, 1]) \rightarrow I_{0+}^{H+1/2}(L^2([0, 1])) \quad (2.3)$$

which is defined by

$$\mathcal{K}_H \varphi := \begin{cases} c_H \cdot I_{0+}^1 (t^{H-1/2} I_{0+}^{H-1/2} (s^{1/2-H} \varphi(s))(t)), & H \in (1/2, 1); \\ c_H \cdot I_{0+}^{2H} (t^{1/2-H} I_{0+}^{1/2-H} (s^{H-1/2} \varphi(s))(t)), & H \in (0, 1/2]. \end{cases}$$

Here c_H is a universal constant depending only on H . In addition, one has $\bar{\mathcal{H}} = I_{0+}^{H+1/2}(L^2([0, 1]; \mathbb{R}^d))$ and

$$\|h\|_{\bar{\mathcal{H}}} = \|\mathcal{K}_H^{-1}h\|_{L^2([0, 1]; \mathbb{R}^d)}$$

for all $h \in \bar{\mathcal{H}}$.

By using the above explicit formula for $\mathcal{K}_H$, it is not hard to write down the kernel $K_H(t, s)$ ($s < t$) such that

$$(\mathcal{K}_H \varphi)(t) = \int_0^t K_H(t, s) \varphi(s) ds \quad (2.4)$$

for any $\varphi \in L^2([0, 1])$ (see [5, Section 1.1]). We will not give the expression of $K_H(t, s)$ since it is not important to us. A useful observation from Theorem 2.5 and (2.4) is that an fBM with Hurst parameter H can be represented as

$$B_t = \int_0^t K_H(t, s) dW_s, \quad (2.5)$$

where $(W_t)_{0 \leq t \leq 1}$ is a d -dimensional Brownian motion.

We present a lemma on the equivalence of the Cameron-Martin norm in the $H > 1/2$ case. This lemma will be very useful to us and is also of independent interest for other purposes. Its proof relies on various existing results from different references and it is helpful to include the argument here for completeness.

Lemma 2.6. *Let $H \in (1/2, 1)$. Then one has*

$$\|h\|_{\bar{\mathcal{H}}}^2 \asymp_H \|D_{0+}^{H-1/2} \dot{h}\|_{L^2([0,1])}^2 \asymp_H \|\dot{h}\|_{L^2([0,1])}^2 + \int_0^1 \int_0^1 \frac{|\dot{h}_t - \dot{h}_s|^2}{|t-s|^{2H}} ds dt \quad (2.6)$$

for all $h \in \bar{\mathcal{H}}$. Here $a \asymp_H b$ means $c_H b \leq a \leq C_H b$ for some positive constants c_H, C_H depending only on H .

Proof. (i) The equivalence

$$\|h\|_{\bar{\mathcal{H}}} \asymp_H \|D_{0+}^{H-1/2} \dot{h}\|_{L^2([0,1])}$$

is contained in [30, Theorem 19] (with obvious restriction from $\mathbb{R}_+$ to $[0, 1]$). The equivalence

$$\|D_{0+}^{H-1/2} u\|_{L^2([0,1])}^2 \asymp_H \|u\|_{L^2([0,1])}^2 + \int_0^1 \int_0^1 \frac{|u_t - u_s|^2}{|t-s|^{2H}} ds dt \quad (2.7)$$

on $I_{0+}^{H-1/2}(L^2([0, 1]))$ requires some effort which we now address. We set $\mu := H - 1/2$. It is important to note that $\mu \in (0, 1/2)$.

(ii) We claim that $I_{0+}^\mu : L^2([0, 1]) \rightarrow L^2([0, 1])$ is a bounded linear operator, i.e.

$$\|I_{0+}^\mu f\|_2 \leq C_\mu \|f\|_2. \quad (2.8)$$

Indeed, one can write $I_{0+}^\mu f = k_\mu * f$ where

$$k_\mu(t) := \frac{t^{\mu-1}}{\Gamma(\mu)} \mathbf{1}_{(0,1)}(t).$$

The estimate (2.8) follows from Young's convolution inequality with $C_\mu := \|k_\mu\|_1$. Equivalently, one has

$$\|f\|_2 \leq C_\mu \|D_{0+}^\mu f\|_2$$

for all $f \in I_{0+}^\mu(L^2([0, 1]))$.

(iii) Define the following norms (cf. [15]):

$$\|f\|_{J^\mu(0,1)} := \|f\|_{L^2([0,1])} + \|D_{0+}^\mu f\|_{L^2([0,1])}, \quad f \in C_c^\infty((0, 1));$$

$$\|g\|_{H^\mu(\mathbb{R})} := \|g\|_{L^2(\mathbb{R})} + \|[\omega \mapsto |\omega|^\mu \hat{g}(\omega)]\|_{L^2(\mathbb{R})}, \quad g \in C_c^\infty(\mathbb{R}),$$

where $\hat{g}$ denotes the Fourier transform of g . Let $J_0^\mu(0, 1)$ (respectively, $H^\mu(0, 1)$) be the closure of $C_c^\infty((0, 1))$ under $\|\cdot\|_{J^\mu(0,1)}$ (respectively, $\|\cdot\|_{H^\mu(\mathbb{R})}$); here

functions on $(0, 1)$ are extended by zero to the whole $\mathbb{R}$). According to [15, Theorem 2.13], one has $J_0^\mu(0, 1) = H^\mu(0, 1)$ and the two norms $\|\cdot\|_{J^\mu(0, 1)}$, $\|\cdot\|_{H^\mu(\mathbb{R})}$ are equivalent.

(iv) We claim that

$$J_0^\mu(0, 1) = I_{0+}^\mu(L^2([0, 1])), \quad (2.9)$$

where the latter space is equipped with the same norm. One only needs to show that $I_{0+}^\mu(L^2([0, 1])) \subseteq J_0^\mu(0, 1)$ as the other direction is trivial. To this end, let $u = I_{0+}^\mu f$ with some $f \in L^2([0, 1])$. Pick a sequence $f_n \in C_c^\infty((0, 1))$ with $f_n \rightarrow f$ in L^2 and set $v_n := I_{0+}^\mu f_n$. By (2.8), one has $v_n \rightarrow u$ under $\|\cdot\|_{J^\mu(0, 1)}$.

It remains to show that $v_n \in J_0^\mu(0, 1)$. Indeed, given small $\varepsilon > 0$, let $\chi_\varepsilon \in C^\infty([0, 1])$ be such that

$$\chi_\varepsilon(t) = 1 \quad \text{for } t \leq 1 - 2\varepsilon, \quad \chi_\varepsilon(t) = 0 \quad \text{for } t \geq 1 - \varepsilon.$$

$$0 \leq \chi_\varepsilon \leq 1, \quad |\dot{\chi}_\varepsilon| \leq C\varepsilon^{-1}.$$

Define $v_{n,\varepsilon} := \chi_\varepsilon v_n \in C_c^\infty((0, 1))$ and set

$$w_{n,\varepsilon} := v_n - v_{n,\varepsilon} = (1 - \chi_\varepsilon)v_n.$$

It is apparent that $\|w_{n,\varepsilon}\|_2 \rightarrow 0$ as $\varepsilon \rightarrow 0^+$ (n is fixed). In addition, since

$$\|\dot{w}_{n,\varepsilon}\|_\infty \leq \frac{C_{v_n}}{\varepsilon} \quad \text{and} \quad w_{n,\varepsilon}(t) = 0 \quad \text{for } t \leq 1 - 2\varepsilon,$$

explicit calculation based on the formula (2.2) shows that

$$\|D_{0+}^\mu w_{n,\varepsilon}\|_2 \leq C_{\mu, v_n} \varepsilon^{1/2-\mu}.$$

Since $\mu \in (0, 1/2)$, it follows that $\|D_{0+}^\mu w_{n,\varepsilon}\|_2 \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. Consequently, $v_{n,\varepsilon} \rightarrow v_n$ under $\|\cdot\|_{J^\mu(0, 1)}$ and thus $v_n \in J_0^\mu(0, 1)$. One therefore concludes that $J_0^\mu(0, 1) = I_{0+}^\mu(L^2([0, 1]))$.

As a consequence of Parts (ii), (iii) and the relation (2.9), one arrives at the following equivalence of norms

$$\|D_{0+}^\mu u\|_{L^2([0, 1])} \asymp_H \|\tilde{u}\|_{H^\mu(\mathbb{R})}$$

for all $u \in I_{0+}^\mu(L^2([0, 1]))$. Here $\tilde{u}$ denotes the extension of u by zero to the whole $\mathbb{R}$.

(v) Define the fractional Sobolev norm (cf. [14])

$$\|g\|_{W^{\mu, 2}(\mathbb{R})}^2 := \|g\|_{L^2(\mathbb{R})}^2 + \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|g_t - g_s|^2}{|t - s|^{1+2\mu}} ds dt, \quad g \in C_c^\infty(\mathbb{R}).$$

According to [14, Proposition 3.4],

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|g_t - g_s|^2}{|t - s|^{1+2\mu}} ds dt = c_\mu \|[\omega \mapsto |\omega|^\mu \hat{g}(\omega)]\|_{L^2(\mathbb{R})}^2$$

with some explicit constant c_μ . As a result, the H^μ - and $W^{\mu,2}$ -norms are also equivalent. To complete the proof of (2.7), it remains to show that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\tilde{u}_t - \tilde{u}_s|^2}{|t-s|^{1+2\mu}} ds dt \leq C_\mu \left[\|u\|_{L^2([0,1])}^2 + \int_0^1 \int_0^1 \frac{|u_t - u_s|^2}{|t-s|^{1+2\mu}} ds dt \right] \quad (2.10)$$

for all $u \in I_{0+}^\mu(L^2([0,1]))$.

To this end, one first writes

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\tilde{u}_t - \tilde{u}_s|^2}{|t-s|^{1+2\mu}} ds dt \\ &= \mathcal{I}(u) + 2 \int_0^1 u_t^2 \left(\int_{-\infty}^0 \frac{ds}{(t-s)^{1+2\mu}} + \int_1^\infty \frac{ds}{(s-t)^{1+2\mu}} \right) dt \\ &= \mathcal{I}(u) + \frac{1}{\mu} (A(u) + B(u)), \end{aligned} \quad (2.11)$$

where

$$\begin{aligned} \mathcal{I}(u) &:= \int_0^1 \int_0^1 \frac{|u_t - u_s|^2}{|t-s|^{1+2\mu}} ds dt, \\ A(u) &:= \int_0^1 t^{-2\mu} u_t^2 dt, \quad B(u) := \int_0^1 (1-t)^{-2\mu} u_t^2 dt. \end{aligned}$$

We only estimate $A(u)$. The integral $B(u)$ can be handled in the same way by a change of variables $s = 1 - t$.

Let $\delta \in (0, 1/3)$ and $\eta > 0$. Define

$$A_\delta := \int_\delta^1 t^{-2\mu} u_t^2 dt.$$

It is obvious that

$$\int_{1/3}^1 t^{-2\mu} u_t^2 dt \leq 3^{2\mu} \|u\|_2^2. \quad (2.12)$$

We now estimate the $\int_\delta^{1/3}$ -integral. First of all, one has the following elementary inequality

$$u_t^2 \leq (1+\eta)u_s^2 + (1+\eta^{-1})(u_t - u_s)^2 \quad (2.13)$$

for all s, t . By averaging (2.13) over $s \in (2t, 3t)$ (i.e. applying $t^{-1} \int_{2t}^{3t} ds$ to both sides) and then integrating against $t^{-2\mu} dt$ over $(\delta, 1/3)$, one finds that

$$\int_\delta^{1/3} t^{-2\mu} u_t^2 dt \leq (1+\eta)I_\delta + (1+\eta^{-1})J_\delta, \quad (2.14)$$

where

$$\begin{aligned} I_\delta &:= \int_\delta^{1/3} t^{-1-2\mu} \int_{2t}^{3t} u_s^2 ds dt, \\ J_\delta &:= \int_\delta^{1/3} t^{-1-2\mu} \int_{2t}^{3t} (u_t - u_s)^2 ds dt. \end{aligned}$$

By interchanging the iterated integrals, one has

$$I_\delta \leq \int_{2\delta}^1 u_s^2 \left(\int_{s/3}^{s/2} t^{-1-2\mu} dt \right) ds = a_\mu \int_{2\delta}^1 s^{-2\mu} u_s^2 ds \leq a_\mu A_\delta, \quad (2.15)$$

where

$$a_\mu := \int_2^3 r^{2\mu-1} dr < 1 \quad (2.16)$$

and the last inequality follows from the fact that $(2\delta, 1) \subseteq (\delta, 1)$. In addition, by using the inequality

$$t^{-1-2\mu} \leq \frac{2^{1+2\mu}}{(s-t)^{1+2\mu}} \quad \text{for all } s \in (2t, 3t),$$

one easily finds that

$$J_\delta \leq 2^{1+2\mu} \mathcal{I}(u). \quad (2.17)$$

By substituting (2.15, 2.17) into (2.14) and noting (2.12), it follows that

$$A_\delta \leq (1+\eta)a_\mu A_\delta + (1+\eta^{-1})2^{1+2\mu} \mathcal{I}(u) + 3^{2\mu} \|u\|_2^2. \quad (2.18)$$

Since $a_\mu < 1$ (see (2.16)), one can fix a small η so that $(1+\eta)a_\mu < 1$. The estimate (2.18) becomes

$$A_\delta \leq \frac{1}{1 - (1+\eta)a_\mu} ((1+\eta^{-1})2^{1+2\mu} \mathcal{I}(u) + 3^{2\mu} \|u\|_2^2).$$

Note the the right hand side does not depend on δ . By taking $\delta \downarrow 0$, one concludes that

$$A(u) \leq C_\mu (\|u\|_2^2 + \mathcal{I}(u)).$$

The same estimate holds for $B(u)$. By applying these inequalities to (2.11), one obtains the desired estimate (2.10).

Now the proof of Lemma 2.6 is complete. $\square$

We also recall a formula from [28, Equation (13)] that will be useful for a later estimate of the Cameron-Martin norm in the $H < 1/2$ case.

Lemma 2.7. *Let $H \in (0, 1/2)$. Then one has*

$$(K_H^{-1}h)(t) = \int_0^t Q(t, s) \dot{h}_s ds \quad (2.19)$$

for any absolutely continuous h , where

$$Q(t, s) := c_H t^{H-1/2} s^{1/2-H} (t-s)^{-(1/2+H)}$$

and c_H is an explicit constant depending only on H .

Finally, we discuss the SDE (1.1) driven by an fBM with Hurst parameter $H \in (1/4, 1)$. We always assume that the vector fields $V_0, V_1, \dots, V_d$ are of class C_b^∞ . There are two basic regularity regimes to be considered: *Young's regime* $H \in (1/2, 1)$ and *the rough regime* $H \in (1/4, 1/2]$. It is classical that with probability one, sample paths of fBM are α -Hölder continuous for any $\alpha < H$ and have finite p -variation for any $p > 1/H$. We use $C^{p\text{-var}}([0, 1]; \mathbb{R}^d)$ to denote the space of paths $w \in \mathcal{W}$ with finite p -variation.

Young's regime. If $H \in (1/2, 1)$, by using Young integration theory one can define pathwise integrals of the type $\int_s^t f(x_s) dy_s$ for any smooth function f and any $x, y \in C^{p\text{-var}}([0, 1]; \mathbb{R})$. The SDE (1.1) is well-posed with the stochastic integral being interpreted in the sense of Young (see [25]).

Rough regime. If $H \in (1/4, 1/2]$, one needs rough path theory to make sense of the SDE (1.1). The moral is that with probability one, B_t can be canonically regarded (more precisely, enhanced) as a geometric rough path with roughness $p \in (1/H, 4)$. Under the framework of rough path theory, the SDE (1.1) is well-posed in the space of geometric rough paths provided that the stochastic integral is interpreted as a rough path integral (see [12, 26]). If $H \leq 1/4$, one can no longer regard B_t as a geometric rough path in any canonical way. We therefore restrict our study to the setting of $H > 1/4$.

3 General lower tail estimates in the rough regime

In this section, we consider the SDE (1.1) in the rough regime of $H \in (1/4, 1/2)$. The vector fields V_i are assumed to be of class C_b^∞ . Recall from the Cass-Litterer-Lyons estimate (1.6) (see also (1.7)) that the solution to *any* system with C_b^∞ -coefficients always possesses a $(2H+1)$ -Weibull upper tail in the rough regime of $H \in (1/4, 1/2)$. In addition, the example of [7] demonstrates the sharpness of this estimate (see (1.8)). It was conjectured in the same paper that the matching lower estimate (1.8) should be a generic phenomenon for uniformly elliptic, noncommutative systems in the rough regime.

Our aim in this section is to confirm the aforementioned conjecture for a broad class of vector fields. As we will see, ellipticity is not a relevant condition while noncommutativity plays an essential role. We first introduce the following quantitative noncommutativity condition (NC). We define the vector field

$$V_I := [V_{i_1}, [V_{i_2}, \dots [V_{i_{k-1}}, V_{i_k}] \dots]]$$

for any word $I = (i_1, \dots, i_k)$. For $l \leq r$, we use $\mathcal{V}_l^r$ to denote the collection of vector fields of the form V_I , where $I = (i_1, \dots, i_k)$ with $l \leq k \leq r$ and $i_j = 1, \dots, d$.

(NC) There exist an integer $r \geq 2$, a positive constant κ and a smooth function $F : \mathbb{R}^N \rightarrow \mathbb{R}$ with uniformly bounded derivatives $DF, D^2F, \dots, D^{r+1}F$, such that

$$\inf_{x \in \mathbb{R}^N} \max_{I \in \mathcal{V}_2^r} |(V_I F)(x)| \geq \kappa. \quad (3.1)$$

The main result of this section is stated as follows.

Theorem 3.1. *Let X_t be the solution to the SDE (1.1) with $H \in (1/4, 1/2)$ and $V_i \in C_b^\infty$. Suppose that the vector fields $\{V_1, \dots, V_d\}$ satisfy Condition (NC). Then there exist positive constants C_1, C_2 depending only on H, V and dimensions, such that*

$$\mathbb{P}(|X_1 - x_0| > R) \geq C_1 e^{-C_2 R^{2H+1}}$$

for all large R .

Remark 3.2. For the above theorem to hold, it is critical that $|I| \geq 2$ in Condition (NC). The condition therefore captures the noncommutativity of the vector fields $V_1, \dots, V_d$. If one allows $|I| = 1$ (i.e., including the original vector fields V_i), one can at best obtain a Gaussian lower bound

$$\mathbb{P}(|X_1 - x_0| > R) \geq C_1 e^{-C_2 R^2},$$

which is consistent with the trivial example $(V_1, \dots, V_d) = \text{id}$ ($X_t = B_t$) (cf. Theorem 5.3).

Remark 3.3. A sufficient condition for (NC) is the following uniform nondegeneracy condition on V_I .

(NC') There exist an integer $r \geq 2$, a positive constant κ and a unit vector $\eta \in \mathbb{R}^N$, such that

$$\inf_{x \in \mathbb{R}^N} \max_{I \in \mathcal{V}_2^r} |\langle V_I(x), \eta \rangle_{\mathbb{R}^N}| \geq \kappa.$$

In fact, Condition (NC) follows by taking $F(x) = \langle \eta, x \rangle_{\mathbb{R}^N}$.

3.1 A general criterion

In this subsection, we first prove an abstract theorem under the following more general structural Condition (G) (see Theorem 3.4 below). In the next subsection, we will prove that Condition (NC) implies Condition (G).

(G) There exist a globally Lipschitz function $F : \mathbb{R}^N \rightarrow \mathbb{R}$, a measurable map $q : \mathbb{R}^N \rightarrow W^{1,\infty}([0, 1]; \mathbb{R}^d)$ (absolutely continuous paths with a.e. bounded derivatives) and positive constants K, a , such that the following properties hold true:

(G1) $q_x(0) = q_x(1)$ and $\|\dot{q}_x\|_{L^\infty} \leq K$;

(G2) $F(\Psi_{q_x}(x)) \geq F(x) + a$ for all $x \in \mathbb{R}^N$. Here $\Psi_q : \mathbb{R}^N \rightarrow \mathbb{R}^N$ denotes the time-one solution map for the skeleton ODE

$$dz_t = \sum_{i=1}^d V_i(z_t) dq_t^i, \quad z_0 = x$$

given any BV path $q : [0, 1] \rightarrow \mathbb{R}^d$, namely, $\Psi_q(x) := z_1$.

Theorem 3.4. *Let X_t be the solution to the SDE (1.1) with $H \in (1/4, 1/2)$ and $V_i \in C_b^\infty$. Suppose that Condition (G) holds. Then there exist positive constants C_1, C_2 depending only on H, V, F, K, a , such that*

$$\mathbb{P}(|X_1 - x_0| > R) \geq C_1 e^{-C_2 R^{2H+1}} \quad (3.2)$$

for all large R .

In what follows, we develop the main ingredients towards the proof of Theorem 3.4.

3.1.1 Local continuity estimate

This part is just a collection of known estimates from rough path theory. To set up the lemma properly, let $I = [s, s + \delta]$ be a given interval. Let $(\mathbf{b}_t)_{t \in I}$ be a given weakly geometric p -rough path over $\mathbb{R}^d$. Recall from [27] that the p -variation of $\mathbf{b}$ is defined by

$$\|\mathbf{b}\|_{p\text{-var}} := \left(\sum_{i=1}^{\lfloor p \rfloor} \sup_{\mathcal{P}} \sum_{t_j \in \mathcal{P}} |\mathbf{b}_{t_{j-1}, t_j}^i|^{p/i} \right)^{1/p},$$

where $\mathcal{P}$ ranges over all finite partitions of I . Let $q^I : I \rightarrow \mathbb{R}^d$ be a BV path. For each $x \in \mathbb{R}^N$, we define $(Y_t^x)_{t \in I}$ to be the solution to the RDE

$$dY_t^x = \sum_{i=1}^d V_i(Y_t^x) d(T_{q^I} \mathbf{b})^i + V_0(Y_t^x) dt, \quad Y_s^x = x. \quad (3.3)$$

Here $T_{q^I} \mathbf{b}$ denotes the translation of $\mathbf{b}$ along direction q^I as a weakly geometric p -rough path (see [11]).

Lemma 3.5. *Let $K > 0$ be a given number. There exists a positive constant $C_{p,V,K}$ depending only on p, V, K , such that the following estimate*

$$|Y_{s+\delta}^x - \Psi_{q^I}(x)| \leq C_{p,V,K} (\|\mathbf{b}\|_{p\text{-var}} + \delta)$$

holds uniformly for all $s \geq 0, \delta > 0, x \in \mathbb{R}^N$, all weakly geometric p -rough paths $\mathbf{b}$ and BV paths q^I satisfying $\|\mathbf{b}\|_{p\text{-var}} \leq 1$ and $\|q^I\|_{1\text{-var}} \leq K$.

Proof. First of all, recall from [17, Theorem 9.33] that

$$d_{p\text{-var}}(T_{q^I} \mathbf{b}, T_{q^I} \mathbf{1}) \leq C_{p,K} \|\mathbf{b}\|_{p\text{-var}} \quad (3.4)$$

uniformly for all $\mathbf{b}$ and q^I satisfying $\|\mathbf{b}\|_{p\text{-var}} \leq 1$ and $\|q^I\|_{1\text{-var}} \leq K$. Let $(Y_t^0)_{t \in I}$ be the solution to the RDE

$$d\bar{Y}_t^x = \sum_{i=1}^d V_i(\bar{Y}_t^x) d(T_{q^I} \mathbf{b})^i, \quad \bar{Y}_s^x = x. \quad (3.5)$$

Respectively, let z_t be the solution to the ODE

$$dz_t^x = \sum_{i=1}^d V_i(z_t^x) d(q_t^I)^i, \quad z_s^x = x.$$

Note that $z_{s+\delta}^x = \Psi_{q^I}(x)$. According to the standard RDE continuity estimate (see [17, Theorem 10.38]) and (3.4), one has

$$|\bar{Y}_{s+\delta}^x - \Psi_{q^I}(x)| \leq C_{p,V,K} \|\mathbf{b}\|_{p\text{-var}}. \quad (3.6)$$

On the other hand, the equation (3.3) of Y_t^x can be written as

$$dY_t^x = \sum_{i=1}^d V_i(Y_t^x) d(T_{q^I} \mathbf{b})^i + \delta V_0(Y_t^x) d\tau_t^I, \quad (3.7)$$

where $\tau_t^I \triangleq \frac{t-s}{\delta}$ satisfies $\|\tau^I\|_{1\text{-var}} = 1$. By applying the RDE continuity estimate with drift (see [17, Theorem 12.10]) to the equations (3.5, 3.7), one finds that

$$|Y_{s+\delta}^x - \bar{Y}_{s+\delta}^x| \leq C_{p,V,K} \delta. \quad (3.8)$$

The desired estimate follows by combining (3.6) and 3.8. $\square$

3.1.2 Linear growth under iterated loop controls

We use $\Phi(t; s, x, \mathbf{b})$ ($t \geq s$) to denote the time- t solution to the RDE

$$dy_t = \sum_{i=1}^d V_i(y_t) d\mathbf{b}_t^i + V_0(y_t) dt$$

with starting time s and starting location x .

Let $n \geq 1$. Under Condition (G), we construct two (random) sequences $\{X_{t_k}^{(n)}\}_{k=1}^n$ and $\{q_k^{(n)}\}_{k=1}^n$ in the following inductive way. Let $I_k := [t_{k-1}, t_k]$ where $t_k := k/n$ ($0 \leq k \leq n$). Define

$$X_{t_1}^{(n)} := \Phi(t_1; t_0, x_0, T_{q_1^{(n)}} \mathbf{B}|_{I_1}),$$

where $q_1^{(n)}$ is the loop q_{x_0} arising from Condition (G) but reparametrised on the interval I_1 , namely,

$$q_1^{(n)}(t) := q_{x_0}(nt), \quad t \in I_1.$$

Inductively, suppose that $X_{t_{k-1}}^{(n)}$ and $q_{k-1}^{(n)}$ have been constructed. We then define

$$X_{t_k}^{(n)} := \Phi(t_k; t_{k-1}, X_{t_{k-1}}^{(n)}, T_{q_k^{(n)}} \mathbf{B}|_{I_k}),$$

where $q_k^{(n)}$ is the loop $q_{X_{t_{k-1}}^{(n)}}$ from Condition (G) but reparametrised on I_k , namely,

$$q_k^{(n)}(t) := q_{X_{t_{k-1}}^{(n)}}(nt - (k-1)), \quad t \in I_k.$$

It is clear that

$$X_{t_k}^{(n)} = \Phi(t_k; 0, x_0, T_{h^{(n)}} \mathbf{B})$$

for all $k = 1, \dots, n$, where

$$h^{(n)} := q_1^{(n)} \sqcup \dots \sqcup q_n^{(n)} \tag{3.9}$$

with $\sqcup$ denoting path concatenation.

For each $\varepsilon > 0$, we also define the event

$$A_n(\varepsilon) := \left\{ \max_{1 \leq k \leq n} \|\mathbf{B}\|_{p\text{-var}; I_k} \leq \varepsilon \right\}. \tag{3.10}$$

Here $p \in (2, 4)$ is a fixed number such that $pH > 1$.

Lemma 3.6. *Under Condition (G), there exist constants $\Lambda, \varepsilon_0, n_0$ depending only on p, V, F, K, a , such that the following estimate*

$$|X_1^{(n)} - x_0| \geq \Lambda n$$

holds on the event $A_n(\varepsilon)$, uniformly for all $n > n_0$ and $\varepsilon < \varepsilon_0$.

Proof. By the previous construction and Property (G2), one has

$$F\left(\Psi_{q_k^{(n)}}(X_{t_{k-1}}^{(n)})\right) \geq F(X_{t_{k-1}}^{(n)}) + a$$

for every k . On the other hand, since F is globally Lipschitz (say, with Lipschitz constant C_F) and $\|q_k^{(n)}\|_{1\text{-var}} \leq K$ by assumption, one knows from Lemma 3.5 that

$$\left|F(X_{t_k}^{(n)}) - F\left(\Psi_{q_k^{(n)}}(X_{t_{k-1}}^{(n)})\right)\right| \leq C_F C_{p,V,K} \left(\|\mathbf{B}\|_{p\text{-var}; I_k} + \frac{1}{n}\right),$$

provided that $\|\mathbf{B}\|_{p\text{-var}; I_k} \leq 1$. It follows that

$$F(X_{t_k}^{(n)}) \geq F(X_{t_{k-1}}^{(n)}) + a - C_{p,V,K,F} \left(\|\mathbf{B}\|_{p\text{-var}; I_k} + \frac{1}{n}\right),$$

provided that $\|\mathbf{B}\|_{p\text{-var}; I_k} \leq 1$. Now one chooses

$$\varepsilon_0 \triangleq \frac{a}{4C_{p,V,K,F}} \wedge 1, \quad n_0 \triangleq \frac{4C_{p,V,K,F}}{a}.$$

Then

$$F(X_{t_k}^{(n)}) \geq F(X_{t_{k-1}}^{(n)}) + \frac{a}{2}$$

holds on the event $A_n(\varepsilon)$ uniformly for all $n > n_0$, $\varepsilon < \varepsilon_0$ and $k = 1, \dots, n$. The conclusion of the lemma thus follows by summing over k and applying the global Lipschitz property of F . $\square$

3.1.3 A Cameron-Martin norm estimate

In this part, we regard the previously constructed $h^{(n)}$ (see (3.9)) as a Cameron-Martin path in $\tilde{\mathcal{H}}$ and estimate its Cameron-Martin norm. We first state a general lemma.

Lemma 3.7. *Let $K > 0$ be a given number. There exists a positive constant $C_{H,K}$, such that the following estimate*

$$\|h^{(n)}\|_{\tilde{\mathcal{H}}} \leq C_H K n^{H+1/2} \quad (3.11)$$

holds uniformly for all $n \geq 1$ and all absolutely continuous paths $h^{(n)}$ satisfying $\|\dot{h}^{(n)}\|_{L^\infty} \leq Kn$ and $h_{t_{k-1}}^{(n)} = h_{t_k}^{(n)}$ for every $k = 1, \dots, n$.

Proof. Now let $h^{(n)}$ be a given absolutely continuous path such that $\|\dot{h}^{(n)}\|_{L^\infty} \leq Kn$ and $h_{t_{k-1}}^{(n)} = h_{t_k}^{(n)}$ for every $k = 1, \dots, n$. To prove the lemma, it suffices to show that

$$|(K_H^{-1} h^{(n)})(t)| \leq C_H K n^{H+1/2} \quad (3.12)$$

for all $t \in [0, 1]$. To this end, let $t \in I_j$ for some j . We decompose the s -integral on the RHS of (2.19) into four parts:

$$\int_{I_1} + \sum_{k=2}^{j-2} \int_{I_k} + \int_{I_{j-1}} + \int_{t_{j-1}}^t$$

We estimate each part separately.

Firstly, one has by assumption that

$$\begin{aligned} \left| \int_{t_{j-1}}^t Q(t, s) \dot{h}_s^{(n)} ds \right| &\leq C_H K n \int_{t_{j-1}}^t (t-s)^{-(1/2+H)} (s/t)^{1/2-H} ds \\ &\leq C_H K n \int_{t_{j-1}}^t (t-s)^{-(1/2+H)} ds \\ &\leq C_H K n (t-t_{j-1})^{1/2-H} \leq C_H K n^{H+1/2}. \end{aligned}$$

The integrals $\int_{I_1}$ and $\int_{I_{j-1}}$ are estimated in the same way with the same $n^{H+1/2}$ -growth.

Next, consider the integral $\int_{I_k}$ with $2 \leq k \leq j-2$. Since $h_{t_{k-1}}^{(n)} = h_{t_k}^{(n)}$, one has

$$\begin{aligned} &\left| \int_{I_k} Q(t, s) \dot{h}_s^{(n)} ds \right| \\ &= \left| \int_{I_k} (Q(t, s) - Q(t, t_k)) \dot{h}_s^{(n)} ds \right| \\ &\leq C_H K n \cdot \left(\frac{j-1}{n} \right)^{H-1/2} \cdot \left[\left(\frac{k-1}{n} \right)^{-(1/2+H)} \left(\frac{j-1-k}{n} \right)^{-(1/2+H)} \right. \\ &\quad \left. + \left(\frac{k}{n} \right)^{1/2-H} \left(\frac{j-1-k}{n} \right)^{-(3/2+H)} \right] \cdot \frac{1}{n^2}, \end{aligned}$$

where the last inequality follows by estimating $\partial_s Q(t, s)$ in the obvious way. The exponent of n on the RHS is precisely $H+1/2$. It remains to see that

$$(j-1)^{H-1/2} \sum_{k=2}^{j-2} (k-1)^{-(1/2+H)} (j-1-k)^{-(1/2+H)} \leq C_H, \quad (3.13)$$

$$(j-1)^{H-1/2} \sum_{k=2}^{j-2} k^{1/2-H} (j-1-k)^{-(3/2+H)} \leq C_H \quad (3.14)$$

uniformly for all j . The inequality (3.13) follows by observing that the LHS is a Riemann sum approximation of the Beta integral

$$\int_0^1 x^{-(1/2+H)} (1-x)^{-(1/2+H)} dx.$$

The inequality (3.13) follows by observing that the LHS is bounded above by

$$\sum_{k=2}^{j-2} (j-1-k)^{-(3/2+H)} \leq \sum_{l=1}^{\infty} l^{-(3/2+H)} =: C_H < \infty.$$

Therefore, one concludes that

$$\left| \sum_{k=2}^{j-2} \int_{I_k} Q(t, s) \dot{h}_s^{(n)} ds \right| \leq C_H K n^{H+1/2}.$$

The desired claim (3.12) is easily obtained by combining the above estimates. This completes the proof of the lemma. $\square$

Corollary 3.8. *Let $h^{(n)}$ be the path constructed by (3.9) in Section 3.1.2. Then one has*

$$\|h^{(n)}\|_{\bar{\mathcal{H}}} \leq C_H K n^{H+1/2}$$

for all n .

Proof. One simply observes that $\|\dot{h}^{(n)}\|_{L^\infty} \leq K n$ (because $\|\dot{q}_k^{(n)}\|_{L^\infty} \leq K n$ by Condition (G) and the definition of $q_k^{(n)}$) and $h_{t_{k-1}}^{(n)} = h_{t_k}^{(n)}$ for every k (because all the $q_k^{(n)}$'s are loops by assumption). $\square$

3.1.4 Completing the proof of Theorem 3.4: Girsanov's transform

Recall that $A_n(\varepsilon)$ is the event defined by (3.10).

Lemma 3.9. *There exist positive constants C_1, C_2 depending only on H , such that*

$$\mathbb{P}(A_n(\varepsilon)^c) \leq C_1 n e^{-C_2 \varepsilon^2 n^{2H}}$$

for all $n \geq 1$ and $\varepsilon \in (0, 1)$.

Proof. By definition,

$$\mathbb{P}(A_n(\varepsilon)^c) = \mathbb{P}(\|\mathbf{B}\|_{p\text{-var}; I_k} > \varepsilon \text{ for some } k) \leq n \mathbb{P}(\|\mathbf{B}\|_{p\text{-var}; [0,1]} > n^H \varepsilon),$$

where we used a simple union bound and fBM scaling to reach the last inequality. The result then follows from Fernique's estimate for Gaussian rough paths (cf. [16, Theorem 11.9]). $\square$

We are now in a position to prove Theorem 3.4. The argument is a simple application of Girsanov's theorem together with the exponential martingale inequality.

Proof of Theorem 3.4. Let $h^{(n)} \in \bar{\mathcal{H}}$ be constructed as before and set $\theta^{(n)} := K_H^{-1} \dot{h}^{(n)}$. It is clear from construction that $\theta^{(n)}$ is $\mathcal{F}^W$ -adapted, where W is the BM in the fBM representation. Consider the martingale

$$M_t^{(n)} := \int_0^t \langle \theta_s^{(n)}, dW_s \rangle_{\mathbb{R}^d}.$$

According to Corollary 3.8, one has

$$\langle M^{(n)} \rangle_1 = \|\theta^{(n)}\|_{L^2([0,1])}^2 \leq C_H^2 K^2 n^{2H+1} =: \Gamma_n$$

for all n .

Define the martingale

$$Z_t^{(n)} \triangleq \exp \left(- \int_0^t \langle \theta_s^{(n)}, dW_s \rangle_{\mathbb{R}^d} - \frac{1}{2} \int_0^t |\theta_s^{(n)}|^2 ds \right), \quad 0 \leq t \leq 1.$$

According to Girsanov's theorem, under the new probability measure $d\mathbb{Q}^{(n)} := Z_1^{(n)}d\mathbb{P}$, the process

$$\tilde{W}_t^{(n)} := W_t + \int_0^t \theta_s^{(n)} ds$$

is a BM and hence $\tilde{B}_t^{(n)} := B_t + h_t^{(n)}$ is an fBM. Let X_t be the solution to the SDE (1.1) driven by $\mathbf{B}$ under $\mathbb{P}$. Then one has

$$\begin{aligned} \mathbb{P}(|X_1 - x_0| \geq \Lambda n) &= \mathbb{Q}^{(n)}(|\Phi(1; 0, x_0, \tilde{B}^{(n)}) - x_0| \geq \Lambda n) \\ &= \mathbb{E}^\mathbb{P}[Z_1^{(n)} \mathbf{1}_{\{|\Phi(1; 0, x_0, \tilde{B}^{(n)}) - x_0| \geq \Lambda n\}}] \\ &\geq \mathbb{E}^\mathbb{P}[Z_1^{(n)} \mathbf{1}_{A_n(\varepsilon)}] \end{aligned}$$

for all $n > n_0$, $\varepsilon < \varepsilon_0$, where we used Lemma 3.6 to reach the last inequality with Λ being the constant in that Lemma.

Now let

$$G_n \triangleq \{M_1^{(n)} \leq \Gamma_n\} = \{M_1^{(n)} \leq \Gamma_n, \langle M^{(n)} \rangle_1 \leq \Gamma_n\}.$$

On the event G_n , one knows that $Z_1^{(n)} \geq e^{-3\Gamma_n/2}$. In addition, by the exponential martingale inequality, one has

$$\mathbb{P}(G_n^c) = \mathbb{P}(M_1^{(n)} > \Gamma_n, \langle M^{(n)} \rangle_1 \leq \Gamma_n) \leq 2e^{-\Gamma_n/2}.$$

It follows from Lemma 3.9 that

$$\begin{aligned} \mathbb{P}(A_n(\varepsilon) \cap G_n) &= \mathbb{P}(A_n(\varepsilon)) - \mathbb{P}(A_n(\varepsilon) \cap G_n^c) \\ &\geq 1 - C_1 n e^{-C_2 \varepsilon^2 n^{2H}} - 2e^{-\Gamma_n/2}. \end{aligned}$$

We fix $\varepsilon < \varepsilon_0$. There exists $n_1 > n_0$, such that

$$\mathbb{P}(A_n(\varepsilon) \cap G_n) \geq \frac{1}{2}$$

for all $n > n_1$. As a consequence, one concludes that

$$\mathbb{P}(|X_1 - x_0| \geq \Lambda n) \geq \mathbb{E}^\mathbb{P}[Z_1^{(n)} \mathbf{1}_{A_n(\varepsilon) \cap G_n}] \geq \frac{1}{2} e^{-3\Gamma_n/2} = \frac{1}{2} e^{-3C_H^2 K^2 n^{2H+1}/2}$$

for all $n > n_1$.

The desired estimate (3.2) follows by renaming $R := \Lambda n$. □

3.2 Condition (NC) implies Condition (G)

In this section, we prove Theorem 3.1 by showing that Condition (NC) implies Condition (G). The main result is stated as follows.

Proposition 3.10. *Let $\{V_1, \dots, V_d\}$ be a family of C_b^∞ -vector fields that satisfy Condition (NC). Then Condition (G) holds.*

Before discussing its proof, we first recall some basic notions from [17] on path signatures. The algebra of formal tensor series over $\mathbb{R}^d$ (respectively, the step- m truncated one for any given $m \geq 1$) is denoted as $T((\mathbb{R}^d))$ (respectively, $T^{(m)}(\mathbb{R}^d)$). The map $\pi^{(m)} : T((\mathbb{R}^d)) \rightarrow T^{(m)}(\mathbb{R}^d)$ denotes the canonical projection. The exponential and logarithmic transforms over $T((\mathbb{R}^d))$ are denoted as $\exp$ and $\log$ respectively. The *signature* of a BV path $\gamma : [a, b] \rightarrow \mathbb{R}^d$ is the element of $T((\mathbb{R}^d))$ defined by

$$S(\gamma) = \left(1, \gamma_b - \gamma_a, \int_{a < s < t < b} d\gamma_s \otimes d\gamma_t, \dots, \int_{a < t_1 < \dots < t_n < b} d\gamma_{t_1} \otimes \dots \otimes d\gamma_{t_n}, \dots\right).$$

The signature satisfies the so-called *Chen's identity*, i.e.

$$S(\alpha \sqcup \beta) = S(\alpha) \otimes S(\beta)$$

where $\sqcup$ means path concatenation. If γ is the line segment represented by a vector v , one has $S(\gamma) = \exp(v)$. Given a word $I = (i_1, \dots, i_k)$ with $1 \leq i_j \leq d$, we define

$$e_I := [e_{i_1}, [e_{i_2}, \dots [e_{i_{k-1}}, e_{i_k}] \dots]],$$

where $\{e_1, \dots, e_d\}$ is the canonical basis of $\mathbb{R}^d$ and $[a, b] := a \otimes b - b \otimes a$. The set of all non-empty words over $\{1, \dots, d\}$ is denoted as $\mathcal{W}$.

The proof of Proposition 3.10 relies crucially on the following construction, which is standard from rough path theory.

Lemma 3.11. *There exists a family $\{\gamma^I : I \in \mathcal{W}\}$ of piecewise linear paths such that the following properties hold true:*

- (i) $\pi^{(|I|)} S(\gamma^I) = \mathbf{1} + e_I$ for every $I \in \mathcal{W}$;
- (ii) for each $m \geq 1$, there exists a positive constant L_m depending only on m , such that

$$\|\gamma^I\|_{1-var} \leq L_m$$

for all $I \in \mathcal{W}$ with $|I| \leq m$.

Proof. We prove the lemma by induction on the length of the word. If $I = (i)$, one simply defines $\gamma_t^I = te_i$ ($0 \leq t \leq 1$). Suppose that γ^J has been constructed to satisfy the required properties for every word J with $|J| < m$. Let I be a word with length m and write $I = (i, J)$. One then defines

$$\gamma^I := \gamma^{(i)} \sqcup \gamma^J \sqcup \overleftarrow{\gamma^{(i)}} \sqcup \overleftarrow{\gamma^J},$$

where $\overleftarrow{\cdot}$ denotes the reversal of a path. By the induction hypothesis, one can write

$$\pi^{(m)} \log S(\gamma^J) = e_J + \xi$$

where ξ is some homogeneous Lie polynomial ξ of degree m . According to Chen's identity, one has

$$\begin{aligned}\pi^{(m)}[S(\gamma^I)] &= \pi^{(m)}[\exp(e_i) \otimes \exp(e_J + \xi) \otimes \exp(-e_i) \otimes \exp(-e_J - \xi)] \\ &= \pi^{(m)}\left[\left(\sum_{p=0}^{\infty} \frac{1}{p!} e_i^{\otimes p}\right) \otimes \left(\sum_{q=0}^{\infty} \frac{1}{q!} (e_J + \xi)^{\otimes q}\right) \otimes \left(\sum_{r=0}^{\infty} \frac{1}{r!} (-e_i)^{\otimes r}\right) \right. \\ &\quad \left. \otimes \left(\sum_{s=0}^{\infty} \frac{1}{s!} (-e_J - \xi)^{\otimes s}\right)\right].\end{aligned}$$

After expanding the above tensor products, it is plain algebra to check that

$$\pi^{(m)}S(\gamma^I) = \mathbf{1} + [e_i, e_J] = \mathbf{1} + e_I.$$

This proves Property (i) for γ^I . Property (ii) is obvious because there are at most finitely many words of length $\leq m$. $\square$

We also recall the following stochastic Taylor expansion for ODEs (cf. [17, P. 214]).

Lemma 3.12. *Consider the following ODE:*

$$dy_t = \sum_{i=1}^d V_i(y_t) dq_t^i, \quad y_0 = x \in \mathbb{R}^N,$$

where $V_1, \dots, V_d$ are C_b^∞ -vector fields and $q : [0, 1] \rightarrow \mathbb{R}^d$ is a BV path. Then one has

$$\begin{aligned}f(y_1) &= f(x) + \sum_{k=1}^r \sum_{i_1, \dots, i_k=1}^d (V_{i_1} \cdots V_{i_k} f)(x) \int_{0 < t_1 < \dots < t_k < 1} dq_{t_1}^{i_1} \cdots dq_{t_k}^{i_k} \\ &\quad + \sum_{i_1, \dots, i_{r+1}=1}^d \int_{0 < t_1 < \dots < t_{r+1} < 1} (V_{i_1} \cdots V_{i_{r+1}} f)(y_{t_1}) dq_{t_1}^{i_1} \cdots dq_{t_{r+1}}^{i_{r+1}}\end{aligned}$$

for any $r \geq 1$ and any function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ that is continuously differentiable up to order $r + 1$.

Proof. This is a simple consequence of iterative applications of the chain rule. $\square$

Now we are in a position to prove Proposition 3.10.

Proof of Proposition 3.10. Let r and F be given by Condition (NC). According to Lemma 3.12, one has

$$\begin{aligned}F(\Psi_q(x)) &= F(x) + \sum_{k=1}^l \sum_{i_1, \dots, i_k=1}^d (V_{i_1} \cdots V_{i_k} f)(x) \int_{0 < t_1 < \dots < t_k < 1} dq_{t_1}^{i_1} \cdots dq_{t_k}^{i_k} \\ &\quad + R_{l+1}(x, q)\end{aligned}\tag{3.15}$$

for all $l \leq r$, $x \in \mathbb{R}^N$ and BV paths $q : [0, 1] \rightarrow \mathbb{R}^d$, where the remainder $R_{l+1}(x, q)$ satisfies

$$\sup_{x \in \mathbb{R}^N} |R_{l+1}(x, q)| \leq C_{V, F, r} \|q\|_{1\text{-var}}^{l+1} \quad (3.16)$$

with some positive constant $C_{V, F, r}$ depending only on r , the vector fields and $\|D^l F\|_\infty$ for $1 \leq l \leq r+1$.

Let $\mathcal{D}$ denote the space of smooth differential operators over $\mathbb{R}^d$. Let $\rho : T(\mathbb{R}^d) \rightarrow \mathcal{D}$ be the algebra homomorphism induced by $\rho(e_i) := V_i$. The expansion (3.15) can be rewritten as

$$F(\Psi_q(x)) = (\rho(\pi^{(l)} S(q)) F)(x) + R_{l+1}(x, q). \quad (3.17)$$

In addition, one also notes that $\rho(e_I) = V_I$ for any word I .

According to the assumption (3.1), for each $x \in \mathbb{R}^N$, there exists a word $I(x)$ such that $2 \leq |I(x)| \leq r$ and $|(V_{I(x)} F)(x)| \geq \kappa$. Let

$$\sigma(x) := \text{sgn}[(V_{I(x)} F)(x)]$$

so that

$$|(V_{I(x)} F)(x)| = \sigma(x) (V_{I(x)} F)(x) \geq \kappa.$$

Given $\delta \in (0, 1)$, we define for each $x \in \mathbb{R}^N$ the absolutely continuous path

$$q_x^\delta := (\delta \gamma^{I(x)})^{\sigma(x)}$$

Here α^{-1} means the reversal of the path α . One knows from Lemma 3.11 that

$$\pi^{(|I(x)|)} S(q_x^\delta) = \mathbf{1} + \delta^{|I(x)|} \sigma(x) e_{I(x)}, \quad \|q_x^\delta\|_{1\text{-var}} \leq C_r \delta \quad (3.18)$$

for all x .

It follows from the relation (3.17) along with the properties (3.16, 3.18) that

$$|F(\Psi_{q_x^\delta}(x)) - F(x) - \delta^{|I(x)|} \sigma(x) (V_{I(x)} F)(x)| \leq C'_{V, F, r} \delta^{|I(x)|+1}.$$

Therefore,

$$\begin{aligned} F(\Psi_{q_x^\delta}(x)) &\geq F(x) + \delta^{|I(x)|} \sigma(x) (V_{I(x)} F)(x) - C'_{V, F, r} \delta^{|I(x)|+1} \\ &\geq F(x) + \delta^r (\kappa - C'_{V, F, r} \delta). \end{aligned}$$

We choose $\delta := \kappa / (2C'_{V, F, r})$ and define $a := \kappa \delta^r / 2$. Then one has

$$F(\Psi_{q_x^\delta}(x)) \geq F(x) + a.$$

To complete the proof, one simply chooses the same F and takes $q_x := q_x^\delta$ (assuming that they are all parametrised on $[0, 1]$). Since DF is bounded, F is globally Lipschitz. In addition, since $|I(x)| \geq 2$, one knows from the first relation of (3.18) that q_x is a loop. It is also clear that

$$K \triangleq \sup_x \|\dot{q}_x\|_{L^\infty} < \infty,$$

since the family $\{q_x\}$ only involves paths of the form $(\delta \gamma^I)^{\pm 1}$ ($|I| \leq r$) whose totality is finite. □

3.3 Recovering the line integral estimate of Boedihardjo-Geng

In this subsection, we reproduce the main result of [7] by using the current method. In fact, we will obtain a stronger result under a weaker assumption.

Proposition 3.13. *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a C_b^∞ -function such that $\phi'(w_0) \neq 0$ at some $w_0 \in \mathbb{R}$. Let $B_t = (X_t, Y_t)$ be a two-dimensional fBM with Hurst parameter $H \in (1/4, 1/2)$. Then there exist positive constants C_1, C_2 , such that the following estimate*

$$\mathbb{P}\left(\left|\int_0^1 \phi(X_t) dY_t\right| > R\right) \geq C_1 e^{-C_2 R^{2H+1}}$$

holds for all large R .

Remark 3.14. In [7], the assumption on the nondegeneracy of ϕ' is slightly stronger and the authors only obtained a weaker estimate with Weibull exponent being any $\gamma > 2H + 1$.

Proof. Consider the RDE

$$dW_t = dX_t, \quad dZ_t = \phi(W_t) dY_t \quad (3.19)$$

with $(W_0, Z_0) = (0, 0)$. Then one has $Z_1 = \int_0^1 \phi(X_t) dY_t$. The underlying vector fields are given by $V_1 = \partial_w$, $V_2 = \phi(w)\partial_z$ and they satisfy $[V_1, V_2] = \phi'(w)\partial_z$. Note that the proposition is not an immediate consequence of Theorem 3.1, since we do not impose a uniform nondegeneracy property on ϕ' . Instead, due to the rather special structure of the underlying RDE (3.19), the previous method can be easily adapted to the current setting. We will only point out the essential changes. Without loss of generality, assume that $\phi'(w_0) > 0$.

We consider the function $F(w, z) := z$. Let $\kappa, \theta > 0$ be such that $\phi'(x) \geq \kappa$ for all $x \in [w_0 - 2\theta, w_0 + 2\theta]$. Given $\delta > 0$, consider the loop

$$q^\delta := (\delta e_w) \sqcup (\delta e_z) \sqcup (-\delta e_w) \sqcup (-\delta e_z),$$

where $\{e_w, e_z\}$ is the canonical basis of the (w, z) -plane. It is readily checked that

$$\Psi_{q^\delta}(w, z) = (w, z + \delta(\phi(w + \delta) - \phi(w))).$$

In particular,

$$\begin{aligned} F(\Psi_{q^\delta}(w, z)) - F(w, z) &= \delta(\phi(w + \delta) - \phi(w)) \\ &= \delta \int_w^{w+\delta} \phi'(u) du \geq \kappa \delta^2 =: a \end{aligned} \quad (3.20)$$

for any $w \in [w_0 - \theta, w_0 + \theta]$ and $\delta < \theta$. This essentially gives Condition (G) with the main difference being that Condition (G2) only holds on the strip $\{(w, z) : w \in [w_0 - \theta, w_0 + \theta]\}$.

Let $n \geq 1$. Construct the Cameron-Martin path

$$h^{(n)} := \bar{q} \sqcup q_1^{(n)} \sqcup \dots \sqcup q_n^{(n)}.$$

Here $\bar{q}$ is the horizontal path from the origin to the point $(w_0, 0)$ parametrised on $[0, 1/2]$. The path $q_k^{(n)}$ is q^δ reparametrised on $I_k := [\frac{1}{2} + \frac{k-1}{2n}, \frac{1}{2} + \frac{k}{2n}]$. Let $U_t^{(n)} = (W_t^{(n)}, Z_t^{(n)})$ ($0 \leq t \leq 1$) be the solution to the RDE

$$\begin{cases} dW_t^{(n)} = dX_t + dh_t^{(n),1} \\ dZ_t^{(n)} = \phi(W_t^{(n)})d(Y_t + h_t^{(n),2}) \end{cases}$$

with initial condition $(0, 0)$.

The key observation is that on the event $\{\|\mathbf{B}\|_{p\text{-var};[0,1]} \leq \theta\}$, one has $W_{t_k}^{(n)} \in [w_0 - \theta, w_0 + \theta]$ for all $k = 0, 1, \dots, n$. Indeed, the construction of $h^{(n)}$ yields that

$$W_{t_k}^{(n)} = X_{t_k} + w_0$$

for every k . As a result,

$$|W_{t_k}^{(n)} - w_0| = |X_{t_k} - 0| \leq \|\mathbf{B}\|_{p\text{-var};[0,1]} \leq \theta.$$

It follows from (3.20) that

$$F(\Psi_{q_k^{(n)}}(U_{t_{k-1}}^{(n)})) \geq F(U_{t_{k-1}}^{(n)}) + a$$

for every k .

One can now apply the same argument as in Section 3.1.2 to conclude that

$$|U_1^{(n)} - U_{1/2}^{(n)}| \gtrsim n$$

on the event

$$A'_n(\varepsilon) := \{\|\mathbf{B}\|_{p\text{-var};[0,1]}\} \cap \left\{ \max_{1 \leq k \leq n} \|\mathbf{B}\|_{p\text{-var};I_k} \leq \varepsilon \right\}.$$

In addition, since each $q_k^{(n)}$ is a loop, one can argue in the same way as in the proof of Lemma 3.7 to show that

$$\|h^{(n)}\|_{\mathcal{H}} \lesssim n^{H+1/2}$$

for all n (the additional part $\bar{q}$ only produces a bounded contribution). The desired lower tail estimate is then established by applying the same Girsanov-type argument as before. $\square$

3.4 Gaussian lower tail under uniform ellipticity

We now prove a general Gaussian lower tail estimate under the uniform ellipticity assumption. As we will see, this is a rather trivial corollary of the previous argument for Theorem 3.4.

Theorem 3.15. *Consider the SDE (1.1) with $H \in (1/4, 1/2)$. Suppose that the vector fields are C_b^∞ and uniformly elliptic in the sense of (1.4). Then there exist positive constants depending only on H and the vector fields, such that the estimate*

$$\mathbb{P}(|X_1 - x_0| > R) \geq C_1 e^{-C_2 R^2} \quad (3.21)$$

holds for all large R .

Proof. The argument is identical to the proof of Theorem 3.4. The main observation is that Condition (G) in Section 3.1 remains valid but without requiring q_x to be a loop in (G1). Indeed, let η be a fixed unit vector in $\mathbb{R}^N$ and define $F(x) := \langle x, \eta \rangle_{\mathbb{R}^N}$. Let

$$A(x) := V(x)^T (V(x)V(x)^T)^{-1} \quad (3.22)$$

be the right inverse of $V = (V_1, \dots, V_d)$, which is well-defined due to ellipticity. Given $x \in \mathbb{R}^N$, the control

$$q_x(t) := \int_0^t A(x + s\eta)\eta ds$$

satisfies $\|\dot{q}_x\|_\infty \leq \|A\|_\infty$ and $F(\Psi_{q_x}(x)) = F(x) + 1$. Therefore, Condition (G) without the loop requirement in (G1) holds with $K = \|A\|_\infty$ and $a = 1$.

The only change to the proof of Theorem 3.4 is that the Cameron-Martin estimate (3.11) becomes

$$\|h^{(n)}\|_{\mathcal{H}} \leq C_H K n. \quad (3.23)$$

The estimate (3.23) follows directly from the elementary inequality $\|h\|_{\mathcal{H}} \leq C_H \|\dot{h}\|_\infty$, which can be seen by using the same representation (2.19). With the change of exponent from $H + 1/2$ to 1, one can now carry through exactly the same argument as in the proof of Theorem 3.4 to deduce the desired Gaussian lower bound (3.21). □

Remark 3.16. It is clear from the above argument that the upgrade of a Gaussian lower tail to $(2H + 1)$ -Weibull is precisely due to the additional loop property in Condition (G1). This loop property has led to nice cancellations in the Cameron-Martin estimate, resulting in a smaller Cameron-Martin norm of order $O(n^{H+1/2})$ vs $O(n)$. From a geometric viewpoint, the possibility of moving the dynamics around by using *loop* controls is a consequence of the noncommutativity of the vector fields (Condition (NC)).

4 Examples with $(2H + 1)$ -Weibull upper tail in Young's regime

In this section, we construct an example of a 2×2 uniformly elliptic system with C_b^∞ -coefficients in Young's regime, whose solution admits a $(2H + 1)$ -Weibull upper tail. Since $2H + 1 > 2$, this example demonstrates a striking fact that a uniformly elliptic system with C_b^∞ -coefficients could fail to have a Gaussian lower tail in Young's regime.

4.1 A simple example in dimension one with 2D noise

Before presenting the aforementioned 2×2 example, it is helpful to first discuss a simpler example with dimensions $N = 1$ and $d = 2$. Such an example already encodes the essential mechanism why a Gaussian lower bound could fail in general.

Theorem 4.1. *Let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function such that $\eta(x+1) = \eta(x) + 1$ and $\eta(x) = k$ for all $x \in [k + 1/4, k + 3/4]$ and $k \in \mathbb{Z}$. Consider the solution $\{X_t\}$ to the ODE*

$$dX_t = \cos(\pi\eta(X_t))dB_t^1 + \sin(\pi\eta(X_t))dB_t^2, \quad X_0 = 0,$$

where (B_t^1, B_t^2) is a two-dimensional fBM with Hurst parameter $H > 1/2$. Then there exist positive constants C_1, C_2 depending only on H , such that

$$\mathbb{P}(|X_1| > R) \leq C_1 e^{-C_2 R^{2H+1}}$$

for all large R .

The rest of this section is devoted to the proof of Theorem 4.1.

4.1.1 Reduction to an oscillation event

We begin with the following simple observation.

Lemma 4.2. *The inclusion*

$$\begin{aligned} \{|X_1| > n\} \subseteq & \left\{ \text{there exist } 0 < s_1 < t_1 < \dots < s_n < t_n < 1, \right. \\ & \left. \text{such that } (-1)^{k-1}(B_{t_k}^1 - B_{s_k}^1) = \frac{1}{2} \text{ for every } k \right\} \end{aligned} \quad (4.1)$$

holds for all $n \geq 1$.

Proof. Suppose that $X_1 > n$. Define

$$\begin{aligned} t_k &:= \inf\{t : X_t = k - 1/4\}, \\ s_k &:= \sup\{t < t_k : X_t = k - 3/4\} \end{aligned}$$

for each $1 \leq k \leq n$. By continuity, one must have

$$0 < s_1 < t_1 < \cdots < s_n < t_n < 1.$$

Let $I_k := [k - 3/4, k - 1/4]$. Since $X_t \in I_k$ for $t \in [s_k, t_k]$, one knows from the definition that $\square$

$$dX_t = (-1)^{k-1} dB_t^1$$

on I_k . Therefore,

$$\frac{1}{2} = X_{t_k} - X_{s_k} = (-1)^{k-1} (B_{t_k}^1 - B_{s_k}^1).$$

This proves that the RHS of (4.1) occurs. The same argument applies to the other case $X_1 < -n$.

4.1.2 Cameron-Martin energy of oscillatory paths

Since the problem has been reduced to the analysis of a one-dimensional fBM by Lemma 4.3, from now on we will assume that B_t is a 1D fBM with Hurst parameter $H > 1/2$ and (intrinsic) Cameron-Martin space $\bar{\mathcal{H}}$. The core of the argument lies in the observation that a sufficiently oscillatory Cameron-Martin path cannot have a small Cameron-Martin norm if $H > 1/2$. The main lemma is stated as follows.

Lemma 4.3. *Let $a > 0$ be given fixed. Let $n \geq 1$ and let*

$$0 < s_1 < t_1 < \cdots < s_n < t_n < 1$$

be a given sequence. There exists a positive constant C_H depending only on H (but not on n or the partition points s_i, t_j), such that the estimate

$$\|h\|_{\bar{\mathcal{H}}} \geq C_H a n^{H+1/2}$$

holds uniformly for all those paths $h \in \bar{\mathcal{H}}$ that satisfy

$$(-1)^{k-1} (h_{t_k} - h_{s_k}) \geq a$$

for every $k = 1, \dots, n$.

The rest of this subsection is devoted to the proof of Lemma 4.3. We will always assume that n is an even number without loss of generality (if n is odd, one simply ignores the interval $[s_n, t_n]$ in the lemma).

As the first step, one has the following observation.

Lemma 4.4. *The lower estimate*

$$\|h\|_{\bar{\mathcal{H}}}^2 \geq \frac{a^2 \left(\sum_{k=1}^n l_k^{-1} \right)^2}{\text{Var} \left[\sum_{k=1}^n \frac{\varepsilon_k}{l_k} (B_{t_k} - B_{s_k}) \right]} \quad (4.2)$$

holds uniformly for all $h \in \bar{\mathcal{H}}$ satisfying

$$\varepsilon_k(h_{t_k} - h_{s_k}) \geq a$$

for every $k = 1, \dots, n$, where $\varepsilon_k := (-1)^{k-1}$ and $l_k := t_k - s_k$.

Proof. Consider the linear functional

$$\Lambda : \bar{\mathcal{H}} \rightarrow \mathbb{R}, \quad \Lambda(h) := \sum_{k=1}^n \frac{\varepsilon_k}{l_k} (h_{t_k} - h_{s_k}).$$

We claim that

$$\|\Lambda\|_{\mathcal{H}^*}^2 = \text{Var} \left[\sum_{k=1}^n \frac{\varepsilon_k}{l_k} (B_{t_k} - B_{s_k}) \right]. \quad (4.3)$$

Indeed, under the canonical isometric isomorphisms $\bar{\mathcal{H}}^* \cong \bar{\mathcal{H}} \cong \mathcal{C}_1^B$ (the last denotes the first Wiener chaos of B), the chaos element corresponding to the functional $[h \mapsto h_t]$ is B_t for every given t . By linearity, the chaos element corresponding to Λ is

$$\sum_{k=1}^n \frac{\varepsilon_k}{l_k} (B_{t_k} - B_{s_k}).$$

This proves the claim (4.3). On the other hand, for any $h \in \bar{\mathcal{H}}$ satisfying the assumption of the lemma, one has

$$\Lambda(h) \geq a \sum_{k=1}^n l_k^{-1}.$$

The desired estimate (4.2) follows from the Cauchy-Schwarz inequality and the relation (4.3). $\square$

Now the problem is reduced to an upper estimate of the variance appearing in (4.2). To this end, one first notes from [5, Section 2.1.1] in the $H > 1/2$ regime that

$$\text{Var} \left[\int_0^1 f(r) dB_r \right] = H(2H-1) \int_{\mathbb{R}^2} f(u)f(v)|u-v|^{2H-2} du dv \quad (4.4)$$

for any deterministic and regular $f : [0, 1] \rightarrow \mathbb{R}$ (say, piecewise smooth). Here one extends f to the real line by setting it to be zero outside $[0, 1]$. In our context,

$$f(x) = \sum_{k=1}^n \frac{\varepsilon_k}{l_k} \mathbf{1}_{I_k}(x), \quad (4.5)$$

where $I_k := [s_k, t_k]$.

Next, we derive a basic identity which allows one to analyse the RHS of (4.4) effectively.

Lemma 4.5. *Let $\beta \in (0, 1)$. Then one has*

$$\int_{\mathbb{R}^2} f(u)f(v)|u-v|^{-\beta}dudv = \beta(1+\beta) \int_0^\infty \frac{\|\Delta_r F\|_{L^2(\mathbb{R})}^2}{r^{2+\beta}}dr$$

for all bounded and compactly supported $f : \mathbb{R} \rightarrow \mathbb{R}$, where $F(x) := \int_{-\infty}^x f(u)du$ and $(\Delta_r F)(x) := F(x+r) - F(x)$.

Proof. By definition, one has

$$\begin{aligned} \|\Delta_r F\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}^2} \left(\int_x^{x+r} f(u)du \right) \left(\int_x^{x+r} f(v)dv \right) dx \\ &= \int_{\mathbb{R}^2} f(u)f(v) \left[\int_{\mathbb{R}} \mathbf{1}_{[u-r, u] \cap [v-r, v]}(x) dx \right] dudv \\ &= \int_{\mathbb{R}^2} f(u)f(v)(r - |u-v|)_+ dudv. \end{aligned}$$

It follows that

$$\int_0^\infty \frac{\|\Delta_r F\|_{L^2(\mathbb{R})}^2}{r^{2+\beta}}dr = \int_{\mathbb{R}^2} f(u)f(v) \left[\int_0^\infty \frac{(r - |u-v|)_+}{r^{2+\beta}}dr \right] dudv.$$

The dr -integral is easily evaluated as

$$\int_0^\infty \frac{(r - |u-v|)_+}{r^{2+\beta}}dr = \frac{|u-v|^{-\beta}}{\beta(1+\beta)}.$$

The desired relation thus follows. $\square$

The main upper estimate of the variance appearing in (4.2) is now stated as follows.

Lemma 4.6. *There exists a positive constant C_H depending only on H , such that*

$$\text{Var} \left[\sum_{k=1}^n \frac{\varepsilon_k}{l_k} (B_{t_k} - B_{s_k}) \right] \leq C_H \sum_{k=1}^n l_k^{2H-2}.$$

Proof. According to (4.4) and Lemma 4.5,

$$\text{Var} \left[\sum_{k=1}^n \frac{\varepsilon_k}{l_k} (B_{t_k} - B_{s_k}) \right] = c_H \int_0^\infty \frac{\|\Delta_r F\|_{L^2(\mathbb{R})}^2}{r^{4-2H}}dr, \quad (4.6)$$

where

$$c_H := H(2H-1)(2-2H)(3-2H)$$

and the function $F(x) := \int_{-\infty}^x f(u)du$ with f being defined by (4.5).

To estimate the RHS of (4.6), one further writes

$$F = \sum_{k=1}^{n/2} F_k; \quad F_k := \int_{-\infty}^x \left(\frac{\mathbf{1}_{I_{2k-1}}(u)}{l_{2k-1}} - \frac{\mathbf{1}_{I_{2k}}(u)}{l_{2k}} \right) du.$$

It is easy to see that

$$0 \leq F_k \leq 1, \quad \int_{\mathbb{R}} |F'_k(x)| dx \leq 2. \quad (4.7)$$

In addition, the supports of the F_k 's are disjoint. We claim that

$$\|\Delta_r F_k\|_{L^2(\mathbb{R})}^2 \leq \min\{L_k r^2, 2r\} \quad (4.8)$$

for all $r > 0$, where $L_k := l_{2k-1}^{-1} + l_{2k}^{-1}$. Indeed, one has

$$|(\Delta_r F_k)(x)|^2 = \left| \int_0^r F'_k(x+u) du \right|^2 \leq r \int_0^r |F'_k(x+u)|^2 du.$$

It follows that

$$\|\Delta_r F_k\|_{L^2(\mathbb{R})}^2 \leq r^2 \|F'_k\|_{L^2(\mathbb{R})}^2 = L_k r^2. \quad (4.9)$$

On the other hand, according to (4.7), one also has

$$\begin{aligned} \|\Delta_r F_k\|_{L^2(\mathbb{R})}^2 &\leq \int_{\mathbb{R}} |(\Delta_r F_k)(x)|^2 dx \\ &\leq \int_{\mathbb{R}} \int_0^r |F'_k(x+u)|^2 du dx \leq 2r. \end{aligned} \quad (4.10)$$

The estimate (4.8) thus follows from (4.9) and (4.10).

Now consider the decomposition

$$(\Delta_r F)(x) = \sum_{k=1}^{n/2} (\Delta_r F_k)(x).$$

Since the F_k 's have disjoint supports, for every $x \in \mathbb{R}$ there are at most two nonzero terms in the above summation. As a result, one has

$$|(\Delta_r F)(x)|^2 \leq 2 \sum_{k=1}^{n/2} |(\Delta_r F_k)(x)|^2.$$

It follows from (4.8) that

$$\begin{aligned} \int_0^\infty \frac{\|\Delta_r F\|_{L^2(\mathbb{R})}^2}{r^{4-2H}} dr &\leq 2 \sum_{k=1}^{n/2} \int_0^\infty \frac{\|\Delta_r F_k\|_{L^2(\mathbb{R})}^2}{r^{4-2H}} dr \\ &\leq 2 \sum_{k=1}^{n/2} \int_0^\infty \frac{\min\{L_k r^2, 2r\}}{r^{4-2H}} dr \\ &= \frac{2^{2H-1}}{(2H-1)(1-H)} \sum_{k=1}^{n/2} L_k^{2-2H} \\ &\leq \frac{2^{2H-1}}{(2H-1)(1-H)} \sum_{k=1}^{n/2} (l_{2k-1}^{2H-2} + l_{2k}^{2H-2}). \end{aligned}$$

In view of the relation (4.6), the result thus follows. $\square$

Proof of Lemma 4.3. According to Lemma 4.4 and Lemma 4.6, one has

$$\|h\|_{\mathcal{H}}^2 \geq C_H a^2 \cdot \frac{\left(\sum_{k=1}^n l_k^{-1}\right)^2}{\left(\sum_{k=1}^n l_k^{-(2-2H)}\right)}.$$

Denote $x_k := l_k^{-1}$ and $\beta := 2 - 2H$. Since $\beta \in (0, 1)$, concavity of $x \mapsto x^\beta$ implies that

$$\frac{\left(\sum_{k=1}^n x_k\right)^2}{\sum_{k=1}^n x_k^\beta} = \left(\sum_{k=1}^n x_k\right)^{2-\beta} \frac{\left(\sum_{k=1}^n x_k\right)^\beta}{\sum_{k=1}^n x_k^\beta} \geq n^{\beta-1} \left(\sum_{k=1}^n x_k\right)^{2-\beta}.$$

On the other hand, since

$$\sum_{k=1}^n x_k^{-1} = \sum_{k=1}^n l_k \leq 1,$$

the Cauchy-Schwarz inequality shows that

$$n^2 \leq \left(\sum_{k=1}^n x_k^{-1}\right) \left(\sum_{k=1}^n x_k\right) \leq \sum_{k=1}^n x_k.$$

As a consequence,

$$\|h\|_{\mathcal{H}}^2 \geq C_H a^2 \cdot n^{\beta-1} \cdot n^{2(2-\beta)} = C_H a^2 n^{2H+1}.$$

This completes the proof of Lemma 4.3. □

4.1.3 The fixed-grid probabilistic estimate

Our goal of this subsection is to prove the following estimate. We continue to use the previous notation.

Lemma 4.7. *There exist positive constants C_1, C_2 depending only on H , such that the estimate*

$$\mathbb{P}(\varepsilon_k(B_{t_k} - B_{s_k}) \geq a \text{ for every } k = 1, \dots, n) \leq C_1 e^{-C_2 a^2 n^{2H+1}} \quad (4.11)$$

holds uniformly for all $a > 0$, $n \geq 1$ and $0 < s_1 < t_1 < \dots < s_n < t_n < 1$.

The proof of Lemma 4.7 relies on the following standard result from Gaussian analysis. Recall that the Cameron-Martin subspace of a centered, nondegenerate Gaussian vector $Z = (Z_1, \dots, Z_n)$ is the image $\mathcal{H}_Z$ of the map

$$\text{Span}\{Z_1, \dots, Z_n\} \ni \eta \mapsto z := \mathbb{E}[\eta Z] \in \mathbb{R}^n, \quad (4.12)$$

whose inner product structure is defined by

$$\langle z, w \rangle_{\mathcal{H}_Z} := \mathbb{E}[\eta \zeta],$$

where η (respectively, ζ) is the preimage of z (respectively, w) under (4.12). Of course, $\mathcal{H}_Z = \mathbb{R}^n$ since Z is nondegenerate but the underlying inner product is induced from the covariance structure of Z . More explicitly, one has

$$\langle z, w \rangle_{\mathcal{H}_Z} = z^T Q^{-1} w$$

where Q is the covariance matrix of Z , i.e. $Z \sim N(0, Q)$.

Proposition 4.8. *Let Z be a centered, nondegenerate Gaussian vector. Let K be a closed, convex subset of $\mathbb{R}^n$. Suppose that*

$$r := d_{\mathcal{H}_Z}(o, K) := \inf_{z \in K} \|z\|_{\mathcal{H}_Z} > 0.$$

Then

$$\mathbb{P}(Z \in K) \leq 1 - \Phi(r),$$

where $\Phi(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-u^2/2} du$ is the standard normal distribution function.

Proof. This is essentially the Gaussian isoperimetric inequality (cf. [23]). For completeness, let us provide the short and simple proof in the current finite dimensional setting. Let $Q := \text{Cov}[Z]$ and define $X := Q^{-1/2}Z \sim N(0, \text{Id})$. Also set $L := Q^{-1/2}K$. Note that L is again a closed, convex subset of $\mathbb{R}^n$. In addition, one has

$$d_{\mathbb{R}^n}(o, L) = \inf_{x \in L} |x|_{\mathbb{R}^n} = \inf_{z \in K} |Q^{-1/2}z|_{\mathbb{R}^n} = \inf_{z \in K} \|z\|_{\mathcal{H}_Z} = r.$$

Let $x_* \in L$ be the unique minimiser of $L \ni x \mapsto |x|_{\mathbb{R}^n}$. Convexity implies that

$$\langle x_*, x - x_* \rangle_{\mathbb{R}^n} \geq 0$$

for all $x \in L$. As a result, one has

$$L \subseteq \{x : \langle u, x \rangle_{\mathbb{R}^n} \geq r\},$$

where $u := x_*/|x_*|_{\mathbb{R}^n}$. It follows that

$$\mathbb{P}(Z \in K) = \mathbb{P}(X \in L) \leq \mathbb{P}(\langle u, X \rangle_{\mathbb{R}^n} \geq r) = 1 - \Phi(r),$$

since $\langle u, X \rangle \sim N(0, 1)$. □

Proof of Lemma 4.7. Consider the nondegenerate Gaussian vector

$$Z := \left(\varepsilon_k (B_{t_k} - B_{s_k}) \right)_{k=1}^n.$$

Define the projection

$$P_Z : \bar{\mathcal{H}} \rightarrow \mathbb{R}^n, \quad P_Z(h) := \left(\varepsilon_k (h_{t_k} - h_{s_k}) \right)_{k=1}^n.$$

It is standard that $\mathcal{H}_Z = P_Z(\bar{\mathcal{H}}) = \mathbb{R}^n$ and

$$|z|_{\mathcal{H}_Z} = \inf \{ \|h\|_{\bar{\mathcal{H}}} : P_Z(h) = z \}. \quad (4.13)$$

Let $K := [a, \infty)^n$. According to Lemma 4.3 and (4.13), one has

$$d_{\mathcal{H}_Z}(o, K) \geq C_H a n^{H+1/2},$$

where C_H is a positive constant that does not depend on a, n or the grid points s_i, t_j . It follows from Proposition 4.8 that

$$\mathbb{P}(Z \in K) \leq 2e^{-C'_H a^2 n^{2H+1}}.$$

This yields the desired inequality (4.11). □

4.1.4 The general probabilistic estimate

Finally, we are going to remove the grid points in the estimate (4.11). This is achieved by a simple continuity argument. Given $n \geq 1$, let $\mathcal{P}_n$ denote the set of partitions

$$(\mathbf{s}, \mathbf{t}) : 0 < s_1 < t_1 < \dots < s_n < t_n < 1.$$

For each $a > 0$ and $(\mathbf{s}, \mathbf{t}) \in \mathcal{P}_n$, we define $E_{\mathbf{s}, \mathbf{t}}(a)$ to be the event that

$$\varepsilon_k(B_{t_k} - B_{s_k}) \geq a$$

for every $k = 1, \dots, n$.

Lemma 4.9. *There exist positive constants C_1, C_2 depending only on H and an integer N depending only on H, a , such that the estimate*

$$\mathbb{P}(\{E_{\mathbf{s}, \mathbf{t}}(a) \text{ occurs for some partition } (\mathbf{s}, \mathbf{t}) \in \mathcal{P}_n\}) \leq C_1 e^{-C_2 a^2 n^{2H+1}}$$

holds for all $n > N$.

Proof. Define

$$E_n(a) := \{E_{\mathbf{s}, \mathbf{t}}(a) \text{ occurs for some partition } (\mathbf{s}, \mathbf{t}) \in \mathcal{P}_n\}.$$

Let $\delta_n := n^{-q}$ with q being specified later on. Let $\mathcal{D}(\delta_n)$ be the partition of $[0, 1]$ into equal subintervals of length δ_n . Consider the event

$$G_n(a) := \left\{ m_B(2\delta_n) < \frac{a}{4} \right\},$$

where

$$m_B(\delta) := \sup_{s, t \in [0, 1], |t-s| \leq \delta} |B_t - B_s|$$

is the modulus of continuity of B .

We claim that

$$E_n(a) \cap G_n(a) \subseteq \bigcup_{\substack{(\hat{\mathbf{s}}, \hat{\mathbf{t}}) \in \mathcal{P}_n \\ \hat{s}_i, \hat{t}_j \in \mathcal{D}(\delta_n)}} E_{\hat{\mathbf{s}}, \hat{\mathbf{t}}}(a/2).$$

Indeed, let $\omega \in \text{LHS}$. Then

$$\varepsilon_k(B_{t_k} - B_{s_k}) \geq a \quad (4.14)$$

holds with some $2n$ -tuple $(\mathbf{s}, \mathbf{t})$ for every k . It is obvious that $t_k - s_k > 2\delta_n$ for every k , for otherwise the relation (4.14) would contradict the fact that $m_\omega(2\delta_n) < a/4$. Define the $2n$ -tuple $(\hat{\mathbf{s}}, \hat{\mathbf{t}})$ where $\hat{s}_k > s_k$ (respectively, $\hat{t}_k < t_k$) is the partition point in $\mathcal{D}(\delta_n)$ that is closest to s_k from the right (respectively, closest to t_k from the left). The points $\hat{s}_k, \hat{t}_k$ exist because $t_k - s_k > 2\delta_n$. It follows that

$$\begin{aligned} \varepsilon_k(B_{\hat{t}_k} - B_{\hat{s}_k}) &\geq \varepsilon_k(B_{t_k} - B_{t_k}) - |B_{s_k} - B_{\hat{s}_k}| - |B_{t_k} - B_{\hat{t}_k}| \\ &\geq a - 2\omega_B(\delta_n) > a - \frac{a}{2} = \frac{a}{2} \end{aligned}$$

for every k . This proves the claim.

Note that the cardinality of $2n$ -tuples $(\hat{\mathbf{s}}, \hat{\mathbf{t}})$ chosen from the partition $\mathcal{D}(\delta_n)$ is at most

$$|\mathcal{D}(\delta_n)|^{2n} = e^{2nq \log n}.$$

According to Lemma 4.7, one obtains that

$$\mathbb{P}(E_n(a) \cap G_n(a)) \leq e^{2nq \log n} \times C_1 e^{-C_2 a^2 n^{2H+1}}. \quad (4.15)$$

The first term is negligible comparing to the second term for all $n > N(H, a, q)$.

Now it remains to estimate $\mathbb{P}(G_n(a)^c)$. To this end, we first claim that

$$\mathbb{P}(m_B(r) > \lambda) \leq c_H r^{-1} e^{-C_H \lambda^2 / r^{2H}} \quad (4.16)$$

for all $r \in (0, 1)$ and $\lambda > 0$. Indeed, note that

$$m_B(r) \leq 2 \max_{0 \leq k \leq r^{-1}-2} \sup_{t \in J_k} |B_t - B_{kr}|$$

where $J_k := [kr, (k+2)r]$. By applying a simple union bound, the fractional Brownian scaling and Borel's inequality (see [1, Theorem 2.1]), one finds that

$$\mathbb{P}(m_B(r) > \lambda) \leq r^{-1} \mathbb{P}\left(\sup_{t \in [0, 2]} |\tilde{B}_t| > \frac{u}{2r^H}\right) \leq c_H r^{-1} e^{-C_H u^2 / r^{2H}}.$$

This proves the estimate (4.16). It follows that

$$\mathbb{P}(G_n(a)^c) \leq C_3 n^q e^{-C_4 a^2 n^{2Hq}}.$$

By choosing $q > 1 + \frac{1}{2H}$, the RHS of the above inequality is negligible comparing to the main estimate (4.15). As a consequence, one concludes that

$$\mathbb{P}(E_n(a)) \leq \mathbb{P}(E_n(a) \cap G_n(a)) + \mathbb{P}(G_n(a)^c) \leq C_5 e^{-C_6 a^2 n^{2H+1}}$$

for all $n > N(H, a)$. $\square$

According to Lemma 4.2 and Lemma 4.9, the proof of Theorem 4.1 is now complete.

4.2 A genuine 2D example

A major unsatisfactory point about the 1D example of Theorem 4.1 is that it is not a genuine elliptic system (the dimension of noise is larger than the dimension of solution). The availability of additional noise dimension has made the underlying oscillation mechanism quite transparent. On the other hand, any naive attempt of adding an additional state dimension to the system will turn it back into the Gaussian lower tail regime.

In this section, we present a genuine 2×2 example for which a $(2H + 1)$ -Weibull upper tail remains valid. The main result is stated as follows.

Theorem 4.10. *Let $\eta : [0, \infty) \rightarrow [0, \infty)$ be a smooth function that is constant near the origin and such that $\eta(r) = r$ for $r \geq 2$. Let $B_t = (B_t^1, B_t^2)$ be a fractional Brownian motion with Hurst parameter $H > 1/2$. Consider the SDE*

$$dX_t = V_1(X_t)dB_t^1 + V_2(X_t)dB_t^2, \quad X_0 = 0 \in \mathbb{R}^2, \quad (4.17)$$

where

$$(V_1(x), V_2(x)) = \begin{pmatrix} \cos \eta(|x|) & -\sin \eta(|x|) \\ \sin \eta(|x|) & \cos \eta(|x|) \end{pmatrix} \quad (4.18)$$

Then there exist positive constants C_1, C_2 , such that

$$\mathbb{P}\left(\sup_{0 \leq t \leq 1} |X_t| > R\right) \leq C_1 e^{-C_2 R^{2H+1}}$$

for all large R .

Our main idea of proving Theorem 4.10 can be vaguely summarised as follows. Consider the solution map $w \mapsto (X_t^w)_{0 \leq t \leq 1}$. The key insight is that if h is a Cameron-Martin path that drives the dynamics to a distance of order R , i.e.

$$\sup_{0 \leq t \leq 1} |X_t^h| \geq R,$$

the path h has to be either longer than $O(R^{H+1/2})$ or it has to make at least $O(R)$ number of “effective turns”. The latter case forces its Cameron-Martin norm to be at least $O(R^{H+1/2})$. A technical part of the argument is to show that such a coercivity property is robust under perturbation of h by $w \in C^{p\text{-var}}([0, 1]; \mathbb{R}^2)$ with bounded p -variation norm. This perturbative coercivity estimate is needed to transfer the deterministic estimate to a probabilistic one by using the Gaussian isoperimetric inequality. The mechanism here can be viewed as a two-dimensional analogue of Theorem 4.1.

4.2.1 A perturbative coercivity estimate

The essential ingredient for the proof of Theorem 4.10 is the following coercivity estimate. This is a more delicate two-dimensional version of Lemma 4.3, which is also robust under perturbation. Given $w \in C^{p\text{-var}}([0, 1]; \mathbb{R}^2)$ ($1/H < p < 2$), we use X_t^w to denote the solution to the differential equation (4.17) driven by the path w .

Proposition 4.11. *Let $p \in (H^{-1}, 2)$. For any $M > 0$, there exist positive constants c_* and R_* depending only on p, H, V and M , such that the following implication*

$$\sup_{0 \leq t \leq 1} |X_t^{w+h}| \geq R, \quad R > R_* \implies \|h\|_{\bar{\mathcal{H}}} \geq c_* R^{H+1/2} \quad (4.19)$$

holds uniformly over all $h \in \bar{\mathcal{H}}$ and all $w \in C^{p\text{-var}}([0, 1]; \mathbb{R}^2)$ satisfying $\|w\|_{p\text{-var}} \leq M$.

The rest of this subsection is devoted to the proof of Proposition 4.11.

A lower estimate of Cameron-Martin norm

In this part, we establish an estimate which controls the Cameron-Martin norm from below by the number of “effective” turns. We first introduce the definition of an effective turn. Let $a_0 > 0$ and $\theta_0 \in (0, \pi)$ be given fixed numbers. Later on, we will specify their values (see (4.28)). For the current discussion, this is not important. Given two unit vectors $v, w \in \mathbb{R}^2$, we use $\angle(v, w) := \arccos\langle v, w \rangle \in [0, \pi]$ to denote their angle.

Definition 4.12. Let $h : [0, 1] \rightarrow \mathbb{R}^2$ be an absolutely continuous path. Let $\mu_h(dt) := |\dot{h}_t|dt$ denote its arclength measure. An interval $I = [u, v] \subseteq [0, 1]$ is said to *contain an (a_0, θ_0) -effective turn* of h if there are measurable subsets $A, B \subseteq I$ such that

$$\mu_h(A) \geq a_0, \quad \mu_h(B) \geq a_0$$

and

$$\angle\left(\frac{\dot{h}_s}{|\dot{h}_s|}, \frac{\dot{h}_t}{|\dot{h}_t|}\right) \geq \theta_0$$

for $\mu_h \otimes \mu_h$ -a.e. $(s, t) \in A \times B$ at which $\dot{h}_s \neq 0$ and $\dot{h}_t \neq 0$. We define $\mathcal{N}_{\text{eff}}(h)$ to be the maximal number of disjoint intervals in $[0, 1]$ that contain an (a_0, θ_0) -effective turn of h .

Remark 4.13. Heuristically, an effective turn consists of two pieces of h on I which have arclength at least a_0 and have angle separately uniformly by at least θ_0 .

The main Cameron-Martin estimate in this part is stated as follows.

Lemma 4.14. *There exists a positive constant C_H depending only on H , such that*

$$\|h\|_{\bar{\mathcal{H}}} \geq C_H a_0 \sin\left(\frac{\theta_0}{2}\right) \cdot \mathcal{N}_{\text{eff}}(h)^{H+1/2}$$

for all $h \in \bar{\mathcal{H}}$.

Proof. In particular,

$$\|h\|_{\bar{\mathcal{H}}}^2 \geq C_H \int_0^1 \int_0^1 \frac{|\dot{h}_t - \dot{h}_s|^2}{|t-s|^{2H}} ds dt. \quad (4.20)$$

Next, let $I \subseteq [0, 1]$ be an interval which contains an (a_0, θ_0) -effective turn (A, B) of h . By definition, it is not hard to see that

$$|\dot{h}_s - \dot{h}_t|^2 \geq 4 \sin^2\left(\frac{\theta_0}{2}\right) \cdot |\dot{h}_s| \cdot |\dot{h}_t|$$

for $(\mu_h \otimes \mu_h)$ -a.e. $(s, t) \in A, B$. As a consequence,

$$\begin{aligned} \int_{I \times I} \frac{|\dot{h}_t - \dot{h}_s|^2}{|t-s|^{2H}} ds dt &\geq 4 \sin^2\left(\frac{\theta_0}{2}\right) \int_{A \times B} \frac{|\dot{h}_s| \cdot |\dot{h}_t|}{|t-s|^{2H}} ds dt \\ &\geq 4 \sin^2\left(\frac{\theta_0}{2}\right) \cdot |I|^{-2H} \cdot a_0^2. \end{aligned} \quad (4.21)$$

Now suppose that $I_1, \dots, I_N$ are N given disjoint intervals, each containing an (a_0, θ_0) -effective turn. It follows from (4.21) that

$$\begin{aligned} \int_0^1 \int_0^1 \frac{|\dot{h}_t - \dot{h}_s|^2}{|t-s|^{2H}} ds dt &\geq \sum_{k=1}^N \int_{I_k \times I_k} \frac{|\dot{h}_t - \dot{h}_s|^2}{|t-s|^{2H}} ds dt \\ &\geq 4a_0^2 \sin^2\left(\frac{\theta_0}{2}\right) \cdot \sum_{k=1}^N |I_k|^{-2H}. \end{aligned} \quad (4.22)$$

On the other hand, by using the convexity of $x \mapsto x^{-2H}$ and the fact that $\sum_k |I_k| \leq 1$, one has

$$\frac{1}{N} \sum_{k=1}^N |I_k|^{-2H} \geq \left(\frac{1}{N} \sum_{k=1}^N |I_k| \right)^{-2H} \geq N^{2H}. \quad (4.23)$$

After combining (4.20), (4.22) and (4.23), one arrives at

$$\|h\|_{\bar{\mathcal{H}}}^2 \geq 4C_H a_0^2 \sin^2\left(\frac{\theta_0}{2}\right) \cdot N^{2H+1}.$$

The result follows from the definition of $\mathcal{N}_{\text{eff}}(h)$. $\square$

A non-perturbative coercivity estimate

In order to prove Proposition 4.11, one needs to control the quantity $\mathcal{N}_{\text{eff}}(h)$ from below. It is helpful to first establish a non-perturbative lower estimate of $\mathcal{N}_{\text{eff}}(h)$. This part contains the essential idea and the perturbative version follows easily by controlling an additional Young integration error. The main lemma is stated as follows.

Lemma 4.15. *There exist universal positive constants C_1, C_2 , such that the estimate*

$$\mathcal{N}_{\text{eff}}(h) \geq C_1 R - \frac{C_2 \|h\|_{1\text{-var}}}{R} \quad (4.24)$$

holds uniformly for all $R > 4$ and all $h \in \bar{\mathcal{H}}$ satisfying $\sup_{0 \leq t \leq 1} |X_t^h| \geq R$.

The rest of this part is devoted to the proof of Lemma 4.15.

From now on, we fix three parameters $a_0 > 0$, $\theta_0 \in (0, \pi)$ and $m_0 \in \mathbb{N}$ (their exact values will be chosen by (4.28) later on). We divide the unit circle S^1 into m_0 equal arcs $P_1, P_2, \dots, P_{m_0}$. Let $h : [0, 1] \rightarrow \mathbb{R}^2$ be an absolutely continuous path and let $I = [u, v] \subseteq [0, 1]$ be a given interval. An arc P_k is said to be *heavy* with respect to $h|_I$ if

$$\mu_h \left(\left\{ t \in I : \dot{h}_t \neq 0, \frac{\dot{h}_t}{|\dot{h}_t|} \in P_k \right\} \right) \geq a_0,$$

where $\mu_h(dt) := |\dot{h}_t| dt$ is the arclength measure associated with h . An arc that is not heavy is said to be *light*.

The following observation is important for the proof of Lemma 4.15.

Lemma 4.16. *Suppose that I does not contain an (a_0, θ_0) -effective turn of h and there is at least one heavy arc with respect to $h|_I$. Then there exists a measurable set $E \subseteq I$ and a unit vector v_* , such that*

$$\mu_h(E) \leq m_0 a_0 \quad (4.25)$$

and

$$\angle \left(\frac{\dot{h}_t}{|\dot{h}_t|}, v_* \right) \leq \theta_0 + \frac{4\pi}{m_0} \quad (4.26)$$

for μ_h -a.e. $t \in I \setminus E$ with $\dot{h}_t \neq 0$.

Proof. This is a simple consequence of the definition of effective turn. Let w_k be the center of P_k for every k . We claim that

$$\angle(w_k, w_l) \leq \theta_0 + \frac{2\pi}{m_0} \quad (4.27)$$

for any pair P_k, P_l of heavy arcs. Indeed, suppose that (4.27) is not true for some heavy pair P_k, P_l . Let

$$A := \left\{ t \in I : \dot{h}_t \neq 0, \frac{\dot{h}_t}{|\dot{h}_t|} \in P_k \right\}, \quad B := \left\{ t \in I : \dot{h}_t \neq 0, \frac{\dot{h}_t}{|\dot{h}_t|} \in P_l \right\}.$$

Then one has

$$\begin{aligned} \angle\left(\frac{\dot{h}_s}{|\dot{h}_s|}, \frac{\dot{h}_t}{|\dot{h}_t|}\right) &\geq \angle(w_k, w_l) - \angle\left(\frac{\dot{h}_s}{|\dot{h}_s|}, \frac{\dot{h}_t}{|\dot{h}_t|}\right) \\ &> \theta_0 + \frac{2\pi}{m_0} - \frac{2\pi}{m_0} = \theta_0 \end{aligned}$$

for $\mu_h \otimes \mu_h$ -a.e. $(s, t) \in A \times B$. Since P_k, P_l are heavy, it follows that (A, B) provides an (a_0, θ_0) -effective turn of h in I . This is a contradiction. Therefore, the estimate (4.27) holds.

Now let v_* be the center of a given heavy arc (which exists by assumption). Define

$$E := \left\{ t \in I : \dot{h}_t \neq 0, \frac{\dot{h}_t}{|\dot{h}_t|} \in P_{\text{light}} \right\},$$

where P_{light} denotes the union of all light arcs. Since there are at most m_0 light arcs, one has by definition that $\mu_h(E) \leq m_0 a_0$. This proves the first claim (4.25). In addition, according to the estimate (4.27), all heavy arcs lie inside the cone centered at v_* with S^1 -radius

$$\theta_0 + \frac{2\pi}{m_0} + 2 \times \frac{\pi}{m_0} = \theta_0 + \frac{4\pi}{m_0}.$$

This proves the second claim (4.26). $\square$

Now we fix the following four parameters:

$$a_0 = \frac{1}{256}\pi, \quad \theta_0 = \frac{1}{8}\pi, \quad m_0 = 32, \quad \gamma_0 = \frac{1}{8}\pi. \quad (4.28)$$

Given $h \in \bar{\mathcal{H}}$, recall that X_t^h is the solution to the control differential equation

$$\dot{X}_t^h = V(X_t^h) \cdot \dot{h}_t dt, \quad X_0^h = 0,$$

where V is the vector field given by (4.18). Note that V is orthonormal at every point. Let $r_t := |X_t^h|$.

Lemma 4.17. *Let $I = [u, v] \subseteq [0, 1]$ and $a > 2$. Suppose that I does not contain an (a_0, θ_0) -effective turn of h . Suppose further that*

$$r_u = a, \quad r_v = a + 3\pi$$

and $r_t \in [a, a + 3\pi]$ for all $t \in I$. Then $\mu_h(I) \geq \gamma_0 a$.

Proof. We first claim that there is at least one heavy arc with respect to $h|_I$. Indeed, since V is everywhere orthonormal, one has

$$|\dot{X}_t^h| = |\dot{h}_t| \quad \text{for every } t \implies \mu_h(I) = \|h\|_{1\text{-var}; I} = \|X^h\|_{1\text{-var}; I} \geq 3\pi$$

by assumption. On the other hand, if all arcs were light, according to (4.28) one would obtain that

$$\mu_h(I) \leq m_0 a_0 = \frac{\pi}{8},$$

which is a contradiction. Therefore, there is at least one heavy arc. It follows from Lemma 4.16 that there exists a measurable subset $E \subseteq I$ and a unit vector v_* , such that (4.25) and (4.26) hold.

To prove the lemma, let us write

$$X_t^h = r_t e^{i\phi_t}, \dot{h}_t = |\dot{h}_t| e^{i\alpha_t}, \quad t \in I. \quad (4.29)$$

Here ϕ_t is any continuous determination of the angular component of X_t^h ; one can fix any $\phi_u \in \arg X_u^h$ and then determine a unique lifted path $(\phi_t)_{t \in I}$. The path α_t is any measurable choice of $\arg(\dot{h}_t/|\dot{h}_t|)$, which is μ_h -a.e. defined on $\{t : \dot{h}_t \neq 0\}$. Since

$$V(x) = \begin{pmatrix} \cos |x| & -\sin |x| \\ \sin |x| & \cos |x| \end{pmatrix}$$

when $|x| \geq 2$, elementary computation shows that

$$\dot{r}_t = |\dot{h}_t| \cos(r_t + \alpha_t - \phi_t), \quad \dot{\phi}_t = \frac{1}{r_t} |\dot{h}_t| \sin(r_t + \alpha_t - \phi_t). \quad (4.30)$$

In particular, one has

$$|\dot{r}_t| \leq |\dot{h}_t|, \quad |\dot{\phi}_t| \leq \frac{1}{a} |\dot{h}_t| \quad (4.31)$$

for all $t \in I$.

Now suppose on the contrary that $\mu_h(I) < \gamma_0 a$. The second relation in (4.31) implies that

$$\sup_{s, t \in I} |\phi_t - \phi_s| \leq \frac{1}{a} \mu_h(I) \leq \gamma_0.$$

In addition, writing $v_* = e^{i\alpha_*}$ one knows from (4.26) that

$$\angle(e^{i\alpha_t}, e^{i\alpha_*}) \leq \theta_0 + \frac{4\pi}{m_0}$$

for μ_h -a.e. $t \in I \setminus E$. As a result, one has

$$\text{dist}_{2\pi}(\alpha_t - \phi_t, \alpha_* - \phi_u) \leq \theta_0 + \frac{4\pi}{m_0} + \gamma_0 \quad (4.32)$$

for μ_h -a.e. $t \in I \setminus E$. Here $\text{dist}_{2\pi}$ denotes the distance modulo 2π (equivalently, the S^1 -distance). Let

$$\beta_* \in [a + m_0 a_0, a + 3\pi - m_0 a_0]$$

be a point satisfying

$$\beta_* = \pi - (\alpha_* - \phi_u) \pmod{2\pi}. \quad (4.33)$$

Such a β_* clearly exists since $3\pi - 2m_0 a_0 > 2\pi$ (see (4.28)). It follows that

$$\begin{aligned} \text{dist}_{2\pi}(r_t + \alpha_t - \phi_t, \pi) &= \text{dist}_{2\pi}(r_t + \alpha_t - \phi_t, \beta_* + (\alpha_* - \phi_u)) \\ &\leq m_0 a_0 + \theta_0 + \frac{4\pi}{m_0} + \gamma_0 \\ &= \frac{\pi}{8} + \frac{\pi}{8} + \frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{2} \end{aligned}$$

whenever $t \in I \setminus E$ and $r_t \in [\beta_* - m_0 a_0, \beta_* + m_0 a_0]$ for μ_h -a.e. In particular,

$$\dot{r}_t = |\dot{h}_t| \cos(r_t + \alpha_t - \phi_t) \leq 0 \quad (4.34)$$

for all such t .

To reach a contradiction, let

$$\begin{aligned} t_+ &:= \inf\{t \in I : r_t = \beta_* + m_0 a_0\}, \\ t_- &:= \sup\{t \in I : t < t_+, r_t = \beta_* - m_0 a_0\}. \end{aligned} \quad (4.35)$$

Both $t_{\pm}$ are well-defined since

$$[\beta_* - m_0 a_0, \beta_* + m_0 a_0] \subseteq [a, a + 3\pi]$$

by the construction of β_* . According to the first relation of (4.31) along with the estimates (4.34) and (4.25), one obtains that

$$\begin{aligned} 2m_0 a_0 &= r_{t_+} - r_{t_-} = \int_{t_-}^{t_+} \dot{r}_s ds \\ &= \int_{E \cap [t_-, t_+]} \dot{r}_s ds + \int_{(I \setminus E) \cap [t_-, t_+]} \dot{r}_s ds \\ &\leq \int_{E \cap [t_-, t_+]} \dot{r}_s ds \leq \mu_h(E) \leq m_0 a_0. \end{aligned}$$

This is absurd. Therefore, $\mu_h(I) \geq \gamma_0 a$. $\square$

We are now in a position to prove the non-perturbative coercivity estimate (4.24).

Proof of Lemma 4.15. Let $R > 4$. Let $h \in \bar{\mathcal{H}}$ be such that $r_t := |X_t^h| \geq R$ for some $t \in [0, 1]$. We decompose the annulus $\{x : R/2 \leq |x| \leq R\}$ into disjoint concentric annuli

$$A_j := \{x : a_j \leq |x| \leq b_j\}, \quad a_j := \frac{R}{2} + 3\pi(j-1), \quad b_j := a_j + 3\pi.$$

There are $N_R = \frac{R}{6\pi}$ of them. Using first and last hitting, one can find a sequence of times

$$0 < s_1 < t_1 < \cdots < s_{N_R} < t_{N_R} < 1,$$

such that $r_{s_j} = a_j, r_{t_j} = b_j$ and $r_t \in [a_j, b_j]$ for every j . Let $I_j := [s_j, t_j]$. Denote

$$\mathcal{B} := \{j : I_j \text{ does not contain an } (a_0, \theta_0)\text{-effective turn of } h\}.$$

According to Lemma 4.17, one has

$$\mu_h(I_j) \geq \gamma_0 a_j \geq \frac{\gamma_0 R}{2}$$

for every $j \in \mathcal{B}$. It follows that

$$\|h\|_{1\text{-var}} \geq \sum_{j \in \mathcal{B}} \mu_h(I_j) \geq \frac{\gamma_0 R}{2} \cdot |\mathcal{B}|.$$

Therefore,

$$|\mathcal{B}| \leq \frac{2\|h\|_{1\text{-var}}}{\gamma_0 R}.$$

The number of j 's such that I_j contains an (a_0, θ_0) -effective turn is thus bounded below by

$$N_R - |\mathcal{B}| \geq \frac{R}{6\pi} - \frac{2\|h\|_{1\text{-var}}}{\gamma_0 R}.$$

The desired estimate (4.24) now follows from the definition of $\mathcal{N}_{\text{eff}}(h)$. $\square$

The perturbative estimate

In this part, we obtain a perturbative version of Lemma 4.15 which will easily lead to a proof of Proposition 4.11.

In what follows, we fix

$$p = (1/H)^+, \quad p' := \frac{p}{1-p}, \quad q = (p')^-, \quad \theta := \frac{p}{q}. \quad (4.36)$$

It is important to note that $p - 1 < \theta < 1$ and θ can be chosen such that

$$\theta(H + 1/2) < 1.$$

This is possible precisely when $H > 1/2$. Our main perturbative estimate for the number $\mathcal{N}_{\text{eff}}(h)$ of effective turns is stated as follows.

Lemma 4.18. *There exist positive constants C_1, C_2, R_0 depending only on p, θ, V , such that the estimate*

$$\mathcal{N}_{\text{eff}}(h) \geq C_1 R - C_2 \left[M^p + M^{1+\theta} + MR^{1-1/p} + \frac{\|h\|_{1\text{-var}}}{R} + M\|h\|_{1\text{-var}}^\theta \right] \quad (4.37)$$

holds uniformly over all $R > R_0$, all $h \in \bar{\mathcal{H}}$ and all $w \in C^{p\text{-var}}([0, 1]; \mathbb{R}^2)$ satisfying

$$\sup_{0 \leq t \leq 1} |X_t^{w+h}| \geq R$$

and $\|w\|_{p\text{-var}} \leq M$.

Proof of Proposition 4.11 assuming Lemma 4.18. Let $R > R_0$ and let $(w, h) \in C^{p\text{-var}}([0, 1]; \mathbb{R}^2) \times \bar{\mathcal{H}}$ be given such that

$$\sup_{0 \leq t \leq 1} |X_t^{w+h}| \geq R, \quad \|w\|_{p\text{-var}} \leq M.$$

If $\|h\|_{1\text{-var}} > R^{H+1/2}$, one can use the simple inequality (cf. (2.6))

$$\|h\|_{1\text{-var}} \leq C_H \|h\|_{\mathcal{H}}$$

to deduce that

$$\|h\|_{\mathcal{H}} \geq C_H^{-1} R^{H+1/2}.$$

Now consider the case when $\|h\|_{1\text{-var}} \leq R^{H+1/2}$. According to Lemma 4.14 and Lemma 4.18,

$$\begin{aligned} \|h\|_{\mathcal{H}} \geq C_H \Big[& C_1 R - C_2 \left(M^p + M^{1+\theta} + MR^{1-1/p} \right. \\ & \left. + R^{H-1/2} + MR^{\theta(H+1/2)} \right) \Big]^{H+1/2}. \end{aligned}$$

Since $\theta(H+1/2) < 1$, one has

$$C_2 \left(M^p + M^{1+\theta} + MR^{1-1/p} + R^{H-1/2} + MR^{\theta(H+1/2)} \right) \leq \frac{C_1 R}{2}$$

whenever $R > R_*$ for some constant R_* depending additionally on M . It follows that

$$\|h\|_{\mathcal{H}} \geq c_* R^{H+1/2}$$

with a suitable constant $c_* > 0$ for all $R > R_M$. □

Remark 4.19. It is clear from the above proof that the constant c_* does not depend on M .

The rest of this subsection is devoted to the proof of Lemma 4.18.

Given $I \subseteq [0, 1]$, we set

$$\omega_I := \|w\|_{p\text{-var}; I}, \quad l_I := \|h\|_{1\text{-var}; I}. \quad (4.38)$$

From time to time, we will just write $X_t := X_t^{w+h}$ to ease notation. The first tool we need is the following simple continuity estimate.

Lemma 4.20. *There exist positive constants C, δ depending only on p and V , such that the estimate*

$$\|X^{w+h}\|_{p\text{-var}; I} \leq C(\omega_I + l_I) \quad (4.39)$$

holds uniformly over all subintervals $I \subseteq [0, 1]$, all $h \in C^{1\text{-var}}([0, 1]; \mathbb{R}^2)$ and all $w \in C^{p\text{-var}}([0, 1]; \mathbb{R}^2)$ satisfying $\omega_I \leq \delta$.

Proof. Let $I = [u, v]$. One has

$$X_{s,t} = \int_s^t V(X_r) dw_r + \int_s^t V(X_r) dh_r$$

for all $s \leq t \in I$. The dh -integral is easily estimated as

$$\left\| \int_u^\cdot V(X_t) dh_t \right\|_{p\text{-var}; I} \leq \|V\|_\infty l_I.$$

For the dw -integral, one has from [17, Theorem 6.8] (Young's integral estimate) that

$$\left\| \int_u^\cdot V(X_r) dw_r \right\|_{p\text{-var}; I} \leq C_p(\|V\|_\infty + \|DV\|_\infty \omega_I) \omega_I.$$

Therefore,

$$\begin{aligned} \|X\|_{p\text{-var}; I} &\leq \|V\|_\infty l_I + C_p(\|V\|_\infty + \|DV\|_\infty \|X\|_{p\text{-var}; I}) \omega_I \\ &\leq C_{1;p,V}(l_I + \omega_I) + C_{2;p,V} \omega_I \|X\|_{p\text{-var}; I}. \end{aligned} \quad (4.40)$$

Choose δ to be such that

$$C_{2;p,V} \delta \leq \frac{1}{2}.$$

It follows from (4.40) that

$$\|X\|_{p\text{-var}; I} \leq C_{1;p,V}(l_I + \omega_I) + \frac{1}{2} \|X\|_{p\text{-var}; I} \iff \|X\|_{p\text{-var}; I} \leq 2C_{1;p,V}(l_I + \omega_I),$$

provided that $\omega_I \leq \delta$. This yields the desired estimate (4.39). $\square$

The next result provides the main Young perturbative estimates needed for the proof of Lemma 4.18.

Lemma 4.21. *Let p, q, θ be given exponents satisfying (4.36). Let F be a uniformly bounded, Lipschitz function with Lipschitz constant L_F . Then there exists a positive constant C depending only on p, θ, V , such that the following estimate*

$$\sup_{J \subseteq I} \left| \int_J F(X_t^{w+h}) dw_t \right| \leq C(\|F\|_\infty + \|F\|_\infty^{1-\theta} L_F^\theta (\omega_I + l_I)^\theta) \omega_I, \quad (4.41)$$

holds uniformly over all subintervals $I \subseteq [0, 1]$, all $h \in C^{1\text{-var}}([0, 1]; \mathbb{R}^2)$ and all $w \in C^{p\text{-var}}([0, 1]; \mathbb{R}^2)$ satisfying $\omega_I \leq \delta$, where δ is the constant appearing in Lemma 4.20.

Proof. According to [17, Theorem 6.8], one has

$$\left| \int_J F(X_t) dw_t \right| \leq C_{p,q}(\|F\|_\infty + \|F(X)\|_{q\text{-var}; I}) \omega_I \quad (4.42)$$

for all $J \subseteq I$, where $q \in (p, p')$ is given fixed (see (4.36)). On the other hand, by using the simple estimate

$$\begin{aligned} |F(X_t) - F(X_s)|^q &= |F(X_t) - F(X_s)|^{q-p} |F(X_t) - F(X_s)|^p \\ &\leq (2\|F\|_\infty)^{q-p} (L_F |X_t - X_s|)^p \end{aligned}$$

where L_F denotes the Lipschitz constant for F , one finds that

$$\|F(X)\|_{q\text{-var};I} \leq (2\|F\|_\infty)^{1-\theta} L_F^\theta \|X\|_{p\text{-var};I}^\theta.$$

It follows from Lemma 4.20 that

$$\|F(X)\|_{q\text{-var};I} \leq C_{p,V} (2\|F\|_\infty)^{1-\theta} L_F^\theta (\omega_I + l_I)^\theta, \quad (4.43)$$

provided that $\omega_I \leq \delta$. By substituting (4.43) into (4.42), one obtains the desired estimate (4.41). $\square$

To proceed further, one needs to examine the radial and angular dynamics of X_t^{w+h} . For each $x = re^{i\phi} \in \mathbb{R}^2$, we define

$$e_1(x) := \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}, \quad e_2(x) := \begin{pmatrix} -\sin \phi \\ \cos \phi \end{pmatrix}$$

and set

$$A(x) := e_1(x)^T \cdot V(x), \quad B(x) := e_2(x)^T \cdot V(x).$$

As a convention, we take $\phi = 0$ if $r = 0$. In other words, $A(x)$ (respectively, $B(x)$) is the coordinates of $e_1(x)$ (respectively, $e_2(x)$) with respect to the orthonormal frame $V(x)$.

Given a control pair (w, h) with $w \in C^{p\text{-var}}([0, 1]; \mathbb{R}^2)$ and $h \in \bar{\mathcal{H}}$, let $r_t := X_t^{w+h}$. Suppose that $r_t \geq 2$ for all $t \in I = [u, v]$. Similar to the equations (4.29) and (4.30), one can write

$$X_t^{w+h} = r_t e^{i\phi_t}, \quad \dot{h}_t = |\dot{h}_t| e^{i\alpha_t},$$

where (r_t, ϕ_t) satisfies the controlled differential equations

$$dr_t = A(X_t)dw_t + \cos(r_t + \alpha_t - \phi_t)|\dot{h}_t|dt, \quad (4.44)$$

$$d\phi_t = \frac{1}{r_t} (B(X_t)dw_t + \sin(r_t + \alpha_t - \phi_t)|\dot{h}_t|dt). \quad (4.45)$$

Corollary 4.22. *Define*

$$\Xi_I := \omega_I (1 + (\omega_I + l_I)^\theta)$$

for each $I = [u, v] \subseteq [0, 1]$. There exists a positive constant K depending only on p, θ, V , such that the following two estimates

$$\sup_{J \subseteq I} \left| \int_J A(X_t^{w+h})dw_t \right| \leq K\Xi_I,$$

$$\sup_{J \subseteq I} \left| \int_J \frac{1}{r_t} B(X_t^{w+h})dw_t \right| \leq \frac{K}{a}\Xi_I$$

hold uniformly over all $I \subseteq [0, 1]$, $a > 2$, $h \in \bar{\mathcal{H}}$ and $w \in C^{p\text{-var}}([0, 1]; \mathbb{R}^2)$ satisfying $\omega_I \leq \delta$ and $r_t \geq a$ for all $t \in I$.

Proof. Take $F(x) = A(x)$ or $B(x)/|x|$ in Lemma 4.21 (in the latter case, one shall also apply a smooth cut-off near the origin which is harmless since the estimate only concerns with the region $|x| \geq 2$). Then F is uniformly bounded and Lipschitz on $\{x : |x| \geq 2\}$. $\square$

Now we establish the perturbative version of Lemma 4.17. We will continue to use the same definitions, notation and exact values of parameters (e.g. $a_0, \theta_0, m_0, \gamma_0$ satisfying (4.28)) as in Section 4.2.1. Recall that $\delta = \delta_{p,V}$ is the constant appearing in Lemma 4.20.

Lemma 4.23. *There exist positive constants A_0, η_0 depending only on p, θ, V , such that the following statement holds. Given any control pair (w, h) , any $I = [u, v] \subseteq [0, 1]$, any $a > 2$, whenever (w, h, X^{w+h}) satisfies*

$$r_u = a, r_v = a + 3\pi, \quad r_t \in [a, a + 3\pi] \text{ for all } t \in I, \quad (4.46)$$

$$a \geq A_0, \omega_I \leq \delta, \Xi_I \leq \eta_0 \quad (4.47)$$

and

$$\mu_h(I) < \frac{\gamma_0 a}{2}, \quad (4.48)$$

the interval I must contain an (a_0, θ_0) -effective turn of h .

Proof. By choosing η_0 small, one can always ensure that there is at least a heavy arc with respect to $h|_I$. Indeed, since $\|A\|_\infty = 1$, one finds from the equation (4.44) of r_t that

$$3\pi = r_v - r_u \leq K\Xi_I + l_I \leq K\eta_0 + l_I,$$

provided that $\Xi_I \leq \eta_0$. Suppose that

$$\eta_0 < \frac{1}{K}(3\pi - m_0 a_0) = \frac{23\pi}{8K}. \quad (4.49)$$

Then one has

$$l_I \geq 3\pi - K\eta_0 > m_0 a_0.$$

This implies by definition that there must be at least one heavy arc (otherwise, $l_I \leq m_0 a_0$).

Now suppose that the conditions (4.46, 4.47, 4.48) are all satisfied while I does not contain an (a_0, θ_0) -effective turn of h . We will show that this leads to a contradiction if A_0 and η_0 are chosen properly. First of all, by using the given conditions, the ϕ -equation (4.45) and Corollary 4.22, one finds that

$$\sup_{s, t \in I} |\phi_t - \phi_s| \leq \frac{K}{a}\Xi_I + \frac{1}{a}\mu_h(I) \leq \frac{K\eta_0}{A_0} + \frac{\gamma_0}{2}.$$

By requiring that

$$\frac{K\eta_0}{A_0} < \frac{\gamma_0}{2}, \quad (4.50)$$

one has

$$\sup_{s, t \in I} |\phi_t - \phi_s| < \gamma_0.$$

In addition, according to Lemma 4.16, there exist a measurable subset $E \subseteq I$ and a unit vector $v_* = e^{i\alpha_*}$ such that (4.25) and (4.26) holds. One can then carry the argument of Lemma 4.17 word by word to show that

$$\text{dist}_{2\pi}(r_t + \alpha_t - \phi_t, \pi) \leq \frac{\pi}{2} \quad (4.51)$$

for μ_h -a.e. $t \in I \setminus E$ such that $\dot{h}_t \neq 0$ and $r_t \in [\beta_* - m_0 a_0, \beta_* + m_0 a_0]$, where β_* is defined according to the relation (4.33). Define $t_{\pm}$ as in (4.35). It follows from the r -equation (4.44), Corollary 4.22, the estimates (4.25) and (4.51) that

$$\begin{aligned} 2m_0 a_0 &= r_{t_+} - r_{t_-} = \int_{t_-}^{t_+} dr_s \\ &= \int_{t_-}^{t_+} A(X_s) dw_s + \int_{t_-}^{t_+} \cos(r_s + \alpha_s - \phi_s) |\dot{h}_s| ds \\ &\leq K\Xi_I + \mu_h(E) \leq K\eta_0 + m_0 a_0. \end{aligned}$$

By requiring η_0 to satisfy

$$K\eta_0 < m_0 a_0, \quad (4.52)$$

one reaches a contradiction.

To summarise, one first chooses η_0 that satisfies (4.49, 4.52) and then chooses A_0 to satisfy (4.50). With such choices, for any $(I, a; w, h)$ satisfying the conditions (4.46, 4.47, 4.48), the interval I must contain an (a_0, θ_0) -effective turn of h . This completes the proof of the lemma. $\square$

Finally, we are in a position to prove Lemma 4.18.

Proof of Lemma 4.18. We continue to use the same notation as in the proof of Lemma 4.15 (the annuli decomposition $[a_j, b_j]$ and radial crossing $I_j = [s_j, t_j]$). Let $R > 2A_0$ (A_0 is the constant appearing in Lemma 4.23 so that $a_j > A_0$ for every j). Our goal is to estimate the number of I_j 's which do not contain an (a_0, θ_0) -effective turn of h . According to Lemma 4.23, at least one of the constraints in (4.47, 4.48) fails for such I_j 's. We discuss the possibilities as follows.

(i) $\omega_{I_j} > \delta$.

Since $\|w\|_{p\text{-var}} \leq M$, one has

$$\delta^p \cdot |\{j : \omega_{I_j} > \delta\}| \leq \sum_{j=1}^{N_R} \omega_{I_j}^p \leq M^p.$$

As a result,

$$|\{j : \omega_{I_j} > \delta\}| \leq \delta^{-p} M^p. \quad (4.53)$$

(ii) $l_{I_j} > \gamma_0 a_j / 2$.

Since $a_j \geq R/2$, one has

$$\frac{\gamma_0 R}{4} \cdot |\{j : l_{I_j} > \gamma_0 a_j/2\}| \leq \sum_{j=1}^{N_R} l_{I_j} \leq \|h\|_{1\text{-var}}.$$

Therefore,

$$|\{j : l_{I_j} > \gamma_0 a_j/2\}| \leq \frac{4\|h\|_{1\text{-var}}}{\gamma_0 R}. \quad (4.54)$$

(iii) $\Xi_{I_j} > \eta_0$.

By definition, one has

$$\sum_{j=1}^{N_R} \Xi_{I_j} \leq \sum_{j=1}^{N_R} \omega_{I_j} + \sum_{j=1}^{N_R} \omega_{I_j} l_{I_j}^\theta + \sum_{j=1}^{N_R} \omega_{I_j}^{1+\theta}. \quad (4.55)$$

The first sum in (4.55) is estimated as

$$\sum_{j=1}^{N_R} \omega_{I_j} \leq N_R^{1-1/p} \left(\sum_{j=1}^{N_R} \omega_{I_j}^p \right)^{1/p} \leq M N_R^{1-1/p}.$$

For the second sum in (4.55), one can first use Hölder's inequality to obtain that

$$\sum_{j=1}^{N_R} \omega_{I_j} l_{I_j}^\theta \leq M \left(\sum_{j=1}^{N_R} l_{I_j}^{\theta p'} \right)^{1/p'}.$$

Next, observe that $\theta p' > 1$. Since the l^p -norm is nonincreasing in p , one has

$$\left(\sum_{j=1}^{N_R} l_{I_j}^{\theta p'} \right)^{1/p'} \leq \left(\sum_{j=1}^{N_R} l_{I_j} \right)^\theta \leq \|h\|_{1\text{-var}}^\theta.$$

As a consequence,

$$\sum_{j=1}^{N_R} \omega_{I_j} l_{I_j}^\theta \leq M \|h\|_{1\text{-var}}^\theta.$$

The last sum in (4.55) again easily estimated by monotonicity of l^p -norm; since $1 + \theta > p$, one has

$$\sum_{j=1}^{N_R} \omega_{I_j}^{1+\theta} \leq \left(\sum_{j=1}^{N_R} \omega_{I_j}^p \right)^{\frac{1+\theta}{p}} \leq M^{1+\theta}.$$

To summarise, one concludes that

$$\sum_{j=1}^{N_R} \Xi_{I_j} \leq M \left(\frac{R}{6\pi} \right)^{1-1/p} + M \|h\|_{1\text{-var}}^\theta + M^{1+\theta}.$$

It follows that

$$|\{j : \Xi_{I_j} > \eta_0\}| \leq \eta_0^{-1} \left[M \left(\frac{R}{6\pi} \right)^{1-1/p} + M \|h\|_{1\text{-var}}^\theta + M^{1+\theta} \right]. \quad (4.56)$$

The number of j 's such that I_j does not contain an (a_0, θ_0) -effective turn is therefore bounded above by the sum of the RHS of (4.53, 4.54, 4.56). Since $N_R = \frac{R}{6\pi}$, this immediately yields a lower bound of $\mathcal{N}_{\text{eff}}(h)$ in the form of (4.37). The proof of Lemma 4.18 is now complete. $\square$

4.2.2 The $(2H+1)$ -Weibull upper estimate

We now proceed to prove Theorem 4.10 by using the coercivity estimate given by Proposition 4.11. The argument is a simple application of Borell's isoperimetric inequality.

Proof of Theorem 4.10. Let $(\mathcal{W}, \bar{\mathcal{H}}, \mu)$ be the abstract Wiener space associated with the 2D fBM (in particular, μ is the law of fBM). Fix $p \in (1/H)^+$ as before and fix $M > 0$ (its precise value is of no importance). Define

$$G_M := \{w \in \mathcal{W} : \|w\|_{p\text{-var}} \leq M\}.$$

For each $R > 0$, we also set

$$A_R := \left\{ w \in \mathcal{W} : \sup_{0 \leq t \leq 1} |X_t^w| > R \right\}$$

According to Proposition 4.11, there exist positive constants c_0, R_0 , such that

$$G_M + c_0 R^{H+1/2} \mathcal{K} \subseteq A_R^c$$

for all $R > R_0$, where $\mathcal{K}$ denotes the closed unit ball of $\bar{\mathcal{H}}$. Let

$$\kappa := \Phi^{-1}(\mu(G_M)) \in \mathbb{R},$$

where $\Phi(u) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du$. It follows from Borell's isoperimetric inequality (see [23, Theorem 4.3]) that

$$\begin{aligned} \mu(A_R) &= 1 - \mu(A_R^c) \leq 1 - \mu(G_M + c_0 R^{H+1/2} \mathcal{K}) \\ &\leq 1 - \Phi(\kappa + c_0 R^{H+1/2}) \leq C_1 e^{-C_2 R^{2H+1}} \end{aligned}$$

for all $R > R_0$ with suitable constants $C_1, C_2 > 0$. This completes the proof of Theorem 4.10. $\square$

5 General lower tail estimates in Young's regime

In this section, we consider the SDE (1.1) with $H \in (1/2, 1)$. Our aim is to provide a simple sufficient condition on the vector fields, under which the solution admits a Gaussian lower tail. The main result is Theorem 5.2. As a quite straightforward application of the technique developed in the proof, we will also establish a general $(2H + 1)$ -Weibull lower tail for uniformly elliptic systems. This is the content of Theorem 5.3.

5.1 A sufficient condition for Gaussian lower tail in Young's regime

Let X_t be the solution to the SDE (1.1) with $H \in (1/2, 1)$. Our main sufficient condition on the vector fields to ensure Gaussian lower tail is formulated as follows.

(H) There exists a function $F \in C^2(\mathbb{R}^N)$ with uniformly bounded first two derivatives and a unit vector $\eta \in S^{d-1}$, such that

$$m := \inf_{x \in \mathbb{R}^N} (W_\eta F)(x) > 0 \quad (5.1)$$

where $W_\eta(x) := V(x)\eta := \sum_{i=1}^d V_i(x)\eta^i$.

Remark 5.1. Geometrically, Condition (H) asserts that level sets of the function F are uniformly transversal to integral curves of the vector field W_η .

The main result of this subsection is now stated as follows.

Theorem 5.2. *Let $H \in (1/2, 1)$ and $V_0, V_1, \dots, V_d \in C_b^2$. Suppose that Condition (H) holds. Then there exist positive constants C_1, C_2 depending only on H and the vector fields, such that the estimate*

$$\mathbb{P}(|X_1 - x_0| > R) \geq C_1 e^{-C_2 R^2}$$

holds for all large R .

Proof. As usual, we use X_t^w to denote the solution to the same equation (1.1) with B_t replaced by w_t . Given $R > 0$, we set $h_t^R := Rt\eta$ ($0 \leq t \leq 1$) where η is the unit vector given by Condition (H). This particular Cameron-Martin path will play a critical role in the argument. Let F be the function given by Condition (H).

(i) The key claim is that there exist positive constants M, C_0, R_0 depending only on p, H, V, F , such that

$$\|w\|_{p\text{-var}} \leq M \implies |X_1^{w+h^R} - x| \geq C_0 R \quad (5.2)$$

for all $R > R_0$.

To prove this claim, let us write $X_t := X_t^{w+h^R}$ and $V = (V_1, \dots, V_d)$ for simplicity. Note that X_t satisfies the following equation:

$$dX_t = RW_\eta(X_t)dt + V(X_t)dw_t + V_0(X_t)dt.$$

Let F be the function given by Condition (H). By applying the chain rule to $F(X_t)$, one obtains that

$$\begin{aligned} F(X_1) - F(x_0) &= R \int_0^1 (W_\eta F)(X_t)dt + \int_0^1 (\nabla F(X_t))^T V(X_t)dw_t \\ &\quad + \int_0^1 (\nabla F(X_t))^T V_0(X_t)dt =: A + B + C. \end{aligned}$$

According to Condition (H), one has $A \geq mR$. It is also obvious that $|C| \leq \|DF\|_\infty \|V_0\|_\infty$.

To estimate B , let us set

$$x_I := \|X\|_{p\text{-var}; I}, \quad \omega_I := \|\omega\|_{p\text{-var}; I}$$

for any given interval $I \subseteq [0, 1]$. By using [17, Theorem 6.8], one easily finds that

$$x_I \leq (R\|V\|_\infty + \|V_0\|_\infty)|I| + C_p(\|V\|_\infty + \|DV\|_\infty x_I)\omega_I.$$

Let M be chosen such that

$$C_p\|DV\|_\infty M < \frac{1}{2}.$$

Whenever $\|w\|_{p\text{-var}} \leq M$, one has

$$x_I \leq 2(R\|V\|_\infty + \|V_0\|_\infty)|I| + 2C_p\|V\|_\infty M. \quad (5.3)$$

Now we divide the interval $[0, 1]$ into equal subintervals $I_1, \dots, I_N$ ($N \asymp R$) each having length $1/R$. According to (5.3), on each subinterval I_j one has

$$x_{I_j} \leq C_{p,V} \quad (5.4)$$

with some constant $C_{p,V}$ depending on p, V . It follows again from [17, Theorem 6.8] and (5.4) that

$$\begin{aligned} |B| &\leq \sum_{j=1}^N \left| \int_{I_j} (\nabla F(X_t))^T V(X_t)dw_t \right| \\ &\leq C_p(\|(\nabla F)^T \cdot V\|_\infty + C_{p,V}\|D((\nabla F)^T \cdot V)\|_\infty) \sum_{j=1}^N \omega_{I_j} \\ &\leq C'_{p,V,F} R^{1-1/p} \left(\sum_{j=1}^N \omega_{I_j}^p \right)^{1/p} \\ &\leq C'_{p,V,F} M R^{1-1/p}. \end{aligned}$$

As a consequence,

$$\begin{aligned} \|\nabla F\|_\infty |X_1 - x_0| &\geq F(X_1) - F(x_0) \\ &\geq mR - C'_{p,V,F} M R^{1-1/p} - \|DF\|_\infty \|V_0\|_\infty. \end{aligned}$$

Let R_0 be chosen such that the RHS is larger than $mR/2$ for all $R > R_0$. The claim (5.2) thus follows with $C_0 := \frac{m}{2\|\nabla F\|_\infty}$.

(ii) The claim (5.2) shows that

$$G_M := \{w : \|w\|_{p\text{-var}} \leq M\} \subseteq \left\{ |X_1^{w+h^R} - x_0| \geq C_0 R \right\}$$

for all $R > R_0$. As a consequence,

$$\mathbb{P}(|X_1^B - x_0| \geq C_0 R) \geq \mu(h^R + G_M),$$

where μ is the law of fBM on path space. By exactly the same Cameron-Martin transformation argument as in the proof of Theorem 3.4, one finds that

$$\mu(h^R + G_M) = e^{-\|h^R\|_{\mathcal{H}}^2/2} \mathbb{E}[e^{-\int_0^1 \langle \theta_t^R, dW_t \rangle} \mathbf{1}_{G_M}(B)].$$

Here $\theta^R := K_H^{-1} h^R \in L^2([0, 1]; \mathbb{R}^d)$ (K_H is the isomorphism defined by (2.3)) and W is the Wiener process in the fBM representation (2.5). By the definition of h^R , it is obvious that

$$\|h^R\|_{\mathcal{H}} = R \| [t \mapsto t\eta] \|_{\mathcal{H}} =: \kappa R. \quad (5.5)$$

In addition, due to the reflection symmetry of the set G_M , one has

$$\mathbb{E}[e^{-\int_0^1 \langle \theta_t^R, dW_t \rangle} \mathbf{1}_{G_M}(B)] = \mathbb{E}[e^{\int_0^1 \langle \theta_t^R, dW_t \rangle} \mathbf{1}_{G_M}(B)].$$

In particular,

$$\begin{aligned} \mathbb{E}[e^{-\int_0^1 \langle \theta_t^R, dW_t \rangle} \mathbf{1}_{G_M}(B)] &= \mathbb{E}\left[\cosh\left(\int_0^1 \langle \theta_t^R, dW_t \rangle\right) \mathbf{1}_{G_M}(B)\right] \\ &\geq \mu(G_M) := \delta. \end{aligned}$$

As a result, one concludes that

$$\mathbb{P}(|X_1^B - x_0| \geq C_0 R) \geq \delta e^{-\kappa^2 R^2/2}$$

for all $R > R_0$.

Now the proof of Theorem 5.2 is complete. $\square$

5.2 General $(2H + 1)$ -Weibull lower tail for uniformly elliptic systems

We now prove a general $(2H + 1)$ -Weibull lower tail estimate under the uniform ellipticity assumption. Parallel to the rough regime in Section 3.4, this part is also a simple adaptation of the proof of Theorem 5.2.

Theorem 5.3. *Consider the SDE (1.1) with $H \in (1/2, 1)$. Suppose that the vector fields are C_b^∞ and uniformly elliptic. Then there exist positive constants C_1, C_2 depending only on H and the vector fields, such that the estimate*

$$\mathbb{P}(|X_1 - x_0| > R) \geq C_1 e^{-C_2 R^{2H+1}} \quad (5.6)$$

holds for all large R .

Proof. The argument is a simple adaptation of the proof of Theorem 5.2. It is more convenient to use Hölder norms here. Fix a unit vector $\eta \in \mathbb{R}^{N-1}$ and $\rho \in (1/2, H)$. Given $R > 1$, let X_t^R denote the solution to the differential equation

$$dX_t^R = R\eta dt + V(X_t^R)dB_t + V_0(X_t^R)dt, \quad X_0^R = x_0.$$

Similar to Part (i) of the proof of Theorem 5.2 with p -variation replaced by ρ -Hölder norm (cf. Lemma 3.6), there exist positive constants M, C_1, C_2, R_0 , such that on the event

$$A_M := \{\|B\|_{\rho\text{-Hölder};[0,1]} \leq M\},$$

the estimates

$$|X_t^R - X_s^R| \leq C_1(R|t-s| + M|t-s|^\rho + |t-s|) \quad (5.7)$$

and

$$|X_1^{R,w} - x_0| \geq C_2 R \quad (5.8)$$

hold for all $R > R_0$ and $s, t \in [0, 1]$.

Now let us define the (random) Cameron-Martin path

$$h_t^R := R \int_0^t a(X_s^R) ds,$$

where $a(x) := A(x)\eta$ and $A(x)$ is the right inverse of $V(x)$ defined by (3.22). Note that h^R is $\mathcal{F}^W$ -adapted (cf. (2.5)). According to the norm-equivalence relation (2.6), one has

$$\begin{aligned} \|h^R\|_{\mathcal{H}}^2 &\leq C_H \left(\|\dot{h}^R\|_{L^2}^2 + \int_0^1 \int_0^1 \frac{|\dot{h}_t^R - \dot{h}_s^R|^2}{|t-s|^{2H}} ds dt \right) \\ &= C_H R^2 \left(\|u^R\|_{L^2}^2 + \int_0^1 \int_0^1 \frac{|\dot{u}_t^R - \dot{u}_s^R|^2}{|t-s|^{2H}} ds dt \right) \end{aligned}$$

where $u_t^R := a(X_t^R)$. The L^2 -norm in the first term is uniformly bounded by $\|A\|_\infty$. For the second term, by using (5.7) one easily finds that

$$|u_t^R - u_s^R| \leq C_3(\min\{1, R|t-s|\} + M|t-s|^\rho)$$

for all $R > R_0$ and $s, t \in [0, 1]$. It follows from explicit analysis that

$$\begin{aligned} & \int_0^1 \int_0^1 \frac{|\dot{u}_t^R - \dot{u}_s^R|^2}{|t - s|^{2H}} ds dt \\ & \leq C_3 \int_0^1 \int_0^1 \left(\frac{\min\{1, R^2|t - s|^2\}}{|t - s|^{2H}} + M|t - s|^{2\rho - 2H} \right) ds dt \\ & \leq C_4 R^{2H-1} \end{aligned}$$

on the event A_M . As a consequence,

$$\|h^R\|_{\mathcal{H}}^2 \leq C_H R^2 (\|A\|_\infty^2 + C_4 R^{2H-1}) \leq C_5 R^{2H+1} \quad (5.9)$$

on A_M .

Let

$$\tau := \inf \{t \geq 0 : \|B\|_{\rho\text{-H\"ol};[0,t]} > M\} \wedge 1$$

and define

$$\hat{h}_t^R := R \int_0^{t \wedge \tau} a(X_s^R) ds.$$

Note that $\hat{h}^R$ is also $\mathcal{F}^W$ -adapted and $\hat{h}^R = h^R$ on A_M . From the above analysis, it is not hard to see that with probability one,

$$\|\dot{\hat{h}}^R\|_{\mathcal{H}} \leq C_6 R^{2H+1} \quad (5.10)$$

for all R . With the global Cameron-Martin estimate (5.10), one can now carry through the Cameron-Martin transformation argument as in the proof of Theorem 5.2 (cf. the proof of Theorem 3.4) to conclude the desired $(2H+1)$ -Weibull lower bound (5.6). $\square$

6 2D periodic elliptic systems always have Gaussian lower tail

In this section, we prove that a 2×2 periodic, uniformly elliptic system always possesses a Gaussian lower tail, which is a striking result in contrast to the radially periodic counterexample of Theorem 4.10.

Theorem 6.1. *Let V_1, V_2 be given smooth vector fields on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Suppose that they are uniformly elliptic, namely,*

$$\dim \text{Span}\{V_1(x), V_2(x)\} = 2$$

for every $x \in \mathbb{T}^2$. Consider the SDE

$$dX_t = \tilde{V}_1(X_t)dB_t^1 + \tilde{V}_2(X_t)dB_t^2, \quad X_0 = x_0 \in \mathbb{R}^2 \quad (6.1)$$

on the universal cover $\mathbb{R}^2$. Here $\tilde{V}_i$ is the canonical periodic lifting of V_i on $\mathbb{R}^2$ and $B_t = (B_t^1, B_t^2)$ is an fBM with Hurst parameter $H \in (1/2, 1)$. Then there exist positive constants C_1, C_2 , such that

$$\mathbb{P}(|X_1 - x_0| > R) \geq C_1 e^{-C_2 R^2} \quad (6.2)$$

for all large R .

6.1 Idea of proof

From the proof of Theorem 5.2, the main mechanism leading to a Gaussian lower tail is the existence of a Cameron-Martin control h^R for each R , such that the following two properties hold true:

- (i) $\|h^R\|_{\mathcal{H}} = O(R)$;
- (ii) Whenever $\|w\|_{p\text{-var}} \leq M$, one has

$$|X_1^{w+h^R} - x_0| \geq C_M \cdot R \quad (6.3)$$

for all large R . In other words, there always exists a control with Cameron-Martin norm $O(R)$ which creates a ballistic tunnel for the dynamics to travel a linear distance that is robust under perturbation.

Our main strategy for establishing the aforementioned mechanism relies on the theory of dynamical systems, in particular, a fundamental classification theorem for torus flows (see Theorem 6.7 below). For each $u \in S^1$, the nonvanishing vector field $W_u := V \cdot u$ generates a 1D foliation $\mathcal{F}_u$ of $\mathbb{T}^2$ (the family of integral curves of W_u). The classification theorem asserts that there are precisely two possibilities for $\mathcal{F}_u$.

Case I: $\mathcal{F}_u$ is conjugate to a suspension (see Definition 6.5 and Definition 6.6). In this case, one can easily construct a closed 1-form η on $\mathbb{T}^2$ with nontrivial cohomology, whose integral along the lifted flow of W_u on the universal cover

$\mathbb{R}^2$ produces global linear displacements. In this regime, one is precisely in the setting of Condition (H) in Theorem 5.2.

Case II: $\mathcal{F}_u$ contains a Reeb component (see Definition 6.4). In this case, W_u must have a periodic orbit with nontrivial homology. The lifting of this orbit to the universal cover $\mathbb{R}^2$ naturally provides a ballistic trajectory for the deterministic dynamics controlled by $h_t^R := Rtu$. However, a major difficulty in this regime is that such a ballistic trajectory may not necessarily create a ballistic tunnel for the stochastic dynamics perturbed by fractional Brownian paths. It is necessary to establish perturbative estimates like (6.3) in order to turn the deterministic Cameron-Martin estimate into a probabilistic Gaussian lower bound (cf. Section 5). For this purpose, one needs some dynamical stability of the periodic orbit.

To overcome the aforementioned difficulty, a key insight is that one can choose another unit vector u^* that is close to u , such that the flow lines of W_{u^*} contain a *hyperbolic* periodic orbit (see Definition 6.10). Hyperbolicity will enforce an attraction property for such an orbit, which provides the exact level of stability that enables one to establish a robust perturbative estimate.

The rest of this section is devoted to the development of the above strategy in a precise mathematical way.

6.2 A classification theorem for 1D foliations on $\mathbb{T}^2$

In this subsection, we recall a fundamental classification theorem for nondegenerate torus flows. We begin by recalling the definition of a foliation (see [24]).

Definition 6.2. Let M be an n -dimensional differentiable manifold. Let $\mathcal{F}$ be a collection of immersed k -dimensional submanifolds. A chart (U, φ) is said to be *flat* for $\mathcal{F}$ if the following two properties hold:

- (i) $\varphi(U)$ is a cube in $\mathbb{R}^n$;
- (ii) for each submanifold $N \in \mathcal{F}$ with nonempty intersection with U , the set $\varphi(N \cap U)$ is a countable union of k -dimensional slices of the form

$$x^{k+1} = c^{k+1}, \dots, x^n = c^n$$

in $\varphi(U)$.

A *k -dimensional (smooth) foliation* on M is a collection $\mathcal{F}$ of disjoint, connected, immersed k -dimensional submanifolds such that the union of members of $\mathcal{F}$ is M and for every $x \in M$, there exists a chart around x which is flat for $\mathcal{F}$. Members of a foliation $\mathcal{F}$ are called *leaves* of $\mathcal{F}$.

Example 6.3. Let M be a closed surface. Let W be a smooth vector field on M which is everywhere nonvanishing. According to the global Frobenius theorem, the integral curves of W form a one-dimensional (1D) foliation on M (see [24, Theorem 19.21]). This is the main example that is relevant to us.

From now on, we restrict ourselves to the case $M = \mathbb{T}^2$. We need two more concepts in order to state the classification theorem of 1D foliations on $\mathbb{T}^2$. The first one is the notion of a Reeb component (see [9, Definition 2.4] and [19, Chapter 1, Section 3.3]).

Definition 6.4. Let $\mathcal{F}$ be a 1D foliation on $\mathbb{T}^2$. We say that $\mathcal{F}$ *has a Reeb component* if there exists an embedded annulus $A \cong S^1 \times [0, 1]$ inside $\mathbb{T}^2$ such that:

- (i) ∂A is the union of two compact leaves of $\mathcal{F}$ that are in opposite homotopy classes;
- (ii) there are no compact leaves of $\mathcal{F}$ in the interior of A .

The next one is the notion of a suspension (see [19, Chapter 1, Section 3.1]). Let $f : S^1 \rightarrow S^1$ be an orientation-preserving diffeomorphism. The integer group $\mathbb{Z}$ acts on $S^1 \times \mathbb{R}$ by

$$k \cdot (x, t) := (f^k(x), t - k), \quad k \in \mathbb{Z}, (x, t) \in S^1 \times \mathbb{R}. \quad (6.4)$$

Let $\Sigma_f := (S^1 \times \mathbb{R}) / \mathbb{Z}$ be the quotient space with canonical projection map $\pi_f : S^1 \times \mathbb{R} \rightarrow \Sigma_f$. It is clear that Σ_f is diffeomorphic to the torus $\mathbb{T}^2$. The image of the vertical leaves $\{x\} \times \mathbb{R}$ under π_f , i.e.

$$\mathcal{F}_f := \{\pi_f(\{x\} \times \mathbb{R}) : x \in S^1\},$$

defines a 1D foliation on Σ_f .

Definition 6.5. The foliation $(\Sigma_f, \mathcal{F}_f)$ is called the *suspension* of the diffeomorphism f .

We also need the concept of conjugated foliations (see [9, Definition 2.3] and [19, Chapter 1, Section 2.2]).

Definition 6.6. Let $\mathcal{F}, \mathcal{F}'$ be k -dimensional foliations on n -dimensional manifolds M, M' respectively. We say that $\mathcal{F}$ and $\mathcal{F}'$ are *conjugate* (or *isomorphic*) if there exists a diffeomorphism $\Psi : M \rightarrow M'$ such that $\Psi(\mathcal{F}) = \mathcal{F}'$.

We now state the main classification theorem for 1D foliations that will be used in our study.

Theorem 6.7. *Let $\mathcal{F}$ be a 1D foliation on $\mathbb{T}^2$. Then either $\mathcal{F}$ has a Reeb component or it is conjugate to the suspension of an orientation-preserving diffeomorphism $f : S^1 \rightarrow S^1$.*

Theorem 6.7 was stated and proved by Hector-Hirsch [19, Proposition 4.3] under the homeomorphism setting. The smooth version was stated by Bonatti-Zhang [9, Theorem 2.6] but the actual theorem the authors referred to was the continuous version of Hector-Hirsch. For completeness, in the appendix we provide a self-contained proof of Theorem 6.7 under the diffeomorphism setting.

6.3 Transverse characteristic number of the Poincaré map

In this subsection, we recall the definition of a hyperbolic periodic orbit, which will play an essential role in our later analysis for the Reeb component regime. To this end, we first need to recall the concepts of a Poincaré map and its transverse characteristic number.

Let W be a smooth, everywhere nonvanishing vector field on $\mathbb{T}^2$. Let

$$\{\phi_t(x) : x \in \mathbb{T}^2, t \in \mathbb{R}\}$$

be its flow of diffeomorphisms.

Definition 6.8. For each $x \in \mathbb{T}^2$, the set $\{\phi_t(x) : t \in \mathbb{R}\}$ is called the *orbit* of W through x . An orbit γ is said to be *periodic* if there exist $T > 0$ and $x \in \gamma$ (equivalently, for all $x \in \gamma$) such that $\phi_T(x) = x$. The smallest positive number satisfying the above relation is called the *period* of γ . A periodic orbit is *noncontractible* if its homology class in $H_1(\mathbb{T}^2; \mathbb{Z})$ is nonzero.

Note that a periodic orbit needs not always exist. However, if it exists it must be noncontractible (see [19, Chapter 1, Proposition 4.2.2] or Proposition A.8).

We now define the (local) Poincaré map. Suppose that γ is a periodic orbit of W through a given point x . Let Σ be a smooth, 1D curve that intersects γ transversally at x (by definition, it means that $T_x \Sigma \oplus T_x \gamma = \mathbb{R}^2$). There is a well-defined map

$$P : U \subseteq \Sigma \rightarrow \Sigma$$

on a sufficiently small neighbourhood $U \subseteq \Sigma$ of x , which is constructed as follows. Let $y \in \Sigma$. Consider the flow line $t \mapsto \phi_t(y)$. Let $\tau(y) > 0$ be the first time that $\phi_t(y)$ returns to Σ . We then define

$$P(y) := \phi_{\tau(y)}(y).$$

Since $P(x) = x$, by continuity the map P is well-defined in a small neighbourhood U of x (see Lemma A.9). In addition, since Σ is one-dimensional, the linear map $(dP)_x : T_x \Sigma \rightarrow T_x \Sigma$ is a scalar multiple of the identity, i.e. $(dP)_x = \lambda_\gamma \text{id}$ for some λ_γ .

Definition 6.9. The map P constructed above is known as the *Poincaré map* at $x \in \gamma$ with respect to the transversal Σ . The number λ_γ is called the *transverse characteristic number* of γ .

Although the Poincaré map depends on the point $x \in \gamma$ and the transversal Σ , it is important to note that λ_γ depends only on the periodic orbit γ (see [29, Sections 3.4, 3.5] or Proposition A.11).

Definition 6.10. Let γ be a periodic orbit of a smooth, everywhere nonvanishing vector field W on $\mathbb{T}^2$. We say that γ is *hyperbolic* if $\lambda_\gamma \neq 1$.

6.4 The suspension case

We now proceed to develop the proof of Theorem 6.1. Let $V = (V_1, V_2)$ be uniformly elliptic, smooth vector fields on $\mathbb{T}^2$. Given fixed $\alpha_0 \in [0, 2\pi)$, we consider the vector field

$$W_{\alpha_0}(x) := V(x) \cdot \begin{pmatrix} \cos \alpha_0 \\ \sin \alpha_0 \end{pmatrix}, \quad x \in \mathbb{T}^2. \quad (6.5)$$

By ellipticity, W_{α_0} is everywhere nonvanishing and thus induces a 1D foliation $\mathcal{F}_{\alpha_0}$ on $\mathbb{T}^2$. According to Theorem 6.7, either $\mathcal{F}_{\alpha_0}$ contains a Reeb component or it is conjugate to a suspension. In this subsection, we deal with the suspension case. The dynamical mechanism for this case turns out to be rather simple; as we will see, one is led precisely to the regime of Section 5. The Reeb component case is much more involved and will be handled in the next two subsections.

Proof of Theorem 6.1: Suspension case. We therefore assume that there exists a diffeomorphism $\Psi : \mathbb{T}^2 \rightarrow \Sigma_f$, where Σ_f is the suspension of an orientation-preserving diffeomorphism $f : S^1 \rightarrow S^1$, such that $\Psi(\mathcal{F}_{\alpha_0}) = \mathcal{F}_f$ (see Definition A.25 and Definition 6.6).

(i) Consider the projection

$$\rho : \Sigma_f \rightarrow S^1 = \mathbb{R}/\mathbb{Z}, \quad \rho([x, t]) \mapsto t \pmod{\mathbb{Z}}.$$

Let $d\theta$ denote the canonical volume form on S^1 such that $\int_{S^1} d\theta = 1$. Define the closed 1-form $\beta := \rho^*(d\theta)$ on Σ_f . Let Y be the suspension vector field on Σ_f ; the vertical vector field ∂_t on $S^1 \times \mathbb{R}$ descends to Y on Σ_f under the quotient of the $\mathbb{Z}$ -action (6.4), which is tangential to the foliation $\mathcal{F}_f$. By definition, one has

$$\langle \beta(\xi), Y(\xi) \rangle = \langle d\theta, (d\rho)(Y) \rangle_{\rho(\xi)} = \langle d\theta, \partial_t \rangle_{\rho(\xi)} = 1$$

for every $\xi \in \Sigma_f$.

(ii) Since $\Psi(\mathcal{F}_{\alpha_0}) = \mathcal{F}_f$, there exists a nonvanishing function $\lambda \in C^\infty(\mathbb{T}^2)$ such that

$$(d\Psi)_x(W_{\alpha_0}(x)) = \lambda(x)Y(\Psi(x))$$

for every $x \in \mathbb{T}^2$. By connectedness, one may assume without loss of generality that

$$m := \inf_{x \in \mathbb{T}^2} \lambda(x) > 0.$$

Let $\eta := \Psi^*(\beta)$. It follows that η is closed and one has

$$\langle \eta(x), W_{\alpha_0}(x) \rangle = \lambda(x) \langle \beta(\Psi(x)), Y(\Psi(x)) \rangle = \lambda(x)$$

for every $x \in \mathbb{T}^2$. In particular,

$$\inf_{x \in \mathbb{T}^2} \langle \eta(x), W_{\alpha_0}(x) \rangle = m > 0. \quad (6.6)$$

(iii) We claim that the (de Rham) cohomology class $[\eta]$ is nontrivial. Indeed, if η were exact, say $\eta = dq$ for some $q \in C^\infty(\mathbb{T}^2)$, one would have $\eta(x^*) = 0$ at the location x^* where q is maximised. This contradicts the fact (6.6). Therefore, $[\eta] \neq 0$.

(iv) Let $[\alpha_1], [\alpha_2]$ denote a basis of $H_1(\Sigma_f; \mathbb{Z})$ such that

$$\int_{[\alpha_1]} [\beta] = 0, \quad \int_{[\alpha_2]} [\beta] = 1. \quad (6.7)$$

One can construct them in the following way. The loop $\alpha_1 = ([x, t])_{x \in S^1}$ is a circle positively winding around the horizontal S^1 -component once. The loop α_2 is defined by a vertical curve $\alpha'_2 := [(x, t)]_{0 \leq t \leq 1}$ concatenated with a horizontal arc α''_2 joining $[(f(x), 0)]$ back to $[(x, 0)]$ along the horizontal S^1 -component. This concatenation is needed since α'_2 may fail to be a closed curve (leaves of $\mathcal{F}_f$ need not be compact). The relation (6.7) is now obvious from the construction.

Let $\{[l_1], [l_2]\}$ be the canonical basis of $H_1(\mathbb{T}^2; \mathbb{Z})$ that is dual to $\{[dx^1], [dx^2]\}$. Let ζ_1, ζ_2 be the preimages of α_1, α_2 under the diffeomorphism Ψ . Then $\{[\zeta_1], [\zeta_2]\}$ is a basis of $H_1(\mathbb{T}^2; \mathbb{Z})$ and one has

$$\int_{[\zeta_1]} [\eta] = 0, \quad \int_{[\zeta_2]} [\eta] = 1.$$

In particular, one can write

$$[\zeta_1] = a[l_1] + b[l_2], \quad [\zeta_2] = c[l_1] + d[l_2]$$

with some matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}).$$

It follows that

$$[\eta] = k_1[dx^1] + k_2[dx^2],$$

where

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} := A^{-1} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{Z}^2 \setminus \{(0, 0)\}.$$

As a result, one has

$$\eta = k_1 dx^1 + k_2 dx^2 + d\varphi$$

for some $\varphi \in C^\infty(\mathbb{T}^2)$.

(v) Consider the function

$$F(x^1, x^2) := k_1 x^1 + k_2 x^2 + \tilde{\varphi}(x^1, x^2), \quad x = (x^1, x^2) \in \mathbb{R}^2,$$

where $\tilde{\varphi}$ is the pullback of φ (i.e. its periodic extension) on $\mathbb{R}^2$. It is clear that F has bounded first and second derivatives on $\mathbb{R}^2$. In addition, let $\eta := (\cos \alpha_0, \sin \alpha_0)^T$ and let $\tilde{V}$ be the periodic extension of V on $\mathbb{R}^2$ (same for $\tilde{W}_{\alpha_0}$). Then one has

$$((\tilde{V} \cdot \eta)F)(x) = \langle dF(x), \tilde{W}_{\alpha_0}(x) \rangle = \langle \eta([x]), W_{\alpha_0}([x]) \rangle \geq m > 0$$

for all $x \in \mathbb{R}^2$ ($[x] \in \mathbb{T}^2$ denotes the projected image of x). As a consequence, Condition (H) in Section 5 is satisfied with the above choices of η and F . The Gaussian lower tail estimate (6.2) now follows from Theorem 5.2. $\square$

6.5 The Reeb component case

We now investigate the Reeb component case. Suppose that the foliation $\mathcal{F}_{\alpha_0}$ of W_{α_0} (see (6.5)) contains a Reeb component. By definition, there is an embedded annulus $A \cong S^1 \times [0, 1]$ inside $\mathbb{T}^2$ with two compact leaves from ∂A (periodic orbits) and all leaves in $\text{int} A$ are diffeomorphic to $\mathbb{R}$. The key insight here is that one can perturb α_0 into a new selection W_{α_*} , whose flow indeed contains a *hyperbolic* periodic orbit. The main lemma is stated as follows.

Lemma 6.11. *Let $\alpha_0 \in [0, 2\pi)$ be such that the foliation of W_{α_0} contains a Reeb component. Then there exists $\alpha_* \in [0, 2\pi)$ such that W_{α_*} admits a hyperbolic periodic orbit.*

We will prove Lemma 6.11 in Section 6.6. In what follows, we complete the proof of Theorem 6.1 assuming the correctness of this lemma. We first state a simple continuity estimate for Young's differential equations.

Lemma 6.12. *Let $p \in (1/H, 2)$. Consider the differential equation*

$$\begin{cases} dY_s = \tilde{W}(Y_s)ds + \tilde{V}(Y_s)dv_s, & 0 \leq s \leq T, \\ Y_0 = y \in \mathbb{R}^2, \end{cases}$$

where $\tilde{W}, \tilde{V}$ are the periodic extension of $W, V \in C^\infty(\mathbb{T}^2)$ respectively and $v \in C^{p\text{-var}}([0, T]; \mathbb{R}^2)$. For any $M > 0$, there exists $C > 0$ depending only on V, W, p, T and M , such that

$$\|v\|_{p\text{-var}} \leq M \implies \sup_{0 \leq s \leq T} |Y_s - \tilde{\phi}_s(y)| \leq C \|v\|_{p\text{-var}}.$$

Here $\tilde{\phi}_s(y)$ denotes the flow of $\tilde{W}$ on $\mathbb{R}^2$ starting from y .

Proof. This is immediate from standard continuity estimates for Young's differential equations (see e.g. [17, Theorem 10.38]). $\square$

Now let γ be a hyperbolic periodic orbit of $W := W_{\alpha_*}$ from Lemma 6.11. Let T be the period of γ . Recall from Proposition A.11 that the transversal characteristic number λ_γ is strictly positive. Due to hyperbolicity, one has either $\lambda_\gamma \in (0, 1)$ or $\lambda_\gamma > 1$. By changing W to $-W$ if necessary (λ_γ will be changed to its reciprocal), one may assume without loss of generality that $\lambda_\gamma \in (0, 1)$.

For sufficiently small ε , let U_ε be a tubular neighbourhood of γ in $\mathbb{T}^2$. More precisely, there is a diffeomorphism

$$\Phi : \mathbb{R}/T\mathbb{Z} \times (-\varepsilon, \varepsilon) \rightarrow U_\varepsilon, \quad \Phi(t, r) := \gamma_t + r\mathbf{n}(t),$$

where $\mathbf{n}$ is a fixed unit normal vector field along γ . For $x = \Phi(t, r) \in U_\varepsilon$, we denote $z(x) := r$ as the transversal coordinate of x in U_ε . Let $\tilde{\gamma}$ be a given lifting of γ on $\mathbb{R}^2$. The tubular neighbourhood map Φ also lifts to a tubular neighbourhood map $\tilde{\Phi} : \mathbb{R} \times (-\varepsilon, \varepsilon) \rightarrow \tilde{U}_\varepsilon$ of $\tilde{\gamma}$, whose transversal coordinate map is denoted as $\tilde{z} : \tilde{U}_\varepsilon \rightarrow (-\varepsilon, \varepsilon)$.

The following lemma is a consequence of the hyperbolicity of γ , which is crucial to us.

Lemma 6.13. *Let $\lambda^* \in (\lambda_\gamma, 1)$ be a given fixed number. Then there exists $\delta < \varepsilon$, such that*

$$\phi_s(U_\delta) \subseteq U_{\varepsilon/4} \quad (6.8)$$

for all $s \in [0, T]$ and

$$|z(\phi_T(x))| \leq \lambda^* |z(x)| \quad (6.9)$$

for all $x \in U_\delta$. Here ϕ_s denotes the flow of W .

Proof. The property (6.8) is obvious by continuity. To prove the second property, let

$$G(t, r) := z(\phi_T(\Phi(t, r))), \quad (t, r) \times \mathbb{R}/T\mathbb{Z} \times (-\varepsilon, \varepsilon).$$

Note that $\partial_r G(t, 0) = \lambda_\gamma$. By continuity, one has

$$\sup_{\substack{t \in \mathbb{R}/T\mathbb{Z} \\ r \in (-\delta, \delta)}} |\partial_r G(t, r)| \leq \lambda^*,$$

provided that δ is small enough. Since $G(t, 0) = 0$, it follows that

$$|z(\phi_T(x))| = |G(t, r)| \leq \int_0^r |\partial_r G(t, \rho)| d\rho \leq \lambda^* |r| = \lambda^* |z(x)|$$

for any $x = \Phi(t, r) \in U_\delta$. This proves the property (6.9). $\square$

Remark 6.14. Due to periodicity, it is clear that both properties (6.8) and (6.9) remain valid for the lifted objects on $\mathbb{R}^2$.

We are now in a position to prove Theorem 6.1 in the Reeb component case.

Proof of Theorem 6.1: Reeb component case. We divide the argument into several major steps. Let λ^*, U_δ be given as in Lemma 6.13. Let $R > 2T$.

(i) Consider the differential equation

$$\begin{cases} dX_t^{R,w} = R\tilde{W}(X_t^{R,w})dt + \tilde{V}(X_t)dw_t, & 1/2 \leq t \leq 1, \\ X_{1/2}^{R,w} = x \in \tilde{U}_\delta, \end{cases} \quad (6.10)$$

where $w \in C^{p\text{-var}}([0, 1]; \mathbb{R}^2)$. We divide $[1/2, 1]$ into equal subintervals

$$[t_{j-1}, t_j] := [1/2 + (j-1)T/R, 1/2 + jT/R], \quad j = 1, \dots, \lfloor \frac{R}{2T} \rfloor := N_R$$

as well as the last subinterval $[t_{N_R}, 1]$. We claim that there exists $M_1 > 0$ depending on δ , such that

$$\|w\|_{p\text{-var};[1/2,1]} \leq M_1 \implies X_{t_j}^{R,w} \in \tilde{U}_\delta \quad (6.11)$$

for all $j = 1, \dots, N_R$.

To see this, suppose that $X_{t_{j-1}}^{R,w} \in \tilde{U}_\delta$. One can reparametrise the equation (6.10) on $[t_{j-1}, t_j]$ as

$$\begin{cases} dY_s^j = \tilde{W}(Y_s^j)ds + \tilde{V}(Y_s^j)dv_s^j, & 0 \leq s \leq T, \\ Y_0^j = X_{t_{j-1}}^{R,w}, \end{cases}$$

where $v_s^j := w_{t_{j-1}+s/R} - w_{t_{j-1}}$. Note that

$$\|v^j\|_{p\text{-var};[0,T]} = \|w\|_{p\text{-var};[t_{j-1}, t_j]} \leq \|w\|_{p\text{-var};[1/2,1]}.$$

By the relation (6.8) (cf. Remark 6.14), one has

$$\{\tilde{\phi}_s(X_{t_{j-1}}^{R,w}) : 0 \leq s \leq T\} \subseteq \tilde{U}_{\varepsilon/4}.$$

In addition, according to Lemma 6.12, one has

$$\|w\|_{p\text{-var};[1/2,1]} \leq 1 \implies \sup_{0 \leq s \leq T} |Y_s^j - \tilde{\phi}_s(X_{t_{j-1}}^{R,w})| \leq C_1 \|w\|_{p\text{-var};[1/2,1]}.$$

In particular, $X_t^{R,w} \in \tilde{U}_{\varepsilon/2}$ for all $t \in [t_{j-1}, t_j]$, provided that $\|w\|_{p\text{-var};[1/2,1]} \leq M_1$ and $M_1 < \frac{\varepsilon}{4C_1} \wedge 1$.

Note that the transversal coordinate map $z : U_\varepsilon \rightarrow (-\varepsilon, \varepsilon)$ is Lipschitz on $\bar{U}_{\varepsilon/2}$. Let L denote its Lipschitz constant (same constant for $\tilde{z}$). The previous continuity estimate also yields that

$$|\tilde{z}(X_{t_j}^{R,w}) - \tilde{z}(\tilde{\phi}_T(X_{t_{j-1}}^{R,w}))| \leq C_1 L \|w\|_{p\text{-var};[1/2,1]},$$

provided that $\|w\|_{p\text{-var};[1/2,1]} \leq M_1$. It follows from the contraction property (6.9) that

$$\begin{aligned} |\tilde{z}(X_{t_j}^{R,w})| &\leq |\tilde{z}(\tilde{\phi}_T(X_{t_{j-1}}^{R,w}))| + |\tilde{z}(X_{t_j}^{R,w}) - \tilde{z}(\tilde{\phi}_T(X_{t_{j-1}}^{R,w}))| \\ &\leq \lambda^* \delta + C_1 L \|w\|_{p\text{-var};[1/2,1]} \leq \lambda^* \delta + C_1 L M_1. \end{aligned}$$

By further requiring

$$M_1 < \frac{(1 - \lambda^*)\delta}{C_1 L},$$

one obtains that

$$|\tilde{z}(X_{t_j}^{R,w})| < \delta \iff X_{t_j}^{R,w} \in \tilde{U}_\delta.$$

Since $X_{1/2}^{R,w} = x \in \tilde{U}_\delta$ by assumption, this proves the claim (6.11) by induction on j .

(ii) Since γ has nontrivial homology, there exists $k \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that

$$\tilde{\gamma}(s + T) = \tilde{\gamma}(s) + k$$

for all $s \in \mathbb{R}$. Consider the function

$$F(x) := \frac{k \cdot x}{|k|^2}, \quad x \in \mathbb{R}^2.$$

Note that $F(\tilde{\phi}_T(x) - x) = 1$ for all $x \in \tilde{\gamma}$. By shrinking δ if necessary, one can ensure that

$$F(\tilde{\phi}_T(x) - x) \geq \frac{3}{4} \quad (6.12)$$

for all $x \in \tilde{U}_\delta$.

Let $X_t^{R,w}$ be the solution to the equation (6.10). We claim that there exists $M_1 > 0$, such that

$$\|w\|_{p\text{-var};[1/2,1]} \leq M_1 \implies F(X_{t_j}^{R,w} - X_{t_{j-1}}^{R,w}) \geq \frac{1}{2} \quad (6.13)$$

for all $i = 1, \dots, N_R$. Indeed, with M_1 chosen as in Part (i) one has

$$\|w\|_{p\text{-var};[1/2,1]} \leq M_1 \implies |F(X_{t_j}^{R,w} - \tilde{\phi}_T(X_{t_{j-1}}^{R,w}))| \leq \frac{C_1}{|k|} \|w\|_{p\text{-var};[1/2,1]}.$$

Since $X_{t_{j-1}}^{R,w} \in \tilde{U}_\delta$, it follows from (6.12) that

$$F(X_{t_j}^{R,w} - X_{t_{j-1}}^{R,w}) \geq F(\tilde{\phi}_T(X_{t_{j-1}}^{R,w}) - X_{t_{j-1}}^{R,w}) - \frac{C_1}{|k|} \|w\|_{p\text{-var};[1/2,1]} \geq \frac{3}{4} - \frac{C_1 M_1}{|k|}.$$

By further requiring

$$M_1 < \frac{|k|}{4C_1},$$

one concludes that (6.13) holds. As a consequence,

$$F(X_{t_{N_R}}^{R,w} - X_{1/2}^{R,w}) \geq \frac{1}{2} N_R = \frac{1}{2} \lfloor \frac{R}{2T} \rfloor$$

provided that $X_{1/2}^{R,w} \in \tilde{U}_\delta$ and $\|w\|_{p\text{-var};[1/2,1]} \leq M_1$.

Under the same assumptions as above, the last piece $[t_{N_R}, 1]$ is easily estimated as

$$\begin{aligned} |F(X_1^{R,w} - X_{t_{N_R}}^{R,w})| &\leq C_2 \sup_{0 \leq s \leq T} \left[|Y_s^{N_R+1} - \tilde{\phi}_s(X_{t_{N_R}}^{R,w})| \right. \\ &\quad \left. + |\tilde{\phi}_s(X_{t_{N_R}}^{R,w}) - X_{t_{N_R}}^{R,w}| \right] \leq C_3. \end{aligned}$$

By using the Lipschitz property of F again, one therefore obtains that

$$|X_1^{R,w} - X_{1/2}^{R,w}| \geq \frac{|k|}{2} \lfloor \frac{R}{2T} \rfloor - C_3 |k|, \quad (6.14)$$

provided that $X_{1/2}^{R,w} \in \tilde{U}_\delta$ and $\|w\|_{p\text{-var};[1/2,1]} \leq M_1$. The estimate (6.14) is uniform for all large R (with M being chosen as before).

(iii) Recall that $x_0 \in \mathbb{R}^2$ is the given initial condition of the SDE (6.1). Let x be any fixed point on $\tilde{\gamma}$ and let $\beta : [0, 1/2] \rightarrow \mathbb{R}^2$ be any fixed smooth path (e.g. the straight line) such that $\beta_0 = x_0$, $\beta_{1/2} = x$. Define

$$h_{\text{int}}(t) := \int_0^t V^{-1}(\beta_s) \beta_s ds, \quad 0 \leq t \leq \frac{1}{2}.$$

By definition, the control h_{int} drives the dynamics

$$dy_t = V(y_t) \dot{h}_{\text{int}}(t) dt$$

from x_0 to x along β . By the same continuity estimate leading to Lemma 6.12, one has

$$\|w\|_{p\text{-var};[0,1/2]} \leq M_0 \implies X_{1/2}^{w+h_{\text{int}}} \in \tilde{U}_\delta$$

for some small M_0 .

(iv) Let us summarise what we have obtained so far. For each $R > 2T$, define

$$h^R(t) \triangleq \hat{h}_{\text{int}}(t) + R \cdot \hat{h}_{\text{main}}(t), \quad 0 \leq t \leq 1,$$

where

$$\hat{h}_{\text{int}}(t) := \begin{cases} h_{\text{int}}(t), & 0 \leq t \leq 1/2, \\ h_{\text{int}}(1/2), & 1/2 \leq t \leq 1, \end{cases} \quad \hat{h}_{\text{main}}(t) := \begin{cases} 0, & 0 \leq t \leq 1/2, \\ (t - 1/2)\eta_*, & 1/2 \leq t \leq 1 \end{cases}$$

and $\eta_* := (\cos \alpha_*, \sin \alpha_*)^T$. Let $M := M_0 \wedge M_1$. What we have shown is that whenever $w \in C^{p\text{-var}}([0, 1]; \mathbb{R}^2)$ satisfies $\|w\|_{p\text{-var};[0,1]} \leq M$, one has

$$|X_1^{w+h^R} - x_0| \geq \frac{|k|}{2} \lfloor \frac{R}{2T} \rfloor - C_3 |k| - |x - x_0|$$

for all large R . On the other hand, it is easy to see that $\hat{h}_{\text{int}}, \hat{h}_{\text{main}} \in \bar{\mathcal{H}}$ and

$$\|h^R\|_{\bar{\mathcal{H}}} \leq \|\hat{h}_{\text{int}}\|_{\bar{\mathcal{H}}} + R \|\hat{h}_{\text{main}}\|_{\bar{\mathcal{H}}}.$$

These are essentially the properties (5.2) and (5.5). Therefore, one can carry through exactly the same Cameron-Martin transformation argument from the proof of Theorem 5.2 to conclude that the solution to the SDE (6.1) at time one admits a Gaussian lower tail.

The proof of Theorem 6.1 is now complete. □

6.6 Proof of the hyperbolic orbit selection lemma

In this subsection, we develop the proof of Lemma 6.11.

6.6.1 Parametric Poincaré map

We first set up some basic definitions. For $\alpha \in [0, 2\pi)$, we define

$$W_\alpha(x) := V(x) \cdot \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \quad x \in \mathbb{T}^2.$$

Suppose that W_{α_0} contains a Reeb component A for some given $\alpha_0 \in [0, 2\pi)$. Let γ be one of the two periodic orbits from ∂A with period $T > 0$ and fix $x \in \gamma$. Let Σ be a smooth curve intersecting γ transversally at x . Since our argument is local, we assume that Σ is parametrised by $\sigma : (-\varepsilon, \varepsilon) \rightarrow \Sigma$ with $\sigma(0) = x$. We also assume that the orientation of Σ is chosen such that $\sigma((0, \varepsilon))$ is contained in the interior of the Reeb component A .

Recall that the Poincaré map is defined in Section A.1.2. For our purpose, we need a parametric version of the Poincaré map with respect to the parameter α near α_0 . More specifically, we define two functions

$$P : (-\delta, \delta) \times (\alpha_0 - \theta, \alpha_0 + \theta) \rightarrow (-\varepsilon, \varepsilon),$$

$$\tau : (-\delta, \delta) \times (\alpha_0 - \theta, \alpha_0 + \theta) \rightarrow (0, \infty)$$

with small δ, θ in the following way. Given $u \in (-\delta, \delta)$ and $\alpha \in (\alpha_0 - \theta, \alpha_0 + \theta)$, the number $\tau(u, \alpha)$ is the first return time to Σ of the flow line $t \mapsto \phi_t^\alpha(\sigma(u))$, where $\phi_t^\alpha(y)$ denotes the flow of W_α starting from y . Correspondingly, $P(u, \alpha)$ is the location of first return, i.e.

$$\sigma(P(u, \alpha)) = \phi_{\tau(u, \alpha)}^\alpha(\sigma(u)). \quad (6.15)$$

Since

$$P(0, \alpha_0) = 0, \quad \tau(0, \alpha_0) = T,$$

by continuity the maps P, τ are both well-defined for sufficiently small $\delta < \varepsilon$ and θ . In fact, P and τ are smooth in both variables. We define the displacement $D(u, \alpha) := P(u, \alpha) - u$.

A key observation is the following lemma, which is a consequence of the special structure of the family $\alpha \mapsto W_\alpha$.

Lemma 6.15. *One has $\partial_\alpha D(0, \alpha_0) \neq 0$.*

Proof. Recall that $\gamma : [0, T] \rightarrow \mathbb{T}^2$ ($\gamma_0 = x$) is the chosen periodic orbit of W_{α_0} from a Reeb component. Consider the function

$$d(t) := \det(W_{\alpha_0}(\gamma_t), \xi(t))$$

where $\xi(t) := \partial_\alpha \phi_t^\alpha(x)|_{\alpha=\alpha_0}$. Let $Z(y) := \partial_\alpha W_\alpha(y)|_{\alpha=\alpha_0}$ for $y \in \mathbb{T}^2$. By using the flow equation of ϕ_t^α , it is easily seen that

$$\dot{\xi}(t) = DW_{\alpha_0}(\gamma_t)\xi(t) + Z(\gamma_t), \quad \xi(0) = 0.$$

Therefore, one has

$$\begin{aligned} \dot{d}(t) &= \det((DW_{\alpha_0} \cdot W_{\alpha_0})(\gamma_t), \xi(t)) + \det(W_{\alpha_0}(\gamma_t), DW_{\alpha_0}(\gamma_t)\xi(t)) \\ &\quad + \det(W_{\alpha_0}(\gamma_t), Z(\gamma_t)) \\ &= (\operatorname{div} W_{\alpha_0})(\gamma_t) \cdot d(t) + \det(W_{\alpha_0}(\gamma_t), Z(\gamma_t)). \end{aligned} \quad (6.16)$$

To reach the last equality, we used the simple relation that

$$\det(A \cdot \mathbf{u}, \mathbf{v}) + \det(\mathbf{u}, A \cdot \mathbf{v}) = \operatorname{tr}[A] \det(\mathbf{u}, \mathbf{v})$$

for any 2×2 matrix A and 2×1 row vectors $\mathbf{u}, \mathbf{v}$. Since $d(0) = 0$, it follows from (6.16) that

$$d(T) = \int_0^T \exp \left[\int_s^T (\operatorname{div} W_{\alpha_0})(\gamma(r)) dr \right] \det(W_{\alpha_0}(\gamma_s), Z(\gamma_s)) ds.$$

Explicit computation shows that

$$\det(W_{\alpha_0}(y), Z(y)) = \det V(y)$$

for all $y \in \mathbb{T}^2$. Since V is uniformly elliptic, the above determinant has a constant sign. As a consequence, $d(T) \neq 0$.

On the other hand, by differentiating the relation (6.15) with respect to α at $(u, \alpha) = (0, \alpha_0)$, one finds that

$$\xi(T) + \partial_\alpha \tau(0, \alpha_0) W_{\alpha_0}(x) = \partial_\alpha P(0, \alpha_0) \cdot \sigma'(0).$$

By taking determinant with $W_{\alpha_0}(\gamma_T) = W_{\alpha_0}(x)$, it follows that

$$d(T) = \partial_\alpha P(0, \alpha_0) \cdot \det(W_{\alpha_0}(x), \sigma'(0)).$$

Since Σ intersects γ transversally at x , one knows that

$$\det(W_{\alpha_0}(x), \sigma'(0)) \neq 0.$$

Therefore,

$$\partial_\alpha D(0, \alpha_0) = \partial_\alpha P(0, \alpha_0) \neq 0.$$

This completes the proof of the lemma. $\square$

6.6.2 Hyperbolic orbit selection

Recall that $D(0, \alpha_0) = 0$ and $\partial_\alpha D(0, \alpha_0) \neq 0$ by Lemma 6.15. According to the implicit function theorem, there exists $\delta_1 < \delta$ and a smooth map

$$A : (-\delta_1, \delta_1) \rightarrow (\alpha_0 - \theta, \alpha_0 + \theta),$$

such that

$$D(u, A(u)) = 0 \quad (6.17)$$

for all $u \in (-\delta_1, \delta_1)$. In other words, the orbit of $W_{A(u)}$ passing through $\sigma(u)$ is periodic for every $u \in (-\delta_1, \delta_1)$. The following lemma shows that these orbits must contain a hyperbolic one.

Lemma 6.16. *There exists a $u_* \in (0, \delta_1)$, such that the orbit of $W_{A(u_*)}$ passing through $\sigma(u_*)$ is hyperbolic.*

Proof. By differentiating (6.17), one finds that

$$\partial_u D(u, A(u)) + \partial_\alpha D(u, A(u)) A'(u) = 0$$

for all $u \in (-\delta_1, \delta_1)$. For $u \in (0, \delta_1)$, recall that the orbit of W_{α_0} through $\sigma(u)$ is nonperiodic by assumption (because it is a leaf inside the interior of the Reeb component A). In particular, one has $D(u, \alpha_0) \neq 0$ and therefore, $A(u) \neq \alpha_0$ for any $u \in (0, \delta_1)$. Since $A(0) = \alpha_0$, it follows that for any $\eta > 0$, there exists $u_\eta \in (0, \eta)$ such that $A'(u_\eta) \neq 0$. On the other hand, one knows from Lemma 6.15 and continuity that $\partial_\alpha D(u, A(u)) \neq 0$ for all sufficiently small u . As a consequence, there must exist $u_* \in (0, \delta_1)$, such that

$$\partial_u D(u_*, A(u_*)) \neq 0,$$

or equivalently, $\partial_u P(u_*, \alpha_*) \neq 1$ where $\alpha_* := A(u_*)$. By definition, one concludes that the periodic orbit of W_{α_*} through $\sigma(u_*)$ is hyperbolic. $\square$

The proof of Lemma 6.11 is now complete.

Remark 6.17. Although the suspension case is relatively simple, one cannot expect avoiding the Reeb component case by hoping that “there always exists $\alpha \in [0, 2\pi)$ such that W_α does not contain a Reeb component”. For instance, consider the elliptic system

$$V(x) = \begin{pmatrix} \cos 2\pi x^1 & -\sin 2\pi x^1 \\ \sin 2\pi x^1 & \cos 2\pi x^1 \end{pmatrix}, \quad x = (x^1, x^2) \in \mathbb{T}^2.$$

One has

$$W_\alpha(x) = \begin{pmatrix} \cos(2\pi x^1 + \alpha) \\ \sin(2\pi x^1 + \alpha) \end{pmatrix}$$

for every $\alpha \in [0, 2\pi)$. In particular, the foliation of W_α contains precisely two Reeb components bounded by the compact leaves (periodic orbits)

$$L_+ : x^1 = \frac{\pi/2 - \alpha}{2\pi}, \quad L_- : x^1 = \frac{3\pi/2 - \alpha}{2\pi}$$

respectively, regardless of the value of α .

Remark 6.18. Both Theorem 4.10 and Theorem 6.1 are concerned with periodic systems. The essential reason behind their qualitatively different tail behaviours can be summarised as follows.

For the radially periodic example, when the system propagates through annuli, one encounters new geometry and there is no simple ballistic tunnel that the system could use. Any large distance propagation has to cause sufficiently many effective turns for the control.

On the other hand, for the torus system, there is always either a globally monotone direction (suspension case) or a periodic orbit on the fundamental

domain $\mathbb{T}^2$ with nontrivial homology, which can be reused by the system precisely due to the repeating geometry the system encounters when it propagates through different fundamental domains. Both cases lead to a global ballistic tunnel with a small growth of Cameron-Martin norm.

A Classification of nondegenerate torus flows

In this appendix, we provide a self-contained proof of Theorem 6.7. The argument is essentially an adaptation of the proof given by Hector-Hirsch [19] to the smooth setting.

A.1 Nondegenerate torus flow as 1D foliation

Let $M = \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$. We always fix a smooth, nonvanishing vector field W on $\mathbb{T}^2$. Given $x \in \mathbb{T}^2$, define the flow $t \mapsto \phi_t(x)$ by

$$\frac{d}{dt}\phi_t(x) = W(\phi_t(x)), \quad \phi_0(x) = x.$$

Since $\mathbb{T}^2$ is compact, the flow $\phi_t(x)$ is well-defined for all $t \in \mathbb{R}$. Let $\mathcal{F}_W$ denote the collection of integral curves (flow lines) of W , namely,

$$\mathcal{F}_W := \{\{\phi_t(x)\}_{t \in \mathbb{R}} : x \in \mathbb{T}^2\}.$$

The starting point of the whole discussion is the following basic fact.

Proposition A.1. *The family $\mathcal{F}_W$ defines a 1D foliation on $\mathbb{T}^2$ in the sense of Definition 6.2.*

Proof. It is clear that $\mathcal{F}_W$ is a family of immersed 1D manifolds whose union is $\mathbb{T}^2$. To prove the local flatness property, let $x \in \mathbb{T}^2$. Since $W(x) \neq 0$, one can pick a nonzero vector $v \in T_x \mathbb{T}^2$ that is linearly independent of $W(x)$. Let $\sigma : (-\delta, \delta) \rightarrow \mathbb{T}^2$ be a smooth arc such that $\sigma(0) = x$, $\dot{\sigma}(0) = v$. Define

$$F : \mathbb{R} \times (-\delta, \delta) \rightarrow \mathbb{T}^2, \quad F(t, r) := \phi_t(\sigma(r)).$$

It is clear that $F(0, 0) = x$ and $(dF)_{(0,0)}$ is nonsingular. Therefore, F is a local diffeomorphism from $V := (-\varepsilon, \varepsilon) \times (-\delta', \delta')$ onto $F(V)$ with some small ε and $\delta' < \delta$. The map $F|_V$ provides a flat chart of $\mathbb{T}^2$ around x ; each horizontal arc $s \mapsto (s, r)$ corresponds to a flow line by definition and every flow line inside $F(V)$ must come from a horizontal arc since $W = \partial_s$ in $F(V)$. $\square$

Proposition A.1 asserts that around every point $x \in \mathbb{T}^2$, one can choose a *flow box* $(U; s, r)$ ($s \in (-\delta_1, \delta_1)$, $r \in (-\delta_2, \delta_2)$) such that $x = (0, 0)$ and $W = \partial_s$ on U . In particular, flow lines of W inside U are precisely the horizontal slices $r = c$ ($c \in (-\delta_2, \delta_2)$). This is a basic but important (technical) fact that will be used for several times.

Remark A.2. The actual flow $\{\phi_t(x)\}$ carries more information than the foliation $\mathcal{F}_W$ itself; the foliation is determined by the *1D distribution* $\{\text{Span} W(x) : x \in \mathbb{T}^2\}$ while the flow also records the orientation and the strength of the field.

Definition A.3. The leaf $\mathcal{O}_x := \{\phi_t(x) : t \in \mathbb{R}\}$ is called the *orbit* of W passing through x . An orbit $\mathcal{O}_x$ is said to be *periodic* if it is a compact leaf.

It is a simple exercise that for a periodic orbit $\mathcal{O}_x$, there exist a smallest $T > 0$ such that $\phi_{t+T}(y) = \phi_t(y)$ for all $y \in \mathcal{O}_x$ and $t \in \mathbb{R}$. This number T is called the *period* of $\mathcal{O}_x$ (with respect to W). Note that the period is not a foliation concept; it depends on the actual flow $\phi_t(x)$ or equivalently, on the vector field W (cf. Remark A.2). Another important feature here is that the vector field W orients every orbit $\mathcal{O}_x$ in a natural way.

A.1.1 Periodic orbits

In this subsection, we study basic topological and dynamical properties of periodic orbits. This provides some essential tools for the later proof of the main classification theorem.

Transversal intersection and the intersection number

We first recall some notions from intersection theory (see [10, 18]).

Definition A.4. Let P, Q be two oriented, embedded 1D submanifolds of $\mathbb{T}^2$. We say that P is *transversal* to Q if $T_x P$ and $T_x Q$ (as 1D subspaces of $T_x \mathbb{T}^2$) are linearly independent at every $x \in P \cap Q$. In this case, the *local intersection number* of P and Q at $x \in P \cap Q$ is defined by

$$i_x(P, Q) := \begin{cases} 1, & \text{if } \det(v_x, w_x) > 0; \\ -1, & \text{if } \det(v_x, w_x) < 0, \end{cases}$$

where v_x (respectively, w_x) is any positively oriented vector in $T_x P$ (respectively, $T_x Q$). The *intersection number* of P and Q is defined by

$$I(P, Q) := \sum_{x \in P \cap Q} i_x(P, Q),$$

provided further that $P \cap Q$ is a finite set.

For our purpose, P, Q are always assumed to be smooth, simple and closed curves. Due to compactness, if they intersect transversally their intersection must be a finite set and $I(P, Q)$ is thus well-defined. In particular, $I(P, Q) = 0$ if they do not intersect. It is also known that the intersection number is a topological quantity, i.e. it depends only on the homology classes of P, Q . Let $\{\sigma_1, \sigma_2\}$ be the canonical basis of $H_1(\mathbb{T}^2; \mathbb{Z}) \cong \mathbb{Z}^2$. In other words,

$$\sigma_1 := [(t, 0)_{0 \leq t \leq 1}], \quad \sigma_2 := [(0, t)_{0 \leq t \leq 1}]. \quad (\text{A.1})$$

One has the following formula for the intersection number.

Lemma A.5. *Let P, Q be smooth, simple and closed curves which intersect transversally and whose homology classes are given by*

$$[P] = p_1 \sigma_1 + p_2 \sigma_2, \quad [Q] = q_1 \sigma_1 + q_2 \sigma_2, \quad (\text{A.2})$$

with $p_i, q_j \in \mathbb{Z}$. Then $I(P, Q) = p_1 q_2 - p_2 q_1$.

Proof. Let

$$P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} : S^1 \rightarrow \mathbb{T}^2$$

be parametrised by $S^1 := \mathbb{R}/\mathbb{Z}$. Consider the map $D : S^1 \times S^1 \rightarrow \mathbb{T}^2$ defined by

$$D(s, t) := P(s) - Q(t) \pmod{\mathbb{Z}}.$$

At each intersection point $(s, t) \in D^{-1}(0)$, by the transversal intersection assumption one has

$$\det((dD)_{(s,t)}) = -\det(\dot{P}(s), \dot{Q}(t)) \neq 0.$$

Therefore, 0 is a regular value of D . It follows that $D^{-1}(0)$ is a zero-dimensional submanifold of $S^1 \times S^1$. Hence, it is a finite set of points, say

$$D^{-1}(0) = \{(s_1, t_1), \dots, (s_k, t_k)\}.$$

There exists a neighbourhood U_k of (s_k, t_k) for each k and a neighbourhood V of 0, such that $D|_{U_k}$ is a diffeomorphism from U_k onto V . In addition, one has

$$\operatorname{sgn} \det((dD)_{(s_k, t_k)}) = -\operatorname{sgn} \det(\dot{P}(s_k), \dot{Q}(t_k)) = -i_{x_k}(P, Q),$$

where $x_k := P(s_k) = Q(t_k)$.

Let ω be a smooth 2-form that is compactly supported inside V and satisfies $\int_V \omega = 1$. Since $D|_{U_k} : U_k \rightarrow V$ is a diffeomorphism, one has

$$\int_{U_k} D|_{U_k}^* \omega = \operatorname{sgn} \det((dD)_{(s_k, t_k)}) \int_V \omega = -i_{x_k}(P, Q).$$

Therefore,

$$\int_{S^1 \times S^1} D^* \omega = \sum_{k=1}^m \int_{U_k} D|_{U_k}^* \omega = -I(P, Q).$$

Let $\nu = dx \wedge dy \in \Omega^2(\mathbb{T}^2)$. Since both ω and ν are closed and have the same integral on $\mathbb{T}^2$ (which is equal to one), they differ by an exact form. As a result,

$$\int_{S^1 \times S^1} D^* \nu = \int_{S^1 \times S^1} D^* \omega = -I(P, Q). \quad (\text{A.3})$$

Now it remains to compute the integral of $D^* \nu$.

By using the explicit expressions of D and ν , it is plain to check that

$$D^* \nu = (P'_2(s)Q'_1(t) - P'_1(s)Q'_2(t))ds \wedge dt.$$

Therefore,

$$\begin{aligned} \int_{S^1 \times S^1} D^* \nu &= \int_0^1 \int_0^1 (P'_2(s)Q'_1(t) - P'_1(s)Q'_2(t))dsdt \\ &= \left(\int_0^1 P'_2(s)ds \right) \left(\int_0^1 Q'_1(t)dt \right) - \left(\int_0^1 P'_1(s)ds \right) \left(\int_0^1 Q'_2(t)dt \right). \end{aligned}$$

Let $\tilde{P}, \tilde{Q}$ be some lifts of P, Q on the universal cover $\mathbb{R}^2$. According to the assumption (A.2), one has

$$\begin{aligned}\int_0^1 P'(s)ds &= \int_0^1 \tilde{P}'(s)ds = \tilde{P}(1) - \tilde{P}(0) = (p_1, p_2), \\ \int_0^1 Q'(t)dt &= \int_0^1 \tilde{Q}'(t)dt = \tilde{Q}(1) - \tilde{Q}(0) = (q_1, q_2).\end{aligned}$$

As a consequence,

$$\int_{S^1 \times S^1} D^* \nu = p_2 q_1 - p_1 q_2. \quad (\text{A.4})$$

The conclusion of the lemma follows from the relations (A.3) and (A.4). $\square$

Remark A.6. Lemma A.5 remains valid if $P \cap Q = \emptyset$. In this case, $I(P, Q) = p_1 q_2 - p_2 q_1 = 0$.

Topological properties of periodic orbits

Next, we study periodic orbits of a smooth, nonvanishing vector field W . We first make the following definition.

Definition A.7. A (continuous) closed curve γ in $\mathbb{T}^2$ is said to be *essential* if its (singular) homology class $[\gamma] \in H_1(\mathbb{T}^2; \mathbb{Z})$ is nonzero. It is *primitive* if its homology class cannot be expressed as $[\gamma] = m\sigma$ for some $\sigma \in H_1(\mathbb{T}^2; \mathbb{Z})$ and $m \geq 2$.

Let $\{\sigma_1, \sigma_2\}$ be the canonical basis of $H_1(\mathbb{T}^2; \mathbb{Z})$ defined by (A.1). Write $[\gamma] = p\sigma_1 + q\sigma_2$ for a closed curve γ . Then γ is primitive if and only if $\gcd(|p|, |q|) = 1$. This is also equivalent to saying that $[\gamma]$ can be extended to a basis of $H_1(\mathbb{T}^2; \mathbb{Z})$.

The main topological properties of periodic orbits are summarised in the following result.

Proposition A.8. (i) *Every periodic orbit is essential.*

(ii) *Every essential, simple closed curve is primitive. In particular, every periodic orbit is primitive.*

(iii) *For any pair of periodic orbits γ_1, γ_2 , one has $[\gamma_1] = \pm[\gamma_2]$.*

Proof. (i) Let γ be a periodic orbit of W . Suppose on the contrary that $[\gamma] = 0$. Since γ is simple closed, it must bound a topological disk D in this case. According to the Poincaré-Hopf theorem for the disk D (see [29, Chapter 3, Theorem 2]), one has

$$\text{ind}_W(\partial D) = \sum_{z \in \mathcal{Z}(W) \cap \text{int} D} \text{ind}_W(z).$$

Here $\text{ind}_W(\partial D)$ denotes the index of W on the boundary ∂D and $\text{ind}_W(z)$ denotes the local index of W at the zero z ($\mathcal{Z}(W)$ denotes the set of zeros of

W). It is plain from the definition that $\text{ind}_W(\partial D) = \pm 1$ (the sign depends on the orientation convention of γ). As a result, W must have at least one zero in the interior of D . This is a contradiction. Therefore, $[\gamma] \neq 0$.

(ii) Let γ be an essential, simple closed curve. We claim that $\mathbb{T}^2 \setminus \gamma$ is connected. Suppose on the contrary that $\mathbb{T}^2 \setminus \gamma$ is disconnected. Let $N \cong S^1 \times (-\varepsilon, \varepsilon)$ be a tubular neighbourhood of γ . Then

$$N_- := S^1 \times (-\varepsilon, 0), \quad N_+ := S^1 \times (0, \varepsilon)$$

must be contained in different components of $\mathbb{T}^2 \setminus \gamma$; otherwise, any other component V would intersect this component (because $\gamma \subseteq \bar{V}$) which contradicts disconnectedness. Let U_+ be the component containing N_+ . Since N_- is contained in a different component, $\overline{U_+}$ is a compact surface with boundary γ . As a consequence, $\gamma = \partial c$ for some singular 2-chain. It follows that $[\gamma] = 0$ which leads to a contradiction. Therefore, $\mathbb{T}^2 \setminus \gamma$ is connected.

Pick two points $x_+ \in N_+$ and $x_- \in N_-$. Draw a smooth, simple curve η from x_+ to x_- inside $\mathbb{T}^2 \setminus \gamma$ (which is possible due to connectedness). Let σ be a simple arc which joins x_+ to x_- inside N . By smoothing the corners and removing loops from $\eta \sqcup \sigma$ if necessary, one obtains a simple closed curve β that has precisely one (transversal) intersection with γ . Therefore,

$$I(\gamma, \beta) = \pm 1. \tag{A.5}$$

Suppose that γ is not primitive. Under the canonical basis $\{\sigma_1, \sigma_2\}$ of $H_1(\mathbb{T}^2; \mathbb{Z})$, one can write $[\gamma] = m(p, q)$ for some $m \geq 2$ and $p, q \in \mathbb{Z}$. Let $[\beta] = (r, s)$. According to Lemma A.5, one has

$$I(\gamma, \beta) = m(ps - qr).$$

This contradicts the fact that (A.5). Therefore, γ is primitive.

(iii) Let $\gamma_1 = (p, q)$ and $\gamma_2 = (r, s)$ be two periodic orbits. Since γ_1 is primitive by Part (ii), the integers p, q are coprime. Similarly, r, s is also coprime. Since $\gamma_1 \cap \gamma_2 = \emptyset$, one knows from Lemma A.5 (cf. Remark A.6) that

$$(p, q) = \frac{a}{b}(r, s)$$

for some coprime pair $(a, b) \in \mathbb{Z}^2$. In particular,

$$ar = bp, \quad as = bq.$$

It follows that $a|p$ and $a|q$ which implies that $a = \pm 1$. Similarly, $b = \pm 1$. Therefore, $[\gamma_1] = \pm[\gamma_2]$. $\square$

A.1.2 The Poincaré map

In this subsection, we discuss an elementary but rather useful construction: the Poincaré return map. This construction (or its variant) will be used for several times in the proof of the main classification theorem.

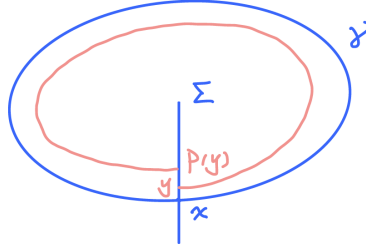
Let $\gamma = \{\phi_t(x)\}_{t \in \mathbb{R}}$ be a periodic orbit of W with period T . Let Σ be a short smooth arc that intersects γ transversally at x . Suppose that $\phi_t(x) \notin \Sigma$ for all $t \in (0, T)$. This is possible by shrinking Σ if necessary; for instance, in a suitable flow box $(U; s, r)$ around x one can assume that $\Sigma = \{s = 0\}$.

We construct a map

$$P : U \subseteq \Sigma \rightarrow \Sigma$$

on a sufficiently small neighbourhood $U \subseteq \Sigma$ of x in the following way. Pick $y \in \Sigma$. Consider the flow line $t \mapsto \phi_t(y)$. Let $\tau(y) > 0$ be the first time that $\phi_t(y)$ returns to Σ . We then define

$$P(y) := \phi_{\tau(y)}(y).$$



Lemma A.9. *There exists a small neighbourhood $U \subseteq \Sigma$ of x , such that $(\tau(y), P(y))$ is well-defined for all $y \in U$. In addition, both $\tau(y)$, $P(y)$ are smooth on U and one has $\tau(x) = T$, $P(x) = x$.*

Proof. First of all, there exists a neighbourhood $U_0 \subseteq \Sigma$ of x , such that U_0 is represented by the level set $\{\rho = 0\}$ of some smooth function ρ near x . Now consider the function

$$F : U_0 \times (T - \delta_0, T + \delta_0) \rightarrow \mathbb{R}, \quad F(y, t) := \rho(\phi_t(y))$$

where δ_0 is small. Note that $F(x, T) = 0$. In addition, since $T_x \Sigma = \ker(d\rho)_x$ and Σ intersects γ transversally at x , one has

$$\partial_t F(x, T) = \langle (d\rho)_x, W(x) \rangle \neq 0. \quad (\text{A.6})$$

According to the implicit function theorem, there exists $U_1 \subseteq U_0$, $\delta_1 < \delta_0$ and a smooth function $\tau : U_1 \rightarrow (T - \delta_1, T + \delta_1)$ such that $\tau(x) = T$ and

$$F(y, \tau(y)) = 0$$

for all $y \in U_1$.

We claim that by shrinking U_1 to some smaller neighbourhood U of x if necessary, one has

$$\tau(y) = \inf\{t > 0 : \phi_t(y) \in \Sigma\}$$

for all $y \in U$. First of all, according to (A.6) there exists $U_2 \subseteq U_1$ and $\delta_2 < \delta_1$, such that $t \mapsto F(y, t)$ is strictly monotone on $[T - \delta_2, T + \delta_2]$ for all $y \in U_2$. As a result, $\tau(y)$ is the unique $t \in [T - \delta_2, T + \delta_2]$ such that $F(y, t) = 0$. We fix this δ_2 in what follows. Next, by using a flow box argument, there exists $U_3 \subseteq U_2$ and $\varepsilon > 0$, such that $\phi_t(y) \notin \Sigma$ for all $y \in U_3$ and $t \in (0, \varepsilon]$. In addition, since

$$\{\phi_t(x) : \varepsilon \leq t \leq T - \delta_2\} \cap \Sigma = \emptyset,$$

one knows from continuity that $\phi_t(y) \notin \Sigma$ for all y in some smaller $U \subseteq U_3$ and all $t \in [\varepsilon, T - \delta_2]$. Combining the above three scenarios, it follows that $\tau(y)$ is the first return time of $\phi_t(y)$ to Σ for every $y \in U$.

Consequently, the map

$$P : U \rightarrow \Sigma, \quad P(y) := \phi_{\tau(y)}(y)$$

is well-defined. Its smoothness follows immediately from the smoothness of τ and the joint smoothness of $(t, y) \mapsto \phi_t(y)$. This completes the proof of the lemma. $\square$

Definition A.10. The map P constructed above is known as the *Poincaré map* at $x \in \gamma$ with respect to the transversal Σ .

Since Σ is 1D and $P(x) = x$, the linear map $(dP)_x : T_x \Sigma \rightarrow T_x \Sigma$ is a scalar multiple of the identity, i.e. $(dP)_x = \lambda_\gamma \text{id}$ for some $\lambda_\gamma \in \mathbb{R}$. Although the Poincaré map P depends on the point $x \in \gamma$ and the transversal Σ , it is a remarkable fact that the number λ_γ is intrinsic to the periodic orbit γ .

Proposition A.11. *The number λ_γ does not depend on the point $x \in \gamma$ or the transversal Σ . In addition, one has $\lambda_\gamma > 0$.*

Proof. We give an alternative intrinsic construction of λ_γ .

(i) Define the function $J_t \in C^\infty(\mathbb{T}^2)$ by

$$\phi_t^* \mu_g = J_t \mu_g,$$

where μ_g denotes the Riemannian volume form on the (oriented) annulus $\mathbb{T}^2$. It is not hard to see that $J_t(x) = \det(d\phi_t)_x$, where the determinant is defined with respect to any positively oriented orthonormal bases of $T_x \mathbb{T}^2$ and $T_{\phi_t(x)} \mathbb{T}^2$. We claim that

$$J_t(x) = \exp \left[\int_0^t (\text{div}_g W)(\phi_s(x)) ds \right], \quad (\text{Liouville's formula}) \quad (\text{A.7})$$

where div_g denotes the Riemannian divergence. To this end, one first recalls the following standard Riemannian geometry fact:

$$\mathcal{L}_W \mu_g = (\text{div}_g W) \mu_g.$$

In addition, by the definition of the Lie derivative, one has

$$\frac{d}{dt} (\phi_t^* \mu_g) = \phi_t^* (\mathcal{L}_W \mu_g).$$

As a consequence,

$$\dot{J}_t(x) = (\operatorname{div}_g W)(\phi_t(x))J_t(x). \quad (\text{A.8})$$

Since $J_0(x) = 1$, the desired relation (A.7) follows by solving the ODE (A.8).

(ii) Suppose that γ is a periodic orbit of W with period T . According to Part (i), one has

$$\det(d\phi_T)_x = \exp \left[\int_0^T (\operatorname{div}_g W)(\phi_s(x)) ds \right]. \quad (\text{A.9})$$

In particular, $\det(d\phi_T)_x$ is independent of x .

(iii) Under the assumption of Part (ii), let Σ be a transversal curve through $x \in \gamma$. Let $v \in T_x \Sigma$ be a nonzero vector. By the definition of the Poincaré map, one has

$$P(y) = \phi_{\tau(y)}(y), \quad y \in U \subseteq \Sigma.$$

It follows from the definition of λ_γ that

$$\lambda_\gamma v = (dP)_x v = (d\phi_T)_x v + \langle d\tau, v \rangle_x W(x).$$

Also note that $(d\phi_T)_x W(x) = W(x)$ since γ is a periodic orbit. Therefore, the matrix of $(d\phi_T)_x : T_x \mathbb{T}^2 \rightarrow T_x \mathbb{T}^2$ with respect to the basis $\{W(x), v\}$ is given by

$$(d\phi_T)_x = \begin{pmatrix} 1 & -\langle d\tau, v \rangle_x \\ 0 & \lambda_\gamma \end{pmatrix}.$$

This shows that $\det(d\phi_T)_x = \lambda_\gamma$. In particular, λ_γ is independent of x and Σ . The positivity of λ_γ follows directly from the relation (A.9). $\square$

Definition A.12. The number λ_γ is called the *transverse characteristic number* of γ . A periodic orbit γ is said to be hyperbolic if $\lambda_\gamma \neq 1$.

Remark A.13. Hyperbolic periodic orbits have strong dynamical stability (exponential attraction of nearby orbits in the case $\lambda_\gamma \in (0, 1)$). Such a stability property is used in the paper [8] to establish a Gaussian lower tail estimate for 2D periodic, uniformly elliptic systems driven by a fractional Brownian motion.

A.1.3 The Poincaré-Bendixson theorem

In general, periodic orbits of W need not always exist. However, if one knows that an end of a nonperiodic orbit is trapped in a compact region *inside a planar domain*, its limit set must be a periodic orbit. This is the content of the *Poincaré-Bendixson theorem*, which provides a general way of identifying periodic orbits.

To state the theorem, we first need to introduce some definitions.

Definition A.14. Let $x \in \mathbb{T}^2$. The *forward orbit* passing through x is defined by

$$\mathcal{O}_x^+ := \{\phi_t(x) : t \geq 0\}.$$

The *forward limit set* of x is defined by

$$\omega(x) := \bigcap_{T>0} \overline{\{\phi_t(x) : t \geq T\}}.$$

One can define the *backward orbit* $\mathcal{O}_x^-$ and the *backward limit set* $\alpha(x)$ in a similar way. A set $\Lambda \subseteq \mathbb{T}^2$ is *W-invariant* if $\phi_t \Lambda \subseteq \Lambda$ for all $t \in \mathbb{R}$.

By compactness, both $\omega(x)$ and $\alpha(x)$ are nonempty, W -invariant compact subsets of $\mathbb{T}^2$. In addition, a point $y \in \omega(x)$ (respectively, $y \in \alpha(x)$) if and only if $\phi_{t_n}(x) \rightarrow y$ for some sequence $t_n \rightarrow \infty$ (respectively, $t_n \rightarrow -\infty$).

Theorem A.15 (The Poincaré-Bendixson Theorem). *Let X be a smooth, nonvanishing vector field on a planar domain $U \subseteq \mathbb{R}^2$. Let $x \in U$ be given fixed. Suppose that the flow $\phi_t(x)$ of X exists for all $t \geq 0$ and*

$$\{\phi_t(x) : t \geq 0\} \subseteq K$$

for some compact set $K \Subset U$. Then there exists a periodic orbit $C \subseteq K$ such that $\omega(x) = C$. A similar assertion holds for $\alpha(x)$.

Remark A.16. The planar assumption is crucial here; it is certainly possible for a vector field on $U = \mathbb{T}^2$ to have no periodic orbits.

Before proving Theorem A.15, we first state a key lemma which indicates the essential role of the planar assumption.

Lemma A.17. *Let X be a smooth, nonvanishing vector field on a planar domain $U \subseteq \mathbb{R}^2$. Let γ be a nonperiodic orbit of X which exists in U for all t . Suppose that*

$$V = \{(s, r) : -\delta_1 < s < \delta_2, -\delta_2 < r < \delta_2\}$$

is a local flow box, so that $X = \partial_s$ in V and

$$\Sigma := \{(s, r) : s = 0, -\delta_2 < r < \delta_2\}$$

is the vertical transversal. Let $0 \leq t_1 < t_2 < \dots$ be a sequence of times such that $z_j := \gamma_{t_j} \in \Sigma$ and $\gamma_t \notin \Sigma$ for all $t \in (t_{j-1}, t_j)$. Then either

$$z_1 < z_2 < z_3 < \dots$$

or

$$z_1 > z_2 > z_3 > \dots.$$

Here we identify Σ with its parametrisation interval $(-\delta_2, \delta_2)$.

Proof. Let $z_0 < z_1$. Suppose on the contrary that $z_2 < z_1$. There are two cases to discuss.

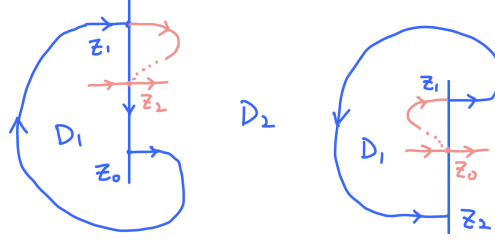
Case I: $z_2 \in (z_0, z_1)$.

Let Γ_{01} be the concatenation of $\gamma_{(t_0, t_1)} \subseteq \Sigma^c$ and the arc connecting z_1 to z_0 along Σ . Then Γ_{01} is a simple closed curve. According to the Jordan curve

theorem, $\Gamma_{0,1}$ divides the plane into a bounded domain D_1 and an unbounded domain D_2 . Since $\gamma_{t_1+\varepsilon} \in D_2$ for small ε , one knows by connectedness that $\gamma|_{(t_1,t_2)} \subseteq D_2$. However, since $z_2 \in (z_0, z_1)$, one must have $\gamma_{t_2-\varepsilon} \in D_1$ due to the flow line pattern inside the flow box V . This is a contradiction.

Case II: $z_2 < z_0$.

Let Γ_{12} be the concatenation of $\gamma_{(t_1,t_2)} \subseteq \Sigma^c$ and the arc connecting z_2 to z_1 along Σ . Then Γ_{12} divides the plane into a bounded domain D_1 and an unbounded one D_2 . Note that $\gamma_{t_1-\varepsilon} \in D_1$ for small ε and thus $\gamma_{(t_0,t_1)} \subseteq D_1$. This is a contradiction since $\gamma_{t_0+\varepsilon} \in D_2$ for small ε .



Therefore, one concludes that $z_2 > z_1$. The case when $z_0 > z_1$ is treated in a similar way. $\square$

Lemma A.17 has the following two important corollaries, which will be used in the proof of Theorem A.15. A point x is said to be *positively recurrent* (respectively, *negatively recurrent*) if $x \in \omega(x)$ (respectively, if $x \in \alpha(x)$).

Lemma A.18. *Let X be a smooth, nonvanishing vector field on a planar domain $U \subseteq \mathbb{R}^2$. Let $y \in U$ be a positively recurrent point of X and assume that $\phi_t(y)$ exists in U for all time. Then $\mathcal{O}_y$ is a periodic orbit.*

Proof. Suppose on the contrary that $\mathcal{O}_y$ is a nonperiodic orbit. Let V be a flow box centered at y . Since $y \in \omega(y)$, there exists a sequence $t_n \rightarrow \infty$ such that $\phi_{t_n}(y) \rightarrow y$. Inside the flow box V , one may adjust the time t_n to some s_n (for all large n , say $n > N$) so that

$$s_n \rightarrow \infty, \quad \phi_{s_n}(y) \in \Sigma \text{ are all distinct,} \quad \phi_{s_n}(y) \rightarrow y.$$

Note that it takes a uniformly positive amount of time for the flow line to leave the flow box once it is on Σ . As a result, there are at most finitely many return times to Σ between 0 and s_{N+1} . It follows from Lemma A.17 that the sequence

$$y = \phi_0(y), \phi_{s_{N+1}}(y), \phi_{s_{N+2}}(y), \dots$$

is monotone on Σ . This contradicts the fact that $\phi_{s_n}(y) \rightarrow y$ as $n \rightarrow \infty$. Therefore, $\mathcal{O}_y$ is a periodic orbit. $\square$

Lemma A.19. *Let X be a smooth nonvanishing vector field on a planar domain $U \subseteq \mathbb{R}^2$. Let $x \in U$ be given fixed. Suppose that the flow $\phi_t(x)$ of X exists for all $t \geq 0$. Let C be a periodic orbit such that $C \cap \omega(x) \neq \emptyset$. Then $\omega(x) = C$.*

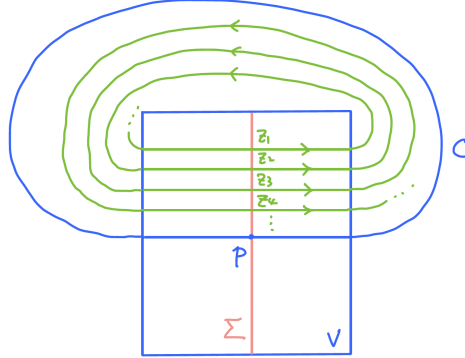
Proof. If $\mathcal{O}_x$ is periodic, one trivially has $\omega(x) = \mathcal{O}_x = C$. We therefore assume that $\mathcal{O}_x$ is nonperiodic. By assumption, let $p \in C \cap \omega(x)$. Then $\phi_{t_n}(x) \rightarrow p$ for some $t_n \rightarrow \infty$. We first claim that $C \subseteq \omega(x)$. Let $q \in C$. One can write $q = \phi_t(p)$ for some $t \geq 0$. It follows that $\phi_{t_n+t}(x) \rightarrow q$, which implies that $q \in \omega(x)$. Therefore, $C \subseteq \omega(x)$.

Conversely, let $q \notin C$. Our aim is to show that $q \notin \omega(x)$. Since C is compact, it suffices to show that

$$\lim_{t \rightarrow \infty} d(\phi_t(x), C) = 0. \quad (\text{A.10})$$

To this end, let V be a flow box centered at p with transversal Σ . Since $\phi_{t_n}(x) \rightarrow p$ and $\mathcal{O}_x$ is nonperiodic, similar to the proof of Lemma A.18, one can find a sequence $s_n \rightarrow \infty$ such that the following properties hold true:

- (i) $z_n := \phi_{s_n}(x) \in \Sigma$ and $\phi_{s_n}(x) \rightarrow p$;
- (ii) $\phi_t(x) \notin \Sigma$ for $t \in (s_{n-1}, s_n)$;
- (iii) the sequence $z_1, z_2, z_3, \dots$ is monotone on Σ .



Consequently, one has

$$z_n \downarrow p \quad \text{or} \quad z_n \uparrow p \quad \text{on } \Sigma. \quad (\text{A.11})$$

Note that $z_{n+1} = P(z_n)$ for all sufficiently large n , where P is the Poincaré map defined in Section A.1.2. By the continuity of the Poincaré map, one has

$$\lim_{z \rightarrow p} \sup_{0 \leq t \leq \tau(z)} d(\phi_t(z), C) = 0, \quad (\text{A.12})$$

where $\tau(z)$ is the first return time of $\phi_t(z)$ to Σ . The claim (A.10) is now a simple consequence of the monotone convergence property (A.11) and the one-iteration asymptotics (A.12). $\square$

Remark A.20. Lemma A.19 remains valid on the torus $\mathbb{T}^2$. In fact, by cutting $\mathbb{T}^2$ along the periodic orbit C , one obtains an annulus $A \cong S^1 \times [0, 1]$ whose two boundary curves are copies of C and $\mathcal{O}_x \subseteq \text{int} A$ (assuming $\mathcal{O}_x$ is nonperiodic). By thickening A across its boundary to an open annulus U (U is diffeomorphic to a planar domain), one is now reduced to the situation of Lemma A.19.

We are now in a position to prove the Poincaré-Bendixson theorem.

Proof of Theorem A.15. Suppose that

$$\mathcal{O}_x^+ \subseteq K \Subset U.$$

Then $\omega(x) \subseteq K$. Since $\omega(x)$ is nonempty, compact and W -invariant, it contains a minimal compact, W -invariant subset (a nonempty, compact W -invariant subset which contains no proper, nontrivial invariant subsets), say C . Let $y \in C$. Then

$$\omega(y) \subseteq C \implies \omega(y) = C$$

since C is minimal. In particular, $y \in \omega(y)$. According to Lemma A.18, $\mathcal{O}_y$ is a periodic orbit. Hence $\omega(x)$ contains a periodic orbit. It follows from Lemma A.19 that $\omega(x) = C$. □

Remark A.21. Here is a simple variant of the Poincaré-Bendixson theorem. Let W be a smooth, nonvanishing vector field on a compact surface M without boundary. Suppose that there exists

$$K \text{ compact, } U \text{ open, } K \Subset U \subseteq M$$

and $t_0 \geq 0$, such that U is diffeomorphic to a planar domain and $\phi_t(x) \in K$ for all $t \geq t_0$. Then $\omega(x)$ is a periodic orbit of W . A similar assertion holds for $\alpha(x)$.

We discuss a few applications of the Poincaré-Bendixson theorem. Let W be a smooth, nonvanishing vector field on $\mathbb{T}^2$.

Lemma A.22. *Let $A \cong S^1 \times [0, 1]$ be an embedded annulus in $\mathbb{T}^2$ whose two boundary curves are compact leaves of $\mathcal{F}_W$. Let $x \in \text{int} A$ and assume that $\mathcal{O}_x$ is nonperiodic. Then $\alpha(x)$ and $\omega(x)$ are distinct periodic orbits.*

Proof. Let $U \supseteq A$ be a slightly larger open annulus which is diffeomorphic to a planar domain. According to the Poincaré-Bendixson theorem, both $\alpha(x)$ and $\omega(x)$ are periodic orbits and they are contained in A . Suppose on the contrary that $\alpha(x) = \omega(x) = C$. Let $p \in C$. Then both tails of $\mathcal{O}_x$ are arbitrarily close to C . In particular, there exists $T_0 > 0$ such that $\phi_t(x) \in U$ for all $|t| > T_0$.

Now choose a small flow box centered at p with transversal Σ . Similar to the proof of Lemma A.18, one can find a two-sided sequence

$$-\infty \downarrow s_{-n} < \cdots < s_{-2} < s_{-1} < s_1 < s_2 < \cdots < s_n \uparrow \infty$$

with the following properties:

- (i) $z_n := \phi_{s_n}(x) \in \Sigma$ and $\phi_t \notin \Sigma$ for all $t \in (s_{n-1}, s_n)$ and for all $n \in \mathbb{Z}$;
- (ii) $z_n \rightarrow p$ as $n \rightarrow \pm\infty$.
- (iii) The sequence $\{z_n\}_{n=-\infty}^{\infty}$ is monotone on Σ .

But the second and third properties force the sequence to be constant, which would imply that $\mathcal{O}_x$ is periodic. This leads to a contradiction. Therefore, $\alpha(x)$ and $\omega(x)$ are distinct periodic orbits. $\square$

Remark A.23. Lemma A.22 may fail on the whole torus $\mathbb{T}^2$; it is certainly possible for a nonperiodic orbit to approach the same periodic orbit C in both tails. However, if one cuts $\mathbb{T}^2$ along C to obtain a compact annulus whose two boundary curves are copies of C (and they are considered as distinct periodic orbits), Lemma A.22 is indeed applicable to this situation.

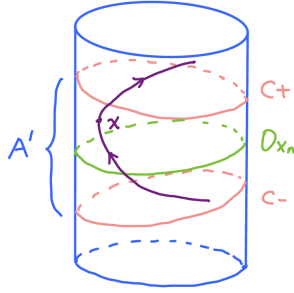
Proposition A.24. (i) Let $A \cong S^1 \times [0, 1]$ be an embedded annulus in $\mathbb{T}^2$ whose two boundary curves are compact leaves of $\mathcal{F}_W$. Let $\mathcal{P}_A := \{x \in A : \mathcal{O}_x \text{ is periodic}\}$. Then $\mathcal{P}_A$ is a closed subset of A .

(ii) Let $\mathcal{P} := \{x \in \mathbb{T}^2 : \mathcal{O}_x \text{ is periodic}\}$. Then $\mathcal{P}$ is closed.

Proof. (i) Let $x_n \in \mathcal{P}_A$ and $x_n \rightarrow x$. Suppose on the contrary that $\mathcal{O}_x$ is nonperiodic. According to Lemma A.22,

$$C_- := \alpha(x), \quad C_+ := \omega(x) \tag{A.13}$$

are distinct periodic orbits on A . Recall from Proposition A.8 that both $C_{\pm}$ are essential, simple closed curves. As a result, they bound a subannulus A' and $x \in \text{int}A'$. Since $x_n \rightarrow x$, there is a large n such that the periodic orbit $\mathcal{O}_{x_n}$ is contained inside $\text{int}A'$, hence dividing $\text{int}A'$ into two connected components. The point x lies inside one of these two components. In order for the orbit $\mathcal{O}_x$ to approach $C_{\pm}$ in both tails (see (A.13)), it must cross the periodic orbit $\mathcal{O}_{x_n}$. Hence $\mathcal{O}_x \cap \mathcal{O}_{x_n} \neq \emptyset$, which is a contradiction. Therefore, $\mathcal{O}_x$ is periodic.



(ii) Let $x_n \in \mathcal{P}$ and $x_n \rightarrow x$. Suppose that $x \notin \mathcal{O}_{x_1}$; otherwise, $\mathcal{O}_x$ is periodic and there is nothing to prove. Cutting $\mathbb{T}^2$ along $\mathcal{O}_{x_1}$ yields a compact annulus A with $x_n \in \mathcal{P}_A$ and $x_n \rightarrow x \in \text{int}A$. The result of Part (i) then implies that $\mathcal{O}_x$ is periodic. $\square$

A.2 Suspension and Reeb component

In this section, we define the concepts of a suspension and a Reeb component, which are the two basic regimes appearing in the main classification theorem.

A.2.1 Suspension

We first define the notion of a suspension. Let $S^1 := \mathbb{R}/\mathbb{Z}$ and let $f : S^1 \rightarrow S^1$ be an orientation-preserving diffeomorphism. We define an equivalence relation on $S^1 \times \mathbb{R}$ by identifying

$$(x, t) \sim (f^k(x), t - k) \quad (\text{A.14})$$

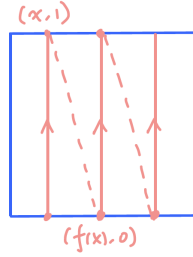
for any $(x, t) \in S^1 \times \mathbb{R}$ and $k \in \mathbb{Z}$. Let $\Sigma_f := (S^1 \times \mathbb{R}) / \sim$ be the quotient space under the above identification, whose quotient map is denoted as $\pi_f : S^1 \times \mathbb{R} \rightarrow \Sigma_f$. It is clear that Σ_f is diffeomorphic to the torus $\mathbb{T}^2$. The image of the vertical leaves $\{x\} \times \mathbb{R}$ under π_f , i.e.

$$\mathcal{F}_f := \{\pi_f(\{x\} \times \mathbb{R}) : x \in S^1\},$$

defines a 1D foliation on Σ_f . This foliation is induced by the vector field

$$\partial_t \big|_{[(x,t)]} := \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} [(x, t + \varepsilon)], \quad [(x, t)] \in \Sigma_f.$$

Definition A.25. The foliation $(\Sigma_f, \mathcal{F}_f)$ is called the *suspension* of the diffeomorphism f . The vector field ∂_t is called the *suspension field*.



Example A.26. (i) Let $f = \text{id}_{S^1}$. The orbits of ∂_t are the vertical circles and they are all periodic with period one.

(ii) (Rational rotation) Let $f(x) := x + p/q \pmod{\mathbb{Z}}$ where p, q are coprime integers ($q > 0$). Then $f^q = \text{id}_{S^1}$. All orbits are periodic with period q .

(iii) (Irrational rotation) Let $f(x) := x + \alpha \pmod{\mathbb{Z}}$ where $\alpha \notin \mathbb{Q}$. Every orbit is diffeomorphic to $\mathbb{R}$ and is dense in $\mathbb{T}^2$.

(iv) Suppose that f has precisely one fixed point $[a] \in S^1$. Then f can be viewed as a diffeomorphism on $(a, a + 1)$ with no fixed points. Suppose further that

$f(x) > x$ for all $x \in (a, a+1)$. Then there is precisely one periodic orbit

$$C : t \mapsto [(a, t)]$$

which has period one. Every orbit other than C is nonperiodic (hence diffeomorphic to $\mathbb{R}$) and approaches C in both ends.

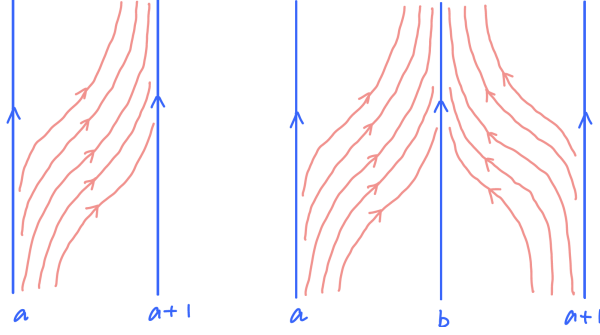


Figure A.1: Suspension examples.

(v) Suppose that f has precisely two fixed points $[a], [b] \in S^1$. With suitable choice of representatives, let us assume that $b \in (a, a+1)$. Suppose that $f(x) > x$ on (a, b) and $f(x) < x$ on $(b, a+1)$. Then there are precisely two periodic orbits

$$C_a : t \mapsto [(a, t)], \quad C_b : t \mapsto [(b, t)].$$

All other orbits are nonperiodic. Figure A.1 sketches the flow lines on the universal cover $\mathbb{R}^2$ for Cases (iv) and (v).

Remark A.27. Periodic orbits of $(\Sigma_f, \mathcal{F}_f)$ correspond to periodic points of f . A point $x \in S^1$ is a *periodic point* of f if there exists a positive integer q such that $f^q(x) = x$. The smallest such q is called the *period* of x . For an orientation-preserving diffeomorphism $f \in \text{Diff}(S^1)$, the period of a periodic point is intrinsic to f and does not depend on the actual point x .

Let W be a smooth, nonvanishing vector field W on $\mathbb{T}^2$. We give a simple criterion under which the foliation $\mathcal{F}_W$ is conjugate to a suspension (see Definition 6.6).

Definition A.28. A smooth, simple closed curve Σ is a *complete transversal* if it intersects every orbit of W transversally.

Proposition A.29. *The foliation $\mathcal{F}_W$ is conjugate to a suspension if and only if there exists a complete transversal.*

Proof. (i) Necessity is obvious. In fact, the horizontal circle

$$S^1 \ni t \mapsto [(t, 0)] \in \Sigma_f$$

is a complete transversal of the suspension foliation. It remains to prove sufficiency.

(ii) Let Σ be a given complete transversal. For each $x \in \Sigma$, we define the first return time

$$\tau(x) := \inf\{t > 0 : \phi_t(x) \in \Sigma\}.$$

We first show that $\tau(x) < \infty$. Suppose on the contrary that $\phi_t(x)$ never meets Σ again at any positive time. We claim that there exists $T_0 > 0$ and a small neighbourhood N of Σ , such that $\phi_t(x) \notin N$ for all $t > T_0$. Indeed, if this were not the case, there would exist a sequence of times $t_n \rightarrow \infty$ such that $\phi_{t_n}(x)$ converges to some $y \in \Sigma$. Draw a flow box $(V; s, r)$ centered at $y = (0, 0)$ so that $\Sigma \cap V = \{s = 0\}$. One can write $\phi_{t_n}(x) = (s_n, r_n)$ under the flow box coordinates and note that $(s_n, r_n) \rightarrow (0, 0)$. Since the flow lines are precisely $s \mapsto (s, r)$ inside V , one has

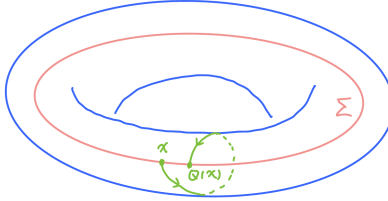
$$\phi_{-s_n}(\phi_{t_n}(x)) = \phi_{t_n - s_n}(x) = (0, r_n) \in \Sigma$$

for all large n . As a result, the time $t_n - s_n$ is a positive return time for large n (because $t_n \rightarrow \infty$ and $s_n \rightarrow 0$) which leads to a contradiction. Therefore, $\{\phi_t(x) : t > 0\}$ is contained outside a tubular neighbourhood N of Σ after some short period of time. Similar to the proof of Proposition A.8 (i), it can be shown that any complete transversal must be essential. As a result, N^c (as a compact annulus) is contained in a planar domain. By applying the Poincaré-Bendixson theorem, one obtains a periodic orbit that is disjoint from Σ . This contradicts the completeness of Σ . Consequently, $\tau(x) < \infty$.

We define the first return map

$$Q : \Sigma \rightarrow \Sigma, \quad Q(x) := \phi_{\tau(x)}(x). \quad (\text{A.15})$$

By realising a neighbourhood of $Q(x)$ on Σ as a level set $\{\rho = 0\}$ for some smooth function ρ , one can apply the same argument as in the proof of Lemma A.9 (construction of the Poincaré map) to conclude that $\tau(x)$ is smooth in x . Therefore, $Q \in C^\infty(\Sigma)$. It is indeed a diffeomorphism since Q^{-1} is defined in the same way by the return-to- Σ map of the inverse flow $-W$. Let $\sigma : S^1 \xrightarrow{\cong} \Sigma$ be an S^1 -parametrisation of Σ . Then $f := \sigma^1 \circ Q \circ \sigma \in \text{Diff}(S^1)$.



(iii) We claim that f is orientation preserving, or more precisely, $f'(r) > 0$ for all r (here we identify $S^1 := \mathbb{R}/\mathbb{Z}$ so that one has the canonical isomorphisms $T_r S^1 \cong T_{f(r)} S^1 \cong \mathbb{R}$). The argument is in spirit similar to the proof of Proposition A.11 (positivity of the transversal characteristic number).

To this end, let Ω be a volume form on $\mathbb{T}^2$ such that

$$g(r) := \Omega(W(\sigma(r)), \dot{\sigma}(r)) > 0$$

for all $r \in S^1$. This is possible since $\mathbb{T}^2$ is orientable and Σ is transversal to W . Let

$$F(s, r) := \phi_{s\tau(\sigma(r))}(\sigma(r)) \quad (\text{A.16})$$

and define

$$J(s, r) := \Omega(W(F(s, r)), \partial_r F(s, r)).$$

It is plain to see that

$$\begin{aligned} \partial_s F(s, r) &= \tau(\sigma(r))W(F(s, r)), \\ \partial_r F(s, r) &= (d\phi_{s\tau(\sigma(r))})\dot{\sigma}(r) + s\tau'(r)W(F(s, r)). \end{aligned} \quad (\text{A.17})$$

Since $\dot{\sigma}(r)$ and $W(\sigma(r))$ are linearly independent, so are

$$(d\phi_{s\tau(\sigma(r))})\dot{\sigma}(r) \quad \text{and} \quad (d\phi_{s\tau(\sigma(r))})W(\sigma(r)) = W(F(s, r)).$$

As a result,

$$J(s, r) = \Omega(W(F(s, r)), (d\phi_{s\tau(\sigma(r))})\dot{\sigma}(r))$$

has constant sign. Note that $J(0, r) = g(r) > 0$. It follows that

$$J(1, r) = \Omega(W(\sigma \circ f(r)), f'(r)\dot{\sigma}(f(r))) = f'(r)g(f(r)) > 0.$$

Therefore, $f'(r) > 0$ and f is orientation preserving.

Now let Σ_f be the suspension of f . Define $\Psi : \Sigma_f \rightarrow \mathbb{T}^2$ by

$$\Psi([r, t]) := \phi_{t\tau(\sigma(r))}(\sigma(r)), \quad r \in S^1, t \in [0, 1].$$

The definition of Ψ is consistent at the junction $t = 0 = 1 \pmod{\mathbb{Z}}$. In addition, Ψ is a homeomorphism and maps the suspension foliation to $\mathcal{F}_W$. The issue is that Ψ may fail to be a diffeomorphism. This is because

$$\frac{\partial}{\partial t} \Big|_{t=1-} \phi_{t\tau(\sigma(r))}(\sigma(r)) \quad \text{and} \quad \frac{\partial}{\partial t} \Big|_{t=0+} \phi_{t\tau(\sigma \circ f(r))}(\sigma \circ f(r))$$

need not coincide. They coincide precisely when $\tau(\sigma(r)) = \tau(\sigma \circ f(r))$, which needs not be true in general.

The above issue can be fixed by a simple leaf reparametrisation. Let

$$\theta(r, t) := ct + (\tau(\sigma(r)) - c)\beta(t), \quad r \in S^1, t \in [0, 1].$$

Here

$$0 < c < \inf_{r \in S^1} \tau(\sigma(r))$$

and $\beta : [0, 1] \rightarrow [0, 1]$ is smooth, nondecreasing and satisfies

$$\beta = 0 \text{ near } 0, \quad \beta = 1 \text{ near } 1.$$

Then one has

$$\begin{aligned}\theta(r, 0) &= 0, & \theta(r, 1) &= \tau(\sigma(r)), \\ \theta(r, t) &= ct \text{ near } t = 0, \\ \theta(r, t) &= \tau(\sigma(r)) + c(t - 1) \text{ near } t = 1.\end{aligned}$$

This ensures that all t -derivatives match at the junction $t = 0 = 1 \pmod{\mathbb{Z}}$. Now we adjust the definition of Ψ to be

$$\hat{\Psi} : \Sigma_f \rightarrow \mathbb{T}^2, \quad \hat{\Psi}([r, t]) := \phi_{\theta(r, t)}(\sigma(r)).$$

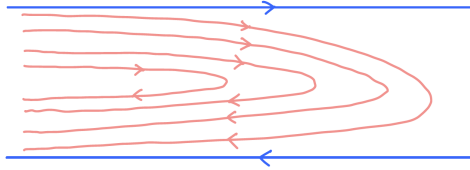
This new map $\hat{\Psi}$ is now easily seen to be a diffeomorphism, which provides a smooth conjugation between Σ_f and $\mathbb{T}^2$. $\square$

A.2.2 Reeb component

Next, we define the notion of a Reeb component.

Definition A.30. We say that $\mathcal{F}_W$ contains a Reeb component if there exists an embedded annulus $A \cong S^1 \times [0, 1]$ inside $\mathbb{T}^2$ such that the following two properties are satisfied:

- (i) ∂A is the union of two compact leaves that are in opposite homology classes (under the W -orientation);
- (ii) there are no compact leaves of $\mathcal{F}_W$ in the interior of A .



The following proposition provides a simple criterion for the existence of a Reeb component.

Proposition A.31. *The foliation $\mathcal{F}_W$ has a Reeb component if and only if there exist two periodic orbits with opposite homology classes.*

Proof. Necessity follows immediate from the definition and Proposition A.8. We now prove sufficiency. Let C_+, C_- be two periodic orbits with opposite homology classes. Let A be a compact annulus bounded by $C_{\pm}$. Define $\mathcal{P}_A^{\pm}$ to be the set of points in A whose orbit is homologous to $C_{\pm}$. According to Proposition A.24, it is easily seen that both $\mathcal{P}_A^{\pm}$ are closed subsets of A (a sequence of periodic orbits with the same homology class cannot converge to a periodic orbit with an opposite direction). Therefore,

$$\inf_{x \in \mathcal{P}_A^+, y \in \mathcal{P}_A^-} d(x, y) > 0.$$

Let $x_* \in \mathcal{P}_A^+$ and $y_* \in \mathcal{P}_A^-$ be two points achieving the above infimum. It follows that there are no periodic orbits between $\mathcal{O}_{x_*}$ and $\mathcal{O}_{y_*}$ inside A . As a consequence, they bound a Reeb component. $\square$

A.2.3 Mutual exclusiveness of the two regimes

We now show that suspension and Reeb component are mutually exclusive.

Proposition A.32. *Suppose $\mathcal{F}_W$ is conjugate to a suspension. Then it cannot contain a Reeb component.*

Proof. Let $\Psi : \mathcal{F}_W \rightarrow \Sigma_f$ be a conjugation to a suspension

$$\Sigma_f = \{[x, t] : x \in S^1, t \in \mathbb{R}, (x, t) \sim (f^k(x), t - k)\}.$$

Let $\eta := \Psi^* dt$, where dt is the vertical form on Σ_f defined by

$$\langle dt, \partial_x \rangle = 0, \quad \langle dt, \partial_t \rangle = 1.$$

The image of a periodic orbit γ of W under Ψ is a reparametrisation of a periodic orbit β of the suspension field ∂_t . The period of β is a positive integer q , whose value is uniquely determined by f and is dependent of the choice of γ (cf. Remark A.27). It follows that

$$\int_{\gamma} \eta = \int_{\beta} dt = q.$$

As a consequence, having two periodic orbits with opposite homology classes is not possible in this case. $\square$

A.3 The main classification theorem

In this section, we state and prove the main classification theorem for nondegenerate torus flows.

Theorem A.33. *Let $\mathcal{F}_W$ be the 1D foliation associated with a smooth, nonvanishing vector field W on $\mathbb{T}^2$. Then precisely one of the following two mutually exclusive possibilities holds:*

- (i) $\mathcal{F}_W$ is conjugate to a suspension;
- (ii) $\mathcal{F}_W$ contains a Reeb component.

In what follows, we develop the proof of Theorem A.33. We will consider two cases separately according to whether W has periodic orbits.

A.3.1 Case I: No periodic orbits

We first assume that W does not have periodic orbits (in particular, there are no Reeb components). Our aim is to prove that $\mathcal{F}_W$ is conjugate to a suspension in this case. According to Proposition A.29, it is equivalent to showing that there exists a complete transversal.

Lemma A.34. *Suppose that there exists a nonperiodic orbit. Then there exists a smooth, simple closed curve Σ that is everywhere transversal to W .*

Proof. Let γ be a nonperiodic orbit. There exists $t_n \rightarrow \infty$ such that γ_{t_n} converges to some y . By drawing a flow box V around y and adjusting t_n if necessary (cf. Proof of Lemma A.18), one can find $s_0 < s_1$ such that $z_i = \gamma_{s_i} \in \Sigma$ ($i = 0, 1$) and $\gamma_t \notin \Sigma$ for all $t \in (s_0, s_1)$. Here Σ denotes the vertical transversal in V passing through y . Note that $z_0 \neq z_1$.

Let us fix a parametrisation $\sigma : (-c, c) \rightarrow \Sigma$ so that $z_0 = \sigma(0)$ and $z_1 = \sigma(d)$ for some $d \in (0, c)$. Define the first return map $f : (-\delta, \delta) \rightarrow (-c, c)$ by the relation

$$\phi_{\tau(r)}(\sigma(r)) = \sigma(f(r)), \quad r \in (-\delta, \delta),$$

where δ is a sufficiently small number and $\tau(r)$ is the first return time of $\phi_t(\sigma(r))$ to Σ (cf. (A.15)). Note that $f(0) = d$. Consider the function

$$F(s, r) := \phi_{s\tau(r)}(\sigma(r))$$

where $s \in [0, 1]$ and $r \in (-\delta, \delta)$ (cf. (A.16)). It is clear that

$$F(0, r) = \sigma(r), \quad F(1, r) = \sigma(f(r)).$$

By shrinking δ if necessary, the map

$$F : [0, 1] \times (-\delta, \delta) \rightarrow \mathbb{T}^2$$

is injective and

$$F((0, 1) \times (-\delta, \delta)) \cap \Sigma = \emptyset.$$

Similar to the proof of Proposition A.29, one can show that $f'(r) > 0$ for all $r \in (-\delta, \delta)$. As a result, there exists $\eta \in (-\delta, 0)$ such that $f(\eta) > 0$.

Fix a function $\rho : [0, 1] \rightarrow [\eta, 0]$ such that $\rho(0) = 0$, $\rho(1) = \eta$ and $\rho'(s) < 0$ for all $s \in [0, 1]$. Define the curve

$$\alpha_s := F(s, \rho(s)), \quad 0 \leq s \leq 1.$$

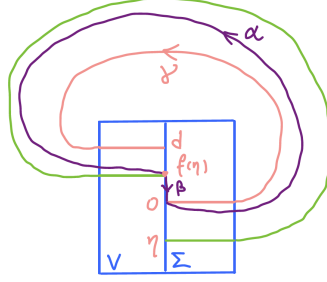
Note that α is a simple curve that joins z_0 to $\sigma(f(\eta))$ outside Σ . By definition, one has

$$\dot{\alpha}_s = \partial_s F(s, \rho(s)) + \rho'(s) \partial_r F(s, \rho(s)).$$

The vector $\partial_s F(s, \rho(s)) \in \text{Span}\{W(\alpha_s)\}$. In addition, since $\rho'(s) < 0$, the vector $\partial_r F(s, \rho(s))$ is linearly independent of $\partial_s F(s, \rho(s))$ (cf. (A.17)). As a consequence, α is everywhere transversal to W . Let $\beta : [1, 2] \rightarrow \Sigma$ be the arc joining $\sigma(f(\eta))$ back to z_0 along Σ . Apparently, β is also everywhere transversal to W . The concatenated curve

$$\zeta_s := \begin{cases} \alpha_s, & 0 \leq s \leq 1; \\ \beta_s, & 1 \leq s \leq 2 \end{cases}$$

is now a simple closed curve, which is smooth and everywhere transversal to W except at the two corners z_0 and $\sigma(f(\eta))$.



It remains to see that the two corners can be smoothed locally without losing the transversal property. Let Ω be a volume form on $\mathbb{T}^2$ such that $\Omega(W(z_0), \dot{\sigma}(0)) > 0$. The function

$$\psi(s) := \Omega(W(\zeta_s), \dot{\zeta}_s)$$

has constant sign on $(0, 1)$ and $(1, 2)$. We claim that

$$\text{sgn}\psi(1-) = \text{sgn}\psi(1+) = \text{sgn}\psi(2-) = \text{sgn}\psi(0+). \quad (\text{A.18})$$

Indeed, since $\rho'(0) < 0$ one has

$$\begin{aligned} \text{sgn}\psi(1-) &= \text{sgn}\psi(0+) = \text{sgn}\Omega(W(z_0), \rho'(0)\dot{\sigma}(0)) \\ &= -\text{sgn}\Omega(W(z_0), \dot{\sigma}(0)). \end{aligned}$$

In addition, by the definition of β one also has

$$\text{sgn}\psi(1+) = \text{sgn}(2-) = -\text{sgn}\Omega(W(z_0), \dot{\sigma}(0))$$

As a consequence, the relation (A.18) holds. This relation shows that at any of the two corners $s_* = 0, 1$, both vectors $\dot{\zeta}_{s_*-}$ and $\dot{\zeta}_{s_*+}$ ($0-$ is the same as $2-$) lie in the cone

$$C_{s_*} := \{v : \Omega(W(\zeta_{s_*}), v) < 0\}.$$

It is clearly possible to smooth the corner ζ_{s_*} locally in a way such that the resulting new smooth arc $\tilde{\zeta}_s$ still satisfies $\Omega(W(\tilde{\zeta}_s), v) < 0$ when s is near s_* . One thus obtains a smooth, simple closed curve which is everywhere transversal to W . $\square$

Proof of Theorem A.33: No periodic orbits case. Suppose that W has no periodic orbits. According to Lemma A.34, there exists a smooth, simple closed transversal curve Σ . If Σ does not intersect some orbit γ , the tails of γ will be trapped outside a tubular neighbourhood of Σ . The Poincaré-Bendixson theorem then produces a periodic orbit, which leads to a contradiction (cf. Part (ii) in the proof of Proposition A.29). Therefore, Σ is complete. It follows from Proposition A.29 that $\mathcal{F}_W$ is conjugate to a suspension. $\square$

A.3.2 Case II: At least one periodic orbit

Now we assume that W has at least one periodic orbit. In this case, we aim at showing that

$$\mathcal{F}_W \text{ has no Reeb component} \implies \mathcal{F}_W \text{ is conjugate to a suspension.}$$

The key lemma for this part is stated as follows.

Lemma A.35. *Let $A \cong S^1 \times [0, 1]$ be an embedded annulus whose boundary consists of two compact leaves C_0 and C_1 . Suppose that A does not contain any Reeb component. Then there exists a smooth, simple curve $c : [0, 1] \rightarrow A$ such that $c_0 \in C_0$, $c_1 \in C_1$ and c is everywhere transversal to W on A .*

Proof. First of all, the periodic orbits C_0, C_1 must have the same homology class; otherwise, there would be a Reeb component inside A according to Proposition A.31, which contradicts the assumption. Let Ω be a volume form on A such that $\Omega(W(x), v) > 0$ when $x \in C_0$ and $v \in T_x \mathbb{T}^2$ is a transversal vector pointing into A . A vector $v \in T_y \mathbb{T}^2$ is said to be *positively transversal* to W at $y \in A$ if $\Omega(W(y), v) > 0$. A smooth curve c_t in A is *positively transversal* to W if $\Omega(W(c_t), \dot{c}_t) > 0$ for all t . Let us define

$$\mathcal{R} := \{y \in A : \exists \text{ a smooth, simple, positively transversal } c : [0, 1] \rightarrow A \text{ s.t. } c_0 \in C_0, c_1 = y\}.$$

Our aim is to show that $\mathcal{R} \cap C_1 \neq \emptyset$. We divide the argument into several steps.

(i) Let N_1 be an open half-tubular neighbourhood of C_1 in A (a lower collar). We claim that $\mathcal{R} \cap N_1 \neq \emptyset$. Suppose this is true and let $y \in \mathcal{R} \cap N_1$. By definition, there is a simple, positively transversal curve c from C_0 to y . By travelling along the fiber in N_1 , one can easily continue c to the boundary C_1 . The resulting curve is smooth and positively transversal to W except at the junction. Similar to the proof of Lemma A.34, one can smooth the corner to obtain a smooth, simple, positively transversal curve from C_0 to C_1 . Now it remains to show that $\mathcal{R} \cap N_1 \neq \emptyset$. In what follows, we assume on the contrary that $\mathcal{R} \subseteq A \setminus N_1$.

(ii) We first show that $\mathcal{R} \cap \text{int} A$ is an open and nonempty subset of A . Indeed, any point inside a half-tubular neighbourhood of C_0 in A (an upper collar) is clearly reachable from C_0 . As a result, $\mathcal{R} \cap \text{int} A \neq \emptyset$. Next, let $y \in \mathcal{R} \cap \text{int} A$. Draw a flow box $(V; s, r) \subseteq \text{int} A$ centered at $y = (0, 0)$. We assume that the transversal coordinate r is chosen such that a curve r_t being positively transversal means $\dot{r}_t > 0$. Let c be a smooth, simple and positively transversal arc from C_0 to y . Let $z \in V \cap c$. By positive transversality, one must have $r(z) < 0$. It follows that all points in

$$V' := \{x \in V : r(x) > r(z)\}$$

is reachable from z by a simple positive transversal. By smoothing the corner if necessary, one concludes that $V' \subseteq \mathcal{R}$. Therefore, $\mathcal{R} \cap \text{int} A$ is an open subset of A .

(iii) Define $\partial_A \mathcal{R} := \bar{\mathcal{R}} \setminus \mathcal{R}$. Note that $\partial_A \mathcal{R} \neq \emptyset$ under the current assumption that $\mathcal{R} \subseteq A \setminus N_1$, since $\mathcal{R} \cap \text{int} A$ is open. We claim that $\partial_A \mathcal{R}$ is a single compact leaf of $\mathcal{F}_W$.

To this end, let $y \in \partial_A \mathcal{R}$. Draw a flow box $(V; s, r)$ centered at $y = (0, 0)$, whose transversal coordinate r is oriented in the same way as in Part (ii). The key observation is that

$$V \cap \mathcal{R} = V \cap \{r > 0\}. \quad (\text{A.19})$$

Let $z \in V \cap \{r > 0\}$. Since $y \in \partial_A \mathcal{R}$ and $\mathcal{R}$ is open in $\text{int} A$, there exists $y' \in V \cap \mathcal{R}$ such that $0 < r(y') < r(z)$. It follows that z is reachable by a positive transversal from y' . Upon smoothing the corner at y' , one concludes that $z \in \mathcal{R}$. To prove the other direction, one observes that $V \cap \{r < 0\}$ is not reachable; otherwise, y would be reachable which is a contradiction. This also implies that $V \cap \{r = 0\}$ is not reachable, since the only way to reach $V \cap \{r = 0\}$ is through $V \cap \{r < 0\}$. As a consequence, $V \cap \{r \leq 0\} \subseteq \mathcal{R}^c$. The relation (A.19) thus follows. An immediate corollary of (A.19) is that $V \cap \partial_A \mathcal{R} = V \cap \{r = 0\}$. In particular, $\partial_A \mathcal{R}$ is a 1D submanifold of A .

The next observation is that $\partial_A \mathcal{R}$ is a disjoint union of leaves. Indeed, let $x \in \partial_A \mathcal{R}$ and define

$$I_x := \{t \in \mathbb{R} : \phi_t(x) \in \partial_A \mathcal{R}\}.$$

Since $\partial_A \mathcal{R}$ is closed in A , it is obvious that I_x is closed. To prove that I_x is open, let $y = \phi_t(x) \in \partial_A \mathcal{R}$. By applying the local relation (A.19) to y , one sees that $\phi_s(x) \in \partial_A \mathcal{R}$ for s near t (because $V \cap \{r = 0\}$ is a flow line). Therefore, I_x is both open and closed, hence being equal to $\mathbb{R}$.

Since $\partial_A \mathcal{R}$ is a closed, embedded 1D submanifold of the compact annulus A , every connected component of $\partial_A \mathcal{R}$ has to be compact. As a consequence, $\partial_A \mathcal{R}$ is a disjoint union of compact leaves. Suppose that there are at least two such leaves, say C', C'' (we assume that C' is closer to C_0). Since a small lower collar of C'' is reachable, any such positive transversal would intersect C' . But $C' \subseteq \partial_A \mathcal{R} \subseteq \mathcal{R}^c$. This leads to a contradiction. Therefore, $\partial_A \mathcal{R}$ has only one connected component, i.e. a compact leaf.

(iv) Let $\partial_A \mathcal{R} = C$ for some compact leaf C according to Part (iii). We claim that C and C_0 are in opposite homology classes. Suppose this is not true. Due to our orientation convention, positively transversal vectors at $y \in C$ are pointing towards the C_1 -side. Let N be a tubular neighborhood of C . Note that C divides N into two halves: N_+ (the component on the C_1 -side) and N_- (the component on the C_0 -side). Let $z \in N_+$ and $y \in C = \partial_A \mathcal{R}$. Then a sufficiently close-to- y element $y' \in N_-$ is reachable by a positive transversal from C_0 . By continuing this transversal towards z inside N in the obvious way (along fiber directions towards C_1), one obtains a positive transversal that will intersect C . This contradicts the fact that $C \subseteq \mathcal{R}^c$. Therefore, C and C_0 must have opposite directions. It follows from Proposition A.31 that there is a Reeb component inside the annulus bounded by C_0 and C . This contradicts the assumption of the lemma.

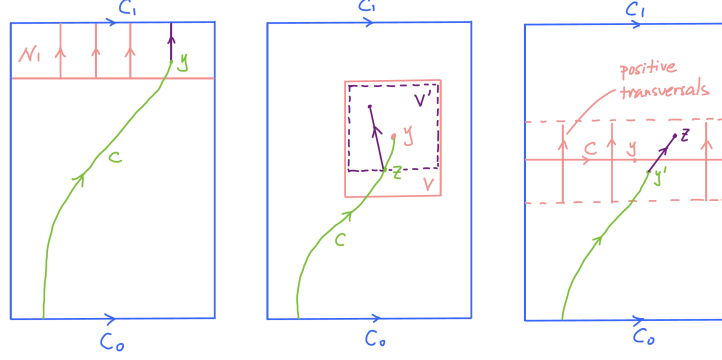
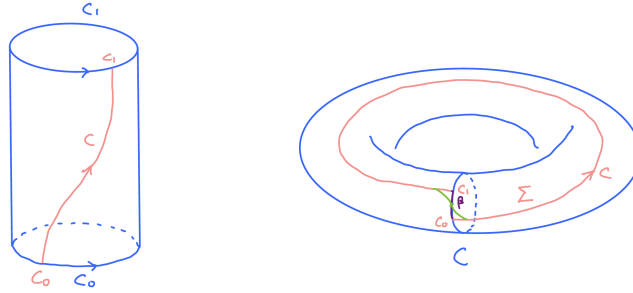


Figure A.2: The left, middle and right figures correspond to Steps (i), (ii) and (iv) respectively.

To summarise, one arrives at the conclusion that the assumption $\mathcal{R} \subseteq A \setminus N_1$ cannot hold. In other words, one must have $\mathcal{R} \cap N_1 \neq \emptyset$. The proof of the lemma is now complete. $\square$

Proof of Theorem A.33: At least one periodic orbit case. Suppose that $\mathcal{F}_W$ has no Reeb component and there is a periodic orbit C of W . Cutting the torus $\mathbb{T}^2$ along C produces a compact annulus A whose boundary consists of two compact leaves C_0 and C_1 (they are copies of C). Since A does not contain any Reeb component, according to Lemma A.35, there exists a smooth, simple transversal $c : [0, 1] \rightarrow A$ from $c_0 \in C_0$ to $c_1 \in C_1$. By gluing back the torus along C , the curve c becomes a smooth, simple transversal such that $c_0, c_1 \in C$ and $c_t \notin C$ for all $t \in (0, 1)$. Let β be the arc joining c_1 to c_0 along C .

Similar to the proof of Lemma A.34, by suitably deforming β along transversal directions (e.g. using the fiber coordinate in a tubular neighbourhood of C) and corner smoothing, one obtains a smooth, simple closed transversal Σ that intersects C precisely once. In particular, $I(\Sigma, C) = \pm 1$.



Suppose that Σ is not complete. The Poincaré-Bendixson theorem then produces a periodic orbit C' which does not intersect Σ . In particular, $I(\Sigma, C') = 0$.

But one knows from Proposition A.8 that $[C'] = \pm[C]$. This leads to a contradiction. Therefore, Σ is a complete transversal. It follows from Proposition A.29 that $\mathcal{F}_W$ is conjugate to a suspension. $\square$

A.4 The dynamical picture of nondegenerate torus flow

Let us summarise the dynamical picture of a nondegenerate torus flow. Let W be a smooth, nonvanishing vector field on $\mathbb{T}^2$. Recall from Proposition A.24 that the set

$$\mathcal{P} := \{x : \mathcal{O}_x \text{ is periodic}\}$$

is closed. As a result, one can decompose $\mathbb{T}^2 \setminus \mathcal{P}$ into an at most countable union of disjoint, connected, open subsets U_α :

$$\mathbb{T}^2 \setminus \mathcal{P} = \bigsqcup_{\alpha} U_{\alpha}.$$

Lemma A.36. *Suppose that $\mathcal{P} \neq \emptyset, \mathbb{T}^2$. Then each U_{α} is an open annulus whose boundary consists of two periodic orbits (which may coincide if $\mathcal{P}$ is a single periodic orbit). All orbits inside U_{α} are nonperiodic.*

Proof. Fix a periodic orbit C_* and cut the torus along C_* . This produces a compact annulus $A \cong S^1 \times [0, 1]$ whose boundary consists of two copies C_0, C_1 of C_* and whose interior is diffeomorphic to $\mathbb{T}^2 \setminus C$ (we view C_0, C_1 as distinct periodic orbits). Let $\mathcal{C}$ be the collection of periodic orbits on A . A given periodic orbit $C \in \mathcal{C} \setminus \{C_0, C_1\}$ divides A into two connected components: $A_-(C)$ (the one containing C_0) and $A_+(C)$ (the one containing C_1). As a convention, $A_-(C_0) = \emptyset$ and $A_+(C_0) = A \setminus C_0$ (respectively, $A_-(C_1) = A \setminus C_1$ and $A_+(C_1) = \emptyset$).

Now let U_{α} be a connected component of $A \setminus \mathcal{P}$. Given $x \in U_{\alpha}$, we define

$$\begin{aligned} \mathcal{P}_-(x) &:= \{C \in \mathcal{C} : x \in A_+(C)\}, \\ \mathcal{P}_+(x) &:= \{C \in \mathcal{C} : x \in A_-(C)\} \end{aligned}$$

respectively. Let

$$S_- := \bigcup_{C \in \mathcal{P}_-(x)} C$$

and define

$$d_- := \inf_{y \in S_-} d(x, y).$$

Note that $d_- \in (0, \infty)$ because S_- is nonempty ($C_0 \in S_-$) and $U_{\alpha} \subseteq \mathcal{P}^c$ is open. Let $y_n \in S_-$ be a sequence such that $d(x, y_n) \rightarrow d_-$. Since A is compact, by passing to a subsequence one can find some z with $y_n \rightarrow z$ and thus $d(x, z) = d_-$. It follows from Proposition A.24 that $C_-(x) := \mathcal{O}_z$ is a periodic orbit and one has $C_-(x) \in \mathcal{P}_-(x)$. One can identify another extremal periodic orbit $C_+(x) \in \mathcal{P}_+(x)$ in a similar way. The two orbits $C_-(x)$ and $C_+(x)$ bound an annulus whose interior is precisely U_{α} . By construction, there are no periodic orbits inside U_{α} . $\square$

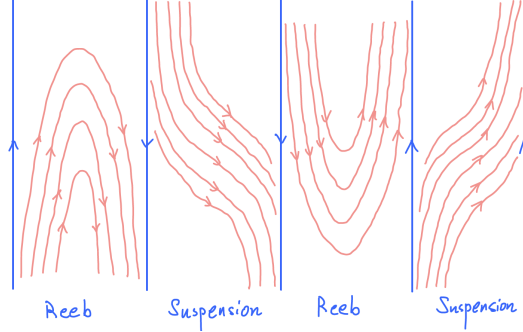


Figure A.3: Dynamical picture of torus flow.

Suppose that $\mathcal{P}$ is not a single periodic orbit nor the whole torus. Let $\partial U_\alpha = C_1 \sqcup C_2$ and let $x \in U_\alpha$. According to Lemma A.22, one has either $\alpha(x) = C_1$, $\omega(x) = C_2$ or $\alpha(x) = C_2$, $\omega(x) = C_1$. In other words, every orbit inside U_α approaches the two distinct boundary leaves in both ends. If C_1, C_2 have opposite homology classes, $\overline{U}_\alpha$ is a Reeb component. In this case, it is clear from the proof of Proposition A.31 that there is at least another Reeb component inside $\overline{U}_\alpha^c$.

On the other hand, if C_1, C_2 have the same orientation, $\overline{U}_\alpha$ is conjugate to a *suspension annulus*. The notion of a suspension annulus is similar to a suspension, except that one does not close the circle. More precisely, let $f : [0, 1] \rightarrow [0, 1]$ be a diffeomorphism such that $f(0) = 0$ and $f(1) = 1$. Define the quotient space

$$\Xi_f := ([0, 1] \times \mathbb{R}) / \sim$$

where the equivalence relation $\sim$ is defined by (A.14). The foliation of vertical leafs $t \mapsto [(x, t)]$ is the suspension annulus with respect to f . Topologically, Ξ_f is a compact annulus whose boundary curves are the two periodic leaves

$$t \mapsto [(0, t)], \quad t \mapsto [(1, t)].$$

Suppose that ∂U_α consists of two periodic orbits with the same homology class. By using the same argument as in Section A.3.1, one can show that $\overline{U}_\alpha$ is conjugate to a suspension annulus.

If $\mathcal{P} = C$ is a single periodic orbit, the torus $\mathbb{T}^2$ is conjugate to a suspension of some $f \in \text{Diff}^+(S^1)$ with exactly one periodic point. Every nonperiodic orbit approaches C in both tails. If $\mathcal{P} = \emptyset$, every orbit is nonperiodic and none of their tails will be trapped in a compact subset of a planar region. In this case, $\mathcal{F}_W$ is also conjugate to a suspension whose suspension map $f \in \text{Diff}^+(S^1)$ does not have periodic points.

A typical dynamical picture of the foliation $\mathcal{F}_W$ (viewed on the universal cover $\mathbb{R}^2$) is sketched in Figure A.3.

Remark A.37. It is possible to have a continuum family of periodic orbits, e.g. $\mathcal{P}$ can be a Cantor-like set.

Acknowledgement

S. Wang gratefully acknowledges the support by the European Research Council (ERC) through the project *Stochastic Quantum Gauge Theories* (SQGT), Grant Agreement No. 101116964.

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